DATA130026.01 Optimization

Assignment 1

Due Time: at the beginning of the class, Mar. 23, 2021

1. Use the definition of convex function ($\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$) to show that the quadratic function $\frac{1}{2}x^T Ax + b^T x + c$ is convex if and only if $A \succeq 0$.

2. Show that both the second order cone, i.e., $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \|x\|_2 \leq t\}$, and the semidefinite cone, i.e., $\{Z \in \mathbb{S}^n : Z \succeq 0\}$, are convex cones.

3. Let $a, b \in \mathbb{R}^n (a \neq b)$. For what values of μ ($\mu > 0$) is the set

$$S_\mu = \{x \in \mathbb{R}^n : \|x - a\|_2 \leq \mu \|x - b\|_2\}$$

convex?

4. Let $C \in \mathbb{R}^n$ be a nonempty convex set. For each $x \in C$ define the normal cone of C at x by

$$N_C(x) = \{w \in \mathbb{R}^n : w^T(y - x) \leq 0 \text{ for all } y \in C\},$$

and define $N_C(x) = \emptyset$ when $x \notin C$. Show that $N_C(x)$ is convex and closed. Particularly, when $x \in \text{int}(C)$, we have $N_C(x) = \{0\}$.

5. *Supporting hyperplanes.*

   (a) Express the closed convex set $\{x \in \mathbb{R}^n_+ | x_1 x_2 \geq 1\}$ as an intersection of halfspaces.

   (b) Let $C = \{x \in \mathbb{R}^n | \|x\|_\infty \leq 1\}$, the l_∞ norm unit $\mathbb{R}^n$ and let $\hat{x}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{x}$ explicitly.

6. **The following two questions are only required for DATA130026h.01.**

   (a) *Set distributive characterization of convexity* [Rockafellar]. Show that $C \in \mathbb{R}^n$ is convex if and only if $(\alpha + \beta)C = \alpha C + \beta C$ for all nonnegative α, β.

   (b) *Composition of linear-fractional functions.* Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^m$ and $\psi : \mathbb{R}^m \to \mathbb{R}^p$ are the linear-fractional functions

$$\phi(x) = \frac{Ax + b}{c^T x + d}, \quad \psi(y) = \frac{Ey + f}{g^T y + h},$$

with domains $\mathbf{dom}(\phi) = \{x : c^T x + d > 0\}$, $\mathbf{dom}(\psi) = \{x : c^T x + d > 0\}$. We associate with ϕ and ψ the matrices

$$\begin{pmatrix} A & b \\ c^T & d \end{pmatrix}, \quad \begin{pmatrix} E & f \\ g^T & h \end{pmatrix}$$

respectively.

Now consider the composition Γ of ϕ and ψ, i.e., $\Gamma(x) = \psi(\phi(x))$, with domain

$$\mathbf{dom}(\Gamma) = \{x : x \in \mathbf{dom}(\psi) = \{x : c^T x + d > 0\} \mid \phi(x) \in \mathbf{dom}(\psi)\}.$$

Show that Γ is linear-fractional, and that the matrix associated with it is the product

$$\begin{pmatrix} A & b \\ c^T & d \end{pmatrix} \begin{pmatrix} E & f \\ g^T & h \end{pmatrix}.$$