DATA130026.01 Optimization
Solution of Assignment 4

For each of the following optimization problems (i) show that it is **convex**, (ii) write a CVX **code** that solves it, and (iii) write down the **optimal solution and optimal value** (by running CVX). You need to write all the above step and publish your codes (using "publish" in MATLAB) in a PDF file. You should upload an electronic version on elearning.

1.

$$\begin{aligned} \min \quad & x_1^2 + 2x_1x_2 + 2x_2^2 + x_3^2 + 3x_1 - 4x_2 \\ \text{s.t.} \quad & \sqrt{2x_1^2 + x_1x_2 + 4x_2^2 + 4} + \frac{(x_1 - x_2 + x_3 + 1)^2}{x_1 + x_2} \le 6 \\ & x \ge 1 \end{aligned}$$

Solution.

(i)
- $f_0(x) = x_1^2 + 2x_1x_2 + 2x_2^2 + x_3^2 + 3x_1 - 4x_2 = x^TAx + b^Tx$, where $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \succeq 0, b = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, is convex.
- $f_{11}(x) = \sqrt{2x_1^2 + x_1x_2 + 4x_2^2 + 4} = \sqrt{(\frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}x_2)^2 + (\sqrt{\frac{3}{2}}x_1)^2 + (\sqrt{\frac{7}{2}}x_2)^2 + 2^2} = \|y\|_2$, where $y = \left(\frac{\sqrt{2}}{2}(x_1 + x_2), \sqrt{\frac{3}{2}}x_1, \sqrt{\frac{7}{2}}x_2, 2\right)^T$, is convex;
  $f_{12}(x) = (x_1 - x_2 + x_3 + 1)^2/(x_1 + x_2)$ is the composition of the quadratic-over-linear function and the affine transformation, which is convex.
  So $f_1(x) = f_{11}(x) + f_{12}(x) - 6$ is convex.
- $f_2(x) = 1 - x$ is convex.
- Thus the optimization problem is convex.

(ii)

```
A=[1,1,0;1,2,0;0,0,1];
cvx_begin
    variable x(3)
    minimize(quad_form(x,A)+3*x(1)-4*x(2))
    subject to
        norm([(x(1)+x(2))/sqrt(2);sqrt(3/2)*x(1);sqrt(7/2)* ...
            x(2);2])+quad_over_lin((x(1)-x(2)+x(3)+1),(x(1)+ ...
            x(2)))<=6
        x>=1
cvx_end
disp(x)
```

(iii)

$$x^* = (1, 1, 1)^T, \quad f^* = 5$$

2.

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 + x_4 \\ \text{s.t.} \quad & (x_1 - x_2)^2 + (x_3 + 2x_4)^4 \le 5 \\ & x_1 + 2x_2 + 3x_3 + 4x_4 \le 6 \\ & x \ge 0 \end{aligned}$$

Solution.

(i)
- $f_0(x) = x_1 + x_2 + x_3 + x_4$ is convex.
- $f_{11}(x) = (x_1 - x_2)^2$ is the composition of the quadratic function and an affine transformation, which is convex;
  $f_{12}(x) = (x_3 + 2x_4)^4$ is also convex.
  So $f_1(x) = f_{11}(x) + f_{12}(x) - 5$ is convex.
- $f_2(x) = x_1 + 2x_2 + 3x_3 + 4x_4 - 6$ is convex.
- $f_3(x) = -x$ is convex.
- Thus the optimization problem is convex.

(ii)

```
cvx_begin
    variable x(4)
    minimize(ones(1,4)*x)
    subject to
        (x(1)-x(2))^2+(x(3)+2*x(4))^4<=5
        [1,2,3,4]*x<=6
        x>=0
cvx_end
disp(x)
```

(iii)

$$x^* = (0,0,0,0)^T, \quad f^* = 0$$

3.

$$\begin{aligned} \min \quad & |2x_1 + 3x_2 + x_3| + \|x\|^2 + \sqrt{2x_1^2 + 4x_1x_2 + 7x_2^2 + 10x_2 + 6} \\ \text{s.t.} \quad & \frac{x_1^2 + 1}{x_2} + 2x_1^2 + 5x_2^2 + 10x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3 \le 7 \\ & \max\{x_1 + x_2, x_3, x_1 - x_3\} \le 19 \\ & x_1 \ge 0 \\ & x_2 \ge 1. \end{aligned}$$

Solution.

(i)
- $f_{01}(x) = |2x_1 + 3x_2 + x_3|$ is the composition of the absolute value function and an affine transformation, which is convex;
  $f_{02}(x) = \|x\|^2$ is convex;
  $f_{03}(x) = \sqrt{2x_1^2 + 4x_1x_2 + 7x_2^2 + 10x_2 + 6} = \sqrt{(\sqrt{2}x_1 + \sqrt{2}x_2)^2 + (\sqrt{5}x_2 + \sqrt{5})^2 + 1^2} = \|y\|_2$, where $y = \left(\sqrt{2}(x_1 + x_2), \sqrt{5}(x_2 + 1), 1\right)^T$, is convex.
  So $f_0(x) = f_{01}(x) + f_{02}(x) + f_{03}(x)$ is convex.

- $f_{11}(x) = \frac{x_1^2+1}{x_2} = \frac{\|y\|_2^2}{x_2}$, where $y = (x_1, 1)^T$, is convex;
  $f_{12}(x) = 2x_1^2 + 5x_2^2 + 10x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3 = x^TAx$, where $A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 10 \end{pmatrix} \succeq 0$, is convex.
  So $f_1(x) = f_{11}(x) + f_{12}(x) - 7$ is convex.
- $f_2(x) = \max\{x_1+x_2, x_3, x_1-x_3\} - 19$ is the composition of the max function and an affine transformation plus a constant, which is convex.
- $f_3(x) = -x_1$, $f_4(x) = 1 - x_2$ are both convex.
- Thus the optimization problem is convex.

(ii)

```
A=[2,2,1;2,5,1;1,1,10];
cvx_begin
    variable x(3)
    minimize(abs([2,3,1]*x)+sum_square(x)+norm([sqrt(2)*(x(1)+ ...
            x(2));sqrt(5)*(x(2)+1);1]))
    subject to
        quad_over_lin([x(1);1],x(2))+quad_form(x,A)<=7
        max([x(1)+x(2);x(3);x(1)-(3)])<=19
        x(1)>=0
        x(2)>=1
cvx_end
disp(x)
```

(iii)

$$x^* = (0, 1, -0.4317)^T, \quad f^* = 8.5505$$

4.

$$\begin{aligned} \min \quad & \sqrt{2x_1^2 + 3x_2^2 + x_3^2 + 4x_1x_2 + 7} + (x_1^2 + x_2^2 + x_3^2 + 1)^2 \\ \text{s.t.} \quad & \frac{(x_1 + x_2)^2}{x_3 + 1} + x_1^8 \le 7 \\ & x_1^2 + x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \le 10 \\ & |x_1 + x_2 - x_3|^2 \le 20 \\ & x \ge 0. \end{aligned}$$

Solution.

(i)

- $f_{01}(x) = \sqrt{2x_1^2 + 3x_2^2 + x_3^2 + 4x_1x_2 + 7} = \sqrt{(\sqrt{2}x_1 + \sqrt{2}x_2)^2 + x_2^2 + x_3^2 + 7} = \|y\|^2$, where $y = (\sqrt{2}x_1 + \sqrt{2}x_2, x_2, x_3, \sqrt{7})^T$, is convex;
  $g(x) = x_1^2+x_2^2+x_3^2+1$ is convex and nonnegative, so $f_{02}(x) = (x_1^2+x_2^2+x_3^3+1)^2$ is the composition of the nondecreasing quadratic function and a convex function, which is convex.
  Thus $f_0(x) = f_{01}(x) + f_{02}(x)$ is convex.
- $f_{11}(x) = \frac{(x_1+x_2)^2}{x_3+1}$ is the composition of the quadratic-over-linear function and an affine transformation, which is convex;

$f_{12} = x_1^8$ is convex.
So $f_1(x) = f_{11}(x) + f_{12}(x) - 7$ is convex.

- $f_2(x) = x_1^2 + x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 - 10 = x^T Ax - 10$, where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 4 \end{pmatrix} \succeq 0$, is convex.
- $f_3(x) = |x_1 + x_2 - x_3|^2 - 20 = (x_1 + x_2 - x_3)^2 - 20$ is the composition of the quadratic function and an affine transformation plus a constant, which is convex.
- $f_4(x) = -x$ is convex.
- Thus the optimization problems is convex.

(ii)

```
A=[1,1,1;1,1,1;1,1,4];
cvx_begin
    variable x(3)
    minimize(norm([sqrt(2)*(x(1)+x(2)),x(2),x(3),sqrt(7)])+ ...
            square_pos(sum_square([x;1])))
    subject to
        quad_over_lin(x(1)+x(2),x(3)+1)+x(1)^8<=7
        quad_form(x,A)<=10
        (x(1)+x(2)-x(3))^2<=20
        x>=0
cvx_end
disp(x)
```

(iii)

$$x^* = (0,0,0)^T \quad f^* = 3.6458$$

5.

$$\begin{aligned} \min \quad & \frac{x_1^4}{x_2^2} + \frac{x_2^4}{x_1^2} + 2x_1x_2 + |x_1+5| + |x_2+5| + |x_3+5| \\ \text{s.t.} \quad & ((x_1^2 + x_2^2 + x_3^2 + 1)^2 + 1)^2 + x_1^4 + x_2^4 + x_3^4 \leq 200 \\ & \max\{x_1^2 + 4x_1x_2 + 9x_2^2, x_1, x_2\} \leq 40 \\ & x_1 \geq 1 \\ & x_2 \geq 1. \end{aligned}$$

Solution.

(i)

- $f_{01}(x) = \frac{x_1^4}{x_2^2} + \frac{x_2^4}{x_1^2} + 2x_1x_2 = \left(\frac{x_1^2}{x_2} + \frac{x_2^2}{x_1}\right)^2$. Since $g_1(x) = \frac{x_1^2}{x_2} + \frac{x_2^2}{x_1}$ is the sum of two quadratic-over-linear function and $x_1, x_2 \geq 1$, it's convex and positive. Thus $f_{01}(x)$ is the composition of the nondecreasing quadratic function and a convex function, which is convex;
$f_{02}(x) = |x_1+5| + |x_2+5| + |x_3+5|$ is the sum of three composition of the absolute value function and affine transformation, which is convex.
Thus $f_0(x) = f_{01}(x) + f_{02}(x)$ is convex.

- $g_2(x) = x_1^2+x_2^2+x_3^2+1$ is convex and nonnegative, so $g_3(x) = (x_1^2+x_2^2+x_3^3+1)^2$ is the composition of the nondecreasing quadratic function and a convex function, which is convex and nonnegative. Then $f_{11}(x) = ((x_1^2+x_2^2+x_3^3+1)^2+1)^2$ is also convex.
  $f_{12}(x) = x_1^4+x_2^4+x_3^4$ is convex.
  Thus $f_1(x) = f_{11}(x) + f_{12}(x)$ is convex.
- $f_2(x) = \max\{x_1^2+4x_1x_2+9x_2^2, x_1, x_2\} - 40 = \max\{(x_1+2x_2)^2+5x_2^2, x_1, x_2\}$-40. Since $g_4(x) = (x_1+2x_2)^2+5x_2^2$ is convex and max function is convex and nondecreasing, $f_2(x)$ is also convex.
- $f_3(x) = -x_1+1$, $f_4(x) = -x_2+1$ are both convex.
- Thus the optimization problems is convex.

(ii)

```
cvx_begin
variable x(3)
    minimize(square_pos(quad_over_lin(x(1),x(2))+ ...
             quad_over_lin(x(2),x(1)))+norm(x+5,1))
    subject to
        square_pos(square_pos(sum_square(x)+1)+1)+x(1)^4+ ...
            x(2)^4+x(3)^4<=200
        max([(x(1)+2*x(2))^2+5*x(2)^2;x(1);x(2)])<=40
        x(1)>=1
        x(2)>=1
cvx_end
disp(x)
```

(iii)

$$x^* = (1, 1, -0.7833)^T, \quad f^* = 20.2167$$

6. Suppose that we are given 40 points in the plane. Each of these points belongs to one of two classes. Specifically, there are 19 points of class 1 and 21 points of class 2. The points are generated and plotted by the MATLAB commands

```
rand('seed',314);
x=rand(40,1);
y=rand(40,1);
class=[2*x<y+0.5]+1;
A1=[x(find(class==1)),y(find(class==1))];
A2=[x(find(class==2)),y(find(class==2))];
plot(A1(:,1),A1(:,2),'*','MarkerSize',6)
hold on
plot(A2(:,1),A2(:,2),'d','MarkerSize',6)
hold off
```

Note that the rows of $A_1 \in \mathbb{R}^{19\times 2}$ are the 19 points of class 1 and the rows of $A_2 \in \mathbb{R}^{21\times 2}$ are the 21 points of class 2. Write a CVX-based code for finding the maximum-margin line separating the two classes of points.

Solution. We have the following convex reformulation of the problem:

$$
\begin{array}{ll}
\min & \|w\|_2 \\
\text{s.t.} & w^T x_i + b \le -1,\ \ x_i \in A_1, \\
& w^T x_i + b \ge 1, \quad x_i \in A_2.
\end{array}
$$

And the following CVX-based code will solve the problem:

```
rand('seed',314);
x=rand(40,1);
y=rand(40,1);
class=[2*x<y+0.5]+1;
A1=[x(find(class==1)),y(find(class==1))];
A2=[x(find(class==2)),y(find(class==2))];

cvx_begin quiet
    variables w(2) b
    minimize(norm(w))
    subject to
        A1*w+b<=-1
        A2*w+b>=1
cvx_end

plot(A1(:,1),A1(:,2),'*','MarkerSize',6)
hold on
plot(A2(:,1),A2(:,2),'d','MarkerSize',6)
hold on
p1 = ezplot(@(x,y) w(1)*x+w(2)*y+b, [0,1]);
p2 = ezplot(@(x,y) w(1)*x+w(2)*y+b+1, [0,1]);
p3 = ezplot(@(x,y) w(1)*x+w(2)*y+b-1, [0,1]);
set(p2,'LineStyle','-.');
set(p3,'LineStyle','-.');
hold off
```

The result can be seen in Figure 1. Besides, the parameters of the maximum-margin line are $(w, b) = \left(\begin{pmatrix} -43.9596 \\ 19.4413 \end{pmatrix}, 12.4170 \right)$.

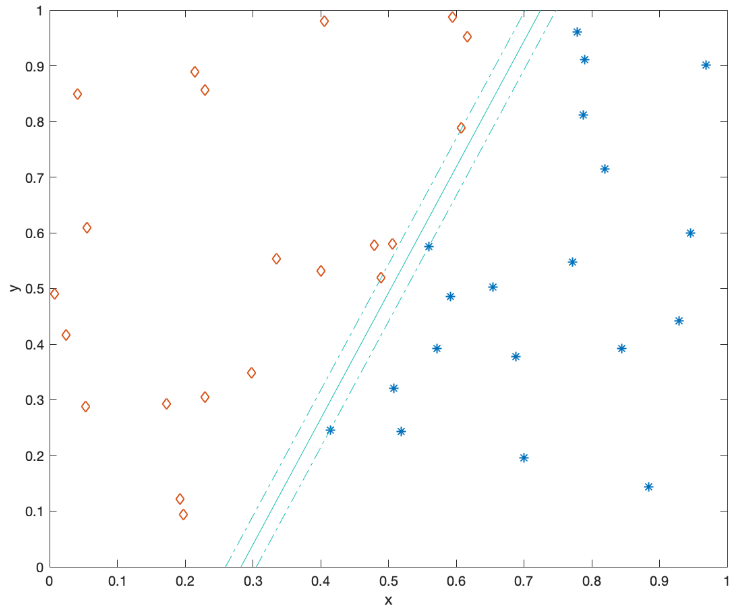

Figure 1