最优化7

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1. (a) Let L^n be the n-dimensional ice-cream cone

$$L^n = \{x \in \mathbb{R}^n : x_n \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2}\}.$$

Prove that L^n is a cone.

(b) Prove that the ice-cream cone is self-dual:

$$(L^n)^* = L^n.$$

(c) Prove that the positive semidefinite cone $S_+^n = \{X : X \succeq 0\}$ is self-dual.

1.

(a). $L^n = \{x \in R^n : x_n \geq \sqrt{x_1^2 + \cdots x_{n-1}^2}\}$

$= \{x \in R^n : x_n^2 \geq x_1^2 + \cdots x_{n-1}^2, x_n \geq 0\}$.

$\forall x \in R^n$. $\alpha \geq 0$.

s.t. αx 有 $\alpha^2 x_n^2 \geq \alpha^2(x_1^2 + \cdots x_{n-1}^2)$. $\alpha x_n \geq 0$.

$\therefore \alpha x \in L^n$.

$\therefore L^n$是一个锥

(b) $(L^n)^* = \{ y\in R^n : \forall x\in L^n . y^T x \ge 0\}$

$(L^n)^* \supseteq L^n$ iff $\forall x_1, x_2 \in R^n$. s.t. $x_1^T x_2 \ge 0$.

而 $x_1^T x_2 \ge 0 \Leftrightarrow x_{1n} x_{2n} \ge -\sum^{n-1} x_{1i} x_{2i}$

$\Leftarrow (\sum^{n-1} x_{1i}^2)\cdot(\sum^{n-1} x_{2i}^2) \ge (\sum x_{1i}x_{2i})^2$

$\Leftrightarrow \sum_{i\ne j}^{n-1} (x_{1i}-x_{2j})^2 \ge 0$.

$\therefore (L^n)^* \supseteq L^n$;

$(L^n)^* \subseteq L^n$ iff $\forall y \in (L^n)^*$ s.t. $y\in L^n$.

~~而 $y_1^T y_2 \ge 0 \Leftrightarrow y_1^T I y_2 \ge 0$.~~

~~$= y_1^T x x^T y_2 \ge 0 \quad \forall x\in L^n$.~~

取 $x_0 = \begin{bmatrix} -y_1 \\ \vdots \\ -y_{n-1} \\ \sqrt{\sum y_i^2} \end{bmatrix}$. 则 $x_0 \in L^n$.

$\Rightarrow y^T x_0 \ge 0 \Leftrightarrow (\sum^{n-1} y_i^2)^{\frac{1}{2}} \cdot y_n - \sum^{n-1} y_i^2 \ge 0$.

$\Rightarrow y_n \ge (\sum^{n-1} y_i^2)^{\frac{1}{2}}$.

$\therefore y\in L^n$

i.e. $(L^n)^*\subseteq L^n$;

综上, $(L^n)^* = L^n$.

(c) $\triangle\ (S_+^n)^* = \{Y : X\cdot Y \ge 0, \forall X\in S_+^n\}$

$\triangle\ (S_+^n)^* \subseteq S_+^n$ iff $\forall Y \in (S_n^+)^*$ s.t. $Y\in S_n^+$.

若 $Y\notin S_+^n$. 则 $\exists a\in R^n/\{0\}. a'Ya<0$.

$\Leftrightarrow Y\cdot(aa')<0$. 但 $aa'\in S_n^+$. 则与对偶条件矛盾.

$\therefore (S_+^n)^*\subseteq S_+^n$;

$S_+^n \subseteq (S_+^n)^*$ iff $\forall X_1, X_2 \in S_+^n$. s.t. $X_1\cdot X_2 \ge 0$.

$X_1\cdot X_2 = tr(X_1X_2) = tr(X_1^{0.5}X_1^{0.5}X_2^{0.5}X_2^{0.5}) = tr(X_1^{0.5}X_2^{0.5}X_2^{0.5}X_1^{0.5}) \ge 0$.

$\therefore S_+^n\subseteq (S_+^n)^*$.

综上 $S_+^n = (S_+^n)^*$

2. Find the Lagrange dual problem of the conic form problem in inequality form

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \preceq_K b \end{array}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and K is a proper cone in $\mathbb{R}^m$. Make any implicit equality constraints explicit.

2. $L(x,\lambda) = c^T x + \lambda^T (Ax-b)$

$$\theta(x) = \inf_x \{ c^T x + \lambda^T A x - \lambda^T b \}$$

$$= \begin{cases} -b^T \lambda. & A^T\lambda + c = 0. \\ -\infty & \text{else} \end{cases}$$

$\Rightarrow$ Dual:

$$\begin{aligned} \max_{\lambda} \quad & -b^T \lambda. \\ \text{s.t.} \quad & A^T\lambda + c = 0. \\ & \lambda \succeq_{K^*} 0 \end{aligned}$$

3. Show that the dual of the SOCP

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \dots, m, \end{array}$$

with variables $x \in \mathbb{R}^n$, can be expressed as

$$\begin{array}{ll} \max & \sum_{i=1}^{m} (b_i^T u_i - d_i v_i) \\ \text{s.t.} & \sum_{i=1}^{m} (A_i^T u_i - c_i v_i) + f = 0 \\ & \|u_i\|_2 \le v_i, \quad i = 1, \dots, m, \end{array}$$

with variables $u_i \in \mathbb{R}^{n_i}$, $v_i \in \mathbb{R}$, $i = 1, \dots, m$. The problem data are $f \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n_i \times n}$, $b_i \in \mathbb{R}^{n_i}$, $c_i \in \mathbb{R}^n$ and $d_i \in \mathbb{R}$, $i = 1, \dots, m$.
Derive the dual in the following two ways.

(a) Introduce new variables $y_i \in \mathbb{R}^{n_i}$ and $t_i \in \mathbb{R}$ and equalities $y_i = A_i x + b_i$, $t_i = c_i^T x + d_i$, and derive the Lagrange dual.

(b) Start from the conic formulation of the SOCP and use the conic dual. Use the fact that the second-order cone is self-dual.

3.

(a) $\Rightarrow \min f^T x$

$$\text{s.t. } \|y_i\|_2 \le t_i. \quad i=1:m$$

$$y_i = A_i x + b_i. \quad i=1:m.$$

$$t_i = c_i^T x + d_i. \quad i=1:m$$

$$L(x, y, t, \lambda, v_1, v_2) = f^T x + \sum \lambda_i (y_i^T y_i - t_i^2) + \sum v_{1i}^T (y_i - A_i x - b_i) + \sum v_{2i} (\overset{t_i -}{c_i^T x + d_i})$$

$$\theta(\lambda, v_1, v_2) = \inf_{x,y,t} \{ f^T x - \sum v_{1i}^T A_i x - \sum v_{2i} c_i^T x + \sum \lambda_i y_i^T y_i + \sum v_{1i}^T y_i - \sum \lambda_i t_i^2 + \sum v_{2i} t_i$$
$$- \sum b_i^T v_{1i} - \sum v_{2i} d_i \}$$

$$= \begin{cases} -\sum (b_i^T v_{1i} + d_i v_{2i}) & f - \sum A_i^T v_{1i} - \sum c_i v_{2i} = 0 \ \& \ \|v_{1i}\|_2 \le \lambda_i \ \& \ v_{2i} = \lambda_i \\ -\infty & \text{else} \end{cases}$$

换元：$u = -v_1$. $v = v_2 = \lambda$.

$\Rightarrow$ Dual

$$\max + \sum (b_i^T u_i - v_i d_i)$$

$$\text{s.t. } f = -\sum A_i^T u_i + \sum v_i c_i$$

$$\|u_i\|_2 \le v_i \quad i=1:m.$$

(b) $\Rightarrow \min f^T x$

$$\text{s.t. } -\begin{pmatrix} A_i x + b_i \\ c_i^T x + d_i \end{pmatrix} \preceq_K 0. \quad K=2 \quad i=1:m$$

$$L(x, u, v) = f^T x - \sum \begin{pmatrix} u_i \\ v_i \end{pmatrix}' \begin{pmatrix} A_i x + b_i \\ c_i^T x + d_i \end{pmatrix}$$

$$\theta(u, v) = \inf_x \{ (f^T - \sum u_i^T A_i - \sum v_i c_i) x - \sum (u_i^T b_i + v_i d_i) \}$$

$$= \begin{cases} -\sum (u_i^T b_i + v_i d_i) & f = \sum A_i^T u_i + \sum v_i c_i \\ -\infty & \text{else} \end{cases}$$

令 $u_i = -u_i$

$\Rightarrow$ Dual

$$\max \sum (b_i^T u_i + d_i v_i)$$

$$\text{s.t. } f = -\sum A_i^T u_i + \sum c_i v_i$$

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} \succeq_{K^*} 0. \quad i=1:m. \quad K^* = K = 2 \iff \|u_i\|_2 \le v_i$$

$\therefore$ 得证

4. Write CVX codes to solve the following SDP and SOCP problems, and show the optimal solutions. (Hint: refer the User's guide for CVX, e.g., from http://cvxr.com/cvx/doc/sdp.html)

   (a) (SDP)

$$\begin{array}{ll} \min & \mathbf{tr}(CX) \\ \text{s.t.} & \mathbf{tr}(A_i X) = b_i, \quad i = 1, \dots, m \\ & X \in \mathbf{S}_+^n, \end{array}$$

   where

$$\mathbf{S}_+^n = \{X \in \mathbb{R}^{n \times n} \mid X \succeq 0, X = X^T\}.$$

   The problem data are generated by the MATLAB commands

```
rand('seed', your student ID);
m = 3;
n = 4;
C = rand(n);
C = C'*C;
for i=1:m
    A{i} = rand(n);
    A{i} = A{i}'*A{i};
    b{i} = rand();
end
```

   Hint. To require that the matrix expression X be symmetric positive semidefinite, we use the syntax `X == semidefinite(n);` or use `cvx_begin sdp;` and use $X >= 0$ to denote a positive semidefinite matrix.

```
rand('seed',18300290007);
m=3;
n=4;
C=rand(n);
C=C'*C;
for i =1:m
    A{i}=rand(n);
    A{i}=A{i}'*A{i};
    b{i}=rand();
end

cvx_begin sdp
    variable X(n,n);
    minimize(trace(C*X));
    subject to
        for i=1:m
            trace(A{i}*X)==b{i};
        end
        X>=0;
cvx_end
disp(X);
```

```
Status: Solved
Optimal value (cvx_optval): +0.171661

     0.1152   -0.3795    0.2274    0.1707
    -0.3795    1.2503   -0.7493   -0.5624
     0.2274   -0.7493    0.4490    0.3370
     0.1707   -0.5624    0.3370    0.2529
```

output:

(b) (SOCP)

$$\begin{aligned} \min \quad & f^T x \\ \text{s.t.} \quad & \begin{pmatrix} A_i x + b_i \\ c_i^T x + d_i \end{pmatrix} \in K_i, \quad i = 1, \ldots, m, \end{aligned}$$

where

$$K_i = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^{n_i+1} \mid \|x\|_2 \le t \right\}.$$

The problem data are generated by the MATLAB commands

```
rand('seed', your student ID);
m = 3;
n = 4;
ni = unidrnd(5,[1,m]);
f = rand(n,1);
for i=1:m
    A{i} = rand(ni(i),n);
    b{i} = rand(ni(i),1);
    c{i} = rand(n,1);
    d{i} = rand();
end
```

```
clear all;
rand('seed',18300290007);
m=3;
n=4;
ni=unidrnd(5,[1,m]);
f=rand(n,1);
for i=1:m
    A{i}=rand(ni(i),n);
    b{i}=rand(ni(i),1);
    c{i}=rand(n,1);
    d{i}=rand();
end
cvx_begin
    variables x(n);
    minimize(f'*x);
    subject to
        for i=1:m
            {A{i}*x+b{i},c{i}'*x+d{i}}==lorentz(ni(i));
```

```
        end
cvx_end
disp(x)
```

```
Status: Solved
Optimal value (cvx_optval): -0.697101
```

output:

```
    0.8250
    1.2435
   -1.8473
   -0.2958
```