TOPOLOGY

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Chapter 1

Preliminaries

1.1 Naive Set Theory

Suppose that A_α, A, B_α, B are sets and I is some index set. We have

- $A \cap \left(\bigcup_{\alpha \in I} B_\alpha \right) = \bigcup_{\alpha \in I} \left(A \cap B_\alpha \right)$, $A \cup \left(\bigcap_{\alpha \in I} B_\alpha \right) = \bigcap_{\alpha \in I} \left(A \cup B_\alpha \right)$.
- $\left(\bigcup_{\alpha \in I} A_\alpha \right)^C = \bigcap_{\alpha \in I} A_\alpha^C$, $\left(\bigcap_{\alpha \in I} A_\alpha \right)^C = \bigcup_{\alpha \in I} A_\alpha^C$
- $E - \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} \left(E - A_\alpha \right)$, $E - \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} \left(E - A_\alpha \right)$

Let $f : X \to Y$ be a map. Suppose that $A_\alpha, A, E \subset X$ and $B_\alpha, B, F \subset Y$. We have

- $f \left(\bigcup_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} f\left(A_\alpha \right)$, $f \left(\bigcap_{\alpha \in I} A_\alpha \right) \subset \bigcap_{\alpha \in I} f\left(A_\alpha \right)$, $f(E - A) \supset f(E) - f(A)$.
- $f^{-1} \left(\bigcup_{\alpha \in I} B_\alpha \right) = \bigcup_{\alpha \in I} f^{-1}\left(B_\alpha \right)$, $f^{-1} \left(\bigcap_{\alpha \in I} B_\alpha \right) = \bigcap_{\alpha \in I} f^{-1}\left(B_\alpha \right)$, $f^{-1}(F - B) = f^{-1}(F) - f^{-1}(B)$.
- $A \subset f^{-1}(f(A))$, $B \supset f(f^{-1}(B))$.

Chapter 2

General Topology

2.1 Topological space

Definition 2.1.1 (topological space) A *topological space* is an ordered pair (X, τ), where X is a set and $\tau \subset \mathcal{P}(X)$ is a collection of subsets of X, satisfying the following axioms:

1. $\varnothing \in \tau$, $X \in \tau$.
2. If $A_i \in \tau$ $(i \in I)$, then $\bigcup\limits_{i \in I} A_i \in \tau$,
3. If $A_i \in \tau$ $(i = 1, 2, \cdots, n)$, then $\bigcap\limits_{i=1}^{n} A_i \in \tau$.

The elements of τ are called *open sets* and the collection τ is called a *topology* on X.

In this chapter we always assume that (X, τ) is a topological space.

Definition 2.1.2 (closed set) Suppose that A is a subset of X. A is a *closed set* if and only if A^C is an open set.

Proposition 2.1.1 Assume that $\mathcal{F} \subset \mathcal{P}(X)$ is a collection of all closed sets of (X, τ).

1. $\varnothing \in \mathcal{F}$, $X \in \mathcal{F}$.
2. If $A_i \in \mathcal{F}$ $(i \in I)$, then $\bigcap\limits_{i \in I} A_i \in \mathcal{F}$,
3. If $A_i \in \mathcal{F}$ $(i = 1, 2, \cdots, n)$, then $\bigcup\limits_{i=1}^{n} A_i \in \mathcal{F}$.

Proof.

1. $\varnothing \in \tau \implies \varnothing^C = X \in \mathcal{F}$. $X \in \tau \implies X^C = \varnothing \in \mathcal{F}$.
2. $A_i \in \mathcal{F}\ (i \in I) \implies A_i^C \in \tau\ (i \in I) \implies \bigcup\limits_{i \in I} A_i^C \in \tau \implies \left(\bigcap\limits_{i \in I} A_i \right)^C \in \tau \implies \bigcap\limits_{i \in I} A_i \in \mathcal{F}$.
3. $A_i \in \mathcal{F}\ (i = 1, \cdots, n) \implies A_i^C \in \tau\ (i = 1, \cdots, n) \implies \bigcap\limits_{i=1}^{n} A_i^C \in \tau \implies \left(\bigcup\limits_{i=1}^{n} A_i \right)^C \in \tau \implies \bigcup\limits_{i=1}^{n} A_i \in \mathcal{F}$.

$\square$

Definition 2.1.3 (neighborhood) Given a point $x \in X$, a set $N \subset X$ is the *neighborhood* of x if there exists an open set $O \in \tau$ such that $x \in O \subset N$. The collection of all neighborhoods of x is denoted by $\mathcal{N}(x)$.

Proposition 2.1.2 Suppose that $x \in X$.

1. $N \in \mathcal{N}(x) \implies x \in N$
2. $N_1, N_2 \in \mathcal{N}(x) \implies N_1 \cap N_2 \in \mathcal{N}(x)$
3. $N \in \mathcal{N}(x) \bigwedge N \subset U \subset X \implies U \in \mathcal{N}(x)$
4. $N \in \mathcal{N}(x) \implies \exists M \in \mathcal{N}(x),\ (M \subset N) \bigwedge (\forall y \in M, M \in \mathcal{N}(y))$

Proof.

1. $N \in \mathcal{N}(x) \implies x \in O \subset N \implies x \in N$.
2. $N_1, N_2 \in \mathcal{N}(x) \implies x \in O_1 \subset N_1 \bigwedge x \in O_2 \subset N_2 \implies x \in O_1 \cap O_2 \subset N_1 \cap N_2$.
3. $N \in \mathcal{N}(x) \bigwedge N \subset U \subset X \implies x \in O \subset N \subset U \implies U \in \mathcal{N}(x)$.
4. If $N \in \mathcal{N}(x)$, then there exists $O \in \tau$ such that $x \in O \subset N$. Note that $O \in \mathcal{N}(x)$ and
$$\forall y \in O, y \in O \subset O \implies \forall y \in O, O \in \mathcal{N}(y).$$
We affirm that O is the set M that we are looking for.

□

Definition 2.1.4 (limit point) Suppose that A is a subset of X. A point $x \in X$ is a *limit point* of A, if every neighbourhood of x contains at least one point of A different from x itself, or alternatively, if for any open set O containing x,
$$O \cap (A - \{x\}) \neq \varnothing.$$
The set of all limit points of A is said to be the *derived set* of A, denoted by A'.

Proposition 2.1.3 Suppose that A and B are subsets of X.

1. $\varnothing' = \varnothing$
2. $(A \cup B)' = A' \cup B'$
3. $A \subset B \Longrightarrow A' \subset B'$
4. $A'' \subset A' \cup A$
5. $a \in A' \Longrightarrow a \in (A - \{a\})'$

Proof.

1. For any point $x \in X$ and for any neighborhood $N \in \mathcal{N}(x)$, we have $N \cap (\varnothing - \{x\}) = \varnothing$. It implies that empty set has no limit points.
2. Omitted.

□

Definition 2.1.5 (closure) Suppose that A is a subset of X. The *closure* of A is the smallest closed set containing A, denoted by $\overline{A}$.

Proposition 2.1.4 The following propositions are equivalent:

1. S is the closure of A.
2. $S = \bigcap_{F_\alpha \in \mathcal{F}: A \subset F_\alpha} F_\alpha$, where $\mathcal{F}$ is a collection of all closed sets of X.
3. $S = A \cup A'$.
4. $S = \{x \in X : \forall N \in \mathcal{N}(x), N \cap A \neq \varnothing\}$.

Proof.

1. Omitted.

□

Proposition 2.1.5 Suppose that A and B are subsets of X.

1. $\overline{\varnothing} = \varnothing$.
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$, $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.
3. $A \subset \overline{A}$.
4. $\overline{(\overline{A})} = \overline{A}$.

Proof.

1. Omitted.

□

Definition 2.1.6 (frontier) Suppose that A is a subset of X. The *frontier* of A denoted by ∂A is defined as

$$\partial A = \overline{A} \cap \overline{A^C}.$$

A point x is called an *boundary point* of A if $x \in \partial A$.

Definition 2.1.7 (interior) Suppose that A is a subset of X. The *interior* of A is the largest open set contained by A, denoted by A°. A point x is called an *interior point* of A if $x \in A^\circ$.

Proposition 2.1.6 The following propositions are equivalent:

1. S is the interior of A.
2. $S = \bigcup_{O_\alpha \in \tau: O_\alpha \subset A} O_\alpha$.
3. $S = A - \partial A$.
4. $S = \{x \in X : \exists N \in \mathcal{N}(x), N_x \subset A\}$.

Proof.

1. Omitted.

□

Proposition 2.1.7 Suppose that A and B are subsets of X.

1. $\varnothing^\circ = \varnothing$.

2. $(A \cup B)^\circ = A^\circ \cup B^\circ$, $(A \cap B)^\circ \supset A^\circ \cap B^\circ$.

3. $A \supset A^\circ$.

4. $(A^\circ)^\circ = A^\circ$.

Proof.

1. Omitted.

□

Proposition 2.1.8 The following propositions are equivalent:

1. F is a closed set.

2. $F' \subset F$.

3. $\overline{F} = F$.

Proof.

1. Omitted.

□

Proposition 2.1.9 The following propositions are equivalent:

1. F is an open set.

2. $\partial F \cap F = \varnothing$.

3. $F^\circ = F$.

4. $\forall x \in F, F \in \mathcal{N}(x)$.

Proof.

1. Omitted.

□

Proposition 2.1.10 Suppose that A is a subset of X.

1. $(\overline{A})^C = (A^C)^\circ$

2. $(A^\circ)^C = \overline{A^C}$

Proof.

1. Omitted.

□

Definition 2.1.8 (dense set) A subset A of a topological space X is called dense in X if $\overline{A} = X$.

Proposition 2.1.11 The following propositions are equivalent:

1. A is dense in X.

2. For any point $x \in X$ and for any neighborhood $N \in \mathcal{N}(x)$, $N \cap A \neq \varnothing$.

Proof.

1. Omitted.

□

Definition 2.1.9 Let $\mathcal{B}$ is a collection of subsets of X, the collection of subsets generated by $\mathcal{B}$ is defined as

$$\langle \mathcal{B} \rangle = \left\{ A \in 2^X : A = \bigcup_{i \in I} B_i, B_i \in \mathcal{B} \right\}.$$

Definition 2.1.10 (base) Let $\mathcal{B}$ be a collection of subsets of the topological space (X, τ). B is a *base* for (X, τ) if $\langle \mathcal{B} \rangle = \tau$.

Proposition 2.1.12 Let $\mathcal{B}$ be a collection of subsets of the topological space (X, τ). The following propositions are equivalent:

1. $\mathcal{B}$ is a base for X.
2. Elements in $\mathcal{B}$ cover X and $\forall B_1, B_2 \in \mathcal{B}$, $\forall x \in B_1 \cap B_2$, $\exists B_3 \in B_1 \cap B_2$ such that $x \in B_3$.

Definition 2.1.11 (second-countable space) A *second-countable* space, also called a completely separable space, is a topological space whose topology has a countable base.

We have several separation axioms in topological spaces denoted by T_r, $r = 0, 1, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, 5, 6$.

Definition 2.1.12 (Hausdorff space/$\mathbf{T_2}$ space) A topological space X is a *Hausdorff space*, separated space or T_2 space if for any two distinct points $x, y \in X$ there exists a neighbourhood U of x and a neighbourhood V of y such that $U \cap V = \varnothing$.

Definition 2.1.13 (normal space/$\mathbf{T_4}$ space) A topological space X is a *normal space* or T_4 space if, given any disjoint closed sets E and F in X, there are neighbourhoods U of E and V of F such that $U \cap V = \varnothing$.

Definition 2.1.14 (compact topological space) Formally, a topological space X is called *compact* if each of its open covers has a finite subcover. That is, X is compact if for every collection $\mathcal{C}$ of open subsets of X such that

$$X = \bigcup_{C \in \mathcal{C}} C$$

there is a finite subset $\mathcal{F}$ of $\mathcal{C}$ such that

$$X = \bigcup_{F \in \mathcal{F}} F$$

A subset K of a topological space X is said to be compact if it is compact as a subspace (in the subspace topology).

2.2 Basic Construction

2.2.1 Subspace Topology

2.2.2 Quotient Topology

Definition 2.2.1 (quotient space) Let (X, τ) be a topological space and $\sim$ be a equivalence relation on X. Let $\pi : X \to X/\sim$ be the projection. The *quotient space* of X is the set $X/\sim$ equiped with the following topology

$$\tau_\sim := \{A \in 2^{X/\sim} \mid \pi^{-1}(A) \in \tau\},$$

which is called *quotient topology*.

It is easy to check that π is a continuous surjection from (X, τ) to $(X/\sim, \tau_\sim)$.

Proposition 2.2.1 For any topological space (Y, σ) and any continuous mapping $f : X \to Y$ such that $a \sim b$ implies $f(a) = f(b)$ for all $a, b \in X$, there exists a unique continuous mapping $\bar{f} : X/\sim \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ X/\sim \arrow[ur, dashed, "\exists! \bar{f}"'] & \end{tikzcd}$$

Proof. Define

$$\begin{aligned} \bar{f} : X/\sim &\longrightarrow Y, \\ [x] &\longmapsto f(x). \end{aligned}$$

It is clear that $\bar{f}$ is well-defined. Since for all $B \in \sigma$,

$$\pi^{-1}\left(\bar{f}^{-1}(B)\right) = \pi^{-1}\left(\{[x] \in X/\sim | f(x) \in B\}\right) = \{x \in X | f(x) \in B\} = f^{-1}(B) \in \tau,$$

we have

$$\bar{f}^{-1}(B) \in \tau_\sim,$$

which implies $\bar{f}$ is continuous. If there exists a function $g : X/\sim \to Y$ such that $f = g \circ \pi$, then for all $[x] \in X/\sim$, we see

$$g([x]) = g(\pi(x)) = f(x) \implies g = \bar{f}.$$

$\square$

2.2.3 Product Topology

Chapter 3

Basic Notions

3.1 CW Complex

A k-cell is a k-dimensional disc

$$D^k = \{x \in \mathbf{R}^k : \ |x| \leq 1\}.$$

Chapter 4

Concrete Examples

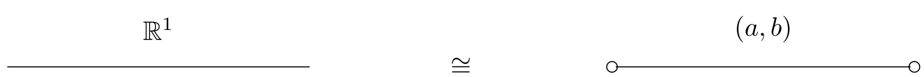
A topological space X is called locally Euclidean if there is a non-negative integer n such that every point in X has a neighbourhood which is homeomorphic to real n-dimensional space $\mathbb{R}^n$.

A topological manifold is a locally Euclidean Hausdorff space. In this chapter we focus on some concrete topological manifolds to develop a topological intuition. Topological manifold will be called manifold in brief whenever we mention it.

4.1 One-dimensional manifolds

4.1.1 Real affine line $\mathbb{R}^1$

The real line $\mathbb{R}^1$ carries a standard topology, which can be introduced from the metric defined above. The real line is homeomorphic to any open interval $(a, b) \in \mathbb{R}$.

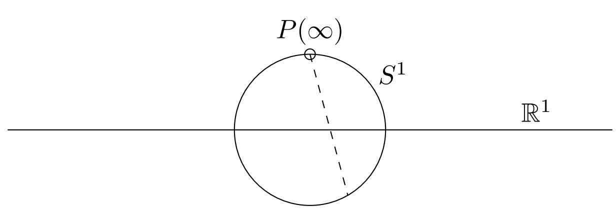


One homeomorphic mapping is

$$f : \mathbb{R}^1 \longrightarrow (a, b), \qquad x \longmapsto \frac{b-a}{\pi} \arctan(x) + \frac{a+b}{2}.$$

4.1.2 Circle S^1

The circle S^1 is defined by $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. S^1 can be described as $S^1 \cong \mathbb{R}^1 \cup \{\infty\}$, which is real line plus a single point representing infinity in both directions. Therefore, if a single point P is removed from a circle, it becomes homeomorphic to $\mathbb{R}^1$.



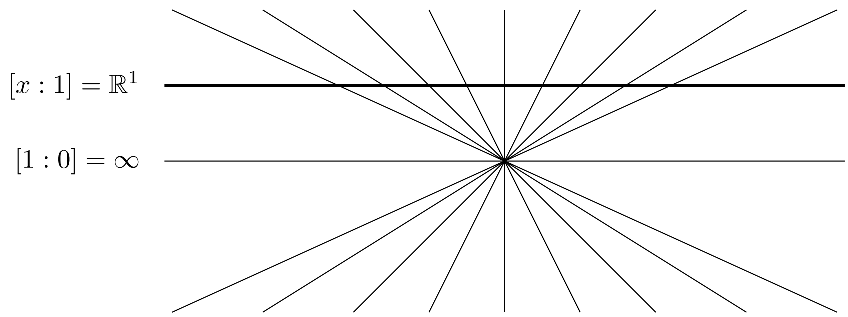
This homeomorphism is exactly the stereographic projection of P. Take $P = (0, 1)$ and the stereographic projection is given by

$$pr : S^1 - P \longrightarrow \mathbb{R}^1, \qquad (x, y) \longmapsto X = \frac{x}{1-y},$$
$$pr^{-1} : \mathbb{R}^1 \longrightarrow S^1 - P, \qquad X \longmapsto (x, y) = \left(\frac{2X}{X^2+1}, \frac{X^2-1}{X^2+1}\right).$$

4.1.3 Real projective line $P^1(\mathbb{R})$

Real projective line $P^1(\mathbb{R})$ is the set of equivalence classes of $\mathbb{R}^2 - (0, 0)$ under the equivalence relation $\sim$ defined by $x \sim y$ if there is a nonzero real number λ such that $x = \lambda y$. Given any representative element $(x, y) \in \mathbb{R}^2 - (0, 0)$, the equivalence class of (x, y) is denoted in the form of homogeneous coordinates $[x : y]$. Clearly we have $[x : y] = [\lambda x : \lambda y]$ for any nonzero $\lambda \in \mathbb{R}$.

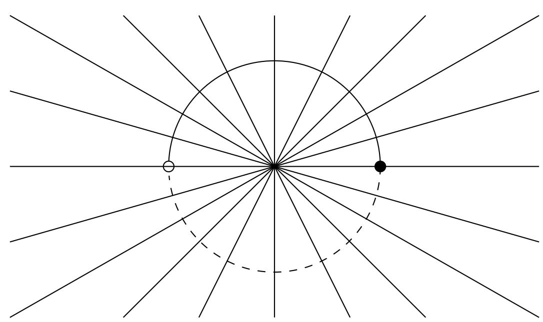
Real projective line consists of all 1-dimensional subspace of $\mathbb{R}^2$.



By introduce the continuous mapping

$$r : P^1(\mathbb{R}) \longrightarrow \mathbb{R}^1 \cup \{\infty\},$$
$$[x : y] \longmapsto \frac{x}{y},$$

we can clearly see $P^1(\mathbb{R}) \cong S^1$. $P^1(\mathbb{R})$ can be constructed by identifying the antipodal points of S^1.



4.2 Two-dimensional manifolds

4.2.1 Real affine plane $\mathbb{R}^2$

The real affine plane $\mathbb{R}^2$ carries a standard topology. $\mathbb{R}^2$ is homeomorphic to any open region $G \in \mathbb{R}$. Since the mapping $(x, y) \mapsto x + \mathrm{i}y$ is a homeomorphism, we have $\mathbb{R}^2 \cong \mathbb{C}$.

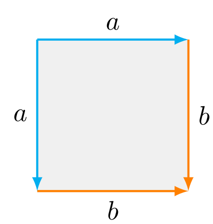
4.2.2 Sphere S^2

The circle S^2 is defined by $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$. Similarly we have $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ and the stereographic projection of $P = (0, 0, 1)$

$$pr : S^2 - P \longrightarrow \mathbb{R}^2, \qquad (x, y, z) \longmapsto (X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right),$$

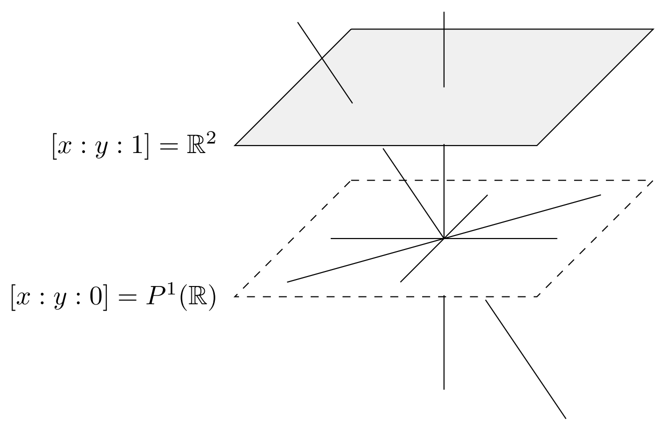
$$pr^{-1} : \mathbb{R}^1 \longrightarrow S^1 - P, \qquad (X, Y) \longmapsto (x, y, z) = \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{X^2+Y^2-1}{X^2+Y^2+1}\right).$$

Sphere S^2 can be described by a square with some edges identified as follows

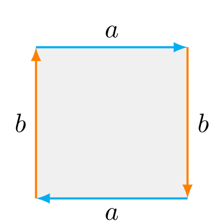


4.2.3 Real projective plane $P^2(\mathbb{R})$

Real projective plane $P^2(\mathbb{R}) = (\mathbb{R}^3 - \{(0, 0, 0)\})/\sim$ where $x \sim y$ whenever there is a nonzero real number λ such that $x = \lambda y$. $P^2(\mathbb{R}) \cong \mathbb{R}^2 \sqcup P^1(\mathbb{R}) \cong \mathbb{R}^2 \sqcup \mathbb{R}^1 \sqcup \{\infty\}$.



$P^2(\mathbb{R})$ can be constructed by identifying the antipodal points of S^2, which can be represented by $S^2/\{1, -1\}$. $P^2(\mathbb{R})$ can also be constructed from a closed unit disk $\overline{D^2}$ by identifying the antipodal points of the boundary $\partial\overline{D^2} = S^1$. Another way to describe $P^2(\mathbb{R})$ is a square with some edges identified as follows

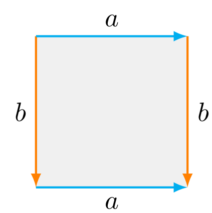


4.2.4 Complex projective line $P^1(\mathbb{C})$

Real projective line $P^1(\mathbb{R})$ is the set of equivalence classes of $\mathbb{R}^2 - (0,0)$ under the equivalence relation $\sim$ defined by $x \sim y$ if there is a nonzero real number λ such that $x = \lambda y$. We have the following homeomorphisms $P^1(\mathbb{C}) \cong \mathbb{C} \cup \{\infty\} \cong \mathbb{R}^2 \cup \{\infty\} \cong S^2$.

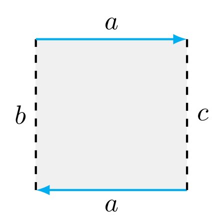
4.2.5 Torus T^2

A torus T^2 is a closed surface defined as the product of two circles $S^1 \times S^1$. Torus can be described by a square with some edges identified as follows



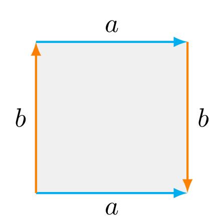
4.2.6 Möbius strip

Möbius strip can be described by a square with some edges identified as follows



4.2.7 Klein bottle

Klein bottle can be described by a square with some edges identified as follows



4.3 Other manifolds

4.3.1 Real projective space $P^n(\mathbb{R})$

Real projective plane $P^n(\mathbb{R}) = (\mathbb{R}^{n+1} - \{\mathbf{0}\})/\sim$ where $x \sim y$ whenever there is a nonzero real number λ such that $x = \lambda y$. In general, we have

$$P^n(\mathbb{R}) \cong S^n/\{\pm 1\} \cong D^n \sqcup (\partial D^n/\pm 1) = D^n \sqcup (S^{n-1}/\{\pm 1\}) \cong D^n \sqcup P^{n-1}(\mathbb{R}) \cong \overline{D^n} \sqcup_{\partial D^n} P^{n-1}(\mathbb{R}),$$

where $D^n := \{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\}$.