Second Course in Analysis

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Contents

1 Multivariable Calculus

1.1 Linear Algebra

Definition 1.1.1 (Matrix transformation)**.** *Suppose $m, n \in \mathbb{N}$ and $A \in \mathcal{M}^{m,n}(\mathbb{R})$. Then, the matrix transformation represented by A is the map $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ defined by*

$$T_A(v) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} v_j e_i \ \forall v \in \mathbb{R}^n,$$

where $v = \sum v_j e_j \in \mathbb{R}^n$ and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

Equivalently, T_A is defined by the matrix multiplication

$$T_A(v) = transpose(A \cdot transpose(v)) \ \forall v \in \mathbb{R}^n,$$

where we treat v strictly as a 1-by-n matrix $\forall v \in \mathbb{R}^n$.

Definition 1.1.2 (Notations: $\mathcal{L}$, $\mathcal{M}$, $\mathbb{F}$, n, m)**.** *Suppose $m, n \in \mathbb{N}$. We denote by*

(i) $\mathcal{M}^{m,n}(\mathbb{R})$ the collection of all m-by-n matrix with real entries;

(ii) $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ the set of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

When no misunderstanding may arise and sufficient context is supplied, we denote, more briefly, $\mathcal{M}^{m,n}(\mathbb{R})$ and $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ by $\mathcal{M}$ and $\mathcal{L}$ respectively.

Unless otherwise stated, we assume that $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $m, n \in \mathbb{N}$.

Proposition 1.1.1. *Matrix transformations are linear transformations.*

Proposition 1.1.2. *Suppose $m, n \in \mathbb{N}$. Then, $\mathcal{M}^{m,n}(\mathbb{R})$ is a vector space with $\dim \mathcal{M}^{m,n}(\mathbb{R}) = mn$.*

Proposition 1.1.3. *Suppose $m, n \in \mathbb{N}$. Then, $\mathcal{M}^{m,n}(\mathbb{R})$ and $\mathbb{R}^{mn}$ are isomorphic.*

Proposition 1.1.4. *Suppose $m, n \in \mathbb{N}$. Then, $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a vector space and $\dim \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = nm$.*

Proposition 1.1.5 (Canonical Isomorphism Induced by Matrix Transformation: $\mathcal{T}$)**.** *Suppose $m, n \in \mathbb{N}$. Then, $\mathcal{T} \colon \mathcal{M}^{m,n}(\mathbb{R}) \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ defined by $\mathcal{T}(A) = T_A \ \forall A \in \mathcal{M}^{m,n}(\mathbb{R})$ is an isomorphism*

Theorem 1.1.1 (Composition of Matrix Transformations)**.** *Suppose $m, k, n \in \mathbb{N}$, $A \in \mathcal{M}^{m,k}(\mathbb{F})$, and $B \in \mathcal{M}^{k,n}(\mathbb{F})$. Then, $T_A \circ T_B = T_{AB}$.*

Definition 1.1.3 (Norm on a vector space)**.** *Suppose V is a vector space over field $\mathbb{F}$. A norm on V is a map $||_V : V \to \mathbb{R}$ satisfying the following properties:*

(i) $|v|_V \geq 0 \ \forall v \in V$ with $|v|_V = 0 \iff v = 0_V$;

(ii) $|kv|_V = |k| \, |v|_V \ \ \forall k \in \mathbb{F}, v \in V$.

(iii) $|v + w|_V \leq |v|_V + |w|_V \ \ \forall v, w \in V$.

When no misunderstanding may arise and sufficient context is supplied, we may denote $||_V$, more briefly, by $||$.

Proposition 1.1.6 (Common Norms on $\mathbb{R}^n$). *Let $n \in \mathbb{N}$. Then, the following maps are norms on $\mathbb{R}^n$:*

(Euclidean Norm or l_2 norm) $||_2 : \mathbb{R}^n \to \mathbb{R}$ *where*

$$|x|_2 = \sqrt{\sum_{i=1}^{n} {x_i}^2} \ \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

(Supremum norm or l_∞ norm) $||_\infty : \mathbb{R}^n \to \mathbb{R}$ *where*

$$|x|_\infty = \max \{|x_i| : i \in \{1, \ldots, n\}\} \ \ \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

(l_1 norm) $||_1 : \mathbb{R}^n \to \mathbb{R}$ *where*

$$|x|_1 = \sum_{i=1}^{n} |x_i| \ \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Definition 1.1.4 (Normed space). *A normed space is a vector space V along with a norm $||$ defined on V.*

Proposition 1.1.7 (Norm-Induced Metric; Normed Spaces are Metric Spaces). *Suppose $(V, ||)$ is a normed space. Then,*

(i) $d \colon V \times V \to \mathbb{R}$ defined by $d(v, w) = |v - w|$, $\forall (v, w) \in V \times V$, is a metric on V.

(ii) (v, d) is a metric space.

Definition 1.1.5 (Banach space). *A vector space is a Banach space if it is a complete normed space.*

Definition 1.1.6 (Operator norm and bounded operator). *Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$. Then, the operator norm of T is*

$$||T|| = \sup \left\{ \frac{|T(v)|_W}{|v|_V} : v \neq 0_V \right\}.$$

An operator is bounded if its operator norm is finite.

Theorem 1.1.2 (Operator Norm Identity). *Suppose V, W are normed spaces where $T \in \mathcal{L}(V, W)$. Then,*

$$\begin{aligned} ||T|| &= \sup \{|T(v)| : |v| < 1\} \\ &= \sup \{|T(v)| : |v| \leq 1\} \\ &= \sup \{|T(v)| : |v| = 1\} \\ &= \inf \{M > 0 : v \in V \implies |T(v)| \leq M |v|\}. \end{aligned}$$

Proof. (INCOMPLETE) □

Proposition 1.1.8 (Operator Norm Properties). *Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$. Then, the following statements hold:*

(i) $||T|| \geq 0$;

(ii) $||T|| = 0 \iff T = 0_{\mathcal{L}(V,W)}$;

(iii) Suppose U is a normed space and $S \in \mathcal{L}(U, V)$. Then, $||T \circ S|| \leq ||T|| \, ||S||$.

Proof. (ii) Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$.

($\implies$) Suppose that $||T|| = 0$. It follows that $\frac{|T(v)|_W}{|v|_V} \leq 0$, $\forall v \in V/\{0_V\}$. We note that, by the definition of a norm, $|T(v)|_W, |v|_V \geq 0 \; \forall v \in V$. Thus, we obtain that, $\forall v \in V/\{0_V\}$,

$$\begin{aligned} \frac{|T(v)|_W}{|v|_V} \geq 0 \implies \frac{|T(v)|_W}{|v|_V} = 0 &\implies |T(v)|_W = 0 \\ &\implies T(v) = 0_W. \qquad \text{(By the definition of norm)} \end{aligned}$$

In addition, certainly $T(0_V) = 0_W$. Hence, we proved that $T(v) = 0_W$, $\forall v \in V$. That is, we show that $T = 0_{\mathcal{L}(V,W)}$, as desired.

($\impliedby$) Suppose $T = 0_{\mathcal{L}(V,W)}$. It follows that $T(v) = 0_W \; \forall v \in V$. Thus, we have that $|T(v)|_W = 0$, $\forall v \in V$, by the definition of a norm. As an immediate result, we obtain that

$$||T|| = \sup\left\{\frac{|T(v)|_W}{|v|_V} : v \neq 0_V\right\} = \sup\{0 : v \neq 0_V\} = 0.$$

□

Theorem 1.1.3 ($\mathcal{L}(V, W)$ is a Normed Space)**.** *Suppose V and W are normed spaces. Then, $\mathcal{L}(V, W)$ along with operator norm $||\;|| : \mathcal{L} \to \mathbb{R}$ is a normed space.*

Definition 1.1.7 (Comparability of norms)**.**

Proposition 1.1.9 (Comparability Induces an Equivalence Relation on the Set of Norms)**.**

Theorem 1.1.4 (All Norms on $\mathbb{R}^n$ are Comparable)**.**

Corollary 1.1.1 (Norms on Finite-Dimensional Normed Space are Comparable)**.**

Theorem 1.1.5 (Finite Operator Norm and Equivalent Conditions)**.** *Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$. Then, the following conditions are equivalent:*

(i) $||T|| < \infty$;

(ii) T is uniformly continuous;

(iii) T is continuous;

(iv) T is continuous at the origin (that is, at 0_V).

Proof. (INCOMPLETE) Suppose V, W are normed spaces and $T \in \mathcal{L}(V, W)$.

($i \implies ii$): Suppose $||T|| < \infty$. By definition, we have that $\exists m > 0$ such that $m = ||T||$

$$\frac{|T(v)|_W}{|v|_V} \leq ||T|| \;\; \forall v \in V/\{0_V\} \implies |T(v)|_W \leq ||T||\,|v|_V \;\; \forall v \in V/\{0_V\}$$

Consider arbitrary $v, w \in V$.

($ii \implies iii$) & ($iii \implies iv$): Trivial.

($iv \implies i$): Suppose T is continuous at 0_V. □

Theorem 1.1.6 (Characteristics of Linear Maps on Normed Spaces)**.** *Suppose $T \in \mathcal{L}(\mathbb{R}^n, W)$, where W is a normed space. Then,*

(i) T is continuous, and

(ii) T is an isomorphism implies T is a homeomorphism.

Proof. (INCOMPLETE) $\square$

Corollary 1.1.2. *Suppose V, W are finite-dimensional normed spaces. Then,*

(i) $T \in \mathcal{L}(V, W) \implies T$ is continuous, and

(ii) $\phi \in \mathcal{L}(V, W)$ is an isomorphism implies ϕ is a homeomorphism.

Proof. (INCOMPLETE) $\square$

Corollary 1.1.3. *(i) Suppose V is a finite-dimensional normed space with norms $||_a$ and $||_b$. Then, the identity map I on V is a homomorphism between the normed spaces $(V, ||_a)$ and $(V, ||_b)$.*

(ii) $\mathcal{T} \colon \mathcal{M} \to \mathcal{L}$ is a homeomorphism.

Proof. (INCOMPLETE) $\square$

Definition 1.1.8 (Conorm)**.**

Exercise 1.1.1 (Determine an Operator Norm)**.** *Consider the dilation map $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (2x, y)$. Prove $||T|| = 2$.*

Proof. Outline: show $||T|| \leq 2$.

Find $(a, b) \in \mathbb{R}^2$ such that $\frac{|T(a,b)|}{|(a,b)|} = 2$.

Use the definition of sup to prove the conclusion, $\square$

1.2 Derivatives

Definition 1.2.1 ((Total) Derivative)**.** *Let $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open. Suppose $f \colon U \to \mathbb{R}^m$.*

(i) The ***derivative (or total derivative)*** *$(Df)_p$* ***of*** *f* ***at*** *$p \in U$ is a map, if it exists, $T \colon \mathbb{R}^n \to \mathbb{R}^m$ such that*

(a) T is a linear map, and

(b) T satisfies

$$f(p+v) = f(p) + T(v) + R(v) \text{ for all sufficiently small } v \in \mathbb{R}^n$$
$$\implies \lim_{|v| \to 0} \frac{R(v)}{|v|} = 0_{\mathbb{R}^m} \tag{Pugh}$$

or, equivalently,

$$f(p+v) = f(p) + T(v) + R(v) \text{ for all sufficiently small } v \in \mathbb{R}^n$$
$$\implies \lim_{v \to 0_{\mathbb{R}^n}} \frac{|R(v)|}{|v|} = 0, \tag{Rudin}$$

or, equivalently,

$$\lim_{v \to 0_{\mathbb{R}^n}} \frac{|f(p+v) - T(v) - f(p)|}{|v|} = 0, \tag{Rudin}$$

where $R(v) \in \mathbb{R}^m$ denotes the Taylor remainder for $f(p+v)$.

(ii) We say that f is ***differentiable at*** *$p \in U$ if $(Df)_p$ exists, and f is* ***differentiable*** *if f is differentiable at p, $\forall p \in U$.*

(iii) Let $E = \{p \in U : (Df)_p \text{ exists}\}$. We call the map $Df \colon E \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, defined by $[Df](p) = (Df)_p$ $\forall p \in E$, the ***derivative (or total derivative) of*** *f.*

Note that we may also denote Df by f'.

Remark 1.2.1. *Recall that $\exists r_p > 0$ such that $q \in \mathbb{R}^n$ and $d(p,q) < r_p \implies q \in E$, since E is open. By sufficiently small $v \in \mathbb{R}^n$, we mean that v is such that $d(p, p+v) < r_p$ so $p + v \in E$.*

Remark 1.2.2. *The choice of T is unique, since a limit is unique, provided it exists. See proof below.*

Definition 1.2.2 (Notations: e_i, u_j, f_j)**.** *Let $n, m \in \mathbb{N}$. Denote the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ by $\{e_1, \ldots, e_n\}$ and $\{u_1, \ldots, u_m\}$, respectively.*

Suppose $U \subset \mathbb{R}^n$, $f \colon U \to \mathbb{R}^m$, and $\exists f_1, \ldots, f_m \colon U \to \mathbb{R}$ such that

$$f(p) = \sum_{j=1}^{m} f_j(p) e_j \ \forall p \in U.$$

Unless otherwise stated, we denote e_i the i-th standard basis vector of $\mathbb{R}^n$, $\forall i \in \{1, \ldots, n\}$, and u_j the j-th standard basis vector of $\mathbb{R}^m$, $\forall j \in \{1, \ldots, m\}$.

Similarly, we denote f_j the j-th component of f, $\forall j \in \{1, \ldots, m\}$.

Definition 1.2.3 (Partial derivative). *Let $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open, and denote $\{e_1, \ldots, e_n\}$ and $\{u_1, \ldots, u_m\}$ the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Suppose $f: U \to \mathbb{R}^m$ and $f(x) = \sum_{i=1}^m f_i(x)u_i \ \forall x \in U$, where $f_j: U \to \mathbb{R} \ \forall j \in \{1, \ldots, m\}$.*

Suppose $p \in U$, $i \in \{1, \ldots, n\}$, and $j \in \{1, \ldots, m\}$. Then, the (i, j)-partial derivative or ij^{th} ***partial derivative of f at*** *$p \in U$ is*

$$\frac{\partial f_i(p)}{\partial x_j} = \lim_{t \to 0} \frac{f_i(p + te_j) - f_i(p)}{t} \in \mathbb{R},$$

provided the limit exists. Let $E = \left\{p \in U : \frac{\partial f_i(p)}{\partial x_j} \text{ exists}\right\}$. We call the map $\frac{\partial f_i}{\partial x_j}: E \to \mathbb{R}$, defined by

$$\left[\frac{\partial f_i}{\partial x_j}\right](p) = \frac{\partial f_i(p)}{\partial x_j} \ \forall p \in E,$$

the ***(i,j)-partial derivative of*** *f.*

We may also denote $\frac{\partial f_i}{\partial x_j}$, the (i, j)-partial derivative of f, more briefly by $D_j f_i$ or $\partial_{x_j} f_i$.

Theorem 1.2.1 (Differentiability Implies Continuity). *Suppose $U \subset \mathbb{R}^n$ is open and $f: U \to \mathbb{R}^m$ is differentiable at $p \in U$. Then, f is continuous at p.*

Proof. Suppose $U \subset \mathbb{R}^n$ is open and $f: U \to \mathbb{R}^m$ is differentiable at $p \in U$. Suppose further that

$$f(p + v) = f(p) + (Df)_p(v) + R(v), \ \forall v \in \mathbb{R}^n \text{ such that } p + v \in U. \tag{*}$$

By definition, we have that $(Df)_p \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $\lim_{|v| \to 0} \frac{R(v)}{|v|} = 0_{\mathbb{R}^m}$. We note that $(Df)_p$ is continuous since linear maps from one normed space to another are continuous. As an immediate result, we have that $||(Df)_p||$ is finite by a previous theorem. We observe that

$$\lim_{|v| \to 0} \left[||(Df)_p|| + \frac{|R(v)|}{|v|}\right] = ||(Df)_p|| \text{ and } \lim_{|v| \to 0} |v| = 0 \implies$$
$$\lim_{|v| \to 0} ||(Df)_p|| \cdot |v| + |R(v)| = \lim_{|v| \to 0} \left(||(Df)_p|| + \frac{|R(v)|}{|v|}\right) \cdot |v| = ||(Df)_p|| \cdot 0 = 0.$$

By definition, we obtain that $\forall \epsilon > 0 \ \exists \delta > 0$ such that $0 < |v| < \delta$ implies that

$$\begin{aligned}
&|||(Df)_p|| \cdot |v| + |R(v)|| = ||(Df)_p|| \cdot |v| + |R(v)| < \epsilon \\
\implies &\epsilon > ||(Df)_p|| \cdot |v| + |R(v)| \geq |(Df)_p(v)| + |R(v)| && \text{(By the definition of } ||(Df)_p||) \\
&\geq |(Df)_p(v) + R(v)|. && \text{(By the Triangle Inequality)}
\end{aligned}$$

Thus, it holds that $\forall \epsilon > 0 \ \exists \delta > 0$ such that $p + v \in U$ and $0 < |(p + v) - p| = |v| < \delta$ implies that

$$|f(p + v) - f(p)| = |(Df)_p(v) + R(v)| < \epsilon. \qquad \text{(By (*))}$$

Hence, f is continuous at p by definition. (We note that if we replace $p + v$ with x, the above statement resembles precisely the familiar $\delta - \epsilon$ definition for the continuity of a function at a point) □

Theorem 1.2.2 (Characterization of Derivative at a Point). *Let $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open. Suppose $f: U \to \mathbb{R}^m$ is differentiable at $p \in U$. Then, $(Df)_p \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is given by*

$$(Df)_p(u) = \lim_{t \to 0} \frac{f(p + tu) - f(p)}{t} \ \forall u \in \mathbb{R}^n.$$

Proof. Let $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open. Suppose $f\colon U \to \mathbb{R}^m$ is differentiable at $p \in U$.

(Case 1): We observe that $(Df)_p(0_{\mathbb{R}^n}) = 0^{\mathbb{R}^m}$ since $(Df)_p$ is linear. It follows that, for $u = 0_{\mathbb{R}^n}$,

$$(Df)_p(u) = 0 = \lim_{t\to 0} \frac{f(p+tu) - f(p)}{t}$$

as desired.

(Case 2): Consider arbitrary $u \in \mathbb{R}^n / \{0_{\mathbb{R}^n}\}$ and suppose $\forall t \in \mathbb{R}$ such that $p + tu \in U$ we have that

$$f(p+tu) = f(p) + (Df)_p(tu) + R(tu).$$

By the differentiability of f at p, we obtain that

$$\lim_{|tu|\to 0} \frac{R(tu)}{|tu|} = 0_{\mathbb{R}^m} \implies \lim_{|t|\to 0} \frac{R(tu)}{|tu|} = 0_{\mathbb{R}^m} \qquad \text{(Since } |u| \text{ is fixed)}$$

$$\implies \forall \epsilon > 0\ \exists \delta > 0 \text{ such that } 0 < |t| < \delta \implies \left| \frac{R(tu)}{|tu|} \right| = \frac{|R(tu)|}{|tu|} < \epsilon. \qquad (*)$$

Consider arbitrary $\epsilon > 0$. Then, $\frac{\epsilon}{|u|} > 0$ since $u \neq 0_{\mathbb{R}^n}$ by assumption. By $(*)$, we obtain that

$$\begin{aligned} \exists \delta > 0 \text{ such that } 0 < |t| < \delta \implies \left| \frac{R(tu)}{t\,|u|} \right| &= \frac{|R(tu)|}{|tu|} < \frac{\epsilon}{|u|} \\ \implies \left| \frac{R(tu)}{t\,|u|} \cdot |u| \right| &< \epsilon. \\ \therefore \lim_{t\to 0} \frac{R(tu)}{t\,|u|}\,|u| &= 0_{\mathbb{R}^m}. \qquad (**) \end{aligned}$$

Now, consider the limit $\lim_{t\to 0} \frac{f(p+tu)-f(p)}{t}$. We observe that

$$\begin{aligned} \lim_{t\to 0} \frac{f(p+tu) - f(p)}{t} &= \lim_{t\to 0} \frac{(Df)_p(tu) + R(tu)}{t} && \text{(By assumption)} \\ &= \lim_{t\to 0} \frac{t(Df)_p(u) + R(tu)}{t} && \text{(Since } (Df)_p \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)) \\ = \lim_{t\to 0} \left[(Df)_p(u) + \frac{R(tu)}{t} \right] &= \lim_{t\to 0} \left[(Df)_p(u) + \frac{R(tu)}{t\,|u|}\,|u| \right] \\ &= (Df)_p(u) && \text{(By } (**)) \end{aligned}$$

That is, we show that $\forall u \in \mathbb{R}^n$, $(Df)_p(u) = \lim_{t\to 0} \frac{f(p+tu)-f(p)}{t}$ and complete this proof. ◻

Corollary 1.2.1 (Uniqueness of Total Derivative)**.** *Suppose $U \subset \mathbb{R}^n$ is open and $f\colon U \to \mathbb{R}^m$ is differentiable at $p \in U$. Then, $T, \tilde{T} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ both satisfy the derivative properties implies that $T = \tilde{T}$.*

Proof. Suppose $U \subset \mathbb{R}^n$ is open and $f\colon U \to \mathbb{R}^m$ is differentiable at $p \in U$. Suppose further that $T, \tilde{T} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ both satisfy the derivative properties. By the preceding theorem, we obtain that

$$T(u) = \lim_{t\to 0} \frac{f(p+tu) - f(p)}{t} = \tilde{T}(u)\ \forall u \in \mathbb{R}^n,$$

which implies, by definition, $T = \tilde{T}$. ◻

Theorem 1.2.3 (Existence of Total Derivative Implies the Existence of Partial Derivative). *Let $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open. Suppose $f\colon U \to \mathbb{R}^m$ is differentiable at $p \in U$. Then, all partial derivatives of f at p exist and they are the entries of the matrix that represents the total derivative $(Df)_p$ at p. That is,*

(i) $\forall (i,j) \in \{1,\dots,m\} \times \{1,\dots,n\}$, $\partial x_j f_i(p) \in \mathbb{R}$, and

(ii) $(Df)_p = T_{A_p}$, where $\forall (i,j) \in \{1,\dots,m\} \times \{1,\dots,n\}$ we have that $(A_p)_{i,j} = \partial_{x_j} f_i(p)$ and

$$A_p = \begin{bmatrix} \partial_{x_1} f_1(p) & \partial_{x_2} f_1(p) & \dots & \partial_{x_n} f_1(p) \\ \partial_{x_1} f_2(p) & \partial_{x_2} f_2(p) & \dots & \partial_{x_n} f_2(p) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} f_m(p) & \partial_{x_2} f_m(p) & \dots & \partial_{x_n} f_m(p) \end{bmatrix} \in \mathcal{M}^{m,n}(\mathbb{R}).$$

Proof. Outline: Apply the preceding theorem and let u be the standard basis vectors in $\mathbb{R}^n$. □

Theorem 1.2.4 (Existence and continuity of Partial Derivatives Imply Existence of Total Derivative). *Suppose $U \subset \mathbb{R}^n$ is open, and $f\colon U \to \mathbb{R}^m$. Suppose further that $\forall (i,j) \in \{1,\dots,m\} \times \{1,\dots,n\}$ it holds that*

(i) $\forall p \in U$, $\partial_{x_j} f_i(p) \in \mathbb{R}$, and

(ii) $\partial_{x_j} f_i$ is continuous.

Then, $\forall p \in U$, f is differentiable at p and $(Df)_p$ exists.

Proof. (INCOMPLETE)

Outline 1 Let A be the matrix whose entries are the partial derivative.

2. Let T be the linear map represented by A

□

Definition 1.2.4 (Bilinear map). *Suppose V, W, Z are vector spaces. Then, a map $B\colon V \times W \to Z$ is bilinear if*

(i) $\forall v \in V$, $B(v,\cdot)\colon W \to Z$ defined by $[B(v,\cdot)](w) = B(v,w)$ $\forall w \in W$ is linear, and

(ii) $\forall w \in W$, $B(\cdot,w)\colon V \to Z$ defined by $[B(\cdot,w)](v) = B(v,w)$ $\forall v \in V$ is linear.

Proposition 1.2.1 (Examples of Bilinear Maps). *The usual multiplication on $\mathbb{R}$, dot product on $\mathbb{R}^n$, and matrix product are bilinear maps.*

Theorem 1.2.5 (Differentiation Rules). ***(Linearity)*** *Suppose $c \in \mathbb{R}$, $U \subset \mathbb{R}^n$ is open, and $f\colon U \to \mathbb{R}^m$ and $g\colon U \to \mathbb{R}^m$ are differentiable at $p \in U$. Then, $f + cg$ is differentiable at p and*

$$(D(f+cg))_p = (Df)_p + c(Dg)_p.$$

(Chain Rule) *Suppose $f\colon U \to \mathbb{R}^m$ is differentiable at $p \in U$, and $W \subset \mathbb{R}^m$ is open with $f(U) \subset W$. Suppose further that $g\colon W \to \mathbb{R}^r$ is differentiable at $f(p)$. Then, $g \circ f\colon U \to \mathbb{R}^r$ is differentiable at p and*

$$(D[g \circ f])_p = (Dg)_{f(p)} \circ (Df)_p \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^r).$$

(Leibniz Rule) *Suppose $\bullet\colon \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^q$ is bilinear, where $f\colon U \to \mathbb{R}^m$ and $g\colon U \to \mathbb{R}^r$ are differentiable at $p \in U$. Then, $T\colon U \to \mathbb{R}^q$ defined by*

$$T(u) = f(u) \bullet g(u) = \bullet(f(u), g(u)),\ \forall u \in U$$

is differentiable at p and

$$[(DT)_p](v) = [Df]_p(v) \bullet g(p) + f(p) \bullet [Dg]_p(v) \ \forall v \in U.$$

*(**Constant Map and Linear Map**) (i) Suppose $U \subset \mathbb{R}^n$ is open and $c \in \mathbb{R}^m$. Define $c_{\mathbb{R}} \colon U \to \mathbb{R}^m$ by $c_{\mathbb{R}}(u) = c \ \forall u \in U$. Then, $c_{\mathbb{R}}$ is differentiable and $\forall p \in U \ (Dc_{\mathbb{R}})_p = 0_{\mathbb{R}}$, where $0_{\mathbb{R}} \colon \mathbb{R}^n \to \mathbb{R}^m$ is defined by $0_{\mathbb{R}}(u) = 0_{\mathbb{R}^m} \ \forall u \in U$.*

(ii) Suppose $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then, T is differentiable and $\forall p \in \mathbb{R}^n \ (DT)_p = T$.

Proof. (INCOMPLETE) □

Definition 1.2.5 (Segment in $\mathbb{R}^n$)**.** *Let $p, q \in \mathbb{R}^n$. Then, the segment $[p, q]$ in $\mathbb{R}^n$ is*

$$[p, q] = \{(1 - \lambda)p + \lambda q : \lambda \in [0, 1]\}.$$

Theorem 1.2.6 (Differentiability of Vector Function $\iff$ Component-wise Differentiability)**.** *Let $n, m \in \mathbb{N}$, $U \subset \mathbb{R}^n$ is open, and $f \colon U \to \mathbb{R}^m$. Then, f is differentiable at $p \in U \iff f_j$ is differentiable at p, $\forall j \in \{1, \ldots, m\}$,*

Proof. (INCOMPLETE) □

Theorem 1.2.7 (General Mean Value Theorem)**.** *Suppose $U \subset \mathbb{R}^n$ is open, $[p, q] \subset U$, and $f \colon U \to \mathbb{R}^m$ is differentiable. Then,*

$$|f(p) - f(q)| \leq M \, |p - q| \,,$$

where $M = sup \{||(Df)_q : q \in U||\}$.

Proof. (INCOMPLETE) □

Definition 1.2.6 (Integrating a Matrix and a derivative at a point)**.** *Suppose $[a, b] \subset \mathbb{R}$, $m, n \in \mathbb{N}$, $A \in \mathcal{M}^{m,n}(\mathbb{R})$. Then, we define*

(i)

$$\int_a^b A dt = \begin{bmatrix} \int_a^b a_{1,1} dt & \int_a^b a_{1,2} dt & \cdots \int_a^b a_{1,n} dt \\ \int_a^b a_{2,1} dt & \int_a^b a_{2,2} dt & \cdots \int_a^b a_{2,n} dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b a_{m,1} dt & \int_a^b a_{m,2} dt & \cdots \int_a^b a_{m,n} dt \end{bmatrix},$$

where $\forall (i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ $a_{i,j}$ is the (i,j)-entry of A.

(ii) Suppose $U \subset \mathbb{R}^n$ and $f \colon U \to \mathbb{R}^m$ is differentiable at $p \in U$ with derivative $(Df)_p$ at p. Then, we define

$$\int_b^a (Df)_p dt = \int_b^a B_p dt,$$

where $B_p \in \mathcal{M}^{m,n}(\mathbb{R})$ is the matrix representing $(Df)_p$.

Definition 1.2.7 (C^n)**.** *Suppose f is a n-th order differentiable map and $\forall i \in \{1, \ldots, n\}$, $f^{(i)}$ is continuous. Then, f is continuous n-th order differentiable (or n-th order continuously differentiable) and we say that $f \in C^n$.*

Theorem 1.2.8 (C^1 Mean Value Theorem)**.** *Suppose $U \subset \mathbb{R}^n$ is open, $[p,q] \subset U$, and $f\colon U \to \mathbb{R}^m \in C^1$. Then,*

$$f(q) - f(p) = T \cdot (q-p)$$

where T is the average derivative of f on the segment $[p,q]$ with

$$T = \int_0^1 (Df)_{p+t(q-p)} dt \in \mathcal{M}^{m,n}(\mathbb{R}).$$

Proof. (INCOMPLETE) □

Corollary 1.2.2 (Connectedness, Differentiability, and Trivial Derivative Implies Constantness)**.** *Suppose $U \subset \mathbb{R}^n$ is open and connected. Suppose further that $f\colon U \to \mathbb{R}^m$ is differentiable and $\forall p \in U$, $(Df)_p = \tilde{0}$, where $\tilde{0}\colon \mathbb{R}^n \to \mathbb{R}^m$ is defined by $\tilde{0}(v) = 0_{\mathbb{R}^m}$ $\forall v \in \mathbb{R}^n$. Then, f is contstant.*

Theorem 1.2.9 (Differentiation Past the Integral)**.** *Suppose $[a,b], (c,d) \subset \mathbb{R}$, $f\colon [a,b] \times (c,d) \to \mathbb{R}$ is continuous, $\frac{\partial f}{\partial y}(x,y) \in \mathbb{R}$ $\forall (x,y) \in [a,b] \times (c,d)$, and $\frac{\partial f}{\partial y}\colon [a,b] \times (c,d) \to \mathbb{R}$ is continuous. Then,*

(i) $F\colon (c,d) \to \mathbb{R}$ defined by

$$F(y) = \int_a^b f(x,y)dx \ \forall y \in (c,d)$$

is of class C^1 and

(ii)

$$F'(y) = \int_a^b \frac{\partial f(x,y)}{\partial y} dx \ \forall y \in (c,d).$$

1.3 Higher Derivatives

Theorem 1.3.1 (Existence of Second Total Derivative Implies the Existence of Other Second Derivatives)**.** *(i) Suppose $U \subset \mathbb{R}^n$ is open, $f\colon U \to \mathbb{R}^m$, and $p \in U$. Suppose that $(D^2 f)_p$ exists. Then, $\forall k \in \{1,\ldots,m\}$*

(a) $(D^2 f_k)_p$ exists,

(b) $\forall i,j \in \{1,\ldots,n\}$ $\frac{\partial^2 f_k}{\partial x_i \partial x_j}(p) \in \mathbb{R}$, and

(c) $\forall i,j \in \{1,\ldots,n\}$

$$(D^2 f_k)_p(e_i, e_j) = \frac{\partial^2 f_k(p)}{\partial_{x_i} \partial_{x_j}}.$$

(ii)

Theorem 1.3.2 (Symmetry of Second Derivative)**.** *Suppose $U \subset \mathbb{R}$ is open, $f\colon U \to \mathbb{R}^m$, and $p \in U$. Suppose further that $(D^2 f)_p$ exists. Then, $(D^2 f)_p$ is symmetric; that is,*

$$\forall v, w \in U \ (D^2 f)_p(v,w) = (D^2 f)_p(w,v).$$

Corollary 1.3.1 (Existence of Second-Derivative Implies Equivalence of Second-Mixed Partials)**.**

Corollary 1.3.2 (Existence of r-th Derivative Implies Symetry and Equivalence of r-th Order Mixed Partials).

Definition 1.3.1 (Class C^r).

Definition 1.3.2 (Smoothness and class C^∞).

Definition 1.3.3 (Uniformly C^r convergent and Uniformly C^r Cauchy).

Theorem 1.3.3 (Euqivalence of Uniformly C^r Convergent and Uniformly C^r Cauchy).

Definition 1.3.4 (C^r norm).

Theorem 1.3.4 (C^r Norm Induced Banach Space).

Theorem 1.3.5 (C^r M-Test).

1.4 Implicit and Inverse Functions

Definition 1.4.1 (Implicit function).

Theorem 1.4.1 (Implicit Function Theorem).

Definition 1.4.2 (Diffeomorphism).

Definition 1.4.3 (C^r Diffeomorphism).

Theorem 1.4.2 (Inverse Function Theorem).

1.5 *The Rank Theorem

Omitted

1.6 *Lagrange Multipliers

Omitted

1.7 Multiple Integrals

Definition 1.7.1 (Multiple Inegrals).

Theorem 1.7.1 (Fubini's Theorem).

Corollary 1.7.1.

Theorem 1.7.2 (Cavalieri's Principal).

Theorem 1.7.3 (Change of Variables).

2 Lebesgue Theory

2.1

2.2

2.3

2.4

2.5