Linear Algebra - HW 6

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1 Question 1

۱) ثابت کنید تساوی زیر برای معکوس ضرب کرونکر بین دو ماتریس برقرار است.

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$$

As we already know : $(A \otimes B)(C \otimes D) = AC \otimes BD$

By using that rule we can say : $(A \otimes B)(A^{-1} \otimes B^{-1}) = AA^{-1} \otimes BB^{-1} = I$

We also know that : $X^{-1}X = I \Rightarrow (A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$

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2 Question 2

۲) ثابت کنید که اگر $\mathbf{A}+\mathbf{B}$ ماتریسی معکوس‌پذیر باشد، تساوی زیر برقرار است:

$$\mathbf{A}-\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}=\mathbf{B}-\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}$$

As we know from a theorem in out book : $(A^{-1}+B^{-1})^{-1}=A-A(B+A)^{-1}A$ by which we can conclude that :

$(X+Y)=X^{-1}-X^{-1}(Y^{-1}+X^{-1})^{-1}X^{-1}$

Now we can check two situations :

I)

$$\begin{cases} X \to A^{-1} \\ Y \to B^{-1} \end{cases}$$

$$\underbrace{(A^{-1}+B^{-1})^{-1}}_{RI}=A-A(B+A)^{-1}A$$

OR

$$\begin{cases} X \to B^{-1} \\ Y \to A^{-1} \end{cases}$$

$$\underbrace{(B^{-1}+A^{-1})^{-1}}_{RII}=B-B(A+B)^{-1}B$$

It’s crystal clear that $\underbrace{(B^{-1}+A^{-1})^{-1}}_{RII}=\underbrace{(A^{-1}+B^{-1})^{-1}}_{RI}$ which would result in :

$A-A(B+A)^{-1}A=B-B(A+B)^{-1}B$

3 Question 3

۳) ثابت کنید اگر $\mathbf{A}^{-1}+\mathbf{B}^{-1}=(\mathbf{A}+\mathbf{B})^{-1}$ باشد آنگاه $\mathbf{AB}^{-1}\mathbf{A}=\mathbf{BA}^{-1}\mathbf{B}$ خواهد بود.

According to what the question is saying, we should assume that $\underbrace{(A^{-1}+B^{-1})}_{X}=\underbrace{(A+B)^{-1}}_{X}$

As we already know $(XX^{-1}=I)$ this would mean that : $\underbrace{(A^{-1}+B^{-1})}_{X}\underbrace{(A+B)}_{X^{-1}}=I$

$$\begin{aligned}(A^{-1}+B^{-1})(A+B) = & A^{-1}A+A^{-1}B+B^{-1}A+B^{-1}B=I \Rightarrow \\ & \cancel{I}+A^{-1}B+B^{-1}A+I=\cancel{I} \Rightarrow \\ & A^{-1}B+B^{-1}A=-I\end{aligned}$$

Now we can have two separate approaches :

$I)$

$A^{-1}B+B^{-1}A=-I \xrightarrow{\text{*B from left}} BA^{-1}B+A=-B \rightarrow BA^{-1}B=-A-B$

$II)$

$A^{-1}B+B^{-1}A=-I \xrightarrow{\text{*A from left}} B+AB^{-1}A=-A \rightarrow AB^{-1}A=-A-B$

Now from both (I) and (II) we can conclude that :

$-A-B=AB^{-1}A=BA^{-1}B \Rightarrow AB^{-1}A=BA^{-1}B$

4 Question 4

۴) در صورتی که $\mathbf{A}$ ماتریس مربعی بوده و $\lim_{n\to\infty} \mathbf{A}^n = 0$ باشد، نشان دهید معکوس ماتریس $(\mathbf{I}-\mathbf{A})$ برابر است با

$$(\mathbf{I}-\mathbf{A})^{-1} = \mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\cdots$$

As we know from the geometrical series : $S_n = I + A + A^2 + ... + A^n, S_0 = I$

$$S_n = \Sigma_{n=0}^{n-1} A^n$$

We also know that for each eigenvalue of λ_i, if $|\lambda_i| < 1$ then the geometric series would converge to :

$$S_n = (I - A^n)(I - A)^{-1}$$

$$\lim_{x\to\infty} S_n = I + A + A^2 + ...$$

$$\Sigma_{n=0}^{\infty} A^n = \lim_{x\to\infty} S_n = \lim_{x\to\infty} (I-A)^{-1}(I-A^n) = I + A + A^2 + ... \xrightarrow{\lim_{x\to\infty} A^n = 0}$$

$$(I-A)^{-1}(I-0) = (I-A)^{-1} = I + A + A^2 + ...$$

5 Question 5

۵) ماتریس مربعی **A** را در نظر بگیرید. ثابت کنید که اگر **X** تنها ماتریسی باشد که در رابطه **XA=I** صدق کند، آنگاه **AX** نیز برابر **I** خواهد بود.

Assume that we have $Y = AX + X - I \xrightarrow{\text{*A from Right Side}} YA = A(XA) + XA - A$
$\xrightarrow{\text{(XA = I) Assumed in the problem description}} YA = A + I - A = I \rightarrow YA = I$

In the description of the question, it's mentioned that "X" is the only matrix that satisfies (XA = I). from this hint we can conclude that Y = X .

$Y = AX + X - I \xrightarrow{\text{Y = X}} X = AX + X - I \rightarrow AX = I$

End of this HomeWork.
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