DATA130026.01 Optimization
Solution of Assignment 2

1. For each of the following sets determine whether they are convex or not (explaining your choice).

   (a) $C_1 = \{x \in \mathbb{R}^n : \|x\|^2 = 1\}$.

   (b) $C_2 = \{x \in \mathbb{R}^n : \max_{i=1,2,\ldots,n} x_i \leq 1\}$.

   (c) $C_3 = \{x \in \mathbb{R}^n : \min_{i=1,2,\ldots,n} x_i \leq 1\}$.

   (d) $C_4 = \{x \in \mathbb{R}^n_{++} : \Pi_{i=1}^n x_i \leq 1\}$, where $\Pi_{i=1}^n x_i = x_1 x_2 \cdots x_n$.

   Solution.

   (a) No. We can choose $x = (1, 0, \ldots, 0), y = (-1, 0, \ldots, 0) \in C_1$ and $\lambda = \frac{1}{2}$, then $\frac{1}{2}(x + y)$ equals to $\mathbf{0} \notin C_1$, which means C_1 isn't convex.

   (b) Yes, since

   $$C_2 = \bigcap_{i=1,2,\ldots,n} \{x \in \mathbb{R}^n : x_i \leq 1\},$$

   which means C_2 can be expressed as the intersection of closed half spaces of $\mathbb{R}^n$, so C_2 is convex.

   (c) (If $n = 1$) Yes, since $C_3 = \{x \in \mathbb{R} : x \leq 1\}$ is obviously convex.
   (If $n \geq 2$) No. We can choose $x = (2, 1, 2, \ldots, 2), y = (1, 2, 2, \ldots, 2) \in C_3$ and $\lambda = \frac{1}{2}$, then $\frac{1}{2}(x + y)$ equals to $(\frac{3}{2}, \frac{3}{2}, 2, \ldots, 2) \notin C_3$, which means C_3 isn't convex.

   (d) (If $n = 1$) Yes, since $C_4 = \{x \in \mathbb{R}_{++} : x \leq 1\}$ is obviously convex.
   (If $n \geq 2$) No. We can choose $x = (\frac{1}{2}, 2, 1, \ldots, 1), y = (2, \frac{1}{2}, 1, \ldots, 1) \in C_4$ and $\lambda = \frac{1}{2}$, then $\frac{1}{2}(x + y)$ equals to $(\frac{5}{4}, \frac{5}{4}, 1, \ldots, 1) \notin C_4$, which means C_4 isn't convex.

2. Let $C \in \mathbb{R}^n$ be a convex set. Let f be a convex function over C, and let g be a strictly convex function over C. Show that the sum function $f + g$ is strictly convex over C.
   Solution. Since $\forall x, y \in C, 0 \leq \theta \leq 1$, we have

   $$f\big(\theta x + (1 - \theta)y\big) \leq \theta f(x) + (1 - \theta)f(y),$$

   $$g\big(\theta x + (1 - \theta)y\big) < \theta g(x) + (1 - \theta)g(y).$$

   Define $h = f + g$, then we obtain

   $$\begin{aligned} h\big(\theta x + (1 - \theta)y\big) &= f\big(\theta x + (1 - \theta)y\big) + g\big(\theta x + (1 - \theta)y\big) \\ &< \theta f(x) + (1 - \theta)f(y) + \theta g(x) + (1 - \theta)g(y) \\ &= \theta\big(f(x) + g(x)\big) + (1 - \theta)\big(f(y) + g(y)\big) \\ &= \theta h(x) + (1 - \theta)h(y). \end{aligned}$$

   Thus $f + g$ is strictly convex over C.

3. Show that the following functions are convex over the specified domain C:

(a) $f(x_1, x_2, x_3) = -\sqrt{x_1 x_2} + 2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$ over $\mathbb{R}^3_{++}$.

(b) $f(x) = \|x\|^4$ over $\mathbb{R}^n$.

(c) $f(x) = \sqrt{x^TQx + 1}$ over $\mathbb{R}^n$, where $Q \succeq 0$ is an $n \times n$ matrix.

Solution.

(a) Since f is twice differentiable and $\mathbf{dom} f = \mathbb{R}^3_{++}$ is convex, we need to show $\nabla^2 f(x) \succeq 0$ for all $x \in \mathbf{dom} f$. The first derivative is

$$\nabla f(x) = \left[-\frac{1}{2}\sqrt{\frac{x_2}{x_1}} + 4x_1 - 2x_2,\ -\frac{1}{2}\sqrt{\frac{x_1}{x_2}} + 4x_2 - 2x_1 - 2x_3,\ 6x_3 - 2x_2\right]^T,$$

and the second derivative is

$$\nabla^2 f(x) = \begin{bmatrix} \frac{1}{4}\sqrt{\frac{x_2}{x_1^3}} + 4 & -\frac{1}{4}\sqrt{\frac{1}{x_1x_2}} - 2 & 0 \\ -\frac{1}{4}\sqrt{\frac{1}{x_1x_2}} - 2 & \frac{1}{4}\sqrt{\frac{x_1}{x_2^3}} + 4 & -2 \\ 0 & -2 & 6 \end{bmatrix}.$$

For all $x_1, x_2, x_3 \in \mathbb{R}^3_{++}$,

$$\frac{1}{4}\sqrt{\frac{x_2}{x_1^3}} + 4 > 0;$$

$$\begin{aligned}
\begin{vmatrix} \frac{1}{4}\sqrt{\frac{x_2}{x_1^3}} + 4 & -\frac{1}{4}\sqrt{\frac{1}{x_1x_2}} - 2 \\ -\frac{1}{4}\sqrt{\frac{1}{x_1x_2}} - 2 & \frac{1}{4}\sqrt{\frac{x_1}{x_2^3}} + 4 \end{vmatrix} &= \sqrt{\frac{x_1}{x_2^3}} + \sqrt{\frac{x_2}{x_1^3}} - \sqrt{\frac{1}{x_1x_2}} + 12 \\
&\geq 2\sqrt{\sqrt{\frac{x_1}{x_2^3}}\sqrt{\frac{x_2}{x_1^3}}} - \sqrt{\frac{1}{x_1x_2}} + 12 \\
&= \sqrt{\frac{1}{x_1x_2}} + 12 \\
&> 0\ ;
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} \frac{1}{4}\sqrt{\frac{x_2}{x_1^3}} + 4 & -\frac{1}{4}\sqrt{\frac{1}{x_1x_2}} - 2 & 0 \\ -\frac{1}{4}\sqrt{\frac{x_1}{x_2}} - 2 & \frac{1}{4}\sqrt{\frac{x_1}{x_2^3}} + 4 & -2 \\ 0 & -2 & 6 \end{vmatrix} &= 6\sqrt{\frac{x_1}{x_2^3}} + 5\sqrt{\frac{x_2}{x_1^3}} - 6\sqrt{\frac{1}{x_1x_2}} + 56 \\
&\geq 2\sqrt{30\sqrt{\frac{x_1}{x_2^3}}\sqrt{\frac{x_2}{x_1^3}}} - 6\sqrt{\frac{1}{x_1x_2}} + 56 \\
&= (2\sqrt{30} - 6)\sqrt{\frac{1}{x_1x_2}} + 56 \\
&> 0.
\end{aligned}$$

So $f(x_1, x_2, x_3)$ is convex over $\mathbb{R}^3_{++}$.

(b) For all $x, y \in \mathbb{R}^n$ and $0 \leq \theta \leq 1$, we have

$$\begin{aligned}
f(\theta x + (1-\theta)y) &= \|\theta x + (1-\theta)y\|^4 \\
&\leq (\|\theta x\| + \|(1-\theta)y\|)^4 \\
&= (\theta\|x\| + (1-\theta)\|y\|)^4 \\
&\leq \theta\|x\|^4 + (1-\theta)\|y\|^4 \\
&= \theta f(x) + (1-\theta)f(y).
\end{aligned}$$

(The first inequality is the triangle inequality, and the second inequality follows from the convexity of x^4.) So $f(x)$ is convex over $\mathbb{R}^n$.

(c) Since $Q \succeq 0$, Q can be written as $Q = C^T C$, where C is lower triangular. Define $y = Cx$, then we can express f as

$$f(x) = \sqrt{x^T Q x + 1} = \sqrt{y^T y + 1} = \sqrt{y_1^2 + \ldots + y_n^2 + 1^2} \triangleq \|\tilde{y}\|_2,$$

where $\tilde{y} = \begin{bmatrix} y \\ 1 \end{bmatrix} = [y_1, y_2, \ldots, y_n, 1]^T$. According to the triangle inequality and the convexity of x^2, $\forall x_1, x_2 \in \mathbb{R}^n$ and $0 \leq \theta \leq 1$,

$$\begin{aligned}
f\big(\theta x_1 + (1-\theta)x_2\big) &= \|\theta\tilde{y_1} + (1-\theta)\tilde{y_2}\|_2 \\
&\leq \|\theta\tilde{y_1}\|_2 + \|(1-\theta)\tilde{y_2}\|_2 \\
&= \theta f(x_1) + (1-\theta)f\big(x_2\big).
\end{aligned}$$

So $f(x)$ is convex over $\mathbb{R}^n$ when $Q \succeq 0$.