DATA130026.01 Optimization
Assignment 7
Due Time: at the beginning of the class, May. 11, 2021

1. (a) Let L^n be the n-dimensional ice-cream cone

$$L^n = \{x \in \mathbb{R}^n : x_n \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2}\}.$$

Prove that L^n is a cone.

(b) Prove that the ice-cream cone is self-dual:

$$(L^n)^* = L^n.$$

(c) Prove that the positive semidefinite cone $S_+^n = \{X : X \succeq 0\}$ is self-dual.

2. Find the Lagrange dual problem of the conic form problem in inequality form

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \preceq_K b \end{array}$$

where $A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m$, and K is a proper cone in $\mathbb{R}^m$. Make any implicit equality constraints explicit.

3. Show that the dual of the SOCP

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m, \end{array}$$

with variables $x \in \mathbb{R}^n$, can be expressed as

$$\begin{array}{ll} \max & \sum_{i=1}^{m} \left(b_i^T u_i - d_i v_i\right) \\ \text{s.t.} & \sum_{i=1}^{m} \left(A_i^T u_i - c_i v_i\right) + f = 0 \\ & \|u_i\|_2 \leq v_i, \quad i = 1, \ldots, m, \end{array}$$

with variables $u_i \in \mathbb{R}^{n_i},\ v_i \in \mathbb{R},\ i = 1, \ldots, m$. The problem data are $f \in \mathbb{R}^n,\ A_i \in \mathbb{R}^{n_i \times n},\ b_i \in \mathbb{R}^{n_i},\ c_i \in \mathbb{R}^n$ and $d_i \in \mathbb{R},\ i = 1, \ldots, m$.
Derive the dual in the following two ways.

(a) Introduce new variables $y_i \in \mathbb{R}^{n_i}$ and $t_i \in \mathbb{R}$ and equalities $y_i = A_i x + b_i,\ t_i = c_i^T x + d_i$, and derive the Lagrange dual.

(b) Start from the conic formulation of the SOCP and use the conic dual. Use the fact that the second-order cone is self-dual.

4. Write CVX codes to solve the following SDP and SOCP problems, and show the optimal solutions. (Hint: refer the User's guide for CVX, e.g., from http://cvxr.com/cvx/doc/sdp.html)

   (a) (SDP)

$$\begin{array}{ll} \min & \mathbf{tr}(CX) \\ \text{s.t.} & \mathbf{tr}(A_i X) = b_i, \quad i = 1, \dots, m \\ & X \in \mathbf{S}_+^n, \end{array}$$

   where

$$\mathbf{S}_+^n = \{X \in \mathbb{R}^{n \times n} \mid X \succeq 0, X = X^T\}.$$

   The problem data are generated by the MATLAB commands

```
rand('seed', your student ID);
m = 3;
n = 4;
C = rand(n);
C = C'*C;
for i=1:m
    A{i} = rand(n);
    A{i} = A{i}'*A{i};
    b{i} = rand();
end
```

   Hint. To require that the matrix expression X be symmetric positive semidefinite, we use the syntax `X == semidefinite(n);` or use `cvx_begin sdp;` and use $X >= 0$ to denote a positive semidefinite matrix.

   (b) (SOCP)

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & \begin{pmatrix} A_i x + b_i \\ c_i^T x + d_i \end{pmatrix} \in K_i, \quad i = 1, \dots, m, \end{array}$$

   where

$$K_i = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^{n_i+1} \mid \|x\|_2 \le t \right\}.$$

   The problem data are generated by the MATLAB commands

```
rand('seed', your student ID);
m = 3;
n = 4;
ni = unidrnd(5,[1,m]);
f = rand(n,1);
for i=1:m
    A{i} = rand(ni(i),n);
    b{i} = rand(ni(i),1);
    c{i} = rand(n,1);
    d{i} = rand();
end
```

Hint. Use the syntax `{..., ...} == lorentz(...);`