Introduction to Complex Analysis

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Introduction

This course [course description]
Additional information regarding this course is provided below:

(A) Instructor: FNameLName
  - (I) Contact:
  - (II) Office Hour:

(B) Course Number/ID: Math 1XX

(C) Course Schedule:
  - (A) Lecture: T,Th 0300PM to 0500PM
  - (B) Discussion: M 1200PM to 0100PM

(D) Textbook:

Note that [additional notes]

Contents

1 Preliminaries to Complex Analysis

1.1 Complex numbers and the complex plane

1.1.1 Basic properties

Proposition 1.1.1 (Positivity of Squared Real Numbers)**.** *Suppose $r \in \mathbb{R}$. Then, $r^2 \geq 0$.*

Corollary 1.1.1. *Suppose $r \in \mathbb{R}$ and $r < 0$. Then, r does not have square roots in $\mathbb{R}$.*

Remark 1.1.1. *We introduce the imaginary numbers to fill the limitations of the real number system.*

Definition 1.1.1 (i)**.** $i = \sqrt{-1}$.

Proposition 1.1.2. *Suppose $z \in \mathbb{C}$. Then, $\exists b \in \mathbb{R}$ such that $z = bi$.*

Proof. Consider $x \in \mathbb{R}$ such that $x \leq 0$. It follows that

$$\begin{aligned} -x \geq 0 &\implies -x = b^2 && \text{(For some } b \in \mathbb{R}) \\ &\implies x = -b^2 = (bi)^2. \end{aligned}$$

□

Definition 1.1.2 (Complex number)**.** *A complex number z is an expression of the form $z = a+bi$ for some $a, b \in \mathbb{R}$.*

We call a the real part of z and denote it by $Re(z)$.

Similarly, we call b the imaginary part of z and denote it by $Im(z)$.

Remark 1.1.2. *We denote the set of complex numbers by $\mathbb{C}$. Furthermore, $\mathbb{C}$ is the algebraic closure (related to the Fundamental Theorem of Algebra) of $\mathbb{R}$.*

Definition 1.1.3 (Complex conjugate)**.** *Suppose $z = a + bi \in \mathbb{C}$. Then, $\overline{z} = a - bi \in \mathbb{C}$ is the complex conjugate of z.*

Definition 1.1.4 (Operations on $\mathbb{C}$)**.** *Suppose $x, y \in \mathbb{C}$ where $x = a+bi$ and $y = c+di$ for some $a, b, c, d \in \mathbb{R}$. Then, we define the*

(i) addition of x and y by $x + y = (a + c) + (b + d)i$;

(ii) multiplication of x and y by $x \cdot y = ac + (bi)(di) + adi + bci = (ac - bd) + (ad + bc)i$.

Remark 1.1.3. *Addition and multiplication on $\mathbb{C}$ satisfies (i) commutativity, (ii) associativity, and (iii) distributivity as in $\mathbb{R}$.*

Proposition 1.1.3. *Suppose $x, y \in \mathbb{C}$ where $x = a + bi$ and $y = c + di$ for some $a, b, c, d \in \mathbb{R}$. Then, x/y is the unique complex number such that $y \cdot (x/y) = x$.*

Example 1.1.1. *Express the following as $a + bi$ for $a, b \in \mathbb{R}$*

(i) $(9 - 12i) + (12i - 16)$.

$$(9 - 12i) + (12i - 16) = (9 - 16) + (-12 + 12)i = -7 + 0i.$$

(ii) $(3+4i)\cdot(3-4i)$.

$$(3+4i)\cdot(3-4i) = 9 - 12i + 12i + 16 = 25.$$

(iii) $\frac{50+50i}{3+4i}$

$$\begin{aligned}\frac{50+50i}{3+4i} = \frac{50+50i}{3+4i}\cdot\frac{3-4i}{3-4i} = \frac{(50+50i)(3-4i)}{25} &= (2+2i)(3-4i)\\ &= 6 - 8i + 6i - 8i^2 = 14 - 2i.\end{aligned}$$

Remark 1.1.4 (Geometric Interpretation of $\mathbb{C}$)**.** *$\mathbb{C}$, geometrically, is a plane where the vertical axis represents the imaginary part of a complex number and the horizontal axis represents the real part of a complex number.*

Proposition 1.1.4 (Re and Im Identity)**.** *Suppose $z \in \mathbb{C}$. Then,*

$$Re(z) = \frac{z+\overline{z}}{2} \text{ and } Im(z) = \frac{z-\overline{z}}{2}$$

Definition 1.1.5 (Modulus (Absolute value) of a complex number)**.** *The modulus of $z = a+bi \in \mathbb{C}$ is*

$$|z| = \sqrt{a^2+b^2},$$

which is the distance between z and 0.

Example 1.1.2 (Subsets of $\mathbb{C}$ in the complex plane)**.** *(i)*

(i)

Theorem 1.1.1 (Taylor Series for Common Functions)**.** *The following statements hold, $\forall z \in \mathbb{C}$:*

(exp) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$;

(cos) $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z)^{2n}}{(2n)!}$;

(sin) $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z)^{2n-1}}{(2n-1)!}$.

Remark 1.1.5 ($\cos, \sin, exp$ are defined on $\mathbb{C}$)**.** *By considering $\cos, \sin,$ and exp in terms of their respective Taylor Series, we observe that it is sensible that the above functions can be defined on $\mathbb{C}$.*

Theorem 1.1.2 (Euler's Theorem)**.** $e^{iz} = \cos(z) + i\sin(z)\ \forall z \in \mathbb{C}$.

Proof. Outline: prove by Taylor series

□

Corollary 1.1.2. *(i)* $e^{i\pi} + 1 = 0$.

(ii) $\sin z = \frac{e^{iz}-e^{-iz}}{2i}\ \forall z \in \mathbb{C}$.

(iii) $\cos z = \frac{e^{iz}+e^{-iz}}{2}\ \forall z \in \mathbb{C}$.

Definition 1.1.6 (Polar form)**.** $(r,\theta) \to (x,y)$

$r = \sqrt{x^2+y^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$

$z = re^{i\theta}$ for $0 \leq r \in \mathbb{R}$ and $\theta \in [0, 2\pi]$. $z = x + iy$ for $x, y \in \mathbb{R}$

By Euler, $x = r\cos(\theta)$, $y = r\sin(\theta)$

Remark 1.1.6 (Geometric Interpretation of Addition and Multiplication in $\mathbb{C}$)**.** *Consider arbitrary $v = re^i\theta, w = \tilde{r}e^{i\tilde{\theta}} \in \mathbb{C}$, where $r, \tilde{r} \geq 0$ and $\theta, \tilde{\theta} \in [0, 2\pi]$. We observe that $v \cdot w = r\tilde{r}e^{i(\theta+\tilde{\theta})}$.*

Note that addition in $\mathbb{C}$ can be viewed as vector addition and multiplication in $\mathbb{C}$ can be deemed as rotation and scaling of the vectors.

For instance, multiplying w by z may be deemed as scaling w by r and rotating w by θ.

Definition 1.1.7 (Argument)**.**

1.1.2 Convergence

Definition 1.1.8 (Convergence of a complex sequence)**.** *A sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ converges to $z \in \mathbb{C}$ and we write $\lim_{n\to\infty} z_n = z$ if*

$$\lim_{n\to\infty} |z_n - z| = 0.$$

or, equivalently,

$$\forall \epsilon > 0 \ \exists N \geq 1 \text{ such that } n > N \implies |z_n - z| < \epsilon$$

Theorem 1.1.3 (Convergence of Complex Sequence $\iff$ Component-wise Convergence)**.** *Suppose $(z_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers, where $\forall j \in \mathbb{N}$ $z_j = x_j + y_j$ for some $x_j, y_j \in \mathbb{R}$. Then,*

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y,$$

where $x, y \in \mathbb{R}$ and $z = x + iy \in \mathbb{C}$.

Proof. ($\implies$) Suppose that $\{z_n\}_{n\in\mathbb{N}}$ converges to $z \in \mathbb{C}$. By definition, we have that $\lim_{n\to\infty} |z_n - z| = 0$, which implies that

$$\begin{aligned}
\forall \epsilon > 0 \ \exists N_\epsilon > 0 \text{ such that } n > N_\epsilon \implies ||z_n - z| - 0| = |z_n - z| < \epsilon \\
\implies \sqrt{(x_n - x)^2 + (y_n - y)^2} = |z_n - z| < \epsilon \\
\implies |x_n - x| = \sqrt{(x_n - x)^2} \leq \sqrt{(x_n - x)^2 + (y_n - y)^2} < \epsilon \\
\text{and } |y_n - y| = \sqrt{(y_n - y)^2} \leq \sqrt{(x_n - x)^2 + (y_n - y)^2} < \epsilon.
\end{aligned}$$

By definition, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ as desired.

($\impliedby$) Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. By definition, we have that

$$\forall \epsilon > 0 \ \exists N_\epsilon, M_\epsilon > 0 \text{ such that } n > N_\epsilon \implies |x_n - x| < \epsilon \text{ and } n > M_\epsilon \implies |y_n - y| < \epsilon.$$

Consider arbitrary $\epsilon > 0$. Then, $E = \sqrt{\frac{\epsilon^2}{2}} > 0$ and, by the above statement, we have that

$$\exists N_E, M_E > 0 \text{ such that } n > N_E \implies |x_n - x| < E \text{ and } n > M_E \implies |y_n - y| < E.$$

Take $N = \max\{N_E, M_E\} > 0$. It follows that

$$\begin{aligned}
n > N \implies |x_n - x|, |y_n - y| < E \implies |x_n - x|^2, |y_n - y|^2 < E^2 \\
\implies (x_n - x)^2 + (y_n - y)^2 = |x_n - x|^2 + |y_n - y|^2 < 2E^2 \\
\implies \sqrt{(x_n - x)^2 + (y_n - y)^2} < \sqrt{2E^2} \\
\implies |z_n - z| = ||z_n - z| - 0| < \sqrt{2\left(\sqrt{\frac{\epsilon^2}{2}}\right)^2} = \epsilon.
\end{aligned}$$

By definition, we obtain that $\lim_{n\to\infty} |z_n - z| = 0$ and, therefore, $\lim_{n\to\infty} z_n = z$. $\square$

Definition 1.1.9 (Cauchy sequence of complex numbers)**.** *Suppose $(z_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers. Then, $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence or Cauchy if $|z_n - z_m| \to 0$ as $n, m \to \infty$ or, equivalently,*

$$\forall \epsilon > 0\ \exists N > 0 \text{ such that } n, m > N \implies |z_n - z_m| < \epsilon.$$

Theorem 1.1.4 (Cauchyness of Complex Sequence $\iff$ Component-wise Cauchy)**.** *Suppose $(z_n)_{n\to\infty}$ is a sequence of complex number, where $\forall j \in \mathbb{N}$ $z_j = x_j + y_j i$ for some $x_j, y_j \in \mathbb{R}$. Then, $(z_n)_{n\to\infty}$ is Cauchy $\iff$ $(x_n)_{n\to\infty}$ and $(y_n)_{n\to\infty}$ are both Cauchy.*

Theorem 1.1.5 ($\mathbb{C}$ is Complete)**.** *$\mathbb{C}$ is Complete; that is, every Cauchy sequence in $\mathbb{C}$ converges to a complex number.*

1.1.3 Sets in the complex plane

Proposition 1.1.5 (Metric On $\mathbb{C}$)**.** *Define $dist\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $dist(z, w) = |z - w|$ $\forall (z, w) \in \mathbb{C} \times \mathbb{C}$. Then, $(\mathbb{C}, dist)$ is a metric space.*

Definition 1.1.10 (Open and closed balls in $\mathbb{C}$)**.** *Suppose $z_0 \in \mathbb{C}$ and $r > 0$. Then, the open ball of center (or around) z_0 with radius r is the set*

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

The closed ball of center (or around) z_0 with radius r is the set

$$\overline{B_r(z_0)} = \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$

Definition 1.1.11 (Open set in $\mathbb{C}$)**.** *$U \subset \mathbb{C}$ is open if*

$$\forall z \in U\ \exists r > 0 \text{ such that } B_r(z) \subset U.$$

Example 1.1.3. *$\mathbb{C}/\mathbb{R}_{\geq 0}, \mathbb{C}, \mathbb{R}$ are open in $\mathbb{C}$.*

Definition 1.1.12 (Boundary)**.**

Definition 1.1.13 (Bounded)**.**

Definition 1.1.14 (Diameter)**.**

Definition 1.1.15 (Compactness)**.**

1.2 Functions on the complex plane

1.2.1 Continuous functions

Remark 1.2.1 (Limit of a complex function)**.** *We define the limit of complex functions as in Real Analysis, except we employ the metric on* $\mathbb{C}$ *instead of the metric on* $\mathbb{R}$*.*

Definition 1.2.1 (Continuous Complex Function)**.** *Suppose* $\Omega \subset \mathbb{C}$*. Then,* $f\colon \Omega \to \mathbb{C}$ *is continuous at* $z_0 \in \Omega$ *if*

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that } z \in \Omega \text{ and } |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$$

or, equivalently, if for any sequence $(z_n)_{n\in\mathbb{N}}$ *of complex numbers such that* $\lim_{n\to\infty} z_n = z$ *it holds that* $\lim_{n\to\infty} f(z_n) = f(z_0)$*.*

Proposition 1.2.1. *Suppose* $\Omega \subset \mathbb{C}$ *and* $f\colon \Omega \to \mathbb{C}$ *is continuous. Then,* $g\colon \Omega \to \mathbb{R}$ *defined by* $g(z) = |f(z)|\ \forall z \in \Omega$ *is continuous.*

Definition 1.2.2 (Maximum and minimum of a complex function)**.**

1.2.2 Holomorphic functions

Definition 1.2.3 (Holomorphic function)**.** *Suppose* $\Omega \subset \mathbb{C}$ *is open and* $f\colon \Omega \to \mathbb{C}$*. Then,* f *is holomorphic (complex differentiable) at* $z_0 \in \Omega$ *if*

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} \in \mathbb{C}$$

or, equivalently, if $\exists Z \in \mathbb{C}$ *such that*

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that } h \in \mathbb{C},\ 0 < |h| < \delta \text{ and } z_0 + \delta \in \Omega \implies \left| \frac{f(z_0 + h) - f(z_0)}{h} - Z \right| < \epsilon$$

or, equivalently, if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}.$$

If such limit exists, we denote it by $f'(z_0)$ *and call it the derivative of* f *at* z_0*. We say* f *is holomorphic*

(i) on $S \subset \Omega$ *if* f *is holomorphic at* p*,* $\forall p \in S$*;*

(ii) if f *is holomorphic on* Ω*;*

(iii) on a closed set $C \subset \mathbb{C}$ *if* f *is holomorphic on an open set* $U \subset \Omega$ *containing* C*.*

We sometimes use the terms differentiable, complex differentiable, or $\mathbb{C}$*-differentiable, and holomorphic interchangeably.*

Definition 1.2.4 (Continiously differentiable map)**.** *A map* f *is continuously differentiable if* f *is differentiable, and its derivative is continuous.*

Remark 1.2.2. *In the definition above,* $h \in \mathbb{C}$ *and* h *is sufficiently close to* z_0*.*

Furthermore, recall that

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} \text{ exist in } \mathbb{C}$$

means that $\exists f'(z_0) \in \mathbb{C}$ *such that*

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that } h \in \mathbb{C}/\{0\} \text{ with } z_0 + h \in U \text{ and } |h| < \delta$$
$$\implies \left| \frac{f(z_0+h) - f(z_0)}{h} - f'(z_0) \right| < \epsilon.$$

We observe that holomorphicity is much stronger than differentiability, as the limits along the path, through which $(z_n)_{n\in\mathbb{N}}$ *approaches* z_0*, are required to be equal.*

As an immediate result, holomorphic functions have strong rigidity properties not shared by real-differentiable functions

Example 1.2.1. *Let* $f(z) = \overline{z}$*. Here,* f *is holomorphic nowhere.*

Outline: Take arb. $z_0 \in z$*. Take a convergent sequence in* $\mathbb{R}$ *and a convergent sequence in* $\mathbb{C}/\mathbb{R}$ *and compare the limit.*

Example 1.2.2. *Comapre the above function with* $f(x, y) = (x, -y)$

Example 1.2.3 (Common Holomorphic Functions)**.**

Theorem 1.2.1 (Equivalent Condition to Holomorphicity)**.** *Suppose* $\Omega \subset \mathbb{C}$*. Then,* $f\colon \Omega \to \mathbb{C}$ *is holomorphic at* $z_0 \in \Omega \iff$

$$f(z_0 + h) = f(z_0) + f'(h) + R(h)\ \forall h \in \mathbb{C} \text{ such that } z_0 + h \in \Omega \implies \lim_{h\to 0} \frac{R(h)}{h} = 0.$$

Theorem 1.2.2 (Holomorphicity Implies Continuity)**.** *Suppose* $\Omega \subset \mathbb{C}$*. Then,* $f\colon \Omega \to \mathbb{C}$ *is holomorphic at* $z_0 \in \Omega$ *implies* f *is continuous at* z_0*.*

Proposition 1.2.2. *Suppose* $U \subset \mathbb{C}$ *is connected,* f, g *are holomorphic on* U*, and* $f = g$ *on a line segment in* U*. Then,* $f(z) = g(z)\ \forall z \in U$*.*

Proposition 1.2.3 (?Check hypothesis)**.** *Suppose* $U \subset \mathbb{C}$ *is connected, and* f *is holomorphic on* U*. Then,* f *is infinitely differentiable on* U*.*

Example 1.2.4. *(a)* $f(z) = z^n$*. Calculate* $\frac{f(z+h)-f(z)}{h}$*...*

Proposition 1.2.4 (Stability Properties of Holomorphic Functions)**.** *Suppose* $U \subset \mathbb{C}$ *is open,* $f, g\colon U \to \mathbb{C}$ *are holomorphic at* $z_0 \in U$*. Then,*

(Addition Rule) $f + g$ *is holomorphic at* z_0 *and*

$$(f + g)'(z_0) = f'(z_0) + g'(z_0);$$

(Product Rule) fg *is holomorphic at* z_0 *and*

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0);$$

(Division Rule) $g(z_0) \neq 0$ *implies* f/g *holomorphic at* z_0 *and*

$$(f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{[g(z_0)]^2}.$$

(Chain Rule) Suppose $U, \Omega \subset \mathbb{C}$ *are open, and* $f\colon \Omega \to \mathbb{C}$ *and* $g\colon U \to \mathbb{C}$*. Suppose further that*

(i) f *is holomorphic at* $z_0 \in \Omega$*, and*

(ii) $f(z_0) \in U$ *and* g *is holomorphic at* $f(z_0)$*.*

Then, $(g \circ f)$ *is holomorphic at* z_0 *and*

$$(g \circ f)'(z_0 = g'(f(z_0)) \cdot f'(z_0).$$

Definition 1.2.5 (Complex Monomial, polynomial, and rational function). *(i) A (Complex) monomial is a map $m\colon \mathbb{C} \to \mathbb{C}$ defined by*

$$az^n \ \forall z \in \mathbb{C}$$

for some $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

(ii) A (complex) polynomial is a map $p\colon \mathbb{C} \to \mathbb{C}$ defined by

$$p(z) = a_0 + a_1 z + \dots a_n z^n = a_0 + \sum_{k=1}^{n} a_k z^k \ \forall z \in \mathbb{C}$$

for some $n \in \mathbb{N}$ and $a_0, \dots, a_n \in \mathbb{C}$.

(iii) A (complex) rational function is a map $r = \frac{p}{q}$ for some (complex) polynomials p, q.

Corollary 1.2.1 (Holomorphicity of Polynomials and Rational Functions). *(i) A polynomial is holomorphic on $\mathbb{C}$.*

(ii) A rational function f/g is holomorphic on $\{z \in \mathbb{C} : g(z) \neq 0\} \subset \mathbb{C}$.

Example 1.2.5. *For each of the following functions, determine $S \subset \mathbb{C}$ on which the function is holomorphic and determine its derivative if it exists.*

(i) $e^{\frac{1}{x^2}}$:

(ii) $\frac{1}{\cos(z)}$:

(iii) $\frac{iz^2+z+9}{z^2+1}$:

(iv) $|z|$:

(v) $\overline{z}$:

Definition 1.2.6 (Partial derivative of complex-valued).

1.2.3 Power series

Definition 1.2.7 (Power series). *A power series is a formal expression*

$$\sum_{n \in \mathbb{W}} a_n z^n \text{ where } a_n \in \mathbb{C} \ \forall n \in \mathbb{W}.$$

Definition 1.2.8 (Operations on power series). *(Addition)*

(Multiplication)

Remark 1.2.3. *A power series defines a function when it converges*

Example 1.2.6 (Geometric series). *Let $a \in \mathbb{C}$ and $a_n = a^N \ \forall n \in \mathbb{W}$.*

$$\sum_{n \in \mathbb{W}} a^n z^n = 1 + az + +a^2 z^2 + \dots$$

converges if the limit of $S_N = \sum_{n \in \mathbb{W}}^{N-1} a^n z^n$ exists.

By the Geometric series identity, we have that

$$S_N = \frac{1-(az)^n}{1-az} \ \forall N \in \mathbb{W}.$$

We observe that $\lim_{N\to\infty} S_N = \frac{1}{1-az}$ *if* $|az| < 1$, *and* S_N *diverges if* $|az| > 1$; *for* $|az| = 1$, s_N *diverge regardless of* az. ...

The geometric series converges absolutely when $|z| < \left|\frac{1}{a}\right|$.

Definition 1.2.9. $\sum_{n\in\mathbb{W}} z_n$, *where* $z_n \in \mathbb{C}\ \forall n \in \mathbb{W}$, *converge absolutely if* $\sum_{n\in\mathbb{W}} |z_n|$ *converges*

Definition 1.2.10 (Radius of convergence)**.** $\sum_{n\in\mathbb{W}} z_n$, *where* $z_n \in \mathbb{C}\ \forall n \in \mathbb{W}$, *has a radius of convergence*

$$r = \frac{1}{\lim_{n\to\infty} sup\, |a_n|^{\frac{1}{n}}} \in \mathbb{R}$$

Example 1.2.7. $a_n = a^n \to r = \frac{1}{|a|}$, *see example above.*

Theorem 1.2.3 (Power Series Test for $\mathbb{C}$)**.** *Let* $r > 0$ *be the raidus of convergence of ...*

(i) $|z| < r \implies f(z)$ *converges absolutely.*

(ii) $|z| > r \implies$, $f(z)$ *diverges*

(iii) If $|z| = r$, $f(z)$ *may converge or diverge*

Proof. (i) □

Remark 1.2.4. *Differentiable function from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *is "smooth-ish". Our goal is to develop geometric intuition for holomorphic function (* $\iff$ *conformal)??*

Remark 1.2.5 (Difference in Holomorphicity and Real Differentiability)**.** *For* $f\colon \mathbb{R} \to \mathbb{R}$, *the derivative of* f *is the "best" linear approximation; that is,* $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ *for* $x_0 \approx x$.

For $f\colon \mathbb{R}^n \to \mathbb{R}^n$, *the derivative of* f *is the "best" linear approximation; that is,* $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ *for* $x_0 \approx x$, *where* $f'(x_0) \in \mathcal{M}^{n,n,}(\mathbb{R})$.

Now, suppose $\Omega \subset \mathbb{C}$ *is open and* $f\colon \Omega \to \mathbb{C}$. *Then,* $f(z) \approx f(z_0) + f'(z_0)(z - z_0)$ *for* $z_0 \approx z$

Above, we have complex multiplication

Now, suppose $\Omega \subset \mathbb{R}^2$ *is open and* $f\colon \Omega \to \mathbb{R}^2$. *Then,* $f(x, y) \approx f(x_0, y_0) + f'(x_0, y_0)\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$

Above we have matrix multiplication

Similarly, the derivative of complex f is a complex limit; the derivative of f in $\mathbb{R}^2$ *is a matrix*

Proposition 1.2.5 (Canonical Bijection Between $\mathbb{C}$ and $\mathbb{R}^2$)**.** $I\colon \mathbb{C} \to \mathbb{R}^2$ *defined by*

$$I(z) = (Re(z), Im(z))\ \forall z \in \Omega.$$

is bijection from $\mathbb{C}$ *to* $\mathbb{R}^2$.

Remark 1.2.6. *To identify a complex function* $f\colon \Omega \to \mathbb{C}$, *where* $\Omega \subset \mathbb{C}$, *as a function on* $\mathbb{R}^2$, *we mean to consider the map* $\tilde{f}\colon \tilde{\Omega} \to \mathbb{R}^2$, *where* $\tilde{\Omega} = \left\{(x, y) \in \mathbb{R}^2 : x + yi \in \Omega\right\}$, *defined by*

$$\tilde{f}(x, y) = (u(x, y), v(x, y))\ \forall (x, y) \in \tilde{\Omega}$$

where u, v *are real-valued functions defined on* $\tilde{\Omega}$ *such that* $f = u + iv$.

Theorem 1.2.4 (Cauchy-Riemann Equation)**.** *Suppose $\Omega \subset \mathbb{C}$ is open, $f\colon \Omega \to \mathbb{C}$ is holomorphic at $z_0 = x_0 + iy_0 \in \mathbb{C}$, and $f = u + iv$ for some real-valued functions u, v defined on $\tilde{\Omega}$. Then,*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \text{ and } \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

Furthermore, the derivative $(D\tilde{f})_{(x_0,y_0)}\colon \mathbb{R}^2 \to \mathbb{R}^2$ of $\tilde{f}$, as defined in the above remark, at (x_0, y_0) is represented by the matrix

$$\begin{bmatrix} \frac{\partial u}{\partial x}(x_0, y_0) & \frac{\partial u}{\partial y}(x_0, y_0) \\ \frac{\partial v}{\partial x}(x_0, y_0) & \frac{\partial v}{\partial y}(x_0, y_0) \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where $a = \frac{\partial u}{\partial x}(x_0, y_0), b = \frac{\partial u}{\partial y}(x_0, y_0) \in \mathbb{R}$.

Definition 1.2.11 (Differential Operators Related to Complex-Valued Functions)**.** *(i) $\frac{\partial}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]$*

(ii) $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right]$

Theorem 1.2.5 (Converse of the Cauchy-Riemann Equation)**.** *Suppose $f = u + iv$ is a complex-valued function defined on an open set $\Omega \subset \mathbb{C}$, where u, v are real-valued functions defined on $\tilde{\Omega} = \left\{(x, y) \in \mathbb{R}^2 : x + iy \in \Omega\right\}$. Suppose further that u, v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω. Then,*

(i) f is holomorphic and

(ii) $\forall z = x + iy \in \Omega$ with $x, y \in \mathbb{R}$, $f'(z) = 2\frac{\partial u(x,y)}{\partial z} = \frac{\partial f(z)}{dz}$.

Proof. Omitted due to the non-triviality. □

Remark 1.2.7. *Conformality: infinitesimally preserve angles of the input or scale the input to* 0.

Corollary 1.2.2 (Converse of the Cauchy-Riemann Equation)**.** *???? Let $n \in \mathbb{N}$. If $f \in C^n$ and the Cauchy-Riemann equation hold at z_0, then f is holomorphic at z_0.*

1.3 Integration along curves

Definition 1.3.1 (Parameterized curve (Parameterization))**.** *Let* $[a, b] \subset \mathbb{R}$*. A parameterized curve is a map* $z\colon [a, b] \to \mathbb{C}$*.*

To mitigate the confusion between the terminologies "parameterized curve" and "curve" (to be subsequently defined), we refer to a parameterized curve more briefly as a ***parameterization****.*

Definition 1.3.2 (Smooth parameterization)**.** *Suppose* $[a, b] \subset \mathbb{R}$ *and* $z\colon [a, b] \to \mathbb{C}$ *is a parameterization. Then,* z *is smooth if*

(i) $z'(t)$ *exists in* $\mathbb{C}$*,* $\forall t \in [a, b]$*,* z' *is continuous on* $[a, b]$*, and*

(ii) $z'(t) \neq 0$ $\forall t \in [a, b]$*.*

That is, z *is smooth if* z *is continuously differentiable on* $[a, b]$ *with non-vanishing derivative on* $[a, b]$*.*

More generally, we say that a map f *is smooth* ***iff it is infinitely differentiable****.*

Definition 1.3.3 (Right-hand and left-hand derivative of a parameterization)**.** *Suppose* $[a, b] \subset \mathbb{R}$ *and* $z\colon [a, b] \to \mathbb{C}$ *is a parameterization. Then,*

(i) the right-hand derivative of z *at* a *is*

$$z'(a) = \lim_{h \to 0^+} \frac{z(a+h) - z(a)}{h};$$

(ii) the left-hand derivative of z *at* b *is*

$$z'(b) = \lim_{h \to 0^-} \frac{z(a+h) - z(a)}{h};$$

Definition 1.3.4 (Piecewise-smooth parameterization)**.** *Suppose* $[a, b] \subset \mathbb{R}$ *and* $z\colon [a, b] \to \mathbb{C}$ *is a parameterization. Then,* z *is piecewise-smooth if*

(i) z *is continuous, and*

(ii) $\exists a_0, a_1, \ldots a_n \in [a, b]$*, where* $a_0 = a$ *and* $a_n = b$*, such that* z *is smooth in the interval* $[a_k, a_{k+1}]$*,* $\forall k \in \{0, \ldots, n-1\}$*.*

That is, we say that z *is piecewise smooth it is continuous and there exist finite subdivisions of* $[a, b]$ *such that* z *is smooth on each of the subdivisions.*

Definition 1.3.5 (Equivalent parameterizations)**.** *Suppose* $[a, b], [c, d] \subset \mathbb{R}$*. Then, two parameterizations* $z\colon [a, b] \to \mathbb{C}$ *and* $\tilde{z}\colon [c, d] \to \mathbb{C}$ *are equivalent if there exists a continuously differentiable bijection* $T\colon [c, d] \to [a, b]$ *such that*

(i) $T'(s) > 0$*,* $\forall s \in [c, d]$ *and*

(ii) $\tilde{z}(s) = z(T(s))$*,* $\forall s \in [c, d]$*.*

We note that (i) *ensures the preservation of orientation; that is, as* $s \in [c, d]$ *travels from* c *to* d*, we have that* $t(s) \in [a, b]$ *travels from* a *to* b*.*

Definition 1.3.6 (Region)**.** $\Omega \subset \mathbb{C}$ *is a region if* Ω *is open and connected.*

Definition 1.3.7 (Curve)**.** *Suppose* $[a, b] \subset \mathbb{R}$ *and* $z\colon [a, b] \to \mathbb{C}$ *is a parameterization. Then, the curve* γ *with parameterization (or parameterized by)* z *is the image of* $[a, b]$ *under* z*,* $\gamma = z([a, b]) \subset \mathbb{C}$*, with orientation given by* z *as* $t \in [a, b]$ *travels from* a *to* b*.*

If sufficient context is supplied and no misunderstandings may arise, we denote a curve with parameterization γ by γ, via some abuse of notation.

Definition 1.3.8 (Smooth and Piecewise-smooth curve??). *Suppose $[a, b] \subset \mathbb{R}$ and $z\colon [a, b] \to \mathbb{C}$ is a parameterization of a curve $\gamma \subset \mathbb{C}$. Then, γ is a*

(i) smooth curve if z is smooth, and

(ii) piecewise-smooth curve if z is piecewise-smooth.

Proposition 1.3.1. *Suppose $[a, b] \subset \mathbb{R}$ and $z\colon [a, b] \to \mathbb{C}$ is a parameterization. Then,*

$$\{k : \ k \text{ is equivalent to } z\}$$

determines a smooth curve $\gamma \subset \mathbb{C}$, where $\gamma = z([a, b])$, with orientation given by z as $t \in [a, b]$ travels from a to b.

Definition 1.3.9 (Reversed curve). *Suppose $[a, b] \subset \mathbb{R}$ and $z\colon [a, b] \to \mathbb{C}$ is a parameterization of a curve $\gamma \subset \mathbb{C}$. Then, the reversed curve of γ is the curve γ^- obtained by reversing the orientation of γ.*

Proposition 1.3.2 (A Parameterization for Reversed Curves). *Suppose $[a, b] \subset \mathbb{R}$ and $z\colon [a, b] \to \mathbb{C}$ is a parameterization of a curve $\gamma \subset \mathbb{C}$. Then, $z^- : [a, b] \to \mathbb{C}$ defined by*

$$z^-(t) = z(b + a - t) \ \forall t \in [a, b]$$

is a parameterization of γ^-.

Definition 1.3.10 (Endpoints of a curve). *Suppose $[a, b] \subset \mathbb{R}$ and $z\colon [a, b] \to \mathbb{C}$ is a parameterization of a curve $\gamma \subset \mathbb{C}$. We say that*

(i) $z(a), z(b)$ are endpoints of γ, and

(ii) γ begins at $z(a)$ and ends at $z(b)$.

We note that the endpoints are independent on the parameterization (??For any parameterization $\tilde{z}$ equivalent to z, $\tilde{z}(a) = z(a)$ and $\tilde{z}(b) = z(b)$)

Definition 1.3.11 (Closed and simple curve). *(i) A smooth or piecewise-smooth curve $\gamma \subset \mathbb{C}$ is closed if for any parameterization $z\colon [a, b] \to \mathbb{C}$ of γ, we have that $z(a) = z(b)$.*

That is, γ is closed if its endpoints overlap.

(ii) A smooth or piecewise-smooth curve $\gamma \subset \mathbb{C}$ is simple if for any parameterization $z\colon [a, b] \to \mathbb{C}$ of γ, we have that $\forall x, y \in [a, b]$, $x \neq y \implies z(x) \neq z(y)$.

That is, γ is simple if it is not self-intersecting.

(iii) Suppose γ is closed. Then, γ is a simple (closed) curve if

$$\forall x, y \in [a, b], \ x \neq y \text{ and } (x \neq a \text{ or } y \neq b) \implies z(x) \neq z(y)$$

or, equivalently, if

$$\forall x, y \in [a, b], \ z(x) = z(y) \implies x = y \text{ or } (x = a \text{ and } y = b).$$

Definition 1.3.12 (Positive and negative orientation).

Remark 1.3.1. *Unless otherwise specified, we hereafter refer to any* ***piecewise-smooth curve*** *more briefly as a* ***curve****.*

Example 1.3.1 (Example of a curve)**.** *Let* $r > 0$ *and* $z_0 \in \mathbb{C}$*. Then,* $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0 = r|\}$*, the circle centered at* z_0 *with radius* r*, is a curve. In addition,*

(i) the parameterization $Z\colon [0, 2\pi] \to \mathbb{C}$*, given by* $Z(t) = z_0 + re^{it}$ $\forall t \in [a, b]$*, of* $C_r(z_0)$ *defines* $C_r(z_0)$ *in the positive orientation (counterclockwise);*

(ii) the parameterization $\tilde{Z}\colon [0, 2\pi] \to \mathbb{C}$*, given by* $\tilde{Z}(t) = z_0 + re^{-it}$ $\forall t \in [a, b]$*, of* $C_r(z_0)$ *defines* $C_r(z_0)$ *in the negative orientation (clockwise).*

Definition 1.3.13 (Notation: C)**.** *We denote* C *a positively oriented circle.*

Definition 1.3.14 (Integral of a complex-valued function along a smooth curve)**.** *Suppose* $\gamma \subset \mathbb{C}$ *is a smooth curve with parameterization* $z\colon [a, b] \to \mathbb{C}$*, and* f *is continuous on* γ*. Then, the integral of* f *along* γ *is*

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

Proposition 1.3.3 (Well-Definedness of an Integral Along a Smooth Curve)**.** *Suppose* $\gamma \subset \mathbb{C}$ *is a smooth curve with parameterization* $z\colon [a, b] \to \mathbb{C}$*, and* f *is continuous on* γ*. Suppose further that* $\tilde{z}\colon [c, d] \to \mathbb{C}$ *is an equivalent parameterization of* γ*. Then,*

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_c^d f(\tilde{z}(s))\tilde{z}'(s).$$

That is, the integral of f *along* γ *is independent of the parameterization of* γ*.*

Proposition 1.3.4 (Integral of a Function Along a Piecewise-Smooth Curve)**.** *Suppose* $\gamma \subset \mathbb{C}$ *is a piecewise-smooth curve with parameterization* $z\colon [a, b] \to \mathbb{C}$*, and* f *is continuous on* γ*. Then,*

$$\int_\gamma f(z)dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t)) \cdot z'(t)dt,$$

where $z([a_k, a_{k+1}])$ *is smooth* $\forall k \in \{0, \ldots, n-1\}$*.*

Definition 1.3.15 (Length of a smooth curve)**.** *Suppose* $\gamma \subset \mathbb{C}$ *is a smooth curve with parameterization* $z\colon [a, b] \to \mathbb{C}$*. Then, the length of* γ *is*

$$length(\gamma) = \int_a^b |z'(t)|\, dt.$$

Proposition 1.3.5 (Properties of Integration of Complex-Valued Functions)**.** *Suppose* $\gamma \subset \mathbb{C}$ *is a smooth curve with parameterization* $z\colon [a, b] \to \mathbb{C}$*, and* $f\colon \gamma \to \mathbb{C}$ *is continuous. Then, the following statements hold.*

(Linearity) Let $\lambda, \mu \in \mathbb{C}$*. Then,*

$$\int_\gamma \lambda f(z) + \mu g(z)dz = \lambda \int_\gamma f(z)dz + \mu \int_\gamma g(z)dz.$$

(Reversed Orientation)

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz.$$

(Inequality)

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z\in\gamma} |f(z)| \cdot length(\gamma)$$

Definition 1.3.16 (Primitive). *Suppose f is a complex-valued function defined on an open subset $\Omega \subset \mathbb{C}$. Then, a complex-valued function F defined on Ω is a primitive of f on Ω if*

(i) F is holomorphic on Ω and

(ii) $F'(z) = f(z)\ \forall z \in \mathbb{C}$.

Theorem 1.3.1 (Application of Fundamental Theorem of Calculus to Complex Integral). *Suppose that $\Omega \subset \mathbb{C}$ is open, and f has a primitive F on Ω. Suppose further that $\gamma \subset \Omega$ is a curve (smooth or piecewise-smooth) with start point $w_1 \in \mathbb{C}$ and endpoint $w_2 \in \mathbb{C}$, and f is a continuous on γ. Then,*

$$\int_\gamma f(z)dz = F(w_2) - F(w_1).$$

[Theorem 3.2 (Stein 22)]

Proof. Suppose $\Omega \subset \mathbb{C}$ is open and f has a primitive F on Ω. Suppose further that $\gamma \subset \Omega$ is a curve with parameterization $g\colon [a,b] \to \mathbb{C}$ where $g(a) = w_1, g(b) = w_2$, and f is continuous on γ.

(Case 1): Suppose γ is smooth. It follows that

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(g(t)) \cdot g'(t)dt && \text{(By defintion (Stein 21))}\\
&= \int_a^b F'(g(t)) \cdot g'(t)dt && \text{(Since } F \text{ is a primitive of } f \text{ on } \Omega)\\
&= \int_a^b \frac{d}{dt}F(g(t)) && \text{(By the Chain Rule)}\\
&= F(g(b)) - F(g(a)) && \text{(By the Fundamental Theorem of Calculus)}\\
&= F(w_2) - F(w_1).
\end{aligned}$$

(Case 2): Suppose γ is piecewise-smooth. By definition (Stein 19),

$$\begin{aligned}
&\exists a_0, a_1, \ldots, a_n \in [a,b] \text{ such that } a = a_0 < a_1 < a_2 < \cdots < a_n = b\\
&\quad \text{and } g([a_k, a_{k+1}]) \text{ is smooth } \forall k \in \{0, 1, \ldots, n-1\},
\end{aligned}$$

and, therefore, we have that

$$\begin{aligned}
\int_\gamma f(z)dz &= \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(g(t)) \cdot g'(t)dt && \text{((Stein 21))}\\
&= \sum_{k=0}^{n-1} F(g(a_{k+1})) - F(g(a_k)) && \text{(By result in Case 1)}\\
&= F(g(a_n)) - F(g(a_0))\\
&= F(g(b)) - F(g(a)) = F(w_2) - F(w_1).
\end{aligned}$$

□

Corollary 1.3.1 (Vanishing Integral Along a Closed Curve). *Suppose that $\Omega \subset \mathbb{C}$ is an open set and f has a primitive on Ω. Suppose further that $\gamma \subset \Omega$ is a closed curve (smooth or piecewise-smooth), and f is continuous on γ. Then,*

$$\int_\gamma f(z)dz = 0.$$

[Corollary 3.3 (Stein 23)]

Proof. Suppose that $\Omega \subset \mathbb{C}$ is an open set and f has a primitive F on Ω. Suppose further that $\gamma \subset \Omega$ is a closed curve (smooth or piecewise-smooth), and f is continuous on γ.

Denote w_2 and w_1 the endpoint and start point of γ respectively. By definition, γ is a closed curve implies that the start point and endpoint of γ coincide. As an immediate result fo the preceding theorem, we then obtain that

$$\int_\gamma f(z)dz = F(w_2) - F(w_1) = F(w_2) - F(w_2) = 0.$$

□

Corollary 1.3.2 (Vanishing Derivative Implies Constantness)**.** *Suppose f is holomorphic on a region $\Omega \subset \mathbb{C}$ and $f'(z) = 0$ $\forall z \in \Omega$. Then, f is constant on Ω.*

[Corollary 3.4 (Stein 23)]

Proof. Suppose f is holomorphic on a region $\Omega \subset \mathbb{C}$ and $f'(z) = 0$ $\forall z \in \Omega$. Fix some $w_0 \in \Omega$. By Exercise 5 (Stein 25), $\forall w \in \Omega$ there exists a continuous path, which is a curve, $\gamma \subset \Omega$ connecting w_0 and w. Now, fix a $w \in \Omega$. By Theorem 3.2 (Stein 22), we obtain that

$$\begin{aligned} &\int_\gamma f'(z)dz = f(w) - f(w_0) \\ \implies &\int_\gamma 0dz = 0 = f(w) - f(w_0) && \text{(By assumption)} \\ &\implies f(w) = f(w_0), \end{aligned}$$

for f' has a primitive f on Ω and certainly f' is continuous on γ since it is constant on Ω. That is, we show that $\forall w \in \Omega$ $f(w) = f(w_0)$, which implies that f is constant on Ω as desired. □

Example 1.3.2.

Theorem 1.3.2. *Suppose $(z_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers. Then, $z_n \to 0 \iff |z_n| \to 0$.*

Corollary 1.3.3 (??)**.** $\int_a^R f(z)dz \to 0$ *as* $R \to \infty \iff \left|\int_a^R f(z)dz\right| \to 0$ *as* $R \to \infty$

Proposition 1.3.6. *https://math.stackexchange.com/questions/2226361/how-do-i-show-that-real-part-of-an-integral-equals-the-integral-of-the-real-part*

2 Cauchy's Theorem and Its Applications

2.1 Goursat's Theorem

Proposition 2.1.1 (Equivalent Conditions for Differentiability)**.** *Suppose f is a real-valued function defined on some open set $\Omega \subset \mathbb{R}$. f is differentiable at $z_0 \in \Omega$ $\iff$*

$$f(z) = f(z_0) + f'(z_0) \cdot (z - z_0) + R(z) \ \forall z \in \Omega / \{z_0\} \implies \lim_{z \to z_0} \frac{R(z)}{z - z_0} = 0$$

$\iff$

$$f(z_0 + h) = f(z_0) + f'(z_0) \cdot h + R(v) \ \forall h \in \mathbb{C} \text{ such that } z_0 + h \in \Omega \implies \lim_{h \to 0} \frac{R(v)}{h} = 0.$$

Theorem 2.1.1 (Goursat's Theorem)**.** *Suppose f is a complex function holomorphic on some open set $\Omega \subset \mathbb{C}$ containing the closure of a triangle T. Then,*

$$\int_T f(z)dz = 0.$$

Corollary 2.1.1 (Goursat's Theorem for Rectangles)**.** *Suppose f is a complex function holomorphic on some open set $\Omega \subset \mathbb{C}$ containing the closure of a rectangle R. Then,*

$$\int_R f(z)dz = 0.$$

2.2 Local existence of primitives and Cauchy's theorem in a disc

Theorem 2.2.1 (Local Holomorphicity Implies Local Existence of Primitive)**.** *Suppose f is a complex function holomorphic on $B_r(z_0)$, for some $r > 0$ and $z_0 \in \mathbb{C}$. Then, f has a primitive on $B_r(z_0)$.*

Theorem 2.2.2 (Cauchy's Theorem on a Disc)**.** *Suppose f is a complex function holomorphic on some open set $\Omega \subset \mathbb{C}$ containing $\overline{B_r(z_0)}$, for some $r > 0$ and $z_0 \in \mathbb{C}$. Then,*

$$\int_{\partial B_r(z_0)} f(z)dz = 0.$$

Theorem 2.2.3 (Generalized Cauchy Theorem)**.** *Suppose f is a complex function holomorphic on some open set $\Omega \subset \mathbb{C}$ (toy contour) containing the closure of a closed curve $\gamma \subset \Omega$. Then,*

$$\int_{\partial\gamma} f(z)dz = 0.$$

2.3 Evaluation of some integrals

Lemma 2.3.1 (Jordan's Inequality). $\frac{2t}{\pi} \leq \sin(t) \leq t \ \forall t \in [0, \pi]$.

Lemma 2.3.2. $\sin(\pi - t) = \sin(t) \ \forall t \in \mathbb{R}$.

Proof. Suppose $t \in \mathbb{R}$. It follows that

$$\begin{aligned} \sin(\pi - t) &= \sin(\pi)\cos(-t) + \sin(-t)\cos(\pi) && \text{(By the addition identity of sin)} \\ &= 0 + -\sin(t) \cdot (-1) = \sin(t). && \text{(Since sin is odd)} \end{aligned}$$

□

Theorem 2.3.1 (Jordan's Lemma). *Let $R > 0$ and $C_R = \{Re^{i\theta} : \theta \in [0, \pi]\}$. Suppose $g \colon C_R \to \mathbb{C}$ is continuous and define $f \colon C_R \to \mathbb{C}$ by*

$$f(z) = e^{iaz} g(z) \ \forall z \in C_R$$

for some $a > 0$. Then,

$$\int_{C_R} f(z)dz \leq \frac{\pi}{a} \cdot \sup_{z \in C_R} |g(z)| .$$

Proof. Let $R > 0$ and $C_R = \{Re^{i\theta} : \theta \in [0, \pi]\}$. Suppose $g \colon C_R \to \mathbb{C}$ is continuous and define $f \colon C_R \to \mathbb{C}$ by

$$f(z) = e^{iaz} g(z) \ \forall z \in C_R$$

for some $a > 0$. By the continuity of g and e^{iaz} on C_r, we have that

$$\int_{C_r} f(z)dz = \int_{C_r} e^{iaz} g(z)dz = \int_0^\pi e^{ia[re^{it}]} g(re^{it}) \cdot ire^{it} dt,$$

which implies that

$$\begin{aligned} \left| \int_{C_r} f(z)dz \right| &= \left| \int_0^\pi e^{ia[r\cos(t) + ir\sin(t)]} g(re^{it}) \cdot ire^{it} dt \right| && \text{(By Euler's Identity)} \\ &\leq \int_0^\pi \left| e^{iar\cos(t)} e^{-ar\sin(t)} g(re^{it}) \cdot ire^{it} \right| dt && \text{(By a property of integration)} \\ &= r \int_0^\pi \left| e^{-ar\sin(t)} \right| \cdot \left| g(re^{it}) \right| dt \\ &\leq rM \int_0^\pi \left| e^{-ar\sin(t)} \right| dt. && \text{(Where } M = \sup_{t \in [0,\pi]} \left| g(re^{it}) \right| ; (*_1)) \end{aligned}$$

We note that $e^{-ar\sin(t)}$ is continuous on $\mathbb{R}$ implies the integral $\int_0^{\frac{\pi}{2}} e^{-ar\sin(t)} dt$ exists. Denote $\int_0^{\frac{\pi}{2}} e^{-ar\sin(t)} dt$ by I. we observe that

$$\begin{aligned} \int_0^\pi e^{-ar\sin(t)} dt &= I + \int_{\frac{\pi}{2}}^\pi e^{-ar\sin(t)} dt && \text{(By a property of integration)} \\ &= I + \int_{\frac{\pi}{2}}^0 e^{-ar\sin(\pi - u)} (-1) du && \text{(By Change of Variable, where } u(t) = \pi - t) \\ &= I + \int_0^{\frac{\pi}{2}} e^{-ar\sin(\pi - u)} du && \text{(By a property of integration)} \\ &= I + \int_0^{\frac{\pi}{2}} e^{-ar\sin(u)} du = 2I. && \text{(By a previous lemma; } (*_2)) \end{aligned}$$

In addition, by Jordan's inequality, we obtain that

$$\begin{aligned} t \in [0,\pi] \implies \sin(t) \geq \frac{2t}{\pi} &\implies -ar\sin(t) \leq -ar\frac{2t}{\pi} \\ &\implies e^{-ar\sin(t)} \leq e^{-ar\frac{2t}{\pi}} \\ \implies 2I = 2\int_0^{\frac{\pi}{2}} e^{-ar\sin(t)}dt &\leq 2\int_0^{\frac{\pi}{2}} e^{-ar\frac{2t}{\pi}}dt \\ &= \frac{\pi}{ar}(1-e^{\frac{ar}{2}}) < \frac{\pi}{ar}. \quad (\text{Since } e^{\frac{ar}{2}} > 0 \implies 1-e^{\frac{ar}{2}} < 1; (*_3)) \end{aligned}$$

Combining $(*_1)$, $(*_2)$, and $(*_3)$, we then yield

$$\left|\int_{C_r} f(z)dz\right| \leq rM\cdot(2I) \leq rM\cdot\left(\frac{\pi}{ar}\right) = \frac{\pi}{a}\cdot \sup_{t\in[0,\pi]}\left|g(re^{it})\right|.$$

□

Theorem 2.3.2 (Real and Imaginary Parts of an Integral)**.** *Suppose $f\colon \Omega \to \mathbb{C}$ for some $\Omega \subset \mathbb{C}$, and $\int_\gamma f(z)dz$ exists for some curve $\gamma \subset \Omega$. Then,*

$$\Im\left[\int_\gamma f(z)dz\right] = \int_\gamma \Im\left[f(z)\right]dz \text{ and } \Re\left[\int_\gamma f(z)dz\right] = \int_\gamma \Re\left[f(z)\right]dz.$$

Remark 2.3.1 (Gaussian Distribution)**.** *We note that the Gaussian Distribution (normal distribution) is given by $f(x) = e^{-\pi x^2}$ and we have*

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

Proposition 2.3.1. *Let $\xi \in \mathbb{R}$. Then*

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-2i\pi x\xi}f(x)dx$$

Proposition 2.3.2. $\int_0^\infty \frac{1-\cos(x)}{x^2}dx = \pi$.

Proof. **[DO: DOUBLE CHECK THE VALIDITY OF INTERCHANGING SUMMATION AND INTEGRAL SIGNS]** Consider arbitrary $\epsilon, r > 0$ such that $r > \epsilon$. Denote γ_1, γ_2, γ_3^-, and γ_4 the curve along the upper half arc of $B_r(0)$ with positive orientation; the line segment connecting $-r$ to $-\epsilon$; the curve along the upper half arc of $B_\epsilon(0)$ with negative orientation; and the line segment connecting ϵ to r.

Let R be the open region enclosed by γ_1, γ_2, γ_3, and γ_4. Define $f\colon \mathbb{C}/\{0\} \to \mathbb{C}$ by $f(z) = \frac{1-e^{iz}}{z^2}$ $\forall z \in \mathbb{C}/\{0\}$. We observe that f is holomorphic on $\mathbb{C}/\{0\}$, which is open and containing $\overline{R}$. By Cauchy's Theorem, we obtain that

$$\begin{aligned} \int_{\partial R} f(z)dz = \int_{\gamma_1} f(z)dz &+ \int_{\gamma_2} f(z)dz + \int_{\gamma_3^-} f(z)dz + \int_{\gamma_4} f(z)dz = 0 \\ \implies \int_{-r}^{-\epsilon} f(t)dt &+ \int_\epsilon^r f(t)dt = -\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz. \qquad (*_1) \end{aligned}$$

We observe that, by the smoothness of γ_1 and continuity of f on γ_1,

$$\left|\int_{\gamma_1} f(z)dz\right| \leq \sup_{z\in\gamma_1}|f(z)|\cdot length(\gamma_1) = \frac{2}{r^2}\cdot\pi r = \frac{2\pi}{r},$$

where $\frac{2\pi}{r} \to 0$ as $r \to \infty$ implies

$$\int_{\gamma_1} f(z)dz \to 0 \text{ as } r \to \infty. \tag{$*_2$}$$

We note that $\sup_{z\in\gamma_1} |f(z)| = \frac{2}{r^2}$ since

$$z = x + iy \in \gamma_1 \implies 0 \le y \le r \implies 1 = e^{-0} \ge e^{-y},$$

which yield that

$$|f(z)| = \left|\frac{1-e^{iz}}{z^2}\right| = \left|\frac{1-e^{ix}e^{-y}}{z^2}\right| = \frac{\left|1-e^{ix}e^{-y}\right|}{|z^2|} \le \frac{1+e^{-y}}{r^2} \le \frac{2}{r^2}.$$

Furthermore, we observe that $z \in \gamma_3$ implies that

$$\begin{aligned} f(z) = \frac{1-e^{iz}}{z^2} &= \frac{1-\sum_{k=0}^{\infty}\frac{(iz)^k}{k!}}{z^2} && \text{(By the Taylor Series expansion of } exp\text{)} \\ = \frac{-\sum_{k=1}^{\infty}\frac{(iz)^k}{k!}}{z^2} &= \frac{-i}{z} + \frac{1}{2} + \sum_{k=3}^{\infty}\frac{i^k z^{k-2}}{k!} \\ &= \frac{-i}{z} + \frac{1}{2} + \sum_{j=0}^{\infty}\frac{i^{j+3}z^{j+1}}{(j+3)!}. && \text{(By re-indexing)} \end{aligned}$$

It follows that **[DO: REWORK THE FOLLOWING; TRY NOT TO INTEGRATE AN INFINITE SERIES]**

$$\begin{aligned} \int_{\gamma_3} f(z)dz &= \int_{\gamma_3} \frac{-i}{z} + \frac{1}{2} + \sum_{j=0}^{\infty}\frac{i^{j+3}z^{j+1}}{(j+3)!}dz \\ &= \int_{\gamma_3} \frac{-i}{z}dz + \int_{\gamma_3}\frac{1}{2}dz + \int_{\gamma_3}\sum_{j=0}^{\infty}\frac{i^{j+3}z^{j+1}}{(j+3)!}dz \\ = -i\int_0^{\pi}\frac{1}{\epsilon e^{it}}\cdot i\epsilon e^{it}dt &+ \int_0^{\pi}\frac{1}{2}\cdot i\epsilon e^{it}dt + \int_0^{\pi}\sum_{j=0}^{\infty}\frac{i^{j+3}(\epsilon e^{it})^{j+1}}{(j+3)!}\cdot i\epsilon e^{it}dt \\ &= \pi + i\epsilon\int_0^{\pi}\frac{1}{2}e^{it}dt + \epsilon^2\int_0^{\pi}\sum_{j=0}^{\infty}\epsilon^j i^{j+4}\frac{e^{it(j+2)}}{(j+3)!}dt \end{aligned}$$

which converges to π as $\epsilon \to 0$, since $\lim_{\epsilon\to 0}\left[i\epsilon\int_0^{\pi}\frac{1}{2}e^{it}dt\right] = 0$ and

$$\forall k \in \mathbb{N}/\{1,2\} \quad \lim_{\epsilon\to 0}\left[i\epsilon^{k-1}\int_0^{\pi}\frac{i^k e^{it(k-1)}}{k!}dt\right] = 0$$

implies the series converges to 0 as $\epsilon \to 0$ by a limit property. Combining $(*_1)$, $(*_2)$, and the above result, we then obtain that $r \to \infty$ and $\epsilon \to 0$ yield that

$$\int_{-r}^{-\epsilon} f(t)dt + \int_{\epsilon}^{r} f(t)dt = -\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz \to 0 + \pi = \pi.$$

That is, we have

$$\int_{-\infty}^{\infty}\frac{1-e^{it}}{t^2}dt = \int_{-\infty}^{\infty} f(t)dt = \lim_{r\to\infty,\epsilon\to 0}\left[\int_{-r}^{-\epsilon} f(t)dt + \int_{\epsilon}^{r} f(t)dt\right] = \pi,$$

implying

$$\Re\left[\int_{-\infty}^{\infty}\frac{1-e^{it}}{t^2}dt\right] = \int_{-\infty}^{\infty}\Re\left[\frac{1-e^{it}}{t^2}\right]dt = \int_{-\infty}^{\infty}\frac{1-\cos(t)}{t^2}dt = \Re(\pi) = \pi$$

and, therefore,

$$\int_0^\infty \frac{1-\cos(t)}{t^2}dt = \frac{\pi}{2}. \qquad \text{(Since } \tfrac{1-\cos(t)}{t^2} \text{ is even)}$$

$\square$

2.4 Cauchy's integral formulas

Theorem 2.4.1 (Cauchy's Integral Formula). *Suppose f is holomorphic on an open subset $\Omega \subset \mathbb{C}$ containing $\overline{B_r(z_0)}$ for some $r > 0$ and $z_0 \in \mathbb{C}$. Then,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{w - z} dw \ \forall z \in B_r(z_0)$$

where we assume $\partial B_r(z)$ has positive orientation.

Corollary 2.4.1 (Generalized Cauchy's Integral Formula). *Suppose f is holomorphic on an open subset $\Omega \subset \mathbb{C}$ containing $\overline{B_r(z_0)} \subset \Omega$ for some $r > 0$ and $z_0 \in \mathbb{C}$. Then, f is infinitely complex differentiable on Ω and*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{(w - z)^{n+1}} dw \ \forall z \in B_r(z_0)$$

where we assume $\partial B_r(z)$ has positive orientation.

Corollary 2.4.2 (Cauchy's Inequality). *Suppose f is holomorphic on an open subset $\Omega \subset \mathbb{C}$ containing $\overline{B_r(z_0)} \subset \Omega$ for some $r > 0$ and $z_0 \in \mathbb{C}$. Then,*

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! \cdot ||f||_C}{r^n} \ \forall n \in \mathbb{W},$$

where $||f||_C = \sup_{z \in \partial B_r(z_0)} |f(z)|$.

Proof. Suppose f is holomorphic on an open subset $\Omega \subset \mathbb{C}$ containing $\overline{B_r(z_0)} \subset \Omega$ for some $r > 0$ and $z_0 \in \mathbb{C}$. Suppose that $n \in \mathbb{W}$. Then, we have that

$$\begin{aligned}
\left| f^{(n)}(z_0) \right| &= \left| \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw \right| && \text{(By Cauchy's Integral Formula)} \\
&= \frac{n!}{2\pi i} \left| \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{(z_0 + Re^{it} - z_0)^{n+1}} \cdot iRe^{it} dt \right| && \text{(Since } f \text{ is continuous on } \partial B_r(z_0)) \\
= \frac{n!}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{(Re^{it})^n} dt \right| &\leq \frac{n!}{2\pi} \int_0^{2\pi} \left| \frac{f(z_0 + Re^{it})}{(Re^{it})^n} \right| dt && \text{(By a property of integral)} \\
= \frac{n!}{2\pi} \int_0^{2\pi} \frac{\left| f(z_0 + Re^{it}) \right|}{R^n} dt &\leq \frac{n!}{2\pi R^n} \int_0^{2\pi} ||f||_C \, dt && \text{(Since } z_0 + Re^{it} \in \partial B_r(z_0)) \\
&= \frac{n!}{2\pi R^n} \cdot ||f||_C \cdot (2\pi) = \frac{n! \, ||f||_C}{R^n}.
\end{aligned}$$

□

Theorem 2.4.2 (Dominated Convergence Theorem). *Suppose $(f_k)_{k \in \mathbb{N}}$ is a sequence of Riemann integrable functions from $\Omega \subset \mathbb{C}$ to $\mathbb{C}$ converging pointwise to a Riemann integrable function $f \colon \Omega \to \mathbb{C}$. Suppose further that $\exists g \colon \Omega \to \mathbb{C}$ such that*

(i) $k \in \mathbb{N}$ and $z \in \Omega \implies |f_k(z)| \leq |g(z)|$, and
(ii) $\int_\gamma |g(z)| \, dz < \infty$, where $\gamma \subset \Omega$ is a curve.

Then,

$$\lim_{n \to \infty} \int_\gamma f_n(z) dz = \int_\gamma \lim_{n \to \infty} f_n(z) dz.$$

Remark 2.4.1. *The above is a special case of Lebesgue's Dominated Convergence Theorem.*

Theorem 2.4.3 (Holomorphicity on Open Set Implies Locally Analytic). *Suppose f is holomorphic on an open subset $\Omega \subset \mathbb{C}$ containing $\overline{B_r(z_0)} \subset \Omega$ for some $r > 0$ and $z_0 \in \mathbb{C}$. Then, f is analytic on $B_r(z_0)$; that is, f has a power series expansion at z_0 such that*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \ \forall z \in B_r(z_0).$$

Definition 2.4.1 (Bounded complex function). *Suppose $\Omega \subset \mathbb{C}$. Then, $f\colon \Omega \to \mathbb{C}$ is bounded if*

$$\exists M > 0 \text{ such that } |f(z)| \leq M \ \forall z \in \Omega.$$

Definition 2.4.2 (Entire function). *A function $f\colon \mathbb{C} \to \mathbb{C}$ is entire if f is holomorphic on $\mathbb{C}$.*

Theorem 2.4.4 (Livouile's Theorem). *Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is bounded and entire. Then, f is constant.*

Proof. Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is bounded and entire. By definition, $\exists B > 0$ such that $\forall z \in \mathbb{C}$ $|f(z)| \leq B$. Consider arbitrary $z_0 \in \mathbb{C}$. It follows that $r > 0$ implies that $B_r(z_0) \subset \mathbb{C}$ and

$$\begin{aligned} |f'(z_0)| &\leq \frac{1! \cdot \sup_{z \in \partial B_r(z_0)} |f(z)|}{R^1} && \text{(By Cauchy's Inequality)} \\ &\leq \frac{B}{R}. \end{aligned}$$

It follows that $|f'(z_0)| = 0$, for if not we would obtain that $|f'(z_0)| > 0 \implies \frac{2B}{|f'(z_0)|} > 0$ implies that

$$|f'(z_0)| \leq \frac{B}{\frac{2B}{|f'(z_0)|}} = \frac{B}{1} \cdot \frac{|f'(z_0)|}{2B} = \frac{|f'(z_0)|}{2},$$

which is a contradiction. Thus, we showed that $\forall z_0 \in \mathbb{C}$, $|f'(z_0)| = 0$ and, therefore, $f'(z_0) = 0$. By Corollary 3.4 (Stein-Shakarchi 23), f is constant on $\mathbb{C}$ as desired. □

Alternative Proof. Outline: Prove via the analyticity of f on $\mathbb{C}$. □

Theorem 2.4.5 (Fundamental Theorem of Algebra; Existence of Root for Non-Constant Complex Polynomial). *Suppose p is a non-constant polynomial with complex coefficients. Then, p has a root in $\mathbb{C}$.*

Proof. Suppose p is a non-constant polynomial with complex coefficients. By definition,

$$\exists a_0, a_1, \ldots, a_n \in \mathbb{C} \text{ such that } p(z) = a_0 + a_1 z + \cdots + a_n z^n \ \forall z \in \mathbb{C}.$$

Since p is non-constant, $\exists m \in \mathbb{N}$ such that $a_m \neq 0$. By re-indexing, we may assume, without loss of generality, that n is the largest such index such that $a_n \neq 0$.

Suppose, to the contrary, that p has no roots in $\mathbb{C}$. It follows that

$$\begin{aligned} \forall z \in \mathbb{C}/\{0\} \ p(z) = a_0 + a_1 z + \cdots + a_n z^n &\implies \frac{p(z)}{z^n} = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} + a_n \\ &\implies \lim_{|z| \to \infty} \frac{a_k}{z^{n-k}} = 0 \ \forall k \in \{0, \ldots, n-1\} \implies \lim_{|z| \to \infty} \frac{p(z)}{z^n} = a_n \neq 0. \end{aligned}$$

By definition, we then obtain that

$$\begin{aligned} \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } |z| > \delta &\implies \left| \frac{p(z)}{z^n} - a_n \right| < \epsilon \\ &\implies \left| \left| \frac{p(z)}{z^n} \right| - |a_n| \right| \leq \left| \frac{p(z)}{z^n} - a_n \right| < \epsilon && \text{(By triangle inequality)} \\ &\implies |z^n| \cdot (-\epsilon + |a_n|) < |p(z)| < |z^n| \cdot (\epsilon + |a_n|). \end{aligned}$$

In particular, we have that

$$\exists R > 0 \text{ such that } |z| > R \implies |z^n| \cdot \left(-\frac{|a_n|}{2} + |a_n| \right) = \frac{|a_n|}{2} \cdot |z^n| < |p(z)| \quad (\text{Since } \tfrac{|a_n|}{2} > 0)$$
$$\implies \frac{1}{|p(z)|} < \frac{2}{|a_n| \cdot |z^n|} = \frac{2}{|a_n| \cdot |z|^n} < \frac{2}{|a_n| \cdot R^n}.$$

Thus, we have that $\left| \frac{1}{p(z)} \right| \leq \frac{2}{|a_n| R^n}\ \forall z \in \mathbb{C}/[\overline{B_R(0)}]$.

We observe that $\frac{1}{p}$ is holomorphic and, hence, continuous on $\mathbb{C}$, since $p(z) \neq 0\ \forall z \in \mathbb{C}$ by assumption. In addition, $\overline{B_R(0)}$ is compact implies $\left[\frac{1}{p}\right] \left(\overline{B_R(0)}\right)$ is also compact, since continuity preserves compactness. By the Heine-Borel Theorem, we then yield that $\left[\frac{1}{p}\right] \left(\overline{B_R(0)}\right)$ is bounded. That is, $\exists M > 0$ such that $M \geq \left| \frac{1}{p(z)} \right|\ \forall z \in \overline{B_R(0)}$. Let $M^* = \max\left\{M, \frac{2}{|a_n| R^n}\right\}$. Then,

$$M^* \geq \left| \frac{1}{p(z)} \right| \ \forall z \in \mathbb{C}.$$

By definition, $\frac{1}{p}$ is bounded. As an immediate result of Liouvile's Theorem, $\frac{1}{p}$ is constant. Therefore, p is also constant, which is a contradiction. Hence, we conclude that p has a root in $\mathbb{C}$. □

Corollary 2.4.3 (Degree of Polynomial and Number of Roots)**.** *Suppose that p is a polynomial of degree $n \in \mathbb{N}$. Then, p has precisely n roots in $\mathbb{C}$ and*

$$p(z) = a_n(z - w_1)(z - w_2) \cdot \dots \cdot (z - w_n)\ \forall z \in \mathbb{C}$$

where a_n is the coefficient of the n-th term of p and $w_1, \dots, w_n$ are the n roots of p.

Proof. Prove via induction. □

Theorem 2.4.6 (Vanishing on a Convergent Sequence Implies Identically the Zero Map)**.** *Suppose $\Omega \subset \mathbb{C}$ is open and connect, $f : \Omega \to \mathbb{C}$ is holomorphic, and $\exists z_0 \in \Omega$ such that z_0 is a limit point of $\{z \in \Omega : f(z) = 0\}$, where $\{z \in \Omega : f(z) = g(z)\} \neq \emptyset$. Then, f is the zero function on Ω.*

Proof. Suppose $\Omega \subset \mathbb{C}$ is open and connect, $f : \Omega \to \mathbb{C}$ is holomorphic, and $z_0 \in \Omega$ is a limit point of $\{z \in \Omega : f(z) = 0\} \neq \emptyset$. By definition, there exists a sequence $(w_k)_{k \in \mathbb{N}}$ of points in $\{z \in \Omega : f(z) = 0\}$ converging to z_0.

By the openness of Ω, $\exists r > 0$ such that $B_r(z_0) \subset \Omega$. Furthermore, by the holomorphicity of f on $B_r(z_0)$, we have, by Theorem 4.4 (Stein-Shakarchi 49), that f is analytic on $B_r(z_0)$ and

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \ \forall z \in B_r(z_0).$$

We claim that $f(z) = 0\ \forall z \in B_r(z_0)$. Suppose, to the contrary, that $f(z) \neq 0$ for some $z \in B_r(z_0)$. Then, $\exists M \in \mathbb{W}$ such that $a_M \neq 0$. Let m be the smallest such whole number. It follows that

$$\begin{aligned}
f(z) = a_0 + a_1 z^1 + \dots + a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \dots \\
= a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \dots \\
= a_m(z - z_0)^m \left(1 + \frac{a_{m+1}}{a_m}(z - z_0)^1 + \dots \right) \\
= a_m(z - z_0)^m \left(1 + \sum_{j=1}^{\infty} \frac{a_{m+j}}{a_m}(z - z_0)^j \right) \\
= a_m(z - z_0)^m \left(1 + g(z) \right),
\end{aligned}$$

where $g\colon B_r(z_0) \to \mathbb{C}$ is defined by $g(z) = \sum_{j=1}^{\infty} \frac{a_{m+j}}{a_m}(z-z_0)^j \ \forall z \in B_r(z_0)$. We observe that that g is continuous since it is a polynomial. In addition, we have that $g(z_0) = 0$ and

$$\lim_{z \to z_0} g(z) = g(z_0) = 0.$$

It follows that $\forall \epsilon > 0 \ \exists \delta > 0$ such that $|z - z_0| < \delta \implies |g(z) - 0| < \epsilon$. In particular, we have

$$\exists \delta > 0 \text{ such that } |z - z_0| < \delta \implies |g(z)| < \frac{1}{2}. \tag{$*$}$$

In addition, by the definition of a convergent sequence, it holds that

$$\forall \epsilon > 0 \exists N > 0 \text{ such that } k > N \implies |w_k - z_0| < \epsilon.$$

In particular, $\exists N > 0$ such that $k > N \implies |w_k - z_0| < \delta$. Take $K > N$ such that $w_K \neq z_0$. Then, we obtain that

$$\begin{aligned} K > N \implies |w_K - z_0| < \delta \implies |g(w_K)| < \frac{1}{2} \qquad & \text{(By $(*)$)} \\ \implies |g(w_K)| \neq 1 \implies g(w_k) \neq -1 \implies 1 + g(w_k) \neq 0 & \\ \implies f(w_K) = a_m(w_K - z_0)^m \left(1 + g(w_K)\right) \neq 0. & \end{aligned}$$

However, by assumption, $w_K \in \{z \in \Omega : f(z) = 0\} \implies f(w_K) = 0$, which is a contradiction. Therefore, $f(z) = 0 \ \forall z \in B_r(z_0)$ as desired. That is, we showed that for any limit point $z_0 \in \Omega$ of $\{z \in \Omega : f(z) = 0\}$, we have that $f(z) = 0$ on some ball B_{z_0} of z_0 and, hence, $B_{z_0} \subset \{z \in \Omega : f(z) = 0\}$.

Denote U the interior of $\{z \in \Omega : f(z) = 0\}$. By definition, U is open and $U \subset \{z \in \Omega : f(z) = 0\}$. Consider arbitrary limit point $u \in \Omega$ of U. We have that there exists a ball B_u of u contained in $\{z \in \Omega : f(z) = 0\}$ by the above conclusion. It follows that $u \in U$ since u is interior to U. By definition, U is also closed and $U \neq \emptyset$.

We observe that Ω/U and U are disjoint, open, and such that $\Omega = U \cup [\Omega/U]$. By the connectedness of Ω, we have that $U = \emptyset$ or $\Omega/U = \emptyset$. It follows that $\Omega/U = \emptyset$ and $\Omega = U \cup [\Omega/U] = U$. Thus, $\forall z \in \Omega = U \subset \{z \in \Omega : f(z) = 0\}$ $f(z) = 0$ as desired. ☐

Theorem 2.4.7 (Principle of Isolated Zeros of Analytic Functions)**.** ***WRITE PROOF PLEASE*** *All zeroes of a (non-identically zero) analytic function on a connected open set U of C are isolated, i.e. for each zero $a \in U$ there exists an open neighbourhood $V \subset U$ of a in which a is the only zero of f.*

Corollary 2.4.4. *Suppose $\Omega \subset \mathbb{C}$ is open and connect, $f, g\colon \Omega \to \mathbb{C}$ are holomorphic, and $\exists z_0 \in \Omega$ such that z_0 is a limit point of $\{z \in \Omega : f(z) = g(z)\}$, where $\{z \in \Omega : f(z) = g(z)\} \neq \emptyset$. Then, $f(z) = g(z) \ \forall z \in \Omega$.*

Definition 2.4.3 (Analytic Continuation of an Analytic Function)**.** *Suppose $f\colon \Omega \to \mathbb{C}$ and $g\colon \Omega' \to \mathbb{C}$ are analytic, where $\Omega, \Omega' \subset \mathbb{C}$ are regions such that $\Omega \subset \Omega'$. Then, g is an analytic continuation of f into Ω' if $g(z) = f(z) \ \forall z \in \Omega$.*

Theorem 2.4.8 (Uniqueness of Analytic Continuation)**.** *If an analytic function has an analytic continuation, then the analytic continuation is unique.*

Proof. Let $\Omega, \Omega' \subset \mathbb{C}$ be regions such that $\Omega \subset \Omega'$. Suppose that $f\colon \Omega \to \mathbb{C}$ is analytic with analytic continuations $g, g'\colon \Omega' \to \mathbb{C}$ into Ω'. **[DO: ELABORATE]**By definition, we have that

$$g(z) = f(z) = g'(z) \ \forall z \in \Omega.$$

It follows, from Corollary 4.9 (Stein-Shakarchi 52), that $g(z) = g'(z) \ \forall z \in \Omega'$. That is, $g = g'$ by definition. ☐

2.5 Further applications

2.5.1 Morera's theorem

Theorem 2.5.1 (Morera's theorem on a Ball)**.** *Suppose f is a complex-valued function continuous in $B_r(z_0)$ for some $r > 0$ and $z_0 \in \mathbb{C}$ such that for any parameterized triangle $T \in B_r(z_0)$*

$$\int_T f(z)dz = 0.$$

Then, f is holomorphic on $B_r(z_0)$.

Theorem 2.5.2 (Morera's theorem on an Open Set)**.** *Suppose f is a complex-valued function continuous in the interior of a toy contour γ such that for any parameterized triangle $T \in$ interior(γ)*

$$\int_T f(z)dz = 0.$$

Then, f is holomorphic in the interior of a toy contour γ

2.5.2 Sequences of holomorphic functions

Theorem 2.5.3 (Uniform Convergence Preserves Holomorphicity)**.** *Suppose $\Omega \subset \mathbb{C}$ is open and $f_n \colon \Omega \to \mathbb{C}$ is holomorphic, $\forall n \in \mathbb{N}$. Suppose further that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f \colon \Omega \to \mathbb{C}$ on every compact subset of Ω. Then, f is holomorphic on Ω.*

Proof. Suppose $\Omega \subset \mathbb{C}$ is open and $f_n \colon \Omega \to \mathbb{C}$ is holomorphic, $\forall n \in \mathbb{N}$. Suppose further that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f \colon \Omega \to \mathbb{C}$ on every compact subset of Ω.

Consider arbitrary $z_0 \in \Omega.$ By the openness of Ω, it follows that $\exists r > 0$ such that $B_r(z_0) \in \Omega$. Let $R = \frac{r}{2}$. Then, $\overline{B_R(z_0)} \subset \Omega$. Consider arbitrary parameterized triangle $T \in B_r(z_0)$ whose interior is contained in $B_R(z_0)$. It follows, from Goursat's Theorem that

$$\forall n \in \mathbb{N} \int_T f_n(z)dz = 0$$

since, $\forall n \in \mathbb{N}$, f_n is holomorphic on $B_R(z_0)$ by assumption. Furthermore, $\overline{B_R(z_0)}$ is compact and $(f_n)_{n\in\mathbb{N}}$ converges to f on every compact subset of Ω implies that f is continuous on $B_R(z_0)$ and

$$\int_T f(z)dz = \int_T \lim_{n\to\infty} f_n(z)dz = \lim_{n\to\infty} \int_T f_n(z)dz = 0. \qquad \text{(Since } T \in \overline{B_R(z_0)}\text{)}$$

As an immediate, f is holomorphic on $B_R(z_0)$ by Morera's Theorem since $\int_T f(z)dz = 0$ for every parameterized triangle $T \subset B_R(z_0)$. In particular, f is holomorphic on z_0.

Since we considered arbitrary $z_0 \in \Omega$, we showed that f is holomorphic on z_0, $\forall z_0 \in \Omega$. Hence, f is holomorphic as desired. ☐

Corollary 2.5.1.

2.5.3 Holomorphic functions defined in terms of integrals

2.5.4 Schwarz reflection principle

Definition 2.5.1 (Symmetry about the real-axis). *Suppose $\Omega \subset \mathbb{C}$. Then, Ω is symmetric about the real-axis if it satisfies the property*

$$z \in \Omega \iff \overline{z} \in \Omega.$$

Definition 2.5.2 (Extends continuously). *Suppose $U \subset \Omega \subset \mathbb{C}$ and $f\colon U \to \mathbb{C}$ is continuous. Then, f extends continuously to Ω if there exists a map $\tilde{f}\colon \Omega \to \mathbb{C}$ such that*

(i) $\tilde{f}(z) = f(z)\ \forall z \in U$, and

(ii) $\tilde{f}$ is continuous.

We say that $\tilde{f}$ is a continuous extension of f

Theorem 2.5.4 (Symmetry Principle). *Let $\Omega \subset \mathbb{C}$ be open, $\Omega^+ = \{z \in \Omega : Im(z) > 0\}$, $\Omega^- = \{z \in \Omega : Im(z) < 0\}$, and $I = \Omega \cap \mathbb{R}$. Suppose that $f^+\colon \Omega^+ \to \mathbb{C}$ and $f^-\colon \Omega^- \to \mathbb{C}$ are holomorphic and extend continuously to I, and satisfies*

$$f^+(x) = f^-(x)\ \forall x \in I.$$

Then, $f\colon \Omega \to \mathbb{C}$, defined by

$$f(z) = \begin{cases} f^+(z) & \text{if } z \in \Omega^+ \\ f^-(z) & \text{if } z \in \Omega^- \\ f^-(z) = f^+(z) & \text{if } z \in I \end{cases},\ \forall z \in \Omega,$$

is holomorphic on Ω.

Theorem 2.5.5 (Schwarz Reflection Principle). *Let $\Omega \subset \mathbb{C}$ be open, $\Omega^+ = \{z \in \Omega : Im(z) > 0\}$, $\Omega^- = \{z \in \Omega : Im(z) < 0\}$, and $I = \Omega \cap \mathbb{R}$. Suppose that $f\colon \Omega^+ \to \mathbb{C}$ is holomorphic, extends continuously to I, and satisfies $f(x) \in \mathbb{R}\ \forall x \in I$. Then, there exists a function $F\colon \Omega \to \mathbb{C}$ that is holomorphic and $F(z) = f(z)\ \forall z \in \Omega^+$.*

In particular, F is defined by

$$F(z) = \begin{cases} f(z) & \text{if } z \in \Omega^+ \cup I \\ \overline{f(\overline{z})} & \text{if } z \in \Omega^- \end{cases},\ \forall z \in \Omega.$$

2.5.5 Runge's approximation theorem

3 Meromorphic Functions and the Logarithm

3.1 Zero and poles

Definition 3.1.1 (Zeros of a complex function). *Suppose f is a complex-valued function defined on $\Omega \subset \mathbb{C}$. Then, f has a zero at $z_0 \in \Omega$ and we say z_0 is a zero of f if $f(z_0) = 0$.*

Definition 3.1.2 (Deleted neighborhood). *A deleted (or punctured) neighborhood of radius $r > 0$ and center $z_0 \in \mathbb{C}$ is the set*

$$N_r^*(z_0) = \{z \in \mathbb{C} : 0 < |z - z_0| < r\} = N_r(z_0) / \{z_0\}.$$

Definition 3.1.3 (Pole of a complex function). *Suppose $r > 0, z_0 \in \mathbb{C}$ and f is a complex valued function defined on $N_r^*(z_0)$. Then, f has a pole at z_0 if $\tilde{f} \colon N_r(z_0) \to \mathbb{C}$ defined by*

$$\tilde{f}(z) = \begin{cases} 0 & \text{if } z = z_0 \\ \frac{1}{f(z)} & \text{if } z \neq z_0 \end{cases} \quad \forall z \in N_r(z_0)$$

is holomorphic on $N_r(z_0)$.

Definition 3.1.4 (Vanishing). *Suppose X is a set. Then, $f \colon X \to \mathbb{C}$ vanishes at $x_0 \in X$ if $f(x_0) = 0$.*

Suppose $A \subset X$. We say that f vanishes on A if $\forall a \in A$ $f(a) = 0$.

Theorem 3.1.1 (Factorization of Holomorphic Function with a Zero). *Suppose $\Omega \subset \mathbb{C}$ is open and connected, $f \colon \Omega \to \mathbb{C}$ is non-vanishing, holomorphic, and has a zero at $z_0 \in \Omega$. Then, there exist $r > 0$, a map $g \colon B_r(z_0) \to \mathbb{C}$, and a unique $n \in \mathbb{N}$ such that*

(i) $B_r(z_0) \subset \Omega$,

(ii) g is non-vanishing and holomorphic, and

(iii) $f(z) = (z - z_0)^n g(z)$ $\forall z \in \Omega$.

Proof. Suppose $\Omega \subset \mathbb{C}$ is open and connected, $f \colon \Omega \to \mathbb{C}$ is non-vanishing, holomorphic, and has a zero at $z_0 \in \Omega$. By Theorem 4.4 (Stein-Shakarchi 49), f is analytic on Ω.

Since f is non-vanishing and analytic on Ω, which is connected and open, the zeros of f are isolated by the Principle of isolated zeros of analytic functions. Hence, $\exists r > 0$ such that $B_r(z_0) \subset \Omega$ and $\forall z \in B_r^*(z_0)$ $f(z) \neq 0$.

By the analyticity of f, f has the expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \ \forall z \in B_r(z_0),$$

where the coefficients are given by

$$a_k = \frac{f^{(k)}(z_0)}{k!} \ \forall k \in \mathbb{W}. \tag{$*$}$$

Since f is not identical zero on $B_r(z_0)$, we have that there exists a smallest positive integer $n \in \mathbb{N}$ such that $a_n \neq 0$. We note that here a_0 must be zero, as $f(z_0) = 0 = a_0(1) + a_1(0) + a_2(0)^2 + \ldots$

implies that $a_0 = 0$. It follows that $\forall z \in B_r(z_0)$

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k = 0(z-z_0)^0 + 0(z-z_0)^1 + \dots + a_n(z-z_0)^n + \dots$$
$$= (z-z_0)^n \left(a_n + a_{n+1}(z-z_0)^{n+1} + \dots\right)$$
$$= (z-z_0)^n \sum_{k=0}^{\infty} a_{n+k}(z-z_0)^k.$$

Define $g\colon B_r(z_0) \to \mathbb{C}$ by $g(z) = \sum_{k=0}^{\infty} a_{n+k}(z-z_0)^k \ \forall z \in B_r(z_0)$. We observe that g is well-defined since the series $\sum_{k=0}^{\infty} a_{n+k}(z-z_0)^k$ converges since it has the same radius of convergence as the series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$, which is the power series expansion of f on $B_r(z_0)$.

To see this, we first denote the radius of convergence of $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ by R. We observe that $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converge on $B_r(z_0)$. Hence, $R \geq r > 0$, for if $R = 0$ the series diverges for every $z \in \mathbb{C}/\{z_0\}$, which is a contradiction. In addition, if $R < r$ we would then obtain that $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ diverges on $\{z \in \mathbb{C} : r > |z-z_0| > R\}$ by the power series test, which is a contradiction.

Hence, we may conclude that $R \neq 0$. If $R = \infty$, then we have that $\lim_{N\to\infty} \sup_{k\in\mathbb{N}_{>N}} |a_k|^{1/k} = 0$. If $R \in \mathbb{R}_{<0}$, then we have that $\lim_{N\to\infty} \sup_{k\in\mathbb{N}_{>N}} |a_k|^{1/k} = \frac{1}{R}$.

We observe that the sequence $\left(\sup_{k\in\mathbb{N}_{>N}} |a_k|^{1/k}\right)_{N\in\mathbb{N}}$ converges and, therefore is bounded. It follows that $\left(\sup_{k\in\mathbb{N}_{>N}} |a_{n+k}|^{1/k}\right)_{N\in\mathbb{N}}$ converges since it is decreasing and bounded, as it is a subsequence of $\left(\sup_{k\in\mathbb{N}_{>N}} |a_k|^{1/k}\right)_{N\in\mathbb{N}}$. We recall from elementary analysis that a convergent subsequence of a convergent sequence converges to the limit of the convergent sequence. Hence, we then have that

$$\lim_{N\to\infty} \sup_{k\in\mathbb{N}_{>N}} |a_{n+k}|^{\frac{1}{k}} = \lim_{N\to\infty} \sup_{k\in\mathbb{N}_{>N}} |a_k|^{\frac{1}{k}}$$

implying that the radius of the series $\sum_{k=0}^{\infty} a_{n+k}(z-z_0)^k$ equals R as claimed. Hence, $\sum_{k=0}^{\infty} a_{n+k}(z-z_0)^k$ converges on $B_r(z_0)$ as desired.

It suffices to show that g is holomorphic and nonvanishing, and n is unique. Suppose, to the contrary, that g is not non-vanishing. Then, $\exists \tilde{z} \in B_r(z_0)$ such that $g(\tilde{z}) = 0$. It follows that

$$\begin{aligned} f(\tilde{z}) = (\tilde{z}-z_0)^n g(\tilde{z}) = 0 & \\ \implies \tilde{z} = z_0 & \qquad \text{(Since } z_0 \text{ is the only zero of } f \text{ in } B_r(z_0)) \\ \implies g(\tilde{z}) = g(z_0) = a_n = 0, & \end{aligned}$$

which is a contradiction since $a_n \neq 0$. Thus, g is indeed nonvanishing on $B_r(z_0)$. In addition, the coefficients of $\sum_{k=0}^{\infty} a_{n+k}(z-z_0)^k$ are holomorphic by $(*)$, which implies that g is also holomorphic.

To prove n is unique, we suppose that there exists $m \in \mathbb{N}$ and a nonvanishing holomorphic function $h\colon B_r(z_0) \to \mathbb{C}$ such that

$$f(z) = (z-z_0)^m h(z) \ \forall z \in B_r(z_0).$$

It follows that $\forall z \in B_r(z_0)$, $(z-z_0)^m h(z) = (z-z_0)^n g(z)$. If $m > n$, we then yield that

$$\forall z \in B_r^*(z_0)\ (z-z_0)^{m-n} h(z) = g(z) \implies g(z_0) = \lim_{z\to z_0} g(z) = \lim_{z\to z_0} (z-z_0)^{m-n} h(z) = 0$$

by the continuity of g since g is holomorphic. Similarly, if $n > m$, we then have that

$$\forall z \in B_r^*(z_0)\ (z-z_0)^{n-m} g(z) = h(z) \implies h(z_0) = \lim_{z\to z_0} h(z) = \lim_{z\to z_0} (z-z_0)^{n-m} g(z) = 0$$

by the continuity of h since h is holomorphic. We obtained a contradiction in both cases since g and h are nonvanishing in $B_r(z_0)$. Therefore, we conclude that $n = m$ and $g = h$. Here, we complete the proof. $\square$

Theorem 3.1.2 (Factorization of Complex Function with a Pole)**.** *Suppose that $\Omega \subset \mathbb{C}$ is open and connected, and $z_0 \in \Omega$. Suppose further that f is defined on a deleted neighborhood $B_R^*(z_0) \subset \Omega$ for some $R > 0$ and has a pole at z_0. Then, there exist $r > 0$, a map $h\colon B_r(z_0) \to \mathbb{C}$, and a unique $n \in \mathbb{N}$ such that*

(i) $B_r(z_0) \subset B_R(z_0)$,

(ii) h is non-vanishing and holomorphic, and

(iii) $f(z) = \frac{h(z)}{(z-z_0)^n}$ $\forall z \in B_r^*(z_0)$.

Proof. Suppose that $\Omega \subset \mathbb{C}$ is open and connected, and $z_0 \in \Omega$. Suppose further that f is defined on some deleted neighborhood $B_R^*(z_0) \subset \Omega$ for some $R > 0$ and has a pole at z_0.

By definition of a pole, we have that $\frac{1}{f}\colon B_R(z_0) \to \mathbb{C}$, defined by $\left[\frac{1}{f}\right](z) = \frac{1}{f(z)}$ $\forall z \in B_R^*(z_0)$ and $\left[\frac{1}{f}\right](z_0) = 0$, is holomorphic and, thus, analytic. By the Principle of isolated zeros of an analytic function, we have that $\exists r' > 0$ such that $\left[\frac{1}{f}\right](z) \neq 0$ $\forall z \in B_{r'}^*(z_0)$ and $B_{r'}(z_0) \subset B_R(z_0)$.

Let $\tilde{f}\colon B_{r'}(z_0) \to \mathbb{C}$ be the restriction of $\frac{1}{f}$ on $B_{r'}(z_0)$. We observe that $\tilde{f}$ has a zero in $B_{r'}(z_0)$, namely z_0, and is not identically zero in $B_{r'}(z_0)$.

Applying Theorem 1.1 (Stein-Shakarchi 73) to $\tilde{f}$, we then obtain that there exist $r > 0$ such that $B_r(z_0) \subset B_{r'}(z_0) \subset B_R(z_0)$, a map $g\colon B_r(z_0) \to \mathbb{C}$, and a unique positive integer n such that g is nonvanishing and holomorphic, and $\forall z \in B_r(z_0)$ $\tilde{f}(z) = (z - z_0)^n g(z)$.

Define $h\colon B_r(z_0) \to \mathbb{C}$ by $h(z) = \frac{1}{g(z)}$ $\forall z \in B_r(z_0)$. We observe that h is well-defined and nonvanishing since g is nonvanishing. Moreover, h is holomorphic since g is holomorphic. Lastly, we have that $\forall z \in B_r^*(z_0)$

$$f(z) = \frac{1}{\left(\frac{1}{f(z)}\right)} = \frac{1}{\tilde{f}(z)} = \frac{1}{(z-z_0)^n g(z)} = \frac{h(z)}{(z-z_0)^n}$$

as desired. Here, we complete the proof. $\square$

Definition 3.1.5 (Order (multiplicity) of a pole)**.** *Suppose $z_0 \in \mathbb{C}$ and f is a complex-valued function defined on some deleted neighborhood $B_R^*(z_0)$ for some $R > 0$ and has a pole at z_0. Then, the order of the pole z_0 is the unique positive integer n such that*

$$f(z) = \frac{h(z)}{(z-z_0)^n} \ \forall z \in B_r^*(z_0)$$

for some $r > 0$ such that $B_r(z_0) \subset B_R(z_0)$, where $h\colon B_r(z_0) \to \mathbb{C}$ is nonvanishing and holomorphic.

Theorem 3.1.3 (Rational Decomposition of Complex Function with a Pole)**.** *Suppose $z_0 \in \mathbb{C}$ and f is a complex-valued function defined on some deleted neighborhood $B_R^*(z_0)$ for some $R > 0$ and has a pole at z_0. Then, there exist $\exists r > 0$ and a map $G\colon B_r(z_0) \to \mathbb{C}$ such that*

(i) $B_r(z_0) \subset B_R(z_0)$,

(ii) G is holomorphic, and

(iii) $\forall z \in B_r^*(z_0)$

$$f(z) = \frac{A_0}{(z-z_0)^n} + \frac{A_1}{(z-z_0)^{n-1}} + \cdots + \frac{A_{n-1}}{(z-z_0)^1} + G(z).$$

Proof. Suppose $z_0 \in \mathbb{C}$ and f is a complex-valued function defined on some deleted neighborhood $B_R^*(z_0)$ for some $R > 0$ and has a pole at z_0.

By Theorem 1.2 (Stein-Shakarchi 74), we obtain that there exist $r > 0$, $h\colon B_r(z_0) \to \mathbb{C}$, and a unique positive integer n such that $B_r(z_0) \subset B_R(z_0)$, a nonvanishing and holomorphic map $h\colon B_r(z_0) \to \mathbb{C}$, and $\forall z \in B_r^*(z_0)$

$$f(z) = \frac{h(z)}{(z-z_0)^n}.$$

Since h is holomorphic, it is analytic and has the expansion in $B_r(z_0)$

$$h(z) = A_0 + A_1(z-z_0) + A_2(z-z_0)^2 + \ldots \ \forall z \in B_r(z_0)$$

by Theorem 4.4 (Stein-Shakarchi 49). Define $G\colon B_r(z_0) \to \mathbb{C}$ by

$$G(z) = A_n + A_{n+1}(z-z_0) + A_{n+2}(z-z_0)^2 + \ldots \ \forall z \in B_r(z_0).$$

Here, G is well-defined since the series $A_n + A_{n+1}(z-z_0) + A_{n+2}(z-z_0)^2 + \ldots$ converges in $B_r(z_0)$ as it has the same radius of convergence as $A_0 + A_1(z-z_0) + A_2(z-z_0)^2 + \ldots$, which converges in $B_r(z_0)$. In addition, by Theorem 4.4 (Stein-Shakarchi 49), the coefficients of the series defining G are holomorphic. Hence, G is also holomorphic.

Lastly, we have that $\forall z \in B_r^*(z_0)$

$$\begin{aligned} f(z) = \frac{h(z)}{(z-z_0)^n} &= \frac{A_0 + A_1(z-z_0) + A_2(z-z_0)^2 + \ldots}{(z-z_0)^n} \\ = \frac{A_0}{(z-z_0)^n} + \frac{A_1}{(z-z_0)^{n-1}} + \cdots + \frac{A_{n-1}}{(z-z_0)^1} &+ A_n + A_{n+1}(z-z_0)^1 + \ldots \\ = \frac{A_0}{(z-z_0)^n} + \frac{A_1}{(z-z_0)^{n-1}} &+ \cdots + \frac{A_{n-1}}{(z-z_0)^1} + G(z). \end{aligned}$$

□

Definition 3.1.6 (Principal part and residue of a complex function with a pole)**.** *Suppose $z_0 \in \mathbb{C}$ and f is a complex-valued function defined on some deleted neighborhood $B_R^*(z_0)$ for some $R > 0$ and has a pole at z_0.*

Then, by Theorem 1.3 (Stein-Shakarchi 75), there exist $r > 0$, a holomorphic map $G\colon B_r(z_0) \to \mathbb{C}$, and a unique positive integer n such that $B_r(z_0) \subset B_R(z_0)$ and $\forall z \in B_r^*(z_0)$

$$f(z) = \frac{A_0}{(z-z_0)^n} + \frac{A_1}{(z-z_0)^{n-1}} + \cdots + \frac{A_{n-1}}{(z-z_0)^1} + G(z).$$

(i) The sum $\frac{A_0}{(z-z_0)^n} + \frac{A_1}{(z-z_0)^{n-1}} + \cdots + \frac{A_{n-1}}{(z-z_0)^1}$ is the principal part of f at the pole z_0, and

(ii) the coefficeint A_{n-1} is the residue of f, which we denote by $res_{z_0}(f)$, at the pole z_0.

Proposition 3.1.1.

Theorem 3.1.4 (Characterization of Residue)**.** *Suppose f is a complex-valued function that has a pole of order n at $z_0 \in \mathbb{C}$. Then,*

$$res_{z_0}(f) = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d}{dz}\right]^{n-1} \left(f(z) \cdot (z-z_0)^n\right).$$

3.2 The Residue Formula

Theorem 3.2.1 (Residue Formula for a Ball Containing One Pole)**.** *Suppose that $B \subset \mathbb{C}$ is a ball, $z_0 \in B$ is the only pole of f in B, and f is holomorphic on an open set $\Omega \subset \mathbb{C}$ containing $\overline{B}/\{z_0\}$. Then,*

$$\int_{\partial B} f(z)dz = 2\pi i res_{z_0}(f).$$

Corollary 3.2.1 (Residue Formula for a Ball Containing Multiple Poles)**.** *Suppose that $B \subset \mathbb{C}$ is a ball, $z_1, \ldots, z_N \in B$ are all the poles of f in B, and f is holomorphic on an open set $\Omega \subset \mathbb{C}$ containing $\overline{B}/\{z_1, \ldots, z_N\}$. Then,*

$$\int_{\partial B} f(z)dz = 2\pi i \sum_{k=1}^{N} res_{z_k}(f).$$

Corollary 3.2.2 (Generalized Residue Formula)**.** *Suppose that $\gamma \subset \mathbb{C}$ is a toy contour with positive orientation, $z_1, \ldots, z_N \in B$ are all the poles of f in $int(\gamma)$, and f is holomorphic on an open set $\Omega \subset \mathbb{C}$ containing $\overline{int(\gamma)}/\{z_1, \ldots, z_N\}$. Then,*

$$\int_{\gamma} f(z)dz = 2\pi i \sum_{k=1}^{N} res_{z_k}(f).$$

3.3 Singularities and meromorphic functions

Theorem 3.3.1 (Riemann's Theorem on Removable Singularities)**.** *Suppose f is a complex-valued function that is holomorphic on an open set $\Omega \subset \mathbb{C}$, except possibly at a point $p_0 \in \Omega$. If f is bounded on $\Omega / \{z_0\}$, then z_0 is a removable singularity.*

Corollary 3.3.1 (Equivalent Statements to Riemann's Theorem on Removable Singularities)**.**

Corollary 3.3.2 (Characterization of Pole)**.** *Suppose f has an isolated singularity at $z_0 \in \mathbb{C}$. Then, z_0 is a pole of $f \iff \lim_{z \to z_0} |f(z)| = \infty$.*

Theorem 3.3.2 (Casorati-Weirstrass)**.** *Suppose $z_0 \in \mathbb{C}$ and f is holomorphic on $B_r^* z_0$ for some $r > 0$ and has an essential singularity at z_0. Then, $f(B_r^* z_0)$ is dense in $\mathbb{C}$.*

Definition 3.3.1 (Meromorphic function)**.**

Definition 3.3.2 (One point compactfication)**.**

Definition 3.3.3 (Isolated singularities at infinity)**.**

3.4 The argument principle and applications

3.5 Homotopies and simply connected domains

Definition 3.5.1 (???Homotopic curves). *Suppose $\gamma, \Gamma \subset \mathbb{C}$ are piecewise smooth curves with the endpoints (that is $\gamma\colon [a,b] \to \mathbb{C}$ and $\Gamma\colon [c,d] \to \mathbb{C}$ implies $\gamma(a) = \Gamma(c)$ and $\gamma(b) = \Gamma(d)$).*

Then, γ and Γ are homotopic curves in an open set $U \subset \mathbb{C}$ if

(i)

(ii)

Definition 3.5.2 (Simply-Connected Region). *A set $\Omega \subset \mathbb{C}$ is a simply-connected region if*

(i) Ω is open, and
(ii) every pair of curves in Ω of the same endpoints are homotopic.

Example 3.5.1 (Simply-Connected Region). *Suppose $R > 0$ and $z_0 \in \mathbb{C}$. Then, $B_R(x_0) \subset B_r(z_0)$ for some $R > 0$ and $x_0 \in B_r(z_0)$ implies that $B_r(z_0)/B_R(x_0)$ is not simply-connected.*

$\mathbb{C}$ and $\mathbb{C}/\mathbb{R}_{\leq 0}$ are simply connected.

Theorem 3.5.1 (Equivalence of Integral Along Homotopic Curves). *Suppose $\Omega \subset \mathbb{C}$ is a open, $f\colon \Omega \to \mathbb{C}$ is holomorphic on Ω, and $\gamma, \Gamma \subset \Omega$ are homotopic curves in Ω. Then,*

$$\int_\gamma f(z)dz = \int_\Gamma f(z)dz.$$

Theorem 3.5.2 (Holomorphicity on Simply-Connected Region Implies Existence of Primitive on the Region). *Suppose $U \subset \mathbb{C}$ is open and simply-connected, and f is a complex function holomorphic on U. Then, f has a primitive on U.*

Corollary 3.5.1 (Holomorphicity in Simply Connected Region Implies Vanishing INtegral Along Closed Curves).

3.6 The complex logarithm

Remark 3.6.1 (Locally and globally defined complex logarithm). *Defining complex logarithms globally on $\mathbb{C}/\{0\}$ leads to non-well-definedness. Thus, we restrict the domain on which we define complex logarithms to simply connected regions $\Omega \subset \mathbb{C}$ such that $1 \in \Omega$ and $0 \notin \Omega$.*

Theorem 3.6.1 (Existence of Complex Logarithm on a Branch).

Definition 3.6.1 (Principle branch).

Proposition 3.6.1 (Property of Complex Logarithm on the Principle Branch).

Remark 3.6.2. *The complex logarithm on the principle branch is an extension of the usual real logarithm.*

Theorem 3.6.2.

3.7 Fourier series and harmonic functions

4 The Fourier Transform

5 Entire Functions

6 The Gamma and Zeta Functions

6.1 The gamma function

6.1.1 Analytic continuation

Definition 6.1.1 (Analytic Continuation of an Analytic Function)**.** *Suppose* $f\colon \Omega \to \mathbb{C}$ *and* $g\colon \Omega' \to \mathbb{C}$ *are analytic, where* $\Omega, \Omega' \subset \mathbb{C}$ *are regions such that* $\Omega \subset \Omega'$*. Then,* g *is an analytic continuation of* f *into* Ω' *if* $g(z) = f(z)\ \forall z \in \Omega$*.*

Theorem 6.1.1 (Uniqueness of Analytic Continuation)**.** *If an analytic function has an analytic continuation, then the analytic continuation is unique.*

Proof. Let $\Omega, \Omega' \subset \mathbb{C}$ be regions such that $\Omega \subset \Omega'$. Suppose that $f\colon \Omega \to \mathbb{C}$ is analytic with analytic continuations $g, g'\colon \Omega' \to \mathbb{C}$ into Ω'. By definition, we have that

$$g(z) = f(z) = g'(z)\ \forall z \in \Omega.$$

It follows, from Corollary 4.9 (Stein-Shakarchi 52), that $g(z) = g'(z)\ \forall z \in \Omega'$. That is, $g = g'$ by definition. □

6.2 The zeta Function

Definition 6.2.1 (Analytic Continuation of an Analytic Function)**.** *Suppose* $f\colon \Omega \to \mathbb{C}$ *and* $g\colon \Omega' \to \mathbb{C}$ *are analytic, where* $\Omega, \Omega' \subset \mathbb{C}$ *are regions such that* $\Omega \subset \Omega'$*. Then,* g *is an analytic continuation of* f *into* Ω' *if* $g(z) = f(z)\ \forall z \in \Omega$*.*

Theorem 6.2.1 (Uniqueness of Analytic Continuation)**.** *If an analytic function has an analytic continuation, then the analytic continuation is unique.*

Proof. Let $\Omega, \Omega' \subset \mathbb{C}$ be regions such that $\Omega \subset \Omega'$. Suppose that $f\colon \Omega \to \mathbb{C}$ is analytic with analytic continuations $g, g'\colon \Omega' \to \mathbb{C}$ into Ω'. By definition, we have that

$$g(z) = f(z) = g'(z)\ \forall z \in \Omega.$$

It follows, from Corollary 4.9 (Stein-Shakarchi 52), that $g(z) = g'(z)\ \forall z \in \Omega'$. That is, $g = g'$ by definition. □

Definition 6.2.2 (Zeta function)**.**

7 The Zeta Function and Prime Number Theorem

8 Conformal Mappings

9 An Introduction to Elliptic Functions

10 Applications of Theta Functions

10.1

10.2

10.3