DATA130026.01 Optimization
Solution of Assignment 8

1. Derive the dual problems of the SDP and SOCP problems in Question 4 in Assignment 7. Write CVX codes to solve the associated dual problems and show the optimal solutions.
(Hint: refer the User's guide for CVX, e.g., from http://cvxr.com/cvx/doc/sdp.html)
Solution.

(a) (SDP)

$$\begin{array}{ll} \min & \mathbf{tr}(CX) \\ \text{s.t.} & \mathbf{tr}(A_i X) = b_i, \quad i = 1, \dots, m \\ & X \in \mathbf{S}_+^n, \end{array}$$

where

$$\mathbf{S}_+^n = \{X \in \mathbb{R}^{n\times n} \mid X \succeq 0, X = X^T\}.$$

The Lagrangian is

$$\begin{aligned} L(X, \lambda, y) &= \mathbf{tr}(CX) - \lambda \bullet X + \sum_{i=1}^n y_i(b_i - \mathbf{tr}(A_i X)) \\ &= \left\langle C - \lambda - \sum_{i=1}^n y_i A_i, X \right\rangle + b^T y \end{aligned}$$

Hence, we have

$$\inf_X L(X, \lambda, y) = \begin{cases} b^T y & C - \lambda - \sum_{i=1}^n y_i A_i = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

So, the dual problem is

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & \sum_{i=1}^n y_i A_i \preceq C \end{array}$$

CVX Code

```
% generate the data
rand('seed',19210980102);
m = 3;
n = 4;
C = rand(n);
C = C'*C;
b = zeros(m,1);
for i = 1:m
    A{i}= rand(n);
    A{i} = A{i}' * A{i};
```

```
    b(i) = rand();
end

% cvx
cvx_begin sdp
    variable y(m);
    variable S(n,n) symmetric;
    maximize(b' * y);
    subject to
        X = 0;
        for i = 1:m
            X = X + y(i) * A{i};
        end
        X + S == C;
        S >= 0;
cvx_end

disp(y)
```

The optimal solution is

$$y^* = \begin{pmatrix} 0.2033 \\ 0.0260 \\ 0.0097 \end{pmatrix}.$$

The optimal value is 0.1717.

(b) (SOCP)

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & \begin{pmatrix} A_i x + b_i \\ c_i^T x + d_i \end{pmatrix} \in K_i, \quad i = 1, \ldots, m, \end{array}$$

where

$$K_i = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^{n_i+1} \mid \|x\|_2 \le t \right\}.$$

We express the SOCP as a conic form problem

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & -\begin{pmatrix} A_i x + b_i \\ c_i^T x + d_i \end{pmatrix} \preceq_{K_i} 0, \quad i = 1, \ldots, m, \end{array}$$

where

$$K_i = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^{n_i+1} | \ \|x\|_2 \le t \right\}.$$

The Lagrangian is

$$\begin{aligned} L(x, u, v) &= f^T x - \sum_{i=1}^m \begin{pmatrix} u_i \\ v_i \end{pmatrix}^T \begin{pmatrix} A_i x + b_i \\ c_i^T x + d_i \end{pmatrix} \\ &= \left(f - \sum_{i=1}^m A_i^T u_i - \sum_{i=1}^m v_i c_i \right)^T x - \sum_{i=1}^m (u_i^T b_i + v_i d_i). \end{aligned}$$

The minimum over x is bounded below if and only if

$$f - \sum_{i=1}^{m} A_i^T u_i - \sum_{i=1}^{m} v_i c_i = 0.$$

So we have

$$g(u,v) = \begin{cases} -\sum_{i=1}^{m} \left(b_i^T u_i + d_i v_i\right) & \sum_{i=1}^{m} \left(A_i^T u_i + v_i c_i\right) = f, \\ \\ -\infty & \text{otherwise.} \end{cases}$$

The conic dual is

$$\begin{aligned} \max \quad & -\sum_{i=1}^{n} \left(b_i^T u_i + d_i v_i\right) \\ \text{s.t.} \quad & \sum_{i=1}^{m} \left(A_i^T u_i + v_i c_i\right) = f \\ & (u_i, v_i) \succeq_{K_{i^*}} 0, \quad i = 1, \ldots, m. \end{aligned}$$

Since the second-order cone is self-dual, we have $K_i^* = K_i$, which means $\|u_i\|_2 \leq v_i$ $(i = 1, \ldots, m)$. Replace u_i with $-u_i$, we obtain the dual problem

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} \left(b_i^T u_i - d_i v_i\right) \\ \text{s.t.} \quad & \sum_{i=1}^{m} \left(A_i^T u_i - c_i v_i\right) + f = 0 \\ & \|u_i\|_2 \leq v_i, \quad i = 1, \ldots, m. \end{aligned}$$

CVX Code

```
rand('seed',19210980102);
m = 3;
n = 4;
ni = unidrnd (5,[1,m]) ;
f = rand(n,1);
for i =1:m
    A{i}= rand(ni(i),n) ;
    b{i}= rand(ni(i),1) ;
    c{i}= rand(n,1) ;
    d{i}= rand() ;
end

% solve
cvx_begin
    variables u1(ni(1)) u2(ni(2)) u3(ni(3)) v1 v2 v3;
    maximize (b{1}' * u1 + b{2}' * u2 + b{3}' * u3 - d{1} * v1 -
d{2} * v2 - d{3} * v3);
    subject to
```

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        A{1}' * u1 + A{2}' * u2 + A{3}' * u3 - c{1} * v1 - c{2} *
    v2 - c{3} * v3 + f == 0;
        {u1, v1} == lorentz(ni(1));
        {u2, v2} == lorentz(ni(2));
        {u3, v3} == lorentz(ni(3));
cvx_end

```

The optimal solution

$$u_1 = \begin{pmatrix} -0.6528 \\ 0.5123 \\ -0.1380 \\ -0.0190 \\ 0.1129 \end{pmatrix}, u_2 = \begin{pmatrix} 0.0751 \\ -0.4128 \\ 0.2383 \\ 0.4266 \\ -0.0532 \end{pmatrix}, u_3 = \begin{pmatrix} 0.3429 \\ -0.2022 \end{pmatrix},$$

$$v_1 = 0.8489, v_2 = 0.6463, v_3 = 0.3981.$$

The optimal value is 0.1717.

2. Prove that

$$\max_z \left\{ p^T z : \|z\|_2^2 \le R^2, \|z\|_\infty \le 1 \right\} = \min_{u,v} \left\{ \|u\|_1 + R\|v\|_2 : u + v = p \right\}$$

Hint: using strong duality and conjugate functions.

Solution.

If $R = 0$, then $z = 0$, the equivalence of two problems is easy to find.

If $R \neq 0$, notice that only when $R > 0$, the RHS is convex problem. Since the objective function is affine and the constraints are norm ball, the LHS is a convex problem. Besides, if $z = 0$, then $\|z\|_2^2 < R^2$ and $\|z\|_\infty < 1$, which means Slater's condition holds. Therefore, strong duality holds for the LHS problem.

Recall that the conjugate of norm is the indicator of unit ball for dual norm:

$$f(x) = \|x\| \quad f^*(y) = \begin{cases} 0 & \|y\|_* \le 1 \\ +\infty & \|y\|_* > 1 \end{cases} \tag{1}$$

Consider the RHS, its Lagrangian is

$$L(u, v, z) = \|u\|_1 + R\|v\|_2 + z^T(p - u - v)$$

Hence

$$\inf_{u,\ v} L(u, v, z) = -\max_u (z^T u - \|u\|_1) - R \max_v (\frac{z^T}{R} - \|v\|_2) + p^T z$$

Recall that the dual norm of 1-norm and 2-norm are ∞- norm and 2-norm, respectively. By conjugate function (1) and strong duality, the equivalence of two problems holds.

3. Demonstrate by an example that the relation $0 \preceq A \preceq B$ does not necessary imply that $A^2 \preceq B^2$.

   Solution. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \succeq 0$ (its eigenvalues are 2.5147 and 19.4853), $B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ (its eigenvalues are 1 and 3) and $B - A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succeq 0$ (its eigenvalues are 0 and 2), that is $0 \preceq A \preceq B$.

   However, $B^2 - A^2 = \begin{pmatrix} 4 & -6 \\ -6 & 8 \end{pmatrix} \not\succeq 0$, since its eigenvalues are -0.3246 and 12.3246.