DATA130026.01 Optimization
Solution of Assignment 3

1. Consider the problem

$$\begin{array}{lll}(\mathrm{P}) & \min & f(x) \\ & \text{s.t.} & g(x) \le 0, \\ & & x \in X,\end{array}$$

where f and g are convex functions over $\mathbb{R}^n$ and $X \subset \mathbb{R}^n$ is a convex set. Suppose that x^* is an optimal solution of (P) that satisfies $g(x^*) < 0$. Show that x^* is also an optimal solution of the problem

$$\begin{array}{lll}(\mathrm{P}) & \min & f(x) \\ & \text{s.t.} & x \in X.\end{array}$$

Solution. Suppose $\exists\, y(\neq x^*) \in X$ s.t. $g(y) > 0$ and $f(y) < f(x^*)$. Consider the line segment L joining x^* to y:

$$L = \{x \in \mathbb{R}^n : x = \lambda y + (1-\lambda)x^*, \lambda \in (0,1]\}.$$

By the convexity property for f, we have

$$f(x) \le \lambda f(y) + (1-\lambda) f(x^*) < f(x^*), \quad \forall x \in L.$$

Since g is convex and a convex function is continuous on the relative interior of its domain, there exists a neighborhood $\mathcal{N}$ of x^* that satisfies $g(x) < 0$ for any $x \in \mathcal{N} \cap X$. If we choose $\tilde{x} \in \mathcal{N} \cap X \cap L$, then

$$g(\tilde{x}) < 0, \quad \text{and } f(\tilde{x}) < f(x^*).$$

This leads to a contradiction, because we know x^* is an optimal solution of (P).

2. Let $f : C \to \mathbb{R}$ be a convex function defined over the convex set $C \subseteq \mathbb{R}^n$. Then the set of optimal solutions of the problem

$$\min\{f(x) : x \in C\},$$

which we denote by X^*, is convex. If, in addition, f is strictly convex over C, then there exists at most one optimal solution of the problem.

Solution.

(a) $\forall x_1, x_2 \in X^*, \theta \in [0,1]$, since C is convex and f is convex, we have

$$\theta x_1 + (1-\theta)x_2 \in C,$$

$$f(\theta x_1 + (1-\theta)x_2) \le \theta f(x_1) + (1-\theta) f(x_2) = f(x_1) = f(x_2).$$

By the definition of minimum, we have

$$f(\theta x_1 + (1-\theta)x_2) = \min f.$$

Therefore $\theta x_1 + (1-\theta)x_2 \in X^*$, X^* is convex.

(b) Proof by contradiction. Suppose that $\exists\, x_1 \neq x_2$ s.t. $x_1, x_2 \in X^*$, since f is strictly convex over C, we have

$$f(\frac{x_1 + x_2}{2}) < \frac{1}{2} f(x_1) + \frac{1}{2} f(x_2) = f(x_1),$$

which leads to a contradiction. Therefore, there exists at most one optimal solution of the problem.