最优化12

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1

1. Compute the projection of a given point x to the second order cone $Q = \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\| \le t\}$.

1. 求 $\mathrm{prox}_Q(x,t)$

先考虑 $\min_{u,v} \frac{1}{2} \|(u,v)-(x,t)\|_2^2$

s.t. $\|u\|_2 \le v$

$L = \frac{1}{2}\|(u,v)-(x,t)\|_2^2 - (\mu,\nu)^T (u,v)$

KKT条件
$$\begin{cases} \begin{bmatrix} u-x-\mu \\ v-t-\nu \end{bmatrix} = 0. \\ -(\mu,\nu) \preceq_Q 0 \\ (u,v) \succeq_Q 0. \\ (\mu,\nu)^T (u,v) = 0 \end{cases}$$

注意到 $\begin{cases} v=0 \Rightarrow u=0. \\ \nu=0 \Rightarrow \mu=0. \end{cases}$

Case 1. $(u,v)=0$. $(\mu,\nu) = -(x,t) \succeq_Q 0$. $\Rightarrow (x,t) \preceq_Q 0$.

2. $(\mu,\nu)=0$. $(u,v) = (x,t) \succeq_Q 0$. $\Rightarrow (x,t) \succeq_Q 0$.

3. $v,\nu \ne 0$. $\frac{\mu}{\nu} = -\frac{u}{v}$. $\mu = -\frac{\nu}{v}u$. & $\|u\|_2 = v$, $\|\mu\|_2 = \nu$

$\Rightarrow x = (1+\frac{\nu}{v})u \Rightarrow \begin{cases} \|x\|_2 = \|u\|_2 + \|\mu\|_2 = v+\nu \\ t = v-\nu \end{cases}$

$\Rightarrow \begin{cases} v = \frac{t+\|x\|_2}{2} \\ \nu = \frac{\|x\|_2 - t}{2} \end{cases} \Rightarrow \begin{cases} u = \frac{x}{1+\frac{\nu}{v}} = \frac{\|x\|_2+t}{2\|x\|_2} \cdot x \\ v = \frac{t+\|x\|_2}{2} \end{cases}$

综上: $\mathrm{prox}_Q(x) = \begin{cases} 0 & \|x\|_2 \le -t. \ t<0 \\ (x,t). & \|x\|_2 \le t. \ t>0 \\ \left(\frac{\|x\|_2+t}{2\|x\|_2}, \frac{\|x\|_2+t}{2}\right) & \|x\|_2 \ge |t| \end{cases}$

2

2. Show that the projection onto the set $C = [l, u] = \{x \in \mathbb{R}^n | l \leq x \leq u\}$ can be written as

$$P_C(x)_i = \begin{cases} l_i & \text{if } x_i \leq l_i, \\ x_i & \text{if } l_i \leq x_i \leq u_i, \\ u_i & \text{if } x_i \geq u_i. \end{cases}$$

2. 先考虑 $\min_u \frac{1}{2}\|u-x\|_2^2 = \frac{1}{2}\sum_1^n (u_i - x_i)^2$

$\text{s.t. } u \in C \quad \text{i.e. } u \geq_C 0.$

对于第i个分量:

易见 $\operatorname{argmin}(u_i - x_i)^2 = \begin{cases} l_i & x_i \leq l_i \\ x_i & l_i \leq x_i \leq u_i \\ u_i & u_i \leq x_i \end{cases}$

$\therefore \operatorname{prox}_C(x) = \begin{cases} l_i & x_i \leq l_i \\ x_i & l_i \leq x_i \leq u_i \\ u_i & x_i \geq u_i \end{cases}$

3

3. Compute the proximal mapping prox_{tf} $(t > 0)$ for the following function f.

- quadratic function $(A \succeq 0)$: $f(x) = \frac{1}{2}x^T Ax + b^T x + c$.
- Euclidean norm: $f(x) = \|x\|_2$.
- logarithmic barrier: $f(x) = -\sum_{i=1}^n \log x_i$.

3.

(a) $\mathrm{prox}_{tf}(x) = \arg\min\limits_u \frac{1}{2}\|u-x\|^2 + f(u)\cdot t$

~~一阶导 $u-x+Ax+b=0$.~~

$= \arg\min\limits_u \frac{1}{2}u'(tA+I)u + (tb-x)'u + tc + x^Tx$

$\because t>0.\ A\succeq 0$

$\therefore tA+I\succ 0$. 凸问题

$\therefore \arg\min\limits_u(\ldots)$ s.t. $(tA+I)u + tb - x = 0$.

$\therefore \mathrm{prox}_{tf}(x) = (tA+I)^{-1}(tb-x)$

(b) $\mathrm{prox}_{tf}(x) = \arg\min\limits_u \frac{1}{2}\|u-x\|^2 + t\|u\|_2^{\circ}$

$=: \arg\min\limits_u g(u)$

need: $0 \in \{u-x\} + t\,\partial\|u\|_2$

$\Rightarrow x-u \in t\,\partial\|u\|_2$

回顾: $t\partial\|u\|_2 = \begin{cases} \frac{tu}{\|u\|_2} & u\neq 0 \\ \{g \mid \|g\|_2 \le t\}. & u=0. \end{cases}$

当 $u\neq 0. \Rightarrow \begin{cases} u = (1-\frac{t}{\|x\|_2})x \\ \|x\|_2 > t \end{cases}$

当 $u=0 \Rightarrow \|x\|_2 \le t$.

$\therefore \mathrm{prox}_{tf}(x) = \begin{cases} (1-\frac{t}{\|x\|_2})x. & \|x\|_2 > t \\ 0 & \|x\|_2 \le t. \end{cases}$

(c) $\mathrm{prox}_{tf}(x) = \arg\min\limits_u \frac{1}{2}\|u-x\|^2 - (\sum_1^n \log u_i)\cdot t$

$= \arg\min\limits_u \sum_1^n[\frac{1}{2}(u_i-x_i)^2 - t\log u_i]$

$=: \arg\min\limits_u g(u)$.

对第i个分量: $\frac{\partial g(u)}{\partial u_i} = u_i - x_i - \frac{t}{u_i} = 0$.

$\Rightarrow u_i^2 - x_iu_i - t = 0$.

$\therefore \begin{cases} u_i = \frac{x_i \pm\sqrt{x_i^2+4t}}{2} \\ u_i > 0 \end{cases}$

$\therefore u_i = \frac{x_i + \sqrt{x_i^2+4t}}{2}$

综上: $\mathrm{prox}_{tf}(x)_i = \frac{x_i + \sqrt{x_i^2+4t}}{2}$ $\quad i=1:n$.

5. Solve the following quadratic programming problem:

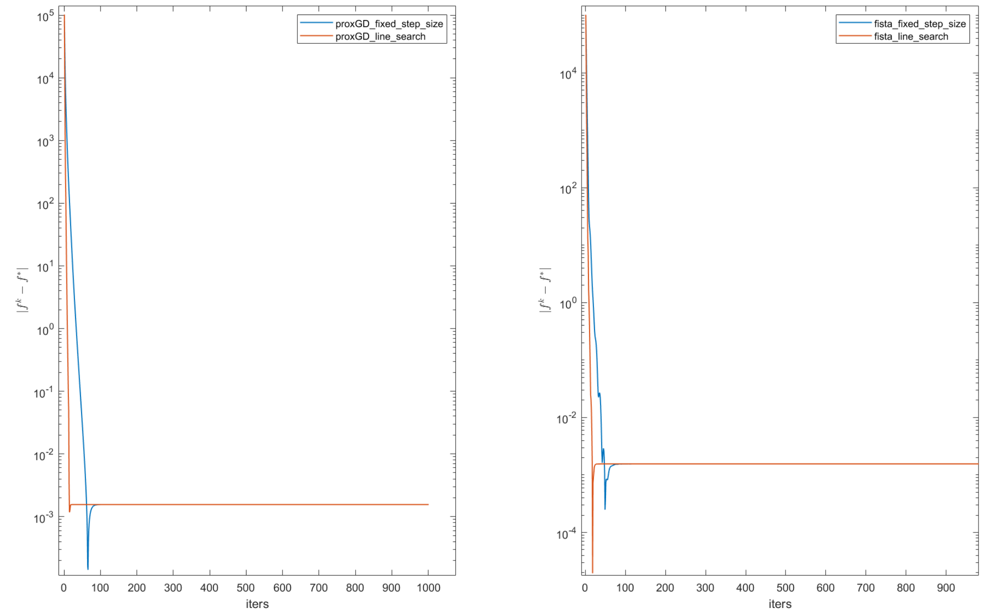
$$\begin{aligned} \min \quad & \frac{1}{2}x^T Q x - b^T x \\ \text{s.t.} \quad & 1 \le x_i \le 2, i = 1, \dots, n. \end{aligned}$$

Use the following Matlab code to generate the data:

```
n = 500; xbar=randn(n,1);
Q = randn(n,n);Q=Q*Q';Q=Q+Q'+eye(n);
b=Q*xbar;e=ones(n,1);
```

Choose initial point `e=ones(n,1)`. Terminate your code after 1000 iterations. Use fixed step size $1/L$, where L is the Lipschitz constant for the smooth part. Implement both the proximal gradient method and the FISTA with i) fixed step size 1/L, and ii) line search (i.e., you need to implement 4 methods). Plot the results (use $f(x_k) - f^*$ as the y-axis, where f^* can be computed by CVX).

运行代码文件夹中的question5.m可得如下：



最后线搜索法回升，通过计算$f(x_{1001}) - f^*$发现值为负，说明回升是因为绝对值翻折而非函数值在迭代中变大，也说明用cvx求解精度没有正确的迭代法高。

6

6. Consider the Lasso problem

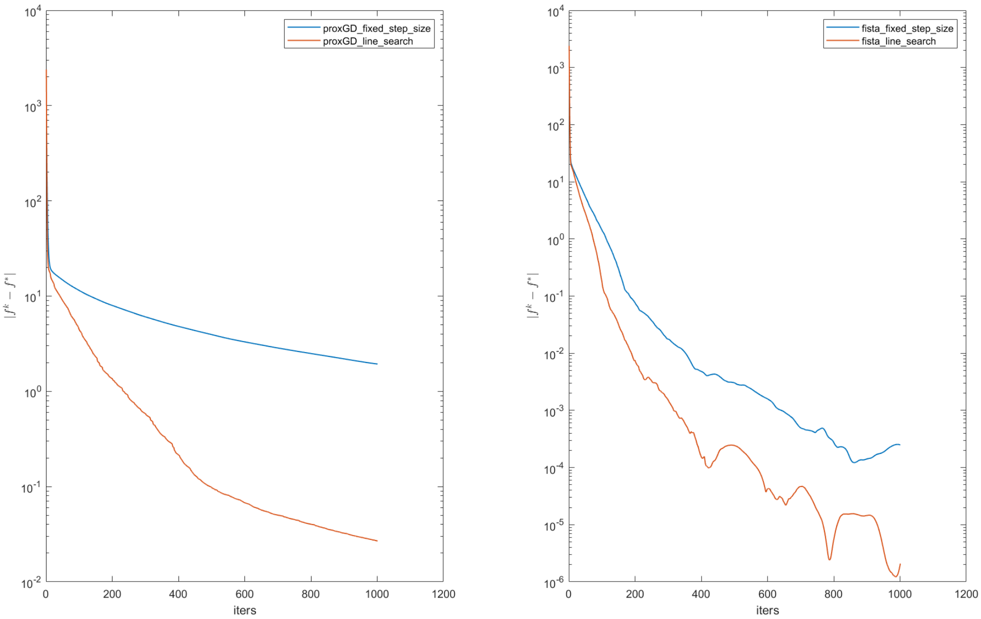
$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - b\|_2^2 + \tau \|x\|_1$$

where $\tau = 1$ is a weighting parameter, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ are given data. Use the following Matlab code to generate the data:

```
m = 100; n = 500; s = 50;
A = randn(m,n);
xs = zeros(n,1); picks = randperm(n); xs(picks(1:s)) = randn(s,1);
b = A*xs;
```

Choose $x = 0$ as the starting point. Terminate your code after 1000 iterations. Use fixed step size $1/L$, where L is the Lipschitz constant for the smooth part. Implement both the proximal gradient method and the FISTA with i) fixed step size 1/L, and ii) line search (i.e., you need to implement 4 methods). Plot the results (use $f(x_k) - f^*$ as the y-axis, where f^* can be computed by CVX).

运行代码文件夹中的question6.m文件可得如下：



实验证明fista收敛更快，但未必单调下降。