MINIMIZATION

aka. Simplification

MINIMIZATION

Number of Gates

Number of Inputs (2-input vs 4-input)

Number of Interconnections

Propagation Time

Cost of Gates

Circuit Area

. . .

A circuit may not satisfy all due to conflicting constraints!

MINIMIZATION

aka. Simplification

SoP (ANDs-OR) → Simplify → NAND

PoS (ORs-AND) → Simplify → NOR

MINIMIZATION

I) Boolean Algebra (algebraically)

MINIMIZATION

II) Map (Karnaugh map, K-map)

ALGEBRA

A set of elements
A set of operators
A set of axioms | postulates | assumptions | definitions

ALGEBRA

A set of elements: e.g.,

$\{0,1\}$

$\{-1,0,1\}$

$\{1,2,3,4,\ldots\}$

$\{\ldots, -2,-1,0,1,2,\ldots\}$

ALGEBRA

A set of operators

$\{+, \times\}$

$\{+, -, \times, \div\}$

$\{\sim, \leq, \geq\}$

ALGEBRA

Unary: !x, -x, x′, …

Binary: x+y, x÷y, x^y, …

Ternary: x?y:z (Elvis), x BETWEEN y AND z (SQL)

George Boole (/buːl/)

Mathematician
Philosopher
Logician

The Laws of Thought (1854)

Boolean Algebra!

Edward Vermilye Huntington
1874 – 1952

In 1904, he put Boolean algebra
on a sound axiomatic foundation

POSTULATE

aka. Axiom, Assumption

https://en.wikipedia.org/wiki/Axiom

A statement that is taken to be TRUE

Serve as a premise or starting point for further reasoning

I. CLOSURE

A set is closed with respect to an operator § if the result of § ∈ S:

Unary: $x \in S$: $\S x \in S$

Binary: $x, y \in S$: $x \S y \in S$

I. CLOSURE

$S = \{\ldots, -2,-1,0,1,2,\ldots\}$

§ = + (addition), - (subtraction), × (multiplication), – (negation), ! (fact)

S is closed with respect to +, -, ×, –, !

I. CLOSURE

S = {..., -2,-1,0,1,2,...}

§ = ^ (power)

S is NOT closed with respect to ^ for 2^(-1)∉S

I. CLOSURE

$S = \{0,1\}$

$\S$ = + (OR), × (AND), ‘ (NOT)

S is closed with respect to +, ×, ‘

I. CLOSURE

$S = \{0,1\}$

§ = – (negation)

S is NOT closed with respect to – for $(-1) \notin S$

II. COMMUTATIVE

A binary operator § on a set S is commutative iff for all x,y ∈ S:

$$x \mathbin{\S} y = y \mathbin{\S} x$$

(x § y may or may not be in S)

II. COMMUTATIVE

S = {…, -2,-1,0,1,2,…}
§ = + (addition), × (multiplication)

+ and × are commutative on S for x+y=y+x and x×y=y×x

II. COMMUTATIVE

$S = \{\ldots, -2,-1,0,1,2,\ldots\}$

§ = - (subtraction)

- is NOT commutative on S for $x-y \neq y-x$

II. COMMUTATIVE

$S = \{0,1\}$

$\S$ = + (OR), $\times$ (AND), $\uparrow$ (NAND), $\downarrow$ (NOR), $\oplus$ (XOR), $\odot$ (XNOR)

All are commutative on S for $x \S y = y \S x$

III. ASSOCIATIVE

A binary operator § on a set S is associative iff for all x,y,z ∈ S:

$x \S (y \S z) = (x \S y) \S z = x \S y \S z$

III. ASSOCIATIVE

$S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

§ = + (addition), × (multiplication), - (subtraction)

+, -, × are associative on S for

$x \pm (y \pm z) = (x \pm y) \pm z = x \pm y \pm z$

$x \times (y \times z) = (x \times y) \times z = x \times y \times z$

III. ASSOCIATIVE

S = {..., -2,-1,0,1,2,...}

§ = ^ (power)

^ is NOT associative on S for 2^(1^3)≠(2^1)^3

III. ASSOCIATIVE

$S = \{0,1\}$

§ = + (OR), × (AND)

+,× are associative on S for

$x+(y+z)=(x+y)+z=x+y+z$

$x\times(y\times z)=(x\times y)\times z=x\times y\times z$

IV. DISTRIBUTIVE

If § and † are two binary operators on a set S, § is distributive over † iff:

Left Distributivity: x § (y † z) = (x § y) † (x § z)
Right Distributivity: (y § x) † (z § x) = (y † z) § x

IV. DISTRIBUTIVE

If § and † are two binary operators on a set S, § is distributive over † iff:

If § Commutative: x § (y † z) = (x § y) † (x § z) = (y † z) § x

IV. DISTRIBUTIVE

$S=\{0,1\}$

$+$ (OR), $\times$ (AND)

$$x + (y \times z) \overset{?}{\Leftrightarrow} (x + y) \times (x + z) \overset{?}{\Leftrightarrow} (y \times z) + x$$

$$x \times (y + z) \overset{?}{\Leftrightarrow} (x \times y) + (x \times z) \overset{?}{\Leftrightarrow} (y + z) \times x$$

V. IDENTITY

$e \in S$ is an identity element w.r.t binary operator § iff for all $x \in S$:

$$x \mathbin{\S} e = x = e \mathbin{\S} x$$

V. IDENTITY

$S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\S = +$ (addition), $\times$ (multiplication)

$e_{+} = 0 : x+0=0+x=x$

$e_{\times} = 1 : x\times 1=1\times x=x$

V. IDENTITY

S = {..., -2,-1,0,1,2,...}

§ = - (subtraction)

e = 0

$x-0 \neq 0-x \neq x$

NO identity element for – (subtraction) in S

V. IDENTITY

$S = \{0,1\}$

$\S = +$ (OR), $\times$ (AND)

$e_+ = 0 : x+0=0+x=x$

$e_\times = 1 : x\times 1=1\times x=x$

VI. INVERSE

For all $x \in S$, there should be $y \in S$ w.r.t binary operator § iff:

$$x \S y = e_{\S} = y \S x$$

We denote $y = x^{-1}$ and $x = y^{-1}$

VI. INVERSE

$S = \{\ldots, -2,-1,0,1,2,\ldots\}$

$\S = +$ (addition)

$x+(-x)=(-x)+x= e_+ = 0$

$x^{-1}=-x$

VI. INVERSE

$S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\S = \times$ (multiplication)

S does not have the inverse property for $\times$ since

$2 \times \frac{1}{2} = \frac{1}{2} \times 2 = e_{\times} = 1$ but $\frac{1}{2} \notin S$

VI. INVERSE

$$S = \{0,1\}$$

$$\S = + \text{ (OR)}, \times \text{ (AND)}$$

$$1+?=?+1 \neq e_{+} = 0$$

$$0\times?=?\times 0 \neq e_{\times} = 1$$

VII. COMPLEMENT

For all $x \in S$, there should be $y \in S$ w.r.t binary operators § and † iff:

$$x \S y = e_{\dagger} = y \S x$$

$$x \dagger y = e_{\S} = y \dagger x$$

We denote $y = x'$ and $x = y'$

VII. COMPLEMENT (NOT)

$S = \{0,1\}$

$\S = +\ (\text{OR}),\ \times\ (\text{AND})$

$1+0=0+1 = e_{\times} = 1$

$0\times1=1\times0 = e_{+} = 0$

$0' = 1,\ 1' = 0$

BOOLEAN ALGEBRA

- A set S with at least two elements x, y and $x \neq y$.
- Two binary operators § and †

- S is closed, commutative, distributive, and complement w.r.t § , †
- $e_{§}$ and $e_{†}$ exist

Claude Elwood Shannon

Mathematician
Electrical Engineer
Cryptographer

M.Sc. Thesis (1937)

A Symbolic Analysis of Relay and Switching Circuits

Switching Algebra!

2-valued Boolean algebra

SWITCHING ALGEBRA

- $S=\{0,1\}$
- $\S = \times$ (AND), $\dagger = +$ (OR)

- S is closed, commutative, distributive, complement w.r.t $\times$, +
- $e_\times = 1$ and $e_+ = 0$

$$0 \times 1 = e_+ = 0$$
$$0 + 1 = e_\times = 1$$
$$0' = 1;\ 1' = 0$$

SWITCHING ALGEBRA
IS-A
BOOLEAN ALGEBRA

It satisfies all conditions of Boolean algebra!

Prove → Book: 2.3 axiomatic definition of Boolean algebra

Another sample of algebra in CS: Relational Algebra (SQL)

Is relational algebra a Boolean algebra? Check this when you take COMP-3150: Database Management Systems!

BASIC THEOREMS

prove by truth table

For equality proof, $F_1=F_2$, for all possibility in the input variables (all rows), both side of equation must have equal value for same input variables.

For inequality proof, $F_1 \neq F_2$, find at least one possibility (a row) that have different values.

BASIC THEOREMS

Prove by postulates