SDE Solution

December 27, 2022

0.1 Compare Euler and Miltstein Solver

(a) Visualization Simulate:

$$dX_t = aX_t dt + bX_t dW_t, X_0 = 1$$

whose exact solution is:

$$X_t = \exp\left[(a - \frac{b^2}{2})t + bW_t\right]$$

Recall Euler scheme:

$$X_{n+1} = X_n + aX_n\Delta t + bX_n(W_{n+1} - W_n), W_{n+1} - W_n = \sqrt{\Delta t}Z, Z \sim \mathcal{N}(0, 1)$$

And Milstein scheme:

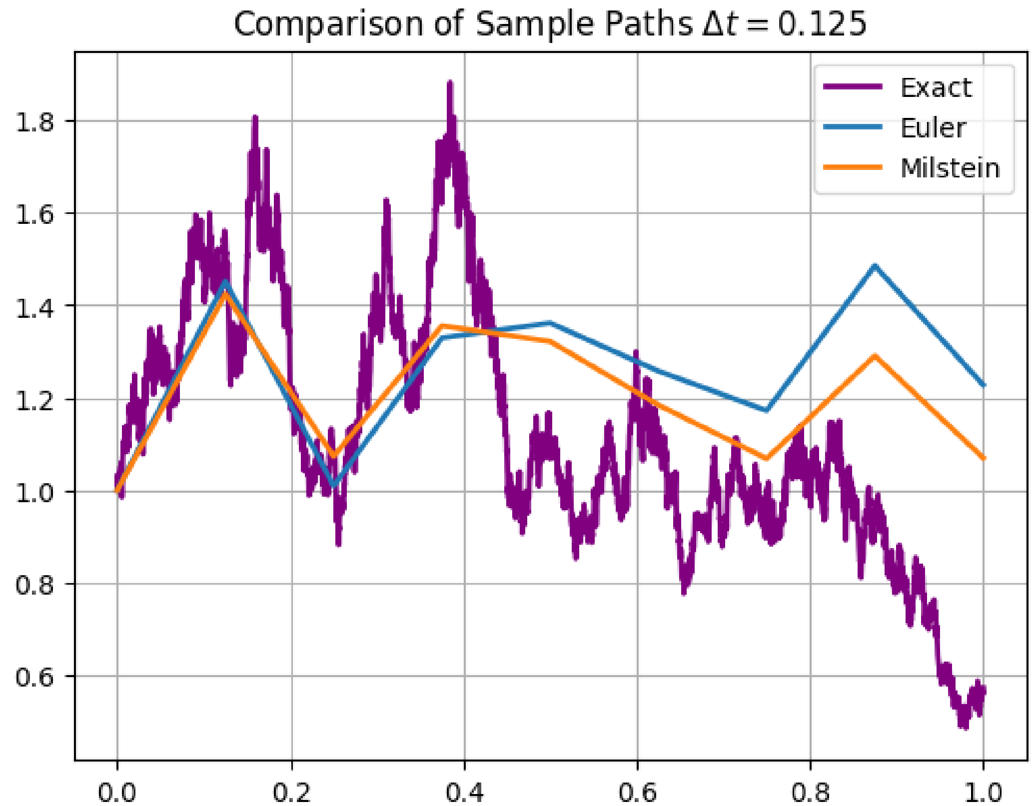
$$X_{n+1} = X_n + aX_n\Delta t + bX_n\Delta W_n + \frac{1}{2}b^2X_n((\Delta W_n)^2 - \Delta t)$$

where $\Delta W_n = W_{n+1} - W_n$.

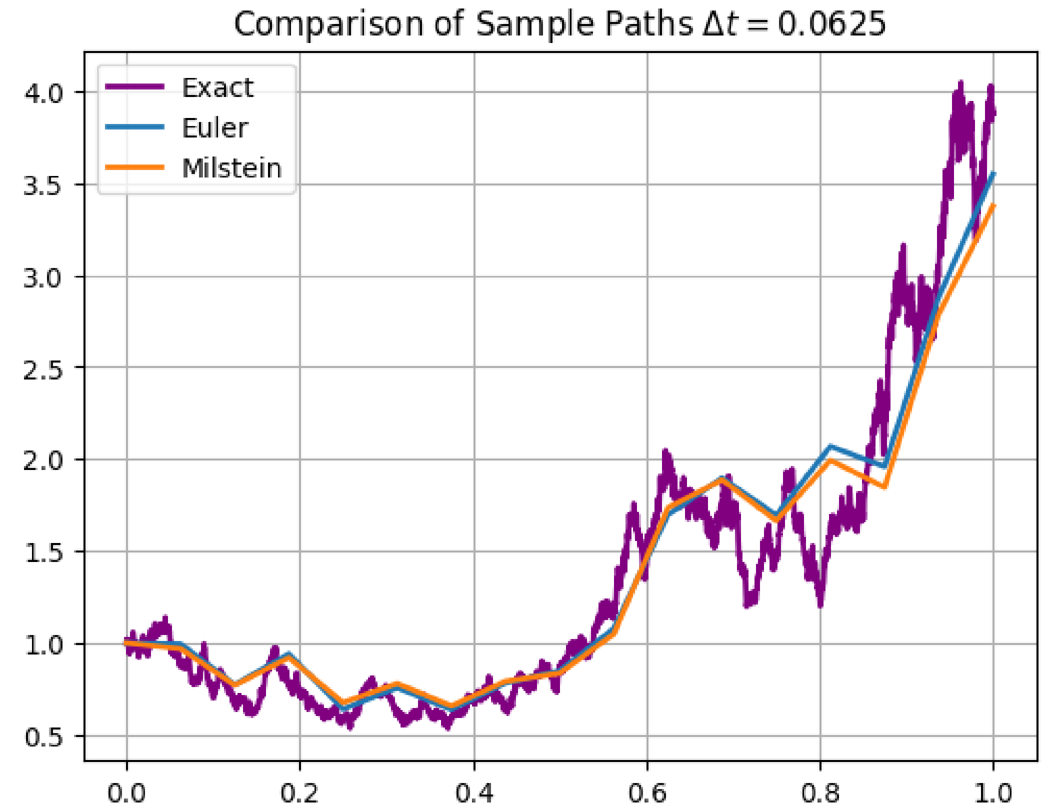
For sufficient resolution, we estimate the exact solution with a small step size, such as $\Delta t = 0.0001$.

[1]:
```python
# numerical libraries
import numpy as np
import scipy
import matplotlib.pyplot as plt
%matplotlib inline

# helper function
def exact_solution(dWt):
    nt = len(dWt)
    dt = 1/nt
    X_exact = [0]
    for i in range(nt):
        X_exact.append(X_exact[-1]+dt+dWt[i])
    X_exact = [np.exp(i) for i in X_exact]
    return X_exact
```

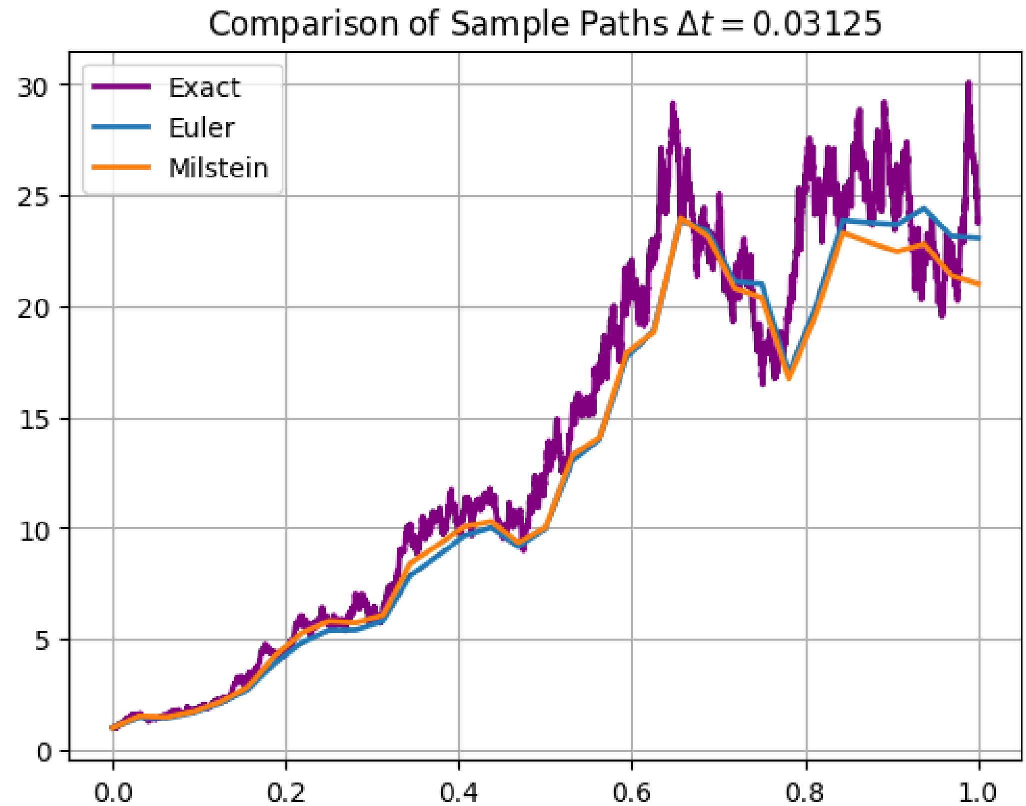
```
# fix seed
np.random.seed(10)
# fixed params
a = 1.5
b = 1
# configure
all_n = np.array([3, 4, 5, 6])
num_trials = len(all_n)
all_dt = np.array([1/(2**s) for s in all_n])
dt_exact = 2**(-12)
t_exact = np.arange(0, 1+dt_exact, dt_exact)
nt_exact = len(t_exact)

for i in range(num_trials):
    dt = all_dt[i]
    # time grid
    t = np.arange(0, 1+dt, dt)
    nt = len(t)

    # simulate a Wiener process using the fine discretization
    dWt = np.sqrt(dt_exact)*np.random.randn(int(1/dt_exact))
    # exact solution
    X_exact = exact_solution(dWt)
    # need to query the Wiener process at coarser leve;
    reduce = int(nt_exact/nt)
    sub_dWt = np.array([np.sum(dWt[(reduce*i):(reduce*(i+1))]) for i in␣
 ↪range(nt)])

    X_euler = np.zeros(nt)
    X_mil = np.zeros(nt)
    X_euler[0] = 1
    X_mil[0] = 1

    for j in np.arange(0, nt-1):
        # Euler scheme
        X_euler[j+1] = X_euler[j] + a*X_euler[j]*dt + b*X_euler[j]*sub_dWt[j]
        # Milstein scheme
        X_mil[j+1] = X_mil[j] + a*X_mil[j] * dt + b*X_mil[j] * sub_dWt[j] + 0.
 ↪5*X_mil[j]*((sub_dWt[j])**2-dt)
    # plotting
    plt.figure(i);
    plt.plot(t_exact, X_exact, lw=2, color='purple')
    plt.plot(t, X_euler, t, X_mil, lw=2);
    plt.legend(['Exact', 'Euler', 'Milstein']); plt.title(r"Comparison of␣
 ↪Sample Paths $\Delta t = {}$".format(dt))
    plt.grid(True);
```



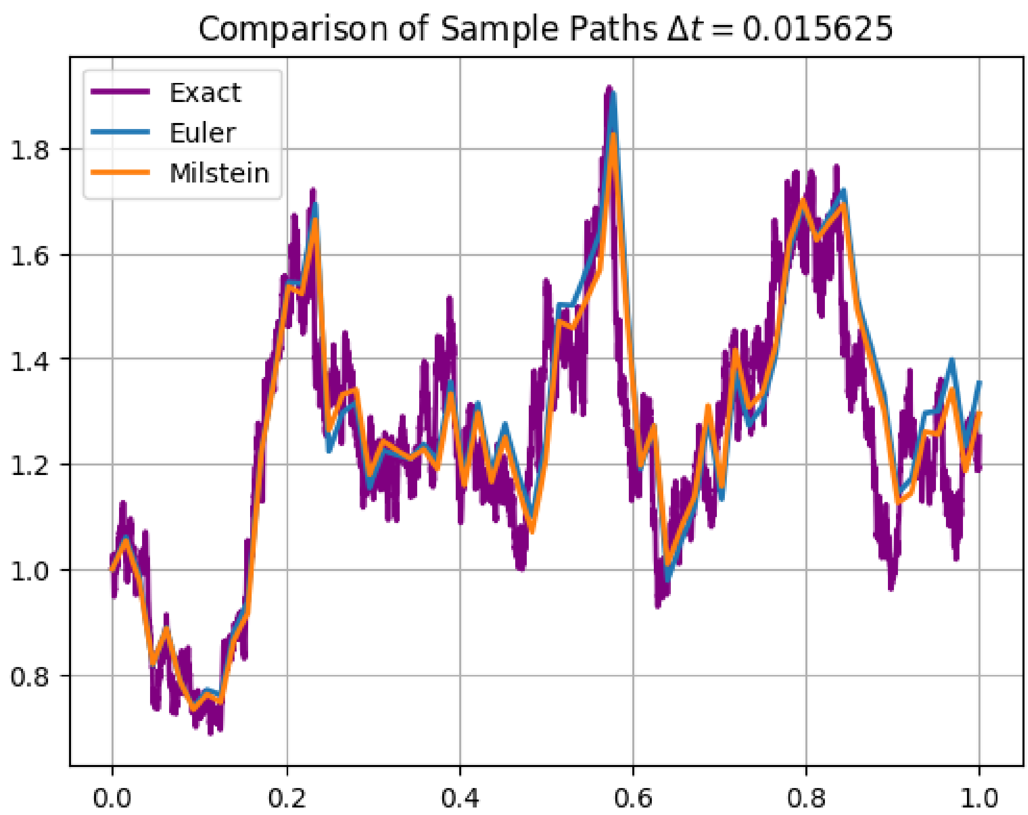
Comparison of Sample Paths $\Delta t = 0.125$
Exact
Euler
Milstein
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.0
0.2
0.4
0.6
0.8
1.0



Comparison of Sample Paths $\Delta t = 0.0625$
Exact
Euler
Milstein
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.2
0.4
0.6
0.8
1.0



Comparison of Sample Paths $\Delta t = 0.03125$
Exact
Euler
Milstein
30
25
20
15
10
5
0
0.0
0.2
0.4
0.6
0.8
1.0



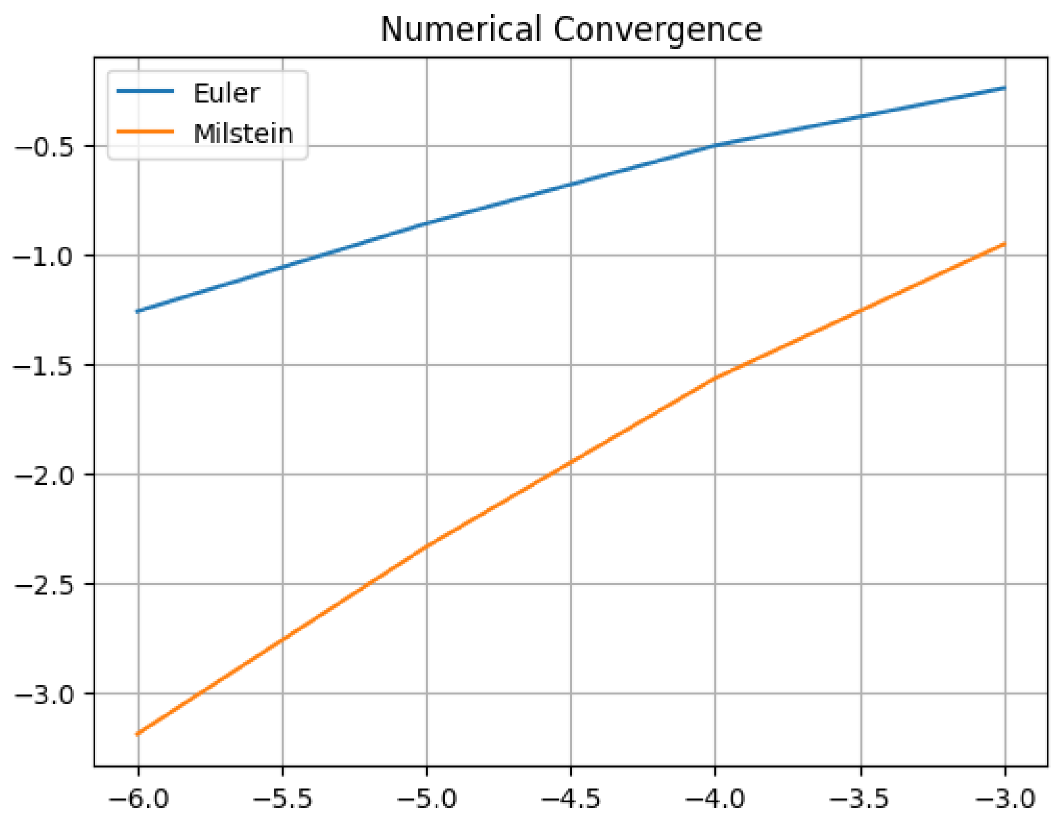
(b) Sample Generation To compare at the final time, the exact solution is computed using a fine discretization, $\Delta t = 2^{-12}$.

[2]:
```
# fix seed
mc = 5000

err_euler = np.zeros([num_trials, mc])
err_mil = np.zeros([num_trials, mc])
all_dt = [2**(-i) for i in range(3, 7)]
dt_exact = 2**(-12)
t_exact = np.arange(0, 1+dt_exact, dt_exact)
nt_exact = len(t_exact)
for i in range(mc):
    # generate exact solution using a fine level
    dWt = np.sqrt(dt_exact)*np.random.normal(loc=0, scale=1, size=nt_exact)
    Wt = np.cumsum(dWt)
    # exact solution
    X_exact = exact_solution(dWt)
    for k in range(3, 7):
        dt = 1/(2**k)
```

```
        t = np.arange(0, 1+dt, dt)
        nt = len(t)
        # compute Euler and Milstein solutions
        reduce = int(nt_exact/nt)
        sub_dWt = np.array([np.sum(dWt[(reduce*zz):(reduce*(zz+1))]) for zz in␣
↪range(nt)])
        # exact solution query points
        X_exact_query = np.array([X_exact[(reduce*zz)] for zz in range(nt)])
        X_euler = np.zeros(nt)
        X_mil = np.zeros(nt)
        X_euler[0] = 1
        X_mil[0] = 1
        for j in np.arange(0, nt-1):
            # Euler scheme
            X_euler[j+1] = X_euler[j] + a*X_euler[j]*dt +␣
↪b*X_euler[j]*sub_dWt[j]
            # Milstein scheme
            X_mil[j+1] = X_mil[j] + a*X_mil[j] * dt + b*X_mil[j] * sub_dWt[j] +␣
↪0.5*X_mil[j]*((sub_dWt[j])**2 - dt)
        # compute final error
        err_euler[k-3, i] = np.max(np.abs(X_euler-X_exact_query))
        err_mil[k-3, i] = np.max(np.abs(X_mil-X_exact_query))
```

Convergence plot

[3]:
```
plt.figure(1);
plt.plot(np.log2(all_dt), np.log2(err_euler.mean(1)), np.log2(all_dt), np.
↪log2(err_mil.mean(1)));
plt.legend(['Euler', 'Milstein']);
plt.title('Numerical Convergence');
plt.grid(True);
```


Numerical Convergence
Euler
Milstein
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
−6.0
−5.5
−5.0
−4.5
−4.0
−3.5
−3.0