Second Course in Analysis

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Contents

1 Multivariable Calculus

1.1 Linear Algebra

Definition 1.1.1 (Matrix transformation)**.** *Suppose $m, n \in \mathbb{N}$ and $A \in \mathcal{M}^{m,n}(\mathbb{R})$. Then, the matrix transformation represented by A is the map $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ defined by*

$$T_A(v) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} v_j e_i \ \forall v \in \mathbb{R}^n,$$

where $v = \sum v_j e_j \in \mathbb{R}^n$ and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

Equivalently, T_A is defined by the matrix multiplication

$$T_A(v) = transpose(A \cdot transpose(v)) \ \forall v \in \mathbb{R}^n,$$

where we treat v strictly as a 1-by-n matrix $\forall v \in \mathbb{R}^n$.

Definition 1.1.2 (Notations: $\mathcal{L}$, $\mathcal{M}$, $\mathbb{F}$, n, m)**.** *Suppose $m, n \in \mathbb{N}$. We denote by*

(i) $\mathcal{M}^{m,n}(\mathbb{R})$ the collection of all m-by-n matrix with real entries;

(ii) $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ the set of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

When no misunderstanding may arise and sufficient context is supplied, we denote, more briefly, $\mathcal{M}^{m,n}(\mathbb{R})$ and $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ by $\mathcal{M}$ and $\mathcal{L}$ respectively.

Unless otherwise stated, we assume that $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $m, n \in \mathbb{N}$.

Proposition 1.1.1. *Matrix transformations are linear transformations.*

Proposition 1.1.2. *Suppose $m, n \in \mathbb{N}$. Then, $\mathcal{M}^{m,n}(\mathbb{R})$ is a vector space with $\dim \mathcal{M}^{m,n}(\mathbb{R}) = mn$.*

Proposition 1.1.3. *Suppose $m, n \in \mathbb{N}$. Then, $\mathcal{M}^{m,n}(\mathbb{R})$ and $\mathbb{R}^{mn}$ are isomorphic.*

Proposition 1.1.4. *Suppose $m, n \in \mathbb{N}$. Then, $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a vector space and $\dim \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = nm$.*

Proposition 1.1.5 (Canonical Isomorphism Induced by Matrix Transformation: $\mathcal{T}$)**.** *Suppose $m, n \in \mathbb{N}$. Then, $\mathcal{T} \colon \mathcal{M}^{m,n}(\mathbb{R}) \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ defined by $\mathcal{T}(A) = T_A \ \forall A \in \mathcal{M}^{m,n}(\mathbb{R})$ is an isomorphism*

Theorem 1.1.1 (Composition of Matrix Transformations)**.** *Suppose $m, k, n \in \mathbb{N}$, $A \in \mathcal{M}^{m,k}(\mathbb{F})$, and $B \in \mathcal{M}^{k,n}(\mathbb{F})$. Then, $T_A \circ T_B = T_{AB}$.*

Definition 1.1.3 (Norm on a vector space)**.** *Suppose V is a vector space over field $\mathbb{F}$. A norm on V is a map $||_V : V \to \mathbb{R}$ satisfying the following properties:*

(i) $|v|_V \geq 0 \ \forall v \in V$ with $|v|_V = 0 \iff v = 0_V$;

(ii) $|kv|_V = |k| \, |v|_V \ \ \forall k \in \mathbb{F}, v \in V$.

(iii) $|v + w|_V \leq |v|_V + |w|_V \ \ \forall v, w \in V$.

When no misunderstanding may arise and sufficient context is supplied, we may denote $||_V$, more briefly, by $||$.

Proposition 1.1.6 (Common Norms on $\mathbb{R}^n$). *Let $n \in \mathbb{N}$. Then, the following maps are norms on $\mathbb{R}^n$:*

(Euclidean Norm or l_2 norm) $||_2 : \mathbb{R}^n \to \mathbb{R}$ where

$$|x|_2 = \sqrt{\sum_{i=1}^{n} {x_i}^2} \; \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

(Supremum norm or l_∞ norm) $||_\infty : \mathbb{R}^n \to \mathbb{R}$ where

$$|x|_\infty = \max \{|x_i| : i \in \{1, \ldots, n\}\} \;\; \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

(l_1 norm) $||_1 : \mathbb{R}^n \to \mathbb{R}$ where

$$|x|_1 = \sum_{i=1}^{n} |x_i| \; \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Definition 1.1.4 (Normed space). *A normed space is a vector space V along with a norm $||$ defined on V.*

Proposition 1.1.7 (Norm-Induced Metric; Normed Spaces are Metric Spaces). *Suppose $(V, ||)$ is a normed space. Then,*

(i) $d \colon V \times V \to \mathbb{R}$ defined by $d(v, w) = |v - w|$, $\forall (v, w) \in V \times V$, is a metric on V.

(ii) (v, d) is a metric space.

Definition 1.1.5 (Banach space). *A vector space is a Banach space if it is a complete normed space.*

Definition 1.1.6 (Operator norm and bounded operator). *Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$. Then, the operator norm of T is*

$$||T|| = \sup \left\{ \frac{|T(v)|_W}{|v|_V} : v \neq 0_V \right\}.$$

An operator is bounded if its operator norm is finite.

Theorem 1.1.2 (Operator Norm Identity). *Suppose V, W are normed spaces where $T \in \mathcal{L}(V, W)$. Then,*

$$\begin{aligned} ||T|| &= \sup \{|T(v)| : |v| < 1\} \\ &= \sup \{|T(v)| : |v| \leq 1\} \\ &= \sup \{|T(v)| : |v| = 1\} \\ &= \inf \{M > 0 : v \in V \implies |T(v)| \leq M |v|\}. \end{aligned}$$

Proof. (INCOMPLETE) □

Proposition 1.1.8 (Operator Norm Properties). *Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$. Then, the following statements hold:*

(i) $||T|| \geq 0$;

(ii) $||T|| = 0 \iff T = 0_{\mathcal{L}(V,W)}$;

(iii) Suppose U is a normed space and $S \in \mathcal{L}(U, V)$. Then, $||T \circ S|| \leq ||T|| \, ||S||$.

Proof. (ii) Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$.

($\implies$) Suppose that $||T|| = 0$. It follows that $\frac{|T(v)|_W}{|v|_V} \leq 0$, $\forall v \in V/\{0_V\}$. We note that, by the definition of a norm, $|T(v)|_W, |v|_V \geq 0\ \forall v \in V$. Thus, we obtain that, $\forall v \in V/\{0_V\}$,

$$\frac{|T(v)|_W}{|v|_V} \geq 0 \implies \frac{|T(v)|_W}{|v|_V} = 0 \implies |T(v)|_W = 0$$
$$\implies T(v) = 0_W. \qquad \text{(By the definition of norm)}$$

In addition, certainly $T(0_V) = 0_W$. Hence, we proved that $T(v) = 0_W$, $\forall v \in V$. That is, we show that $T = 0_{\mathcal{L}(V,W)}$, as desired.

($\impliedby$) Suppose $T = 0_{\mathcal{L}(V,W)}$. It follows that $T(v) = 0_W\ \forall v \in V$. Thus, we have that $|T(v)|_W = 0$, $\forall v \in V$, by the definition of a norm. As an immediate result, we obtain that

$$||T|| = \sup\left\{\frac{|T(v)|_W}{|v|_V} : v \neq 0_V\right\} = \sup\{0 : v \neq 0_V\} = 0.$$

$\square$

Theorem 1.1.3 ($\mathcal{L}(V, W)$ is a Normed Space)**.** *Suppose V and W are normed spaces. Then, $\mathcal{L}(V, W)$ along with operator norm $||\ ||: \mathcal{L} \to \mathbb{R}$ is a normed space.*

Definition 1.1.7 (Comparability of norms)**.**

Proposition 1.1.9 (Comparability Induces an Equivalence Relation on the Set of Norms)**.**

Theorem 1.1.4 (All Norms on $\mathbb{R}^n$ are Comparable)**.**

Corollary 1.1.1 (Norms on Finite-Dimensional Normed Space are Comparable)**.**

Theorem 1.1.5 (Finite Operator Norm and Equivalent Conditions)**.** *Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$. Then, the following conditions are equivalent:*

(i) $||T|| < \infty$;

(ii) T is uniformly continuous;

(iii) T is continuous;

(iv) T is continuous at the origin (that is, at 0_V).

Proof. (INCOMPLETE) Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$.

($i \implies ii$): Suppose $||T|| < \infty$. By definition, we have that $\exists m > 0$ such that $m = ||T||$

$$\frac{|T(v)|_W}{|v|_V} \leq ||T||\ \ \forall v \in V/\{0_V\} \implies |T(v)|_W \leq ||T||\,|v|_V\ \ \forall v \in V/\{0_V\}$$

Consider arbitrary $v, w \in V$.

($ii \implies iii$) & ($iii \implies iv$): Trivial.

($iv \implies i$): Suppose T is continuous at 0_V. $\square$

Theorem 1.1.6 (Characteristics of Linear Maps on Normed Spaces)**.** *Suppose $T \in \mathcal{L}(\mathbb{R}^n, W)$, where W is a normed space. Then,*

(i) T is continuous, and

(ii) T is an isomorphism implies T is a homeomorphism.

Proof. (INCOMPLETE) $\square$

Corollary 1.1.2. *Suppose V, W are finite-dimensional normed spaces. Then,*

(i) $T \in \mathcal{L}(V, W) \implies T$ is continuous, and

(ii) $\phi \in \mathcal{L}(V, W)$ is an isomorphism implies ϕ is a homeomorphism.

Proof. (INCOMPLETE) $\square$

Corollary 1.1.3. *(i) Suppose V is a finite-dimensional normed space with norms $||_a$ and $||_b$. Then, the identity map I on V is a homomorphism between the normed spaces $(V, ||_a)$ and $(V, ||_b)$.*

(ii) $\mathcal{T}\colon \mathcal{M} \to \mathcal{L}$ is a homeomorphism.

Proof. (INCOMPLETE) $\square$

Definition 1.1.8 (Conorm)**.**

Exercise 1.1.1 (Determine an Operator Norm)**.** *Consider the dilation map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (2x, y)$. Prove $||T|| = 2$.*

Proof. Outline: show $||T|| \leq 2$.

Find $(a, b) \in \mathbb{R}^2$ such that $\frac{|T(a,b)|}{|(a,b)|} = 2$.

Use the definition of sup to prove the conclusion, $\square$

1.2 Derivatives

Definition 1.2.1 ((Total) Derivative)**.** *Let $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open. Suppose $f: U \to \mathbb{R}^m$.*

*(i) The **derivative (or total derivative)** $(Df)_p$ **of** f **at** $p \in U$ is a map, if it exists, $T: \mathbb{R}^n \to \mathbb{R}^m$ such that*

(a) T is a linear map, and

(b) T satisfies

$$f(p+v) = f(p) + T(v) + R(v) \text{ for all sufficiently small } v \in \mathbb{R}^n$$
$$\implies \lim_{|v| \to 0} \frac{R(v)}{|v|} = 0_{\mathbb{R}^m} \tag{Pugh}$$

or, equivalently,

$$f(p+v) = f(p) + T(v) + R(v) \text{ for all sufficiently small } v \in \mathbb{R}^n$$
$$\implies \lim_{v \to 0_{\mathbb{R}^n}} \frac{|R(v)|}{|v|} = 0, \tag{Rudin}$$

or, equivalently,

$$\lim_{v \to 0_{\mathbb{R}^n}} \frac{|f(p+v) - T(v) - f(p)|}{|v|} = 0, \tag{Rudin}$$

where $R(v) \in \mathbb{R}^m$ denotes the Taylor remainder for $f(p+v)$.

*(ii) We say that f is **differentiable at** $p \in U$ if $(Df)_p$ exists, and f is **differentiable** if f is differentiable at p, $\forall p \in U$.*

*(iii) Let $E = \{p \in U : (Df)_p \text{ exists}\}$. We call the map $Df: E \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, defined by $[Df](p) = (Df)_p$ $\forall p \in E$, the **derivative (or total derivative) of** f.*

Note that we may also denote Df by f'.

Remark 1.2.1. *Recall that $\exists r_p > 0$ such that $q \in \mathbb{R}^n$ and $d(p,q) < r_p \implies q \in E$, since E is open. By sufficiently small $v \in \mathbb{R}^n$, we mean that v is such that $d(p, p+v) < r_p$ so $p + v \in E$.*

Remark 1.2.2. *The choice of T is unique, since a limit is unique, provided it exists. See proof below.*

Definition 1.2.2 (Notations: e_i, u_j, f_j)**.** *Let $n, m \in \mathbb{N}$. Denote the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ by $\{e_1, \ldots, e_n\}$ and $\{u_1, \ldots, u_m\}$, respectively.*

Suppose $U \subset \mathbb{R}^n$, $f: U \to \mathbb{R}^m$, and $\exists f_1, \ldots, f_m: U \to \mathbb{R}$ such that

$$f(p) = \sum_{j=1}^{m} f_j(p) e_j \ \forall p \in U.$$

Unless otherwise stated, we denote e_i the i-th standard basis vector of $\mathbb{R}^n$, $\forall i \in \{1, \ldots, n\}$, and u_j the j-th standard basis vector of $\mathbb{R}^m$, $\forall j \in \{1, \ldots, m\}$.

Similarly, we denote f_j the j-th component of f, $\forall j \in \{1, \ldots, m\}$.

Definition 1.2.3 (Partial derivative). *Let $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open, and denote $\{e_1, \ldots, e_n\}$ and $\{u_1, \ldots, u_m\}$ the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Suppose $f \colon U \to \mathbb{R}^m$ and $f(x) = \sum_{i=1}^m f_i(x)u_i \ \forall x \in U$, where $f_j \colon U \to \mathbb{R} \ \forall j \in \{1, \ldots, m\}$.*

Suppose $p \in U$, $i \in \{1, \ldots, n\}$, and $j \in \{1, \ldots, m\}$. Then, the (i,j)-partial derivative or ij^{th} ***partial derivative of f at*** *$p \in U$ is*

$$\frac{\partial f_i(p)}{\partial x_j} = \lim_{t \to 0} \frac{f_i(p + te_j) - f_i(p)}{t} \in \mathbb{R},$$

provided the limit exists. Let $E = \left\{ p \in U : \frac{\partial f_i(p)}{\partial x_j} \text{ exists} \right\}$. We call the map $\frac{\partial f_i}{\partial x_j} \colon E \to \mathbb{R}$, defined by

$$\left[\frac{\partial f_i}{\partial x_j}\right](p) = \frac{\partial f_i(p)}{\partial x_j} \ \forall p \in E,$$

the ***(i,j)-partial derivative of*** *f.*

We may also denote $\frac{\partial f_i}{\partial x_j}$, the (i,j)-partial derivative of f, more briefly by $D_j f_i$ or $\partial_{x_j} f_i$.

Theorem 1.2.1 (Differentiability Implies Continuity). *Suppose $U \subset \mathbb{R}^n$ is open and $f \colon U \to \mathbb{R}^m$ is differentiable at $p \in U$. Then, f is continuous at p.*

Proof. Suppose $U \subset \mathbb{R}^n$ is open and $f \colon U \to \mathbb{R}^m$ is differentiable at $p \in U$. Suppose further that

$$f(p + v) = f(p) + (Df)_p(v) + R(v), \ \forall v \in \mathbb{R}^n \text{ such that } p + v \in U. \tag{*}$$

By definition, we have that $(Df)_p \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $\lim_{|v| \to 0} \frac{R(v)}{|v|} = 0_{\mathbb{R}^m}$. We note that $(Df)_p$ is continuous since linear maps from one normed space to another are continuous. As an immediate result, we have that $||(Df)_p||$ is finite by a previous theorem. We observe that

$$\lim_{|v| \to 0} \left[||(Df)_p|| + \frac{|R(v)|}{|v|} \right] = ||(Df)_p|| \text{ and } \lim_{|v| \to 0} |v| = 0 \implies$$
$$\lim_{|v| \to 0} ||(Df)_p|| \cdot |v| + |R(v)| = \lim_{|v| \to 0} \left(||(Df)_p|| + \frac{|R(v)|}{|v|} \right) \cdot |v| = ||(Df)_p|| \cdot 0 = 0.$$

By definition, we obtain that $\forall \epsilon > 0 \ \exists \delta > 0$ such that $0 < |v| < \delta$ implies that

$$\begin{aligned} &|||(Df)_p|| \cdot |v| + |R(v)|| = ||(Df)_p|| \cdot |v| + |R(v)| < \epsilon \\ \implies &\epsilon > ||(Df)_p|| \cdot |v| + |R(v)| \geq |(Df)_p(v)| + |R(v)| && \text{(By the definition of } ||(Df)_p||) \\ &\geq |(Df)_p(v) + R(v)|. && \text{(By the Triangle Inequality)} \end{aligned}$$

Thus, it holds that $\forall \epsilon > 0 \ \exists \delta > 0$ such that $p + v \in U$ and $0 < |(p + v) - p| = |v| < \delta$ implies that

$$|f(p + v) - f(p)| = |(Df)_p(v) + R(v)| < \epsilon. \qquad \text{(By (*))}$$

Hence, f is continuous at p by definition. (We note that if we replace $p + v$ with x, the above statement resembles precisely the familiar $\delta - \epsilon$ definition for the continuity of a function at a point) □

Theorem 1.2.2 (Characterization of Derivative at a Point). *Let $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open. Suppose $f \colon U \to \mathbb{R}^m$ is differentiable at $p \in U$. Then, $(Df)_p \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is given by*

$$(Df)_p(u) = \lim_{t \to 0} \frac{f(p + tu) - f(p)}{t} \ \forall u \in \mathbb{R}^n.$$

Proof. (INCOMPLETE) □

Theorem 1.2.3 (Existence of Total Derivative Implies the Existence of Partial Derivative). *Let $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open. Suppose $f : U \to \mathbb{R}^m$ is differentiable at $p \in U$. Then, all partial derivatives of f at p exist and they are the entries of the matrix that represents the total derivative $(Df)_p$ at p. That is,*

(i) $\forall (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$, $\partial x_j f_i(p) \in \mathbb{R}$, *and*

(ii) $(Df)_p = T_{A_p}$, *where* $\forall (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ *we have that* $(A_p)_{i,j} = \partial_{x_j} f_i(p)$ *and*

$$A_p = \begin{bmatrix} \partial_{x_1} f_1(p) & \partial_{x_2} f_1(p) & \dots & \partial_{x_n} f_1(p) \\ \partial_{x_1} f_2(p) & \partial_{x_2} f_2(p) & \dots & \partial_{x_n} f_2(p) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} f_m(p) & \partial_{x_2} f_m(p) & \dots & \partial_{x_n} f_m(p) \end{bmatrix} \in \mathcal{M}^{m,n}(\mathbb{R}).$$

Proof. Outline: Apply the preceding theorem and let u be the standard basis vectors in $\mathbb{R}^n$. □

Theorem 1.2.4 (Existence and continuity of Partial Derivatives Implies Existence of Total Derivative.). *Suppose $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^m$. Suppose further that $\forall (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ it holds that*

(i) $\forall u \in U$, $\partial_{x_j} f_i(u) \in \mathbb{R}$, *and*

(ii) $\partial_{x_j} f_i(u)$ *is continuous.*

Then, f is differentiable at p and $(Df)_p$ exists.

Proof. (INCOMPLETE)

Outline 1 Let A be the matrix whose entries are the partial derivative.

2. Let T be the linear map represented by A

□

Definition 1.2.4 (Bilinear map).

Proposition 1.2.1 (Examples of Bilinear Maps). *The usual multiplication on $\mathbb{R}$, dot product on $\mathbb{R}^n$, and matrix product are bilinear maps.*

Theorem 1.2.5 (Differentiation Rules). *(Chain Rule)*

(Linearity)

(Leibniz Rule)

Proof. (INCOMPLETE) □

Theorem 1.2.6 (Differentiability of Vector Function $\iff$ Component-wise Differentiability). *Let $n, m \in \mathbb{N}$, $U \subset \mathbb{R}^n$ is open, and $f : U \to \mathbb{R}^m$. Then, f is differentiable at $p \in U \iff f_j$ is differentiable at p, $\forall j \in \{1, \dots, m\}$,*

Proof. (INCOMPLETE) □

Theorem 1.2.7 (General Mean Value Theorem)**.** *Suppose $U \subset \mathbb{R}^n$ is open, $[p, q] \subset U$, and $f : U \to \mathbb{R}^m$ is differentiable. Then,*

$$|f(p) - f(q)| \leq M\,|p - q|\,,$$

where $M = sup\,\{||(Df)_q : q \in U||\}$

Proof. (INCOMPLETE) □

Definition 1.2.5 (Integrating a Matrix and a derivative at a point)**.** *Suppose $[a, b] \subset \mathbb{R}$, $m, n \in \mathbb{N}$, $A \in \mathcal{M}^{m,n}(\mathbb{R})$. Then, we define*

(i)

$$\int_a^b A dt = \begin{bmatrix} \int_a^b a_{1,1}dt & \int_a^b a_{1,2}dt & \cdots \int_a^b a_{1,n}dt \\ \int_a^b a_{2,1}dt & \int_a^b a_{2,2}dt & \cdots \int_a^b a_{2,n}dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b a_{m,1}dt & \int_a^b a_{m,2}dt & \cdots \int_a^b a_{m,n}dt \end{bmatrix},$$

where $\forall (i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ $a_{i,j}$ is the (i,j)-entry of A.

(ii) Suppose $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is differentiable at $p \in U$ with derivative $(Df)_p$ at p. Then, we define

$$\int_b^a (Df)_p dt = \int_b^a B_p dt,$$

where $B_p \in \mathcal{M}^{m,n}(\mathbb{R})$ is the matrix representing $(Df)_p$.

Definition 1.2.6 (C^n)**.** *Suppose f is a n-th order differentiable map and $\forall i \in \{1, \ldots, n\}$, $f^{(i)}$ is continuous. Then, f is continuous n-th order differentiable (or n-th order continuously differentiable) and we say that $f \in C^n$.*

Theorem 1.2.8 (C^1 Mean Value Theorem)**.** *Suppose $U \subset \mathbb{R}^n$ is open, $[p, q] \subset U$, and $f : U \to \mathbb{R}^m \in C^1$. Then,*

$$f(q) - f(p) = T \cdot (q - p)$$

where T is the average derivative of f on the segment $[p, q]$ with

$$T = \int_0^1 (Df)_{p+t(q-p)} dt \in \mathcal{M}^{m,n}(\mathbb{R}).$$

Proof. (INCOMPLETE) □

Corollary 1.2.1 (Connectedness, Differentiability, and Trivial Derivative Implies Constantness)**.**

Theorem 1.2.9 (Interchaning Limits)**.**

1.3 Higher Derivatives

1.4 Implicit and Inverse Functions

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1.7 Multiple Integrals

2 Lebesgue Theory

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