DATA130026.01 Optimization
Assignment 2
Due Time: at the beginning of the class, Mar. 30, 2021

1. For each of the following sets determine whether they are convex or not (explaining your choice).

   (a) $C_1 = \{x \in \mathbb{R}^n : \|x\|^2 = 1\}$.

   (b) $C_2 = \{x \in \mathbb{R}^n : \max_{i=1,2,\ldots,n} x_i \leq 1\}$.

   (c) $C_3 = \{x \in \mathbb{R}^n : \min_{i=1,2,\ldots,n} x_i \leq 1\}$.

   (d) $C_4 = \{x \in \mathbb{R}^n_{++} : \Pi_{i=1}^n x_i \leq 1\}$, where $\Pi_{i=1}^n x_i = x_1 x_2 \cdots x_n$.

2. Let $C \in \mathbb{R}^n$ be a convex set. Let f be a convex function over C, and let g be a strictly convex function over C. Show that the sum function $f + g$ is strictly convex over C.

3. Show that the following functions are convex over the specified domain C:

   (a) $f(x_1, x_2, x_3) = -\sqrt{x_1 x_2} + 2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1 x_2 - 2x_2 x_3$ over $\mathbb{R}^3_{++}$.

   (b) $f(x) = \|x\|^4$ over $\mathbb{R}^n$.

   (c) $f(x) = \sqrt{x^T Q x + 1}$ over $\mathbb{R}^n$, where $Q \succeq 0$ is an $n \times n$ matrix.

4. **[Only required for DATA130026h.01.]**

   (a) Show that the function
   $$f(x) = \frac{\|Ax - b\|_2^2}{1 - x^T x}$$
   is convex on $\{x \mid x^T x \leq 1\}$.

   (b) *A quadratic-over-linear composition theorem.* Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is nonnegative and convex, and $g : \mathbb{R}^n \to \mathbb{R}$ is positive and concave. Show that the function f^2/g, with domain $\mathbf{dom} f \cap \mathbf{dom} g$, is convex.