

Second Course to Real Analysis

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1 Multivariable Calculus

1.1 Linear Algebra

Definition 1.1.1 (Matrix transformation). Suppose $m, n \in \mathbb{N}$ and $A \in \mathcal{M}^{m,n}(\mathbb{R})$. Then, the matrix transformation represented by A is the map $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by

$$T_A(v) = \sum_{i=1}^m \sum_{j=1}^n a_{i,j} v_j e_i \quad \forall v \in \mathbb{R}^n,$$

where $v = \sum v_j e_j \in \mathbb{R}^n$ and $\{e_1, \dots, e_n\}$ is the standard basis of \mathbb{R}^n .

Equivalently, T_A is defined by the matrix multiplication

$$T_A(v) = \text{transpose}(A \cdot \text{transpose}(v)) \quad \forall v \in \mathbb{R}^n,$$

where we treat v strictly as a 1-by- n matrix $\forall v \in \mathbb{R}^n$.

Definition 1.1.2 (Notations: $\mathcal{L}, \mathcal{M}, \mathbb{F}, n, m$). Suppose $m, n \in \mathbb{N}$. We denote by

- (i) $\mathcal{M}^{m,n}(\mathbb{R})$ the collection of all m -by- n matrix with real entries;
- (ii) $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ the set of all linear transformations from \mathbb{R}^n to \mathbb{R}^m .

When no misunderstanding may arise and sufficient context is supplied, we denote, more briefly, $\mathcal{M}^{m,n}(\mathbb{R})$ and $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ by \mathcal{M} and \mathcal{L} respectively.

Unless otherwise stated, we assume that $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $m, n \in \mathbb{N}$.

Proposition 1.1.1. Matrix transformations are linear transformations.

Proposition 1.1.2. Suppose $m, n \in \mathbb{N}$. Then, $\mathcal{M}^{m,n}(\mathbb{R})$ is a vector space with $\dim \mathcal{M}^{m,n}(\mathbb{R}) = mn$.

Proposition 1.1.3. Suppose $m, n \in \mathbb{N}$. Then, $\mathcal{M}^{m,n}(\mathbb{R})$ and \mathbb{R}^{mn} are isomorphic.

Proposition 1.1.4. Suppose $m, n \in \mathbb{N}$. Then, $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a vector space and $\dim \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = nm$.

Proposition 1.1.5 (Canonical Isomorphism Induced by Matrix Transformation: \mathcal{T}). Suppose $m, n \in \mathbb{N}$. Then, $\mathcal{T}: \mathcal{M}^{m,n}(\mathbb{R}) \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ defined by $\mathcal{T}(A) = T_A \quad \forall A \in \mathcal{M}^{m,n}(\mathbb{R})$ is an isomorphism

Theorem 1.1.1 (Composition of Matrix Transformations). Suppose $m, k, n \in \mathbb{N}$, $A \in \mathcal{M}^{m,k}(\mathbb{F})$, and $B \in \mathcal{M}^{k,n}(\mathbb{F})$. Then, $T_A \circ T_B = T_{AB}$.

Definition 1.1.3 (Norm on a vector space). Suppose V is a vector space over field \mathbb{F} . A norm on V is a map $\|\cdot\|_V: V \rightarrow \mathbb{R}$ satisfying the following properties:

- (i) $\|v\|_V \geq 0 \quad \forall v \in V$ with $\|v\|_V = 0 \iff v = 0_V$;
- (ii) $\|kv\|_V = |k| \|v\|_V \quad \forall k \in \mathbb{F}, v \in V$.
- (iii) $\|v + w\|_V \leq \|v\|_V + \|w\|_V \quad \forall v, w \in V$.

When no misunderstanding may arise and sufficient context is supplied, we may denote $\|\cdot\|_V$, more briefly, by $\|\cdot\|$.

Proposition 1.1.6 (Common Norms on \mathbb{R}^n). *Let $n \in \mathbb{N}$. Then, the following maps are norms on \mathbb{R}^n :*

(Euclidean Norm or l_2 norm) $\|\cdot\|_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ where

$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2} \quad \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

(Supremum norm or l_∞ norm) $\|\cdot\|_\infty : \mathbb{R}^n \rightarrow \mathbb{R}$ where

$$\|x\|_\infty = \max \{|x_i| : i \in \{1, \dots, n\}\} \quad \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

(l_1 norm) $\|\cdot\|_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ where

$$\|x\|_1 = \sum_{i=1}^n |x_i| \quad \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Definition 1.1.4 (Normed space). *A normed space is a vector space V along with a norm $\|\cdot\|$ defined on V .*

Proposition 1.1.7 (Norm-Induced Metric; Normed Spaces are Metric Spaces). *Suppose $(V, \|\cdot\|)$ is a normed space. Then,*

(i) $d : V \times V \rightarrow \mathbb{R}$ defined by $d(v, w) = \|v - w\|$, $\forall (v, w) \in V \times V$, is a metric on V .

(ii) (V, d) is a metric space.

Definition 1.1.5 (Banach space). *A vector space is a Banach space if it is a complete normed space.*

Definition 1.1.6 (Operator norm and bounded operator). *Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$. Then, the operator norm of T is*

$$\|T\| = \sup \left\{ \frac{\|T(v)\|_W}{\|v\|_V} : v \neq 0_V \right\}.$$

An operator is bounded if its operator norm is finite.

Theorem 1.1.2 (Operator Norm Identity). *Suppose V, W are normed spaces where $T \in \mathcal{L}(V, W)$. Then,*

$$\begin{aligned} \|T\| &= \sup \{|T(v)| : |v| < 1\} \\ &= \sup \{|T(v)| : |v| \leq 1\} \\ &= \sup \{|T(v)| : |v| = 1\} \\ &= \inf \{M > 0 : v \in V \implies |T(v)| \leq M |v|\}. \end{aligned}$$

Proof. (INCOMPLETE) □

Proposition 1.1.8 (Operator Norm Properties). *Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$. Then, the following statements hold:*

(i) $\|T\| \geq 0$;

(ii) $\|T\| = 0 \iff T = 0_{\mathcal{L}(V, W)}$;

(iii) Suppose U is a normed space and $S \in \mathcal{L}(U, V)$. Then, $\|T \circ S\| \leq \|T\| \|S\|$.

Proof. (ii) Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$.

(\implies) Suppose that $\|T\| = 0$. It follows that $\frac{|T(v)|_W}{|v|_V} \leq 0, \forall v \in V/\{0_V\}$. We note that, by the definition of a norm, $|T(v)|_W, |v|_V \geq 0 \forall v \in V$. Thus, we obtain that, $\forall v \in V/\{0_V\}$,

$$\begin{aligned} \frac{|T(v)|_W}{|v|_V} \geq 0 &\implies \frac{|T(v)|_W}{|v|_V} = 0 \implies |T(v)|_W = 0 \\ &\implies T(v) = 0_W. \quad (\text{By the definition of norm}) \end{aligned}$$

In addition, certainly $T(0_V) = 0_W$. Hence, we proved that $T(v) = 0_W, \forall v \in V$. That is, we show that $T = 0_{\mathcal{L}(V, W)}$, as desired.

(\impliedby) Suppose $T = 0_{\mathcal{L}(V, W)}$. It follows that $T(v) = 0_W \forall v \in V$. Thus, we have that $|T(v)|_W = 0, \forall v \in V$, by the definition of a norm. As an immediate result, we obtain that

$$\|T\| = \sup \left\{ \frac{|T(v)|_W}{|v|_V} : v \neq 0_V \right\} = \sup \{0 : v \neq 0_V\} = 0.$$

□

Theorem 1.1.3 ($\mathcal{L}(V, W)$ is a Normed Space). *Suppose V and W are normed spaces. Then, $\mathcal{L}(V, W)$ along with operator norm $\|\cdot\| : \mathcal{L} \rightarrow \mathbb{R}$ is a normed space.*

Definition 1.1.7 (Comparability of norms).

Proposition 1.1.9 (Comparability Induces an Equivalence Relation on the Set of Norms).

Theorem 1.1.4 (All Norms on \mathbb{R}^n are Comparable).

Corollary 1.1.1 (Norms on Finite-Dimensional Normed Space are Comparable).

Theorem 1.1.5 (Finite Operator Norm and Equivalent Conditions). *Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$. Then, the following conditions are equivalent:*

- (i) $\|T\| < \infty$;
- (ii) T is uniformly continuous;
- (iii) T is continuous;
- (iv) T is continuous at the origin (that is, at 0_V).

Proof. (INCOMPLETE) Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$.

(i \implies ii): Suppose $\|T\| < \infty$. By definition, we have that $\exists m > 0$ such that $m = \|T\|$

$$\frac{|T(v)|_W}{|v|_V} \leq \|T\| \quad \forall v \in V/\{0_V\} \implies |T(v)|_W \leq \|T\| |v|_V \quad \forall v \in V/\{0_V\}$$

Consider arbitrary $v, w \in V$.

(ii \implies iii) & (iii \implies iv): Trivial.

(iv \implies i): Suppose T is continuous at 0_V . □

Theorem 1.1.6 (Characteristics of Linear Maps on Normed Spaces). *Suppose $T \in \mathcal{L}(\mathbb{R}^n, W)$, where W is a normed space. Then,*

- (i) T is continuous, and
- (ii) T is an isomorphism implies T is a homeomorphism.

Proof. (INCOMPLETE)

□

Corollary 1.1.2. *Suppose V, W are finite-dimensional normed spaces. Then,*

(i) $T \in \mathcal{L}(V, W) \implies T$ is continuous, and

(ii) $\phi \in \mathcal{L}(V, W)$ is an isomorphism implies ϕ is a homeomorphism.

Proof. (INCOMPLETE)

□

Corollary 1.1.3. (i) *Suppose V is a finite-dimensional normed space with norms $\| \cdot \|_a$ and $\| \cdot \|_b$. Then, the identity map I on V is a homomorphism between the normed spaces $(V, \| \cdot \|_a)$ and $(V, \| \cdot \|_b)$.*

(ii) $\mathcal{T}: \mathcal{M} \rightarrow \mathcal{L}$ is a homeomorphism.

Proof. (INCOMPLETE)

□

Definition 1.1.8 (Conorm).

Exercise 1.1.1 (Determine an Operator Norm). *Consider the dilation map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $f(x, y) = (2x, y)$. Prove $\|T\| = 2$.*

Proof. Outline: show $\|T\| \leq 2$.

Find $(a, b) \in \mathbb{R}^2$ such that $\frac{|T(a, b)|}{|(a, b)|} = 2$.

Use the definition of sup to prove the conclusion,

□

1.2 Derivatives

Definition 1.2.1 ((Total) Derivative). Let $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open. Suppose $f: U \rightarrow \mathbb{R}^m$.

(i) The **derivative (or total derivative)** $(Df)_p$ of f at $p \in U$ is a map, if it exists, $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

(a) T is a linear map, and

(b) T satisfies

$$f(p+v) = f(p) + T(v) + R(v) \text{ for all sufficiently small } v \in \mathbb{R}^n \\ \implies \lim_{|v| \rightarrow 0} \frac{R(v)}{|v|} = 0_{\mathbb{R}^m} \quad (\text{Pugh})$$

or, equivalently,

$$f(p+v) = f(p) + T(v) + R(v) \text{ for all sufficiently small } v \in \mathbb{R}^n \\ \implies \lim_{v \rightarrow 0_{\mathbb{R}^n}} \frac{|R(v)|}{|v|} = 0, \quad (\text{Rudin})$$

or, equivalently,

$$\lim_{v \rightarrow 0_{\mathbb{R}^n}} \frac{|f(p+v) - T(v) - f(p)|}{|v|} = 0, \quad (\text{Rudin})$$

where $R(v) \in \mathbb{R}^m$ denotes the Taylor remainder for $f(p+v)$.

(ii) We say that f is **differentiable at** $p \in U$ if $(Df)_p$ exists, and f is **differentiable** if f is differentiable at p , $\forall p \in U$.

(iii) Let $E = \{p \in U : (Df)_p \text{ exists}\}$. We call the map $Df: E \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, defined by $[Df](p) = (Df)_p \forall p \in E$, the **derivative (or total derivative) of** f .

Note that we may also denote Df by f' .

Remark 1.2.1. Recall that $\exists r_p > 0$ such that $q \in \mathbb{R}^n$ and $d(p, q) < r_p \implies q \in E$, since E is open. By sufficiently small $v \in \mathbb{R}^n$, we mean that v is such that $d(p, p+v) < r_p$ so $p+v \in E$.

Remark 1.2.2. The choice of T is unique, since a limit is unique, provided it exists. See proof below.

Definition 1.2.2 (Notations: e_i, u_j, f_j). Let $n, m \in \mathbb{N}$. Denote the standard bases of \mathbb{R}^n and \mathbb{R}^m by $\{e_1, \dots, e_n\}$ and $\{u_1, \dots, u_m\}$, respectively.

Suppose $U \subset \mathbb{R}^n$, $f: U \rightarrow \mathbb{R}^m$, and $\exists f_1, \dots, f_m: U \rightarrow \mathbb{R}$ such that

$$f(p) = \sum_{j=1}^m f_j(p) e_j \quad \forall p \in U.$$

Unless otherwise stated, we denote e_i the i -th standard basis vector of \mathbb{R}^n , $\forall i \in \{1, \dots, n\}$, and u_j the j -th standard basis vector of \mathbb{R}^m , $\forall j \in \{1, \dots, m\}$.

Similarly, we denote f_j the j -th component of f , $\forall j \in \{1, \dots, m\}$.

Definition 1.2.3 (Partial derivative). Let $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open, and denote $\{e_1, \dots, e_n\}$ and $\{u_1, \dots, u_m\}$ the standard bases of \mathbb{R}^n and \mathbb{R}^m , respectively. Suppose $f: U \rightarrow \mathbb{R}^m$ and $f(x) = \sum_{i=1}^m f_i(x)u_i \forall x \in U$, where $f_j: U \rightarrow \mathbb{R} \forall j \in \{1, \dots, m\}$.

Suppose $p \in U$, $i \in \{1, \dots, n\}$, and $j \in \{1, \dots, m\}$. Then, the (i, j) -partial derivative or ij^{th} partial derivative of f at $p \in U$ is

$$\frac{\partial f_i(p)}{\partial x_j} = \lim_{t \rightarrow 0} \frac{f_i(p + te_j) - f_i(p)}{t} \in \mathbb{R},$$

provided the limit exists. Let $E = \left\{ p \in U : \frac{\partial f_i(p)}{\partial x_j} \text{ exists} \right\}$. We call the map $\frac{\partial f_i}{\partial x_j}: E \rightarrow \mathbb{R}$, defined by

$$\left[\frac{\partial f_i}{\partial x_j} \right] (p) = \frac{\partial f_i(p)}{\partial x_j} \quad \forall p \in E,$$

the (i, j) -partial derivative of f .

We may also denote $\frac{\partial f_i}{\partial x_j}$, the (i, j) -partial derivative of f , more briefly by $D_j f_i$ or $\partial_{x_j} f_i$.

Theorem 1.2.1 (Differentiability Implies Continuity). Suppose $U \subset \mathbb{R}^n$ is open and $f: U \rightarrow \mathbb{R}^m$ is differentiable at $p \in U$. Then, f is continuous at p .

Proof. Suppose $U \subset \mathbb{R}^n$ is open and $f: U \rightarrow \mathbb{R}^m$ is differentiable at $p \in U$. Suppose further that

$$f(p + v) = f(p) + (Df)_p(v) + R(v), \quad \forall v \in \mathbb{R}^n \text{ such that } p + v \in U. \quad (*)$$

By definition, we have that $(Df)_p \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $\lim_{|v| \rightarrow 0} \frac{R(v)}{|v|} = 0_{\mathbb{R}^m}$. We note that $(Df)_p$ is continuous since linear maps from one normed space to another are continuous. As an immediate result, we have that $\|(Df)_p\|$ is finite by a previous theorem. We observe that

$$\begin{aligned} \lim_{|v| \rightarrow 0} \left[\|(Df)_p\| + \frac{|R(v)|}{|v|} \right] &= \|(Df)_p\| \quad \text{and} \quad \lim_{|v| \rightarrow 0} |v| = 0 \implies \\ \lim_{|v| \rightarrow 0} \|(Df)_p\| \cdot |v| + |R(v)| &= \lim_{|v| \rightarrow 0} \left(\|(Df)_p\| + \frac{|R(v)|}{|v|} \right) \cdot |v| = \|(Df)_p\| \cdot 0 = 0. \end{aligned}$$

By definition, we obtain that $\forall \epsilon > 0 \exists \delta > 0$ such that $0 < |v| < \delta$ implies that

$$\begin{aligned} \|(Df)_p\| \cdot |v| + |R(v)| &= \|(Df)_p\| \cdot |v| + |R(v)| < \epsilon \\ \implies \epsilon > \|(Df)_p\| \cdot |v| + |R(v)| &\geq |(Df)_p(v) + R(v)| \quad (\text{By the definition of } \|(Df)_p\|) \\ &\geq |(Df)_p(v) + R(v)|. \quad (\text{By the Triangle Inequality}) \end{aligned}$$

Thus, it holds that $\forall \epsilon > 0 \exists \delta > 0$ such that $p + v \in U$ and $0 < |(p + v) - p| = |v| < \delta$ implies that

$$|f(p + v) - f(p)| = |(Df)_p(v) + R(v)| < \epsilon. \quad (\text{By } (*))$$

Hence, f is continuous at p by definition. (We note that if we replace $p + v$ with x , the above statement resembles precisely the familiar $\delta - \epsilon$ definition for the continuity of a function at a point) \square

Theorem 1.2.2 (Characterization of Derivative at a Point). Let $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open. Suppose $f: U \rightarrow \mathbb{R}^m$ is differentiable at $p \in U$. Then, $(Df)_p \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is given by

$$(Df)_p(u) = \lim_{t \rightarrow 0} \frac{f(p + tu) - f(p)}{t} \quad \forall u \in \mathbb{R}^n.$$

Proof. (INCOMPLETE) □

Theorem 1.2.3 (Existence of Total Derivative Implies the Existence of Partial Derivative). *Let $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open. Suppose $f: U \rightarrow \mathbb{R}^m$ is differentiable at $p \in U$. Then, all partial derivatives of f at p exist and they are the entries of the matrix that represents the total derivative $(Df)_p$ at p . That is,*

(i) $\forall (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}, \partial_{x_j} f_i(p) \in \mathbb{R}$, and

(ii) $(Df)_p = T_{A_p}$, where $\forall (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ we have that $(A_p)_{i,j} = \partial_{x_j} f_i(p)$ and

$$A_p = \begin{bmatrix} \partial_{x_1} f_1(p) & \partial_{x_2} f_1(p) & \dots & \partial_{x_n} f_1(p) \\ \partial_{x_1} f_2(p) & \partial_{x_2} f_2(p) & \dots & \partial_{x_n} f_2(p) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} f_m(p) & \partial_{x_2} f_m(p) & \dots & \partial_{x_n} f_m(p) \end{bmatrix} \in \mathcal{M}^{m,n}(\mathbb{R}).$$

Proof. Outline: Apply the preceding theorem and let u be the standard basis vectors in \mathbb{R}^n . □

Theorem 1.2.4 (Existence and continuity of Partial Derivatives Implies Existence of Total Derivative). *Suppose $U \subset \mathbb{R}^n$ is open and $f: U \rightarrow \mathbb{R}^m$. Suppose further that $\forall (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ it holds that*

(i) $\forall u \in U, \partial_{x_j} f_i(u) \in \mathbb{R}$, and

(ii) $\partial_{x_j} f_i(u)$ is continuous.

Then, f is differentiable at p and $(Df)_p$ exists.

Proof. (INCOMPLETE)

Outline 1 Let A be the matrix whose entries are the partial derivative.

2. Let T be the linear map represented by A

□

Definition 1.2.4 (Bilinear map).

Proposition 1.2.1 (Examples of Bilinear Maps). *The usual multiplication on \mathbb{R} , dot product on \mathbb{R}^n , and matrix product are bilinear maps.*

Theorem 1.2.5 (Differentiation Rules). *(Chain Rule)*

(Linearity)

(Leibniz Rule)

Proof. (INCOMPLETE) □

Theorem 1.2.6 (Differentiability of Vector Function \iff Component-wise Differentiability). *Let $n, m \in \mathbb{N}$, $U \subset \mathbb{R}^n$ is open, and $f: U \rightarrow \mathbb{R}^m$. Then, f is differentiable at $p \in U \iff f_j$ is differentiable at $p, \forall j \in \{1, \dots, m\}$,*

Proof. (INCOMPLETE) □

Theorem 1.2.7 (General Mean Value Theorem). Suppose $U \subset \mathbb{R}^n$ is open, $[p, q] \subset U$, and $f: U \rightarrow \mathbb{R}^m$ is differentiable. Then,

$$|f(p) - f(q)| \leq M |p - q|,$$

where $M = \sup \{ \|(Df)_q\| : q \in U \}$

Proof. (INCOMPLETE) □

Definition 1.2.5 (Integrating a Matrix and a derivative at a point). Suppose $[a, b] \subset \mathbb{R}$, $m, n \in \mathbb{N}$, $A \in \mathcal{M}^{m,n}(\mathbb{R})$. Then, we define

(i)

$$\int_a^b A dt = \begin{bmatrix} \int_a^b a_{1,1} dt & \int_a^b a_{1,2} dt & \cdots & \int_a^b a_{1,n} dt \\ \int_a^b a_{2,1} dt & \int_a^b a_{2,2} dt & \cdots & \int_a^b a_{2,n} dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b a_{m,1} dt & \int_a^b a_{m,2} dt & \cdots & \int_a^b a_{m,n} dt \end{bmatrix},$$

where $\forall (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ $a_{i,j}$ is the (i, j) -entry of A .

(ii) Suppose $U \subset \mathbb{R}^n$ and $f: U \rightarrow \mathbb{R}^m$ is differentiable at $p \in U$ with derivative $(Df)_p$ at p . Then, we define

$$\int_b^a (Df)_p dt = \int_b^a B_p dt,$$

where $B_p \in \mathcal{M}^{m,n}(\mathbb{R})$ is the matrix representing $(Df)_p$.

Definition 1.2.6 (C^n). Suppose f is a n -th order differentiable map and $\forall i \in \{1, \dots, n\}$, $f^{(i)}$ is continuous. Then, f is continuous n -th order differentiable (or n -th order continuously differentiable) and we say that $f \in C^n$.

Theorem 1.2.8 (C^1 Mean Value Theorem). Suppose $U \subset \mathbb{R}^n$ is open, $[p, q] \subset U$, and $f: U \rightarrow \mathbb{R}^m \in C^1$. Then,

$$f(q) - f(p) = T \cdot (q - p)$$

where T is the average derivative of f on the segment $[p, q]$ with

$$T = \int_0^1 (Df)_{p+t(q-p)} dt \in \mathcal{M}^{m,n}(\mathbb{R}).$$

Proof. (INCOMPLETE) □

Corollary 1.2.1 (Connectedness, Differentiability, and Trivial Derivative Implies Constantness).

Theorem 1.2.9 (Interchanging Limits).

1.3 Higher Derivatives

1.4 Implicit and Inverse Functions

1.5 *The Rank Theorem

1.6 *Lagrange Multipliers

1.7 Multiple Integrals

2 Lebesgue Theory

2.1

2.2

2.3

2.4

2.5