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Notes

For a group we have computed

$$x \sim N(\mu, \Sigma)$$

This matrix can be singular, so we set

$$\Sigma = V \Lambda V^T$$

and we skip the zero eigenvalues / vectors, so that

$$\underset{n \times m}{B} = \left[\sqrt{\lambda_1}\, v_1 \,\vdots\, \sqrt{\lambda_2}\, v_2 \,\vdots\, \cdots \right]$$

where m is the rank of B. Then B defines a mapping to the new variable z, so that

$$x = Bz + \mu$$

and $z|y \sim N_m(0, I)$.

Now we want to remove the contribution from the likelihood.

- The likelihood has the form

$$l(y|x) = \sum l_i(y_i|x_i)$$

so around μ, we get

$$l(y|x) = b^T x - \frac{1}{2} x^T C x$$

where C is diagonal. We need to do this correction for z.

$$\log \pi(z|\sim y) = -\frac{1}{2} z^T z \overset{\text{remove}}{-} l(y|x=\mu)$$

$$= -\frac{1}{2} z^T z - b^T x + \frac{1}{2} x^T C x$$

$$= -\frac{1}{2} z^T z - b^T (Bz + \mu) + \frac{1}{2} (Bz + \mu)^T C (Bz + \mu)$$

$$= -\frac{1}{2} z^T \left[I - B^T C B \right] - \left[B^T b - B^T C \mu \right]^T z$$

so new precision matrix and mean μ^* for $z|-y$ is

$$Q^* = I - B^T C B = (\Sigma^*)^{-1}$$

diag

mean for x

and mean

$$Q^* \mu^* = + B^T(-b + C\mu)$$

$$\mu^* = Q^{*-1} \cdot \underbrace{B^T(-b + \overset{H}{C}\mu)}_{zb}$$

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then the marginal for index x_i is

$$E(x_i|-y) = (B\mu^*)_i + \mu_i$$

$$Var(x_i|-y) = \left(B(Q^*)^{-1}B^T\right)_{ii}$$

The joint approximation

from the last step we have that

$$z \sim N(\mu_z, \Sigma_z) \qquad Q_z = \Sigma_z^{-1}$$

and the mapping is

$$\cancel{x} = Bz + \mu_L \quad \text{Mode with data}$$

we need

$$\pi(y) = \int \pi(y|x)\,\pi(x)\,dx$$

$$= \int \pi(y|\overset{x(z)}{\cancel{z}})\,\pi(z)\,dz$$

$$\approx \left.\frac{\pi(y|x(z))\,\pi(z)}{\pi_G(z|y)}\right|_{z=z^*}$$

where

$\pi_G(z|y)$ is found from

$$\pi(z|y) \propto \pi(z)\,\pi(y|x(z))$$

$$\propto \exp -\frac{1}{2}(z-\mu_z)^T Q_z(z-\mu_z)$$

$$+ b^T x - \frac{1}{2}x^T C x$$

$$= \exp -\frac{1}{2}(z-\mu_z)^T Q_z() + b^T(Bz+\mu)$$

$$-\frac{1}{2}(Bz+\mu)^T C(Bz+\mu)$$

$$= \exp -\frac{1}{2} z^T\left[Q_z + B^TCB\right] + \left[Q_z\mu_z + B^Tb - B^TC\mu\right]^T z$$

so that

$$\Sigma_G = Q_G^{-1}$$

$$Q_G = Q_z + B^TCB$$

$$Q_G\mu_G = Q_z\mu_z + B^T(b - C\mu)$$

$$\mu_G = \Sigma_G\left(Q_z\mu_z + B^T(b - C\mu)\right)$$

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so in the end

$$\pi(y) \approx \frac{\pi(y|x(\mu_G))\,\pi_z(\mu_G)}{\pi_G(\mu_G|\,)}$$

This is for the 2.order approx and how to get the correct constant.

$$a + b(x-\mu) - \frac{1}{2}(x-\mu)^2 c$$

$$= a + bx - b\mu - \frac{1}{2}(x^2 - 2x\mu + \mu^2) c$$

$$= \underbrace{\left[a - b\mu - \frac{1}{2}\mu^2 c\right]}_{A} + x\underbrace{\left[b + c\mu\right]}_{B} - \frac{1}{2}x^2 \underbrace{c}_{C}$$

$$C = c \qquad b = B - C\cdot\mu \qquad a = A + b\mu + \frac{1}{2}\mu^2 C$$

$$= A + (B - C\mu)\mu + \frac{1}{2}\mu^2 C$$

$$= A + B\mu - \frac{1}{2}\mu^2 C$$

correcting $\pi(\theta|y)$

$$\pi(y,\theta) = \pi(\theta|y)\,\pi(y)$$
$$= \pi(\theta, y_G, y_{-G})$$
$$= \pi(y_G|\theta, y_{-G})\,\pi(\theta|y_{-G})\,\pi(y_{-G})$$

since $\pi(y)$ and $\pi(y_{-G})$ are constants, then

$$\pi(\theta|y_{-G}) = \frac{\pi(\theta|y)}{\pi(y_G|\theta, y_{-G})} \cdot \text{constant}$$

but $\pi(y_G|\theta, y_{-G})$ is just the gcpo value we compute for each G and θ. with given integration points, $\{\theta_i\}$, then

$$\sum_i \pi(\theta_i|y) = Z'$$

and

$$\sum_i \frac{\pi(\theta_i|y)}{\pi(y_G|\theta_i, y_{-G})} = Z$$

so that

$$\frac{1}{Z'} \sum \pi(\theta_i|y) = 1$$

and

$$\frac{1}{Z} \sum \pi(\theta_i|y_{-G}) = 1$$

since $\pi(\theta|y)$ is continuous but we only use discrete integration points. same with $\pi(\theta|y_{-G})$

Then

$$\underline{\pi(y_G | y_{-G})} = \sum_i \pi(y_G | y_{-G}, \theta_i) \, \frac{1}{Z} \, \pi(\theta_i | y_{-G})$$

$$= \sum_i \cancel{\pi(y_G | y_{-G}, \theta_i)} \, \frac{1}{Z} \cdot \frac{\pi(\theta_i | y)}{\cancel{\pi(y_G | y_{-G}, \theta_i)}}$$

$$= \frac{1}{Z} \sum \pi(\theta_i | y)$$

$$= \underline{\frac{Z'}{Z}}$$