DATA130026 Optimization
Assignment 9
Due Time: at the beginning of the class, May 25, 2021

1. Let f be a convex and continuous differentiable function over $\mathbb{R}^n$. For a fixed $x \in \mathbb{R}^n$, define the function
$$g_x(y) = f(y) - \nabla f(x)^T y.$$
Suppose ∇f is L Lipschitz continuous, i.e.,
$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \forall x, y \in \mathbb{R}^n.$$
   (a) Prove that x is a minimizer of g_x over $\mathbb{R}^n$.
   (b) Show that for any $x, y \in \mathbb{R}^n$,
   $$g_x(x) \leq g_x(y) - \frac{1}{2L}\|\nabla g_x(y)\|^2.$$
   (c) Show that for any $x, y \in \mathbb{R}^n$,
   $$f(x) + \nabla f(x)^T(y - x) + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|^2 \leq f(y).$$

2. Let $F(x) = Ax + b$ be an affine function, with A an $n \times n$-matrix. What properties of the matrix A correspond to the following conditions (a)-(e) on F? Suppose that A is symmetric, so $F(x)$ is the gradient of a quadratic function
   (a) Monotonicity:
   $$(F(x) - F(y))^T(x - y) \geq 0, \ \forall x, y.$$
   (b) Strict monotonicity:
   $$(F(x) - F(y))^T(x - y) > 0, \ \forall x, y.$$
   (c) Strong monotonicity (for the Euclidean norm):
   $$(F(x) - F(y))^T(x - y) \geq m\|x - y\|_2^2, \ \forall x, y,$$
   where m is a positive constant.
   (d) Lipschitz continuity (for the Euclidean norm):
   $$\|F(x) - F(y)\|_2 \leq L\|x - y\|_2, \ \forall x, y,$$
   where L is a positive constant.
   (e) Co-coercivity (for the Euclidean norm):
   $$(F(x) - F(y))^T(x - y) \geq \frac{1}{L}\|F(x) - F(y)\|_2^2, \ \forall x, y,$$
   where L is a positive constant.