Problem Set 6

Due: Wednesday, October 19, 2016.

Collaboration policy: collaboration is *strongly encouraged*. However, remember that

1. You must write up your own solutions, independently.
2. You must record the name of every collaborator.
3. You must actually participate in solving all the problems. This is difficult in very large groups, so you should keep your collaboration groups limited to 3 people in a given week.
4. **No bibles. This includes solutions posted to problems in previous years.**

Problem 1. Cost-scaling algorithm for minimum-cost flow.

(a) Suppose that you have an optimal integral solution to some minimum-cost circulation problem with integer costs and you then change one edge cost by one unit. Show how you can re-optimize the solution in $\tilde{O}(mn)$ time. Note that this is faster than solving the minimum-cost circulation problem from scratch. **Hint:** which edges can now be involved in negative-cost cycles?

(b) Deduce a cost-scaling algorithm for minimum-cost flow (with integer costs) that makes $O(m \log C)$ (where C is the maximum cost) calls to your solution to part (a) and prove its correctness.

OPTIONAL (c) Design a cost-scaling algorithm for minimum-cost flow that makes $O(n \log C)$ calls to your solution to part (a).

Note: The resulting time bounds are worse than the capacity-scaling algorithm we saw in lecture, but a more careful algorithm along these lines attains a better time bound of $O(mn \log\log U \log_n C)$.

Problem 2. Linear Programming By Hand. Consider the following linear programming problem:

$$\begin{aligned} \text{minimize } cx & \text{ subject to} \\ x_1 + x_2 &\geq 1 \\ x_1 + 2x_2 &\leq 3 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_3 &\geq 0 \end{aligned}$$

For each of the following objectives c, give the optimum value and the set of optimum solutions:

(a) $c = (-1, 0, 0)$

(b) $c = (0, 1, 0)$

(c) $c = (0, 0, -1)$

NONCOLLABORATIVE Problem 3. Let $c, f \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$. Consider the problem of minimizing the ratio $(c^T x)/(f^T x)$ subject to the following constraints:

- $Ax \geq b$,
- $f^T x \geq 1$,
- $-8 \leq c^T x \leq 8$.

Show how linear programming can be used to find an optimal solution within any degree of accuracy. Specifically, assume you have a blackbox solver that solves an input linear program exactly (or reports it is unbounded or infeasible). Show, for any given $\epsilon > 0$, how to find a solution to the above ratio minimization problem whose value is within additive $\pm\epsilon$ of the optimum. What is the running time, as a function of ϵ, in terms of number of calls to the LP solver?

Problem 4. You work for the Short-Term Capital Management company and start the day with D dollars. Your goal is to convert them to Yen through a series of currency trades involving assorted currencies, so as to maximize the amount of Yen you end up with. You are given a list of pending orders: client i is willing to convert up to u_i units of currency a_i into currency b_i at a rate of r_i (that is, he will give you r_i units of currency b_i for each unit of currency a_i). You may also borrow an arbitrary amount of any currency, with zero interest, provided you pay it back in the same currency by the end of the day. Assume that going around any directed cycle of trades, $\prod r_i < 1$—that is, there is no opportunity to make a profit by arbitrage.

(a) Formulate a linear program for maximizing the amount of Yen you have at the end of trading.

(b) Show that it is possible to carry out trades to achieve the objective of the linear program, without ever borrowing currency. (**Hint:** there is a sense in which your solution can be made acyclic.)

(c) Show that there is a sequence of trades that will let you end the day with the optimum amount of Yen and no other currency except dollars.

Problem 5. An internet switch is a transfer point where packets arriving on certain *input lines* are switched over to certain *output lines* where they proceed to the next switch on their journey. More precisely, each incoming packet can be assumed to be labeled with the desired output line to which it must be delivered. In each time step, the switch is typically constrained to transfer a set of packets that form a *matching:* in other words, at most one packet can be drawn from each input line, and at most one packet can be delivered to each output line. Let λ be the maximum of (i) the largest rate at which packets arrive on an input line and (ii) the maximum rate at which packets want to depart from an output line. We wish to prove that so long as the switch can deliver matchings at rate λ, it can deliver the specified traffic.

We will tackle this problem using linear programming and the language of fractional bipartite matchings. The following linear program reflects the minimum cost bipartite matching problem. Given a graph with n vertices per side and m edges, define a variable x_{ij} for each edge ij, let $N(v)$ denote the vertices neighboring vertex v (on the side opposite v), and consider the polytope defined by the constraints

$$\sum_{j \in N(i)} x_{ij} = 1$$
$$\sum_{i \in N(j)} x_{ij} = 1$$
$$x_{ij} \geq 0.$$

Any solution in which the x_{ij} are integers defines a perfect matching. But there can be other solutions, yielding so called *doubly stochastic matrices* in which all the row and column sums are 1, but need not be integers. We will show, however, that every such matrix is actually a *convex combination* of perfect matchings: in other words, a doubly stochastic matrix M can be written as a sum $\sum \lambda_i M_i$, where each M_i is an integer double stochastic matrix (representing a perfect matching) and $\sum \lambda_i = 1$. We also set $M_{ij} = 0$ if $ij \notin E$.

(a) Argue that at any vertex M of the polytope described above, at least $m - 2n + 1$ of the x_{ij} must be equal to 0. **Hint:** the equality constraints in the polytope are not linearly independent.

(b) Conclude that at least one row and at least one column of the matrix M contains a single 1 with all other entries 0, if M represents a vertex.

(c) Use induction to conclude the claim (about convex combinations) and deduce the original claim (about internet switches).

Problem 6. How long did you spend on this problem set? Please answer this question using the Google form that is located on the course website. This problem is mandatory, and thus counts towards your final grade. It is due by the Monday 2:30pm after the pset due date.

Optional problems on Spectral Graph Theory

The following problems are about the material covered in the optional lectures on Monday 10/3 and Wednesday 10/12.

For all the following problems, let $G(V, E)$ a simple, unweighted and undirected graph, with $|V| = n$, and Laplacian matrix L.

OPTIONAL Problem 7. Suppose n equals $2^k - 1$ for some integer $k \geq 1$, and suppose G is the full binary tree on n nodes. Use spectral arguments, similar to the ones presented in class, to give a lower bound (as tight as you can) on the mixing time of G.

OPTIONAL Problem 8. For a subset $S \subset V$, denote by $\bar{S}$ its complement, $\bar{S} := V \setminus S$. The **sparsity** of a cut $(S, \bar{S})$ in G is defined as

$$\phi(S, \bar{S}) := \frac{e(S, \bar{S})}{\min\{|S|, |\bar{S}|\}},$$

where $e(S, \bar{S})$ denotes the number of edges crossing the cut. Intuitively, when we are looking for a *sparse* cut by this definition, we want to have a small number of crossing edges, but also keep both sides of the cut balanced in terms of number of vertices. The **conductance** of G is defined as

$$\Phi(G) := \min_{S \neq \emptyset, V} \phi(S, \bar{S}).$$

Prove that $\phi(G) \geq \frac{1}{2}\lambda_2$.

(**Hint:** In class we saw the proof of the Courant-Fischer theorem, and as a special case we have

$$\lambda_2 = \min_{v \in \mathbb{R}^V : v \neq \mathbf{0}, v \perp \mathbf{1}} \frac{v^T L v}{v^T v},$$

where $\mathbf{0}$ denotes the origin in $\mathbb{R}^V$, and $\mathbf{1}$ denotes the all-1's vector in $\mathbb{R}^V$, which is a Laplacian eigenvector corresponding to $\lambda_1 = 0$.)

OPTIONAL Problem 9. Let $n = 2^k$ for some $k \geq 1$, and let H_n be a graph on n vertices whose vertex set is $\{0,1\}^k$ and two vertices are connected in it iff they differ at exactly one coordinate.

(a) Verify that for any $a \in \{0,1\}^k$, the vector v^a with coordinates given by $v_i^a := (-1)^{a \cdot i}$ is an eigenvector of H_n.[1] What is the eigenvalue that v^a corresponds to?

(b) What is the conductance $\Phi(H_n)$ of this graph (as a function of n)? Provide an upper and lower bound that are as close to each other as you can make them.

[1] Note that coordinates of v^a are indexed by vertices of H_n and these are binary k-dimensional vectors. So, the inner product $a \cdot i$ in the definition of the i-th coordinate of the vector v^a is well-defined.