(3)

Replicated data

$$prob(y) = \sum_{n=N}^{\infty} P_o(n;\lambda) \prod_{i=1}^{d} Bin(y_i; n, p)$$

$$N = max(y)$$

Using recursive formulas:

$$P_o(u;\lambda) \prod_{i=1}^{d} Bin(y_i; u, p)$$

$$= P_o(u-1;\lambda) \prod_{i}^{d} Bin(y_i; u-1, p)$$

$$\cdot \frac{(1-p)^d \lambda}{n} \prod_{i=1}^{d} \frac{n}{n-y_i}$$

so its

$$prob(y) = P_o(N;\lambda) \prod_{i=1}^{d} Bin(y_i; N, p)$$

$$\cdot fac$$

where fac is computed as

$$fac = 1$$

for nn in $nmax : (ymax+1)$

$$fac = 1 + fac \cdot \frac{\lambda(1-p)^d}{nn} \prod_{i=1}^{d} \frac{nn}{nn-y_i}$$

N-mix model : details

Model
$$y \sim Bin(n, p)$$
$$n \sim Pois(\lambda)$$

Normally, so $p = p(z)$, while, $\log(\lambda) = x^{i} \beta$

The likelihood is

$$prob(y) = \sum_{n=y}^{\infty} Pois(n; \lambda) \cdot Bin(y; n, p)$$

There is a nice recursive formula for this density, using that

$$Pois(n; \lambda) = Pois(n-1; \lambda) \frac{\lambda}{n}$$

$$Bin(y; n, p) = Bin(y; n-1, p) \frac{n}{n-y}(1-p)$$

So that

$$Pois(n; \lambda) Bin(y; n, p) = Pois(n-1; \lambda) Bin(y; n-1, p)$$
$$\cdot \frac{\lambda}{n-y}(1-p)$$

Let $f_i = \frac{\lambda(1-p)}{i}$, $i = 1, 2, \ldots, n$ then

then

$$L = \sum_{n=y}^{n_{max}} Pois(n; \lambda) \cdot Bin(y; n, p)$$

$$= Pois(y; \lambda) \cdot Bin(y; y, p) \cdot \left\{ \vphantom{\sum} \right.$$

$$1 + f_1 + f_1 f_2 + f_1 f_2 f_3 + \cdots + f_1 \cdots f_{n_{max}-y} \left.\right\}$$

$$= Pois \cdot Bin \left\{ \underbrace{1 + f_1(1 + f_2(1 + f_3(1 + f_4 \not\!{\not\!{+}} + \cdots)))}_{fac.} \right\}$$

fac. can be computed.

$$fac = 1$$
for i in $i_{max} : 1$
$$fac = \cancel{\not\!{+}} \; 1 + fac \cdot \frac{\lambda(1-p)}{i}$$

then

$$L = Pois(y; \lambda) \cdot Bin(y; y, p) \cdot fac.$$

i_{max} could be $\in \left\lfloor \frac{\lambda(1-p)}{\varepsilon} \right\rfloor$

so that $\frac{\lambda(1-p)}{i_{max}} \lessapprox \varepsilon$

Using negative binomial instead of Poisson.
In this case then

$$n \sim nbinom\,(\lambda, \delta)$$

where λ is the mean $\not{y \mu}$ and δ the dispersion parameter, following the notation in ?dnbinom in R. The density is

$$f(n) = \frac{\Gamma(n+\delta)}{\Gamma(\delta)\,n!}\, q^{\delta}\,(1-q)^{n}$$

where $q = \text{prob} = \dfrac{\delta}{\delta + \lambda}$

Also this has a recursive formulation, as

$$f(n) = \frac{(n+\delta-1)\,\Gamma(n+\delta-1)}{\Gamma(\delta)\,n\,(n-1)!}\, q^{\delta}\,(1-q)^{n-1}\,(1-q)$$

$$= f(n-1) \cdot \frac{n+\delta-1}{n}\,(1-q)$$

This means that

$$nbinom\,(n\,;\lambda,\delta) \cdot Bin(y\,;n,p)$$

$$= nbinom\,(n-1\,;\lambda,\delta)\; Bin(y\,;n-1,p)$$

$$\begin{cases} \cdot\; \dfrac{n+\delta-1}{n}\,(1-q)\cdot\dfrac{n}{n-y}\,(1-p) \\[3mm] = \dfrac{n+\delta-1}{n-y}\,(1-q)(1-p) \end{cases}$$

Again, following the same logic, we can compute the likelihood as:

$$\text{nbinom}(y; \lambda, \delta) \cdot \text{Bin}(y; y, P) \cdot f$$

where f is computed as:

$$f = 1$$
for n in $nmax:(y+1)$
$$f = 1 + f \cdot \frac{n + \delta - 1}{n - y}(1-q)(1-p)$$

For replicated data, we get:

$$\left\{ \text{nbinom}(N; \lambda, \delta) \cdot \prod_{i=1}^{d} \text{Bin}(y_i; N, P) \right\} \cdot f$$

$$f = 1$$
for n in $nmax:(ymax + 1)$
$$f = 1 + f \cdot \frac{n + \delta - 1}{n}(1-q)(1-p)^d \prod_{i=1}^{d} \frac{n}{n - y_i}$$