最优化8

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`手写部分有所修改，本次作业请以本电子版为准。`

1. Derive the dual problems of the SDP and SOCP problems in Question 4 in Assignment 7. Write CVX codes to solve the associated dual problems and show the optimal solutions. (Hint: refer the User's guide for CVX, e.g., from http://cvxr.com/cvx/doc/sdp.html)

SDP可以得到对偶问题如下：

- semidefinite programming

$$\begin{aligned} \inf \quad & C \bullet Z \\ \text{s.t.} \quad & A_j \bullet Z = b_j \quad \text{for } j = 1, \ldots, m, \\ & Z \in X = \mathcal{S}_+^n \end{aligned}$$

dual is

$$\begin{aligned} \sup \quad & b^T w \\ \text{s.t.} \quad & C - \sum_{j=1}^m w_j A_j \in \mathcal{S}_+^n \end{aligned}$$

SOCP在第7次作业中知道对偶问题如下：

$$\begin{aligned} \min \quad & f^T x \\ \text{s.t.} \quad & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \ldots, m, \end{aligned}$$

with variables $x \in \mathbb{R}^n$, can be expressed as

$$\begin{aligned} \max \quad & \sum_{i=1}^m (b_i^T u_i - d_i v_i) \\ \text{s.t.} \quad & \sum_{i=1}^m (A_i^T u_i - c_i v_i) + f = 0 \\ & \|u_i\|_2 \le v_i, \quad i = 1, \ldots, m, \end{aligned}$$

```
clc;clear all;
%% 1 SDP
rand('seed',18300290007);
m=3;
n=4;
C=rand(n);
C=C'*C;
```

```
for i = 1 : m
    A{i} = rand(n);
    A{i} = A{i}' * A{i};
    b{i}=rand();
end
b = cell2mat(b);
cvx_begin sdp
    variable y(3)
    variable s(n, n) semidefinite;
    maximize b*y;
    subject to
        sigma=zeros(n,n);
        for i=1:m
            sigma=sigma+y(i)*A{i};
        end
        sigma+s==C;
        s>=0;
cvx_end
disp(y);

%% 2 SOCP
clear all;
rand('seed',18300290007);
m=3;
n=4;
maxdim=5;
ni=unidrnd(maxdim,[1,m]);
f=rand(n,1);
for i=1:m
    A{i}=rand(ni(i),n);
    b{i}=rand(ni(i),1);
    c{i}=rand(n,1);
    d{i}=rand();
end
d = cell2mat(d);
cvx_begin
    variables u(maxdim, 3) v(3);
    sum=0;
    for i = 1 : m
        sum = sum + b{i}'* u(1:ni(i),i);
    end
    maximize sum - d*v;
    subject to
        A0=zeros(n,1);
        C0=0;
        for i = 1 : m
            A0 = A0 + A{i}' * u(1:ni(i), i);
            C0 = C0 + c{i} * v(i);
        end
        f == -A0 + C0;
        for i = 1 : m
            norm(u(1:ni(i), i)) <= v(i);
        end
cvx_end

```

```
Status: Solved
Optimal value (cvx_optval): +0.171661
```

OUTPUT1:

```
0.2033
0.0260
0.0097
```

```
Status: Solved
Optimal value (cvx_optval): -0.697101
```

OUTPUT2:

```
-0.6528    0.0751    0.3429
 0.5123   -0.4128   -0.2022
-0.1380    0.2383         0
-0.0190    0.4266         0
 0.1129   -0.0532         0

 0.8489
 0.6463
 0.3981
```

与作业7中两题的原问题最优值一致，证明了确实是强对偶。

2. Prove that

$$\max_{z} \left\{ p^T z : \|z\|_2^2 \leq R^2, \|z\|_\infty \leq 1 \right\} = \min_{u,v} \left\{ \|u\|_1 + R\|v\|_2 : u + v = p \right\}.$$

Hint: using strong duality and conjugate functions.

注：其等价问题是凸问题且无不等约束，因此满足slater，因而是强对偶问题

Assignment 8

2. $\min \ \|u\|_1 + R\|v\|_2$

$\text{s.t.} \quad u+v=p.$

$L(u,v,\lambda) = \|u\|_1 + R\|v\|_2 - \lambda^T(u+v-p).$

$\quad = \|u\|_1 - \mu^T\lambda + R\|v\|_2 - v^T\lambda + p^T\lambda$

$\theta(\lambda) = p^T\lambda - \max_u\{u^T\lambda - \|u\|_1\} - \max_v\{v^T\lambda - R\|v\|_2\}.$

$= \begin{cases} p^T\lambda \\ -\infty & \text{else.} \end{cases}$ $\leftarrow f_0(\lambda) = \max_u\{u^T\lambda - \|u\|_1\} = 0 \Leftarrow \|\lambda\|_\infty \le 1$. 否则 $f_0(\lambda)=\infty$

$f_1(\lambda) = \max_v\{v^T\lambda - R\|v\|_2\} = 0 \Leftarrow \|\lambda\|_2 \le |R|$ 否则 $f_1(\lambda)=\infty$

i.e. Dual $\quad \max \ p^T\lambda$

$\text{s.t.} \quad \|\lambda\|_\infty \le 1$

$\|\lambda\|_2^2 \le R^2$

由强对偶同最优值得证。

3. Demonstrate by an example that the relation $0 \preceq A \preceq B$ does not necessary imply that $A^2 \preceq B^2$.

3. $A = \begin{bmatrix} 1 & \\ & 4 \end{bmatrix}$. $B = \begin{bmatrix} 2 & 2 \\ 2 & 9 \end{bmatrix}$ $\quad B-A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \succeq 0$

$A^2 = \begin{bmatrix} 1 & \\ & 16 \end{bmatrix}$. $B^2 = \begin{bmatrix} 8 & 22 \\ 22 & 85 \end{bmatrix}$. $\quad B^2-A^2 = \begin{bmatrix} 7 & 22 \\ 22 & 69 \end{bmatrix} \not\succeq 0$

$\therefore 0 \preceq A \preceq B \not\Rightarrow A^2 \preceq B^2.$