DATA130026.01 Optimization
Solution of Assignment 7

1. (a) Let L^n be the n-dimensional ice-cream cone

$$L^n = \{x \in \mathbb{R}^n : x_n \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2}\}.$$

Prove that L^n is a cone.

(b) Prove that the ice-cream cone is self-dual:

$$(L^n)_* = L^n.$$

(c) Prove that the positive semidefinite cone $S_+^n = \{X : X \succeq 0\}$ is self-dual.

Solution.

(a) $\forall\ \alpha \geq 0,\ x \in L^n$, we have $\alpha x_n \geq \sqrt{(\alpha x_1)^2 + \cdots + (\alpha x_{n-1})^2}$, i.e. $\alpha x \in L^n$.

(b)
- For any fixed $y \in (L^n)_*$, our goal is to prove $y \in L^n$, i.e. $y_n \geq \sqrt{y_1^2 + \cdots + y_{n-1}^2}$. For definition,

$$(L^n)_* = \{y \in \mathbb{R}^n :\ y^T x \geq 0,\ \forall x \in L^n\}.$$

Choose a specific x:

$$x_i = \begin{cases} -\, y_i, & 1 \leq i \leq n-1 \\ \sqrt{y_1^2 + \cdots + y_{n-1}^2}, & i = n. \end{cases} \tag{1}$$

Obviously, $x_n \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2}$, i.e., $x \in L^n$. Meanwhile, we have

$$0 \leq x^T y = -\sum\nolimits_{i=1}^{n-1} y_i^2 + \sqrt{\sum\nolimits_{i=1}^{n-1} y_i^2} \cdot y_n \Leftrightarrow y_n \geq \sqrt{\sum\nolimits_{i=1}^{n-1} y_i^2}.$$

Thus $(L^n)_* \subseteq L^n$.
- For any fixed $x \in L^n$, we then prove $x \in (L^n)_*$, i.e. $x^T y \geq 0, \forall\ y \in L^n$. By Cauchy-Schwartz's inequality,

$$\begin{aligned} x^T y &= x_n y_n + \sum\nolimits_{i=1}^{n-1} x_i y_i \\ &\geq x_n y_n - \sqrt{\sum\nolimits_{i=1}^{n-1} x_i^2}\sqrt{\sum\nolimits_{i=1}^{n-1} y_i^2} \\ &= y_n \left(x_n - \sqrt{\sum\nolimits_{i=1}^{n-1} x_i^2} \right) + \sqrt{\sum\nolimits_{i=1}^{n-1} x_i^2} \left(y_n - \sqrt{\sum\nolimits_{i=1}^{n-1} y_i^2} \right) \\ &\geq 0. \end{aligned}$$

Thus $L^n \subseteq (L^n)_*$.

Therefore, $(L^n)_* = L^n$.

(c) For definition,

$$(S_+^n)_* = \{Y : \; X \bullet Y \geq 0, \; \forall X \in S_+^n\}.$$

- First we prove $S_+^n \subseteq (S_+^n)_*$.
  $\forall\; X, Y \in S_+^n$ (i.e. $X, Y \succeq 0$), $\exists\; A, B \in \mathbb{R}^{n\times n}$, s.t. $X = A^TA,\; Y = B^TB$. Therefore,

  $$X \bullet Y = \mathrm{tr}(X^TY) = \mathrm{tr}\left(A^TAB^TB\right) = \mathrm{tr}((A^TB)^TA^TB) \geq 0,$$

  i.e. $Y \in (S_+^n)_*$. Thus, $S_+^n \subseteq (S_+^n)_*$.
- Then we prove $(S_+^n)_* \subseteq S_+^n$.
  $\forall\; M \notin S_+^n$ (i.e. $M \not\succeq 0$), $\exists\; q \in \mathbb{R}^n$, s.t. $q^TMq = \mathrm{tr}(qq^TM) < 0$. Hence $\exists\; N = qq^T$ satisfies $\mathrm{tr}(NM) < 0$, which means $M \notin (S_+^n)_*$. Therefore, $M \in (S_+^n)^C \Rightarrow M \in (S_+^n)_*^C$, i.e., $(S_+^n)_* \subseteq S_+^n$.

Therefore, $(S_+^n)_* = S_+^n$.

2. Find the Lagrange dual problem of the conic form problem in inequality form

$$\begin{array}{ll} \min & c^Tx \\ \text{s.t.} & Ax \preceq_K b \end{array}$$

where $A \in \mathbb{R}^{m\times n},\; b \in \mathbb{R}^m$, and K is a proper cone in $\mathbb{R}^m$. Make any implicit equality constraints explicit.

Solution. We associate with the inequality a multiplier $\lambda \in \mathbb{R}^m$, and form the Lagrangian

$$L(x, \lambda) = c^Tx + \lambda^T(Ax - b).$$

The dual function is

$$\begin{aligned} g(\lambda) &= \inf_x \left(c^Tx + \lambda^T(Ax - b)\right) \\ &= \begin{cases} -b^T\lambda & A^T\lambda + c = 0 \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

The dual problem is to maximize $g(\lambda)$ over all $\lambda \succeq_{K^*} 0$ or, equivelently,

$$\begin{array}{ll} \max & -b^T\lambda \\ \text{s.t.} & A^T\lambda + c = 0 \\ & \lambda \succeq_{K^*} 0. \end{array}$$

3. Show that the dual of the SOCP

$$\begin{array}{ll} \min & f^Tx \\ \text{s.t.} & \|A_ix + b_i\|_2 \leq c_i^Tx + d_i, \quad i = 1, \ldots, m, \end{array}$$

with variables $x \in \mathbb{R}^n$, can be expressed as

$$\begin{array}{ll} \max & \displaystyle\sum_{i=1}^m \left(b_i^Tu_i - d_iv_i\right) \\ \text{s.t.} & \displaystyle\sum_{i=1}^m \left(A_i^Tu_i - c_iv_i\right) + f = 0 \\ & \|u_i\|_2 \leq v_i, \quad i = 1, \ldots, m, \end{array}$$

with variables $u_i \in \mathbb{R}^{n_i}$, $v_i \in \mathbb{R}$, $i = 1, \ldots, m$. The problem data are $f \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n_i \times n}$, $b_i \in \mathbb{R}^{n_i}$, $c_i \in \mathbb{R}^n$ and $d_i \in \mathbb{R}$, $i = 1, \ldots, m$.
Derive the dual in the following two ways.

(a) Introduce new variables $y_i \in \mathbb{R}^{n_i}$ and $t_i \in \mathbb{R}$ and equalities $y_i = A_i x + b_i$, $t_i = c_i^T x + d_i$, and derive the Lagrange dual.

(b) Start from the conic formulation of the SOCP and use the conic dual. Use the fact that the second-order cone is self-dual.

Solution.

(a) We introduce the new variables, and write the problem as

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & \|y_i\|_2 \le t_i, \quad i = 1, \ldots, m \\ & y_i = A_i x + b_i, \quad i = 1, \ldots, m \\ & t_i = c_i^T x + d_i, \quad i = 1, \ldots, m \end{array}$$

The Lagrangian is

$$\begin{aligned} & L(x, y, t, \lambda, \nu, \mu) \\ = \; & f^T x + \sum_{i=1}^m \lambda_i (\|y_i\|_2 - t_i) + \sum_{i=1}^m \nu_i^T (y_i - A_i x - b_i) + \sum_{i=1}^m \mu_i \left(t_i - c_i^T x - d_i\right) \\ = \; & (f - \sum_{i=1}^m A_i^T \nu_i - \sum_{i=1}^m \mu_i c_i)^T x + \sum_{i=1}^m \left(\lambda_i \|y_i\|_2 + \nu_i^T y_i\right) + \sum_{i=1}^m (-\lambda_i + \mu_i)\, t_i \\ & - \sum_{i=1}^n \left(b_i^T \nu_i + d_i \mu_i\right). \end{aligned}$$

The minimum over x is bounded below if and only if

$$\sum_{i=1}^m \left(A_i^T \nu_i + \mu_i c_i\right) = f.$$

To minimize over y_i, we note that

$$\inf_{y_i} \left(\lambda_i \|y_i\|_2 + \nu_i^T y_i\right) = \begin{cases} 0 & \|\nu_i\|_2 \le \lambda_i \\ -\infty & \text{otherwise.} \end{cases}$$

The minimum over t_i is bounded below if and only if $\lambda_i = \mu_i$. So we have

$$g(\lambda, \nu, \mu) = \begin{cases} -\sum_{i=1}^n \left(b_i^T \nu_i + d_i \mu_i\right) & \sum_{i=1}^m \left(A_i^T \nu_i + \mu_i c_i\right) = f, \\ & \|\nu_i\|_2 \le \lambda_i, \quad \mu = \lambda \\ -\infty & \text{otherwise.} \end{cases}$$

Introduce $u_i = -\nu_i,\ v_i = \mu_i = \lambda_i\ (i = 1, \ldots, m)$, then the dual problem is

$$\begin{array}{ll} \max & \sum_{i=1}^{m} \left(b_i^T u_i - d_i v_i \right) \\ \text{s.t.} & \sum_{i=1}^{m} \left(A_i^T u_i - c_i v_i \right) + f = 0 \\ & \|u_i\|_2 \leq v_i, \quad i = 1, \ldots, m. \end{array}$$

(b) We express the SOCP as a conic form problem

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & -\begin{pmatrix} A_i x + b_i \\ c_i^T x + d_i \end{pmatrix} \preceq_{K_i} 0, \quad i = 1, \ldots, m, \end{array}$$

where

$$K_i = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^{n_i+1} \mid \|x\|_2 \leq t \right\}.$$

The Lagrangian is

$$\begin{aligned} L(x, u, v) &= f^T x - \sum_{i=1}^{m} \begin{pmatrix} u_i \\ v_i \end{pmatrix}^T \begin{pmatrix} A_i x + b_i \\ c_i^T x + d_i \end{pmatrix} \\ &= \left(f - \sum_{i=1}^{m} A_i^T u_i - \sum_{i=1}^{m} v_i c_i \right)^T x - \sum_{i=1}^{m} (u_i^T b_i + v_i d_i). \end{aligned}$$

The minimum over x is bounded below if and only if

$$f - \sum_{i=1}^{m} A_i^T u_i - \sum_{i=1}^{m} v_i c_i = 0.$$

So we have

$$g(u, v) = \begin{cases} -\sum_{i=1}^{m} \left(b_i^T u_i + d_i v_i \right) & \sum_{i=1}^{m} \left(A_i^T u_i + v_i c_i \right) = f, \\ -\infty & \text{otherwise.} \end{cases}$$

The conic dual is

$$\begin{array}{ll} \max & -\sum_{i=1}^{n} \left(b_i^T u_i + d_i v_i \right) \\ \text{s.t.} & \sum_{i=1}^{m} \left(A_i^T u_i + v_i c_i \right) = f \\ & (u_i, v_i) \succeq_{K_i^*} 0, \quad i = 1, \ldots, m. \end{array}$$

Since the second-order cone is self-dual, we have $K_i^* = K_i$, which means $\|u_i\|_2 \le v_i$ $(i = 1, \ldots, m)$. Replace u_i with $-u_i$, we obtain the dual problem

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} \left(b_i^T u_i - d_i v_i\right) \\ \text{s.t.} \quad & \sum_{i=1}^{m} \left(A_i^T u_i - c_i v_i\right) + f = 0 \\ & \|u_i\|_2 \le v_i, \quad i = 1, \ldots, m. \end{aligned}$$

4. Write CVX codes to solve the following SDP and SOCP problems, and show the optimal solutions. (Hint: refer the User's guide for CVX, e.g. from http://cvxr.com/cvx/doc/sdp.html)

   (a) (SDP)

$$\begin{aligned} \min \quad & \mathbf{tr}(CX) \\ \text{s.t.} \quad & \mathbf{tr}(A_i X) = b_i, \quad i = 1, \ldots, m \\ & X \in \mathbf{S}_+^n, \end{aligned}$$

   where

$$\mathbf{S}_+^n = \{X \in \mathbb{R}^{n \times n} \mid X \succeq 0, X = X^T\}.$$

   (b) (SOCP)

$$\begin{aligned} \min \quad & f^T x \\ \text{s.t.} \quad & \begin{pmatrix} A_i x + b_i \\ c_i^T x + d_i \end{pmatrix} \in K_i, \quad i = 1, \ldots, m, \end{aligned}$$

   where

$$K_i = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^{n_i + 1} \mid \|x\|_2 \le t \right\}.$$

Solution.

(a) **Method 1**

```
rand('seed',19210980102);
m = 3;
n = 4;
C = rand(n);
C = C'*C;
for i=1:m
    A{i} = rand(n);
    A{i} = A{i}'*A{i};
    b{i} = rand();
end

cvx_begin
    variable X(n,n);
    minimize(trace(C*X));
    subject to
        for i=1:m
```

```
            trace(A{i}*X) == b{i};
        end
        X == semidefinite(n);
cvx_end

disp(X);
```

Method 2

```
rand('seed',19210980102);
m = 3;
n = 4;
C = rand(n);
C = C'*C;
for i=1:m
    A{i} = rand(n);
    A{i} = A{i}'*A{i};
    b{i} = rand();
end

cvx_begin sdp
    variable X(n,n) symmetric;
    minimize(trace(C*X));
    subject to
        for i=1:m
            trace(A{i}*X) == b{i};
        end
        X >= 0;
cvx_end

disp(X);
```

The optimal solution is

$$X^* = \begin{pmatrix} 0.1152 & -0.3795 & 0.2274 & 0.1707 \\ -0.3795 & 1.2503 & -0.7493 & -0.5624 \\ 0.2274 & -0.7493 & 0.4490 & 0.3370 \\ 0.1707 & -0.5624 & 0.3370 & 0.2529 \end{pmatrix}.$$

The optimal value is 0.1717.

(b)

```
rand('seed',19210980102);
m = 3;
n = 4;
ni = unidrnd(5,[1,m]);
f = rand(n,1);
for i=1:m
    A{i} = rand(ni(i),n);
    b{i} = rand(ni(i),1);
    c{i} = rand(n,1);
    d{i} = rand();
end

cvx_begin
```

```
        variables x(n);
        minimize(f'*x);
        subject to
            for i=1:m
                { A{i}*x+b{i}, c{i}'*x+d{i} } == lorentz(ni(i));
            end
    cvx_end

    disp(x);
```

The optimal solution is

$$x^* = \begin{pmatrix} 0.8250 \\ 1.2435 \\ -1.8473 \\ -0.2958 \end{pmatrix}.$$

The optimal value is -0.6971.