DESIGN RECAP

SoP (ANDs-OR) → NAND
PoS (ORs-AND) → NOR

Given two unsigned numbers x and y, design a logic circuit to see

$$x \overset{?}{\geq} y$$

Solution 1: x – y, if no last borrow

$$x \overset{?}{\geq} y$$

Solution 2: logic circuit

$$x \overset{?}{\geq} y$$

Solution 2: logic circuit
What is the range of x and y?

$$x \overset{?}{\geq} y$$

Solution 2: logic circuit
What is the range of x and y?

$$x, y \in [0, 3]_{10}$$

Solution 2: logic circuit

What is the range of x and y?

$$x, y \in [0, 3]_{10} = [00{,}11]_2$$

Solution 2: logic circuit

What is the range of output?

$$F \in \{0, 1\}$$

Y2	Y1	X2	X1	F(Y2,Y1,X2,X1)=?
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

Y2	Y1	X2	X1	$F(Y2,Y1,X2,X1)=\Sigma\ m(0,1,2,3,5,6,7,10,11,15)$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

Y2	Y1	X2	X1	$F(Y2,Y1,X2,X1)=\prod M(4,8,9,12,13,14)$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

Y2	Y1	X2	X1	F(Y2,Y1,X2,X1)=Σ m(0,1,2,3,5,6,7,10,11,15)	F(Y2,Y1,X2,X1)=Π M(4,8,9,12,13,14)
0	0	0	0	1	1
0	0	0	1	1	1
0	0	1	0	1	1
0	0	1	1	1	1
0	1	0	0	0	0
0	1	0	1	1	1
0	1	1	0	1	1
0	1	1	1	1	1
1	0	0	0	0	0
1	0	0	1	0	0
1	0	1	0	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	0	1	0	0
1	1	1	0	0	0
1	1	1	1	1	1

Which design?

Which design?

SoP and PoS are both effective

SoP and PoS have same efficiency (2-levels)

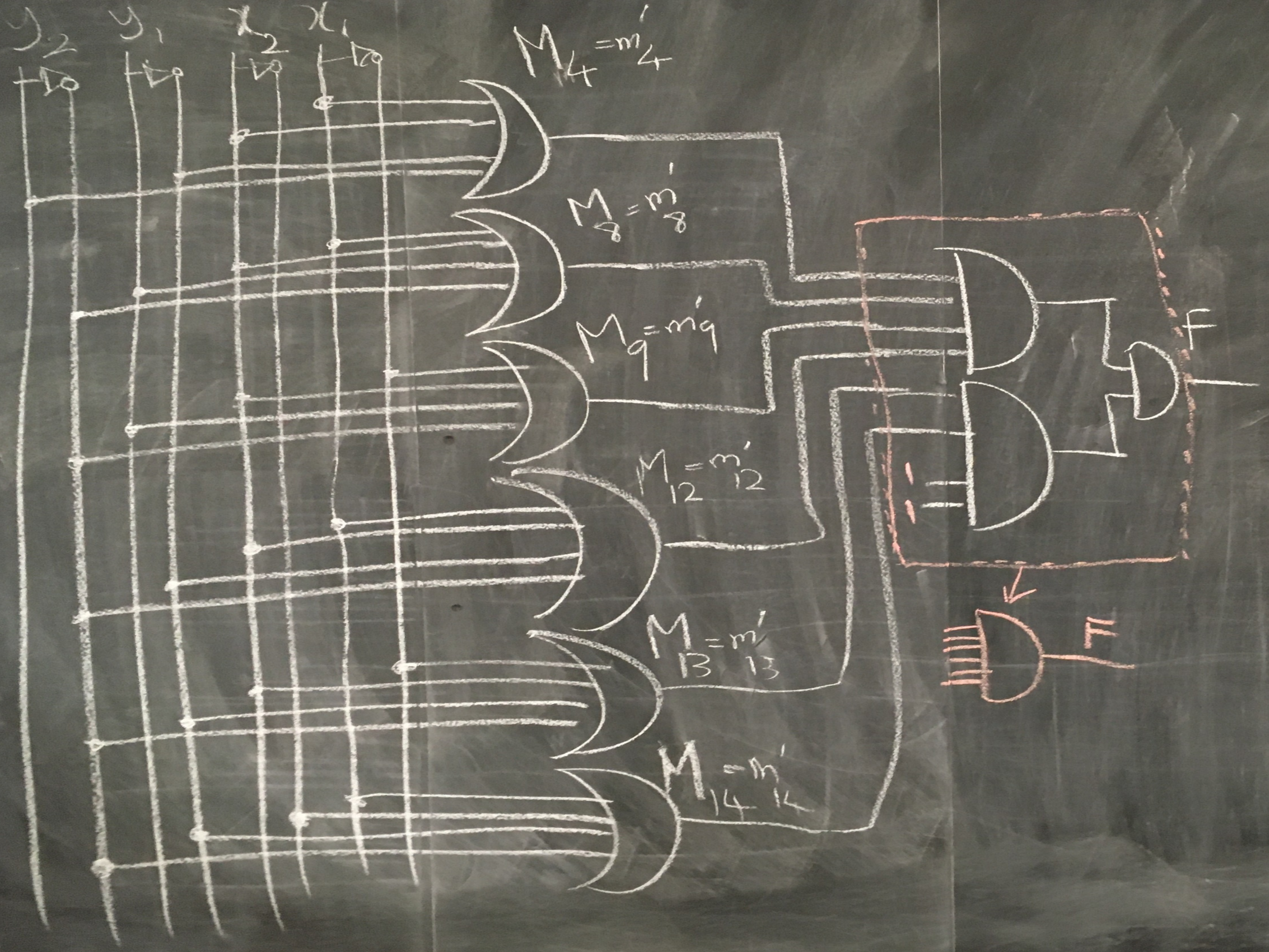
Which design?

Cost: How many gates in SoP vs. PoS?

	$F(Y2,Y1,X2,X1)=$ $\Sigma\ m(0,1,2,3,5,6,7,10,11,15)$	$F(Y2,Y1,X2,X1)=$ $\Pi\ M(4,8,9,12,13,14)$
AND	Each minterm one 4-input-AND	Each MAXTERM one 4-input-OR
OR	One final 10-input-OR	One final 6-input-AND
NOT	Doesn't Matter Much	Doesn't Matter Much

	$F(Y2,Y1,X2,X1)=$ $\Sigma\, m(0,1,2,3,5,6,7,10,11,15)$	$F(Y2,Y1,X2,X1)=$ $\Pi\, M(4,8,9,12,13,14)$
AND	10 $\times$ 4-input-AND	6 $\times$ 4-input-OR
OR	10-input-OR = 3 $\times$ 4-input-OR	6-input-AND = 2 $\times$ 4-input-AND
NOT	Doesn’t Matter Much	Doesn’t Matter Much

	$F(Y2,Y1,X2,X1)=$ $\Sigma m(0,1,2,3,5,6,7,10,11,15)$	$F(Y2,Y1,X2,X1)=$ $\Pi M(4,8,9,12,13,14)$
AND	10 × 4-input-AND	6 × 4-input-OR
OR	10-input-OR = 3 × 4-input-OR	6-input-AND = 2 × 4-input-AND
NOT	Doesn’t Matter Much	Doesn’t Matter Much



$$F = \prod M(4,8,9,12,13,14)$$

$$= M_4 M_8 M_9 M_{12} M_{13} M_{14}$$

$$= m'_4 m'_8 m'_9 m'_{12} m'_{13} m'_{14}$$

$$= (y'_2 y_1 x'_2 x'_1)' \qquad m'_4 \rightarrow y_2 + y'_1 + x_2 + x_1$$

$$(y_2 y'_1 x'_2 x'_1)' \qquad m'_8 \rightarrow y'_2 + y_1 + x_2 + x_1$$

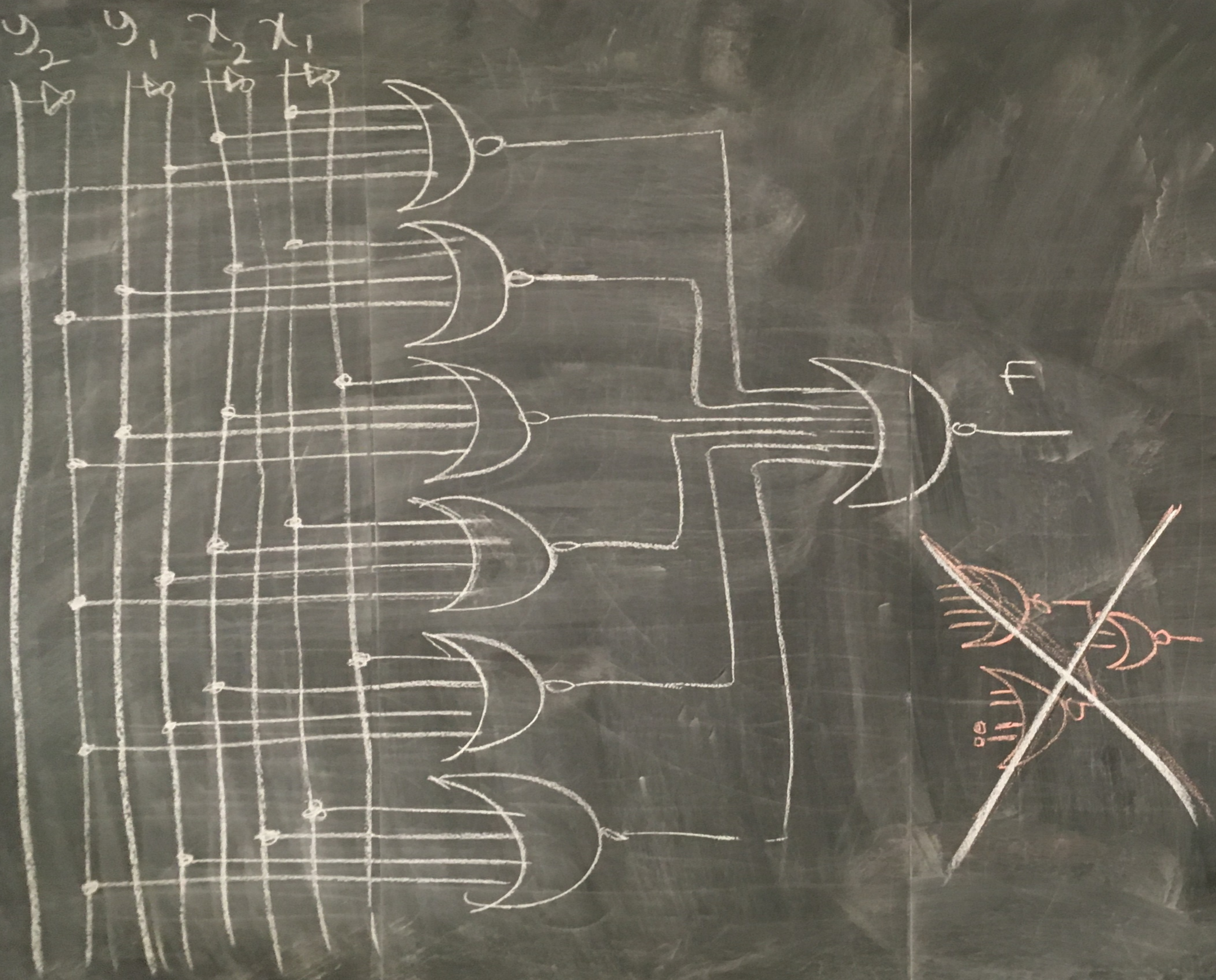
$$(y_2 y'_1 x'_2 x_1)' \qquad m'_9 \rightarrow y'_2 + y_1 + x_2 + x'_1$$

$$(y_2 y_1 x'_2 x'_1)' \qquad m'_{12} \rightarrow y'_2 + y'_1 + x_2 + x_1$$

$$(y_2 y_1 x'_2 x_1)' \qquad m'_{13} \rightarrow y'_2 + y'_1 + x_2 + x'_1$$

$$(y_2 y_1 x_2 x'_1)' \qquad m'_{14} \rightarrow y'_2 + y'_1 + x'_2 + x_1$$

	$F(Y2,Y1,X2,X1)= \Sigma m(0,1,2,3,5,6,7,10,11,15)$	$F(Y2,Y1,X2,X1)= \Pi M(4,8,9,12,13,14)$
NAND	10 × 4-input-NAND + 1 × 10-input-NAND	
NOR		6 × 4-input-NOR + 1 × 6-input-NOR



$$F = \prod M(4,8,9,12,13,14)$$
$$= M_4 M_8 M_9 M_{12} M_{13} M_{14}$$

$$F = (F')'$$
$$= \left(\left(M_4 M_8 M_9 M_{12} M_{13} M_{14}\right)'\right)'$$
$$= \left(M_4' + M_8' + M_9' + M_{12}' + M_{13}' + M_{14}'\right)'$$
$$= \Big((y_2 + y_1' + x_2 + x_1)' + (y_2' + y_1 + x_2 + x_1)' + (y_2' + y_1 + x_2 + x_1')' + (y_2' + y_1' + x_2 + x_1)' + (y_2' + y_1' + x_2 + x_1')' + (y_2' + y_1' + x_2' + x_1)'\Big)'$$

MINIMIZATION

aka. Simplification

MINIMIZATION

Number of Gates

Number of Inputs (2-input vs 4-input)

Number of Interconnections

Propagation Time

Cost of Gates

Circuit Area

. . .

A circuit may not satisfy all due to conflicting constraints!

MINIMIZATION

aka. Simplification

SoP (ANDs-OR) → Simplify → NAND

PoS (ORs-AND) → Simplify → NOR

MINIMIZATION

I) Boolean Algebra (algebraically)

MINIMIZATION

II) Map (Karnaugh map, K-map)

ALGEBRA

A set of elements
A set of operators
A set of axioms | postulates | assumptions | definitions

ALGEBRA

A set of elements: e.g.,

{0,1}

{-1,0,1}

{1,2,3,4,...}

{..., -2,-1,0,1,2,...}

ALGEBRA

A set of operators

$\{+, \times\}$

$\{+, -, \times, \div\}$

$\{\sim, \leq, \geq\}$

ALGEBRA

Unary: !x, -x, x′, …

Binary: x+y, x÷y, x^y, …

Ternary: x?y:z (Elvis), x BETWEEN y AND z (SQL)

George Boole (/buːl/)

Mathematician
Philosopher
Logician

The Laws of Thought (1854)

Boolean Algebra!

Edward Vermilye Huntington
1874 – 1952

In 1904, he put Boolean algebra
on a sound axiomatic foundation

POSTULATE

aka. Axiom, Assumption

https://en.wikipedia.org/wiki/Axiom

A statement that is taken to be TRUE

Serve as a premise or starting point for further reasoning

I. CLOSURE

A set is closed with respect to an operator § if the result of § $\in$ S:

Unary: $x \in S$: $\S x \in S$

Binary: $x, y \in S$: $x \ \S \ y \in S$

I. CLOSURE

S = {…, -2,-1,0,1,2,…}

§ = + (addition), - (subtraction), × (multiplication), – (negation), ! (fact)

S is closed with respect to +, -, ×, –, !

I. CLOSURE

$S = \{\ldots, -2,-1,0,1,2,\ldots\}$

§ = ^ (power)

S is NOT closed with respect to ^ for $2^{(-1)} \notin S$

I. CLOSURE

$S = \{0,1\}$

§ = + (OR), × (AND), ' (NOT)

S is closed with respect to +, ×, '

I. CLOSURE

$S = \{0,1\}$

$\S = -$ (negation)

S is NOT closed with respect to – for $(-1) \notin S$

II. COMMUTATIVE

A binary operator § on a set S is commutative iff for all x,y ∈ S:

x § y = y § x

(x § y may or may not be in S)

II. COMMUTATIVE

$S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

§ = + (addition), × (multiplication)

+ and × are commutative on S for $x+y=y+x$ and $x \times y = y \times x$

II. COMMUTATIVE

$S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$§ = -$ (subtraction)

$-$ is NOT commutative on S for $x-y \neq y-x$

II. COMMUTATIVE

$S = \{0,1\}$

§ = + (OR), × (AND), ↑ (NAND), ↓ (NOR), ⊕ (XOR), ⊙ (XNOR)

All are commutative on S for x § y=y § x

III. ASSOCIATIVE

A binary operator § on a set S is associative iff for all x,y,z ∈ S:

$x \S (y \S z) = (x \S y) \S z = x \S y \S z$

III. ASSOCIATIVE

S = {..., -2,-1,0,1,2,...}

§ = + (addition), × (multiplication), - (subtraction)

+, -, × are associative on S for

$x \pm (y \pm z) = (x \pm y) \pm z = x \pm y \pm z$

$x \times (y \times z) = (x \times y) \times z = x \times y \times z$

III. ASSOCIATIVE

S = {…, -2,-1,0,1,2,…}

§ = ^ (power)

^ is NOT associative on S for 2^(1^3)≠(2^1)^3

III. ASSOCIATIVE

S = {0,1}

§ = + (OR), × (AND)

+,× are associative on S for

$x+(y+z)=(x+y)+z=x+y+z$

$x\times(y\times z)=(x\times y)\times z=x\times y\times z$

IV. IDENTITY

$e \in S$ is an identity element w.r.t binary operator § iff for all $x \in S$:

$$x \mathbin{\S} e = x = e \mathbin{\S} x$$

IV. IDENTITY

$S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

§ = + (addition), × (multiplication)

$e_{+} = 0 : x+0=0+x=x$

$e_{\times} = 1 : x \times 1 = 1 \times x = x$

IV. IDENTITY

$S = \{\ldots, -2,-1,0,1,2,\ldots\}$

§ = - (subtraction)

$e = 0$

$x-0 \neq 0-x \neq x$

NO identity element for – (subtraction) in S

IV. IDENTITY

$S = \{0,1\}$

$\S = +$ (OR), $\times$ (AND)

$e_{+} = 0 : x+0=0+x=x$

$e_{\times} = 1 : x\times1=1\times x=x$

V. INVERSE

For all $x \in S$, there should be $y \in S$ w.r.t binary operator § iff:

$$x \text{ § } y = e_{\text{§}} = y \text{ § } x$$

We denote $y = x^{-1}$ and $x = y^{-1}$

V. INVERSE

$S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$§ = +$ (addition)

$x+(-x)=(-x)+x= e_+ = 0$

$x^{-1}=-x$

V. INVERSE

$S = \{\ldots, -2,-1,0,1,2,\ldots\}$

$§ = \times$ (multiplication)

S does not have the inverse property for $\times$ since

$2\times ½ = ½\times 2 = e_{\times} = 1$ but $½ \notin S$

V. INVERSE

$$S = \{0,1\}$$

$$\S = + \text{ (OR)}, \times \text{ (AND)}$$

$$1+?=?+1 \neq e_{+} = 0$$

$$0\times?=?\times0 \neq e_{\times} = 1$$

VI. DISTRIBUTIVE

If § and † are two binary operators on a set S, § is distributive over † iff:

$$x \ § \ (y \dagger z) = (x \ § \ y) \dagger (x \ § \ z) = (y \dagger z) \ § \ x$$

BOOLEAN ALGEBRA

- A set S with at least two elements x, y and $x \neq y$.
- Two binary operators § and †
- S is closed, commutative, and distributive w.r.t § , †
- $e_{§}$ and $e_{†}$ exist

- Complement: for any $x \in S$, there is $y \in S$ such that
  - $x \; § \; y = e_{†}$
  - $x \dagger y = e_{§}$

we denote $y = x'$