DATA130026.01 Optimization
Solution of Assignment 5

1. Consider the maximization problem

$$\begin{aligned} \max \quad & x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0. \end{aligned}$$

   (i) Is the problem convex?
   (ii) Find all the KKT points of the problem.
   (iii) Find the optimal solution of the problem.

Solution.

(i) Since

$$\begin{aligned} f_0(x) &= -(x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2) \\ &= x^T \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} x + (3 \quad -1)x, \end{aligned}$$

where $\begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} \preceq 0$, it's not a convex problem.

(ii) The Lagrangian is

$$L(x_1, x_2, \lambda_1, \lambda_2, \nu) = -x_1^2 - 2x_1x_2 - 2x_2^2 + 3x_1 - x_2 - \lambda_1 x_1 - \lambda_2 x_2 + \nu(x_1 + x_2 - 1).$$

The KKT conditions are

$$\begin{aligned} -2x_1 - 2x_2 + \nu - \lambda_1 + 3 &= 0, & (1) \\ -2x_1 - 4x_2 + \nu - \lambda_2 - 1 &= 0, & (2) \\ x_1, x_2 &\geq 0, & (3) \\ x_1 + x_2 &= 1, & (4) \\ \lambda_1, \lambda_2 &\geq 0, & (5) \\ \lambda_1 x_1 &= 0, & (6) \\ \lambda_2 x_2 &= 0. & (7) \end{aligned}$$

- If $\lambda_1 = \lambda_2 = 0$, then by (1)−(2), $x_2 = -2$, which is not a feasible solution by (3).
- If $\lambda_1 = 0, \lambda_2 > 0$, then by (7), $x_2 = 0$, and thus $x_1 = 1$ by (4). Plugging the solution into (1), we have $\nu = -1$. It results in $\lambda_2 = -4$ by (2), which is impossible by (5).
- If $\lambda_1 > 0, \lambda_2 = 0$, then by (6), $x_1 = 0$, and thus $x_2 = 1$ by (4). Plugging the solution into (2), we have $\nu = 5$. It results in $\lambda_1 = 6$, then we obtain that $(x_1, x_2, \lambda_1, \lambda_2, \nu) = (0, 1, 6, 0, 5)$ satisfies the KKT system. Hence, $(x_1, x_2) = (0, 1)$ is a KKT point.

- If $\lambda_1, \lambda_2 > 0$, then $x_1 = x_2 = 0$ by (6) and (7), which is not a feasible solution by (4).

To conclude, the only KKT point of the problem is $(x_1, x_2) = (0, 1)$.

(iii) The problem is nonconvex, however, it's a linearly constrained problem. Thus the KKT conditions are necessary, which means the optimal solutions only exist in KKT points. So the optimal solution is $(x_1, x_2) = (0, 1)$.

2. Consider the optimization problem

$$\begin{array}{lll} \text{(P)} & \min & x_1 - 4x_2 + x_3 \\ & \text{s.t.} & x_1 + 2x_2 + 2x_3 = -2 \\ & & x_1^2 + x_2^2 + x_3^2 \leq 1. \end{array}$$

(i) Given a KKT point of problem (P), must it be an optimal solution?

(ii) Find the optimal solution of the problem using the KKT conditions.

Solution.

(i) Yes. Cause it's a convex problem, and the sufficiency of the KKT conditions holds.

(ii) The Lagrangian is

$$L(x_1, x_2, x_3, \lambda, \nu) = x_1 - 4x_2 + x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 1) + \nu(x_1 + 2x_2 + 2x_3 + 2).$$

The KKT conditions are

$$1 + 2\lambda x_1 + \nu = 0, \tag{1}$$
$$-4 + 2\lambda x_2 + 2\nu = 0, \tag{2}$$
$$1 + 2\lambda x_3 + 2\nu = 0, \tag{3}$$
$$x_1^2 + x_2^2 + x_3^2 \leq 1, \tag{4}$$
$$x_1 + 2x_2 + 2x_3 = -2, \tag{5}$$
$$\lambda \geq 0, \tag{6}$$
$$\lambda(x_1^2 + x_2^2 + x_3^2 - 1) = 0. \tag{7}$$

- If $\lambda = 0$, then we have $\nu = -1$ by (1), and $\nu = 2$ by (2), which is a contradiction.
- If $\lambda > 0$, then

$$x_1^2 + x_2^2 + x_3^2 - 1 = 0. \tag{8}$$

Plugging (5) into (8) and (1), we have

$$5x_2^2 + 5x_3^2 + 8x_2x_3 + 8x_2 + 8x_3 + 3 = 0, \tag{9}$$
$$1 - 4\lambda(x_2 + x_3 + 1) + \nu = 0. \tag{10}$$

It results in $\lambda = (9\nu - 5)/4$ by $(10) + 2 * (2) + 2 * (3)$, and plugging λ into (2) and (3), we have

$$x_2 = \frac{-4\nu + 8}{9\nu - 5}, \quad x_3 = -\frac{4\nu + 2}{9\nu - 5}.$$

Plugging the solution into (9), after simplification, we have

$$45\nu^2 - 50\nu - 47 = 0,$$

that is $\nu = 1.7188$ or -0.6077. Since $\lambda = (9\nu - 5)/4$ must be positive, the unique solution is $(x_1, x_2, x_3, \lambda, \nu) = (-0.5194, 0.1074, -0.8477, 2.6173, 1.7188)$, so

$$(x_1, x_2, x_3) = (-0.5194, 0.1074, -0.8477),$$

is the only KKT point. By (i), we know that it's also the optimal solution.

3. Consider the optimization problem

$$\text{(P)} \quad \min\{\mathbf{a}^T\mathbf{x} : \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c \leq 0\},$$

where $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is positive definite, $\mathbf{a}(\neq \mathbf{0}), \mathbf{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

(i) For which values of $\mathbf{Q}, \mathbf{b}, c$ is the problem feasible?

(ii) For which values of $\mathbf{Q}, \mathbf{b}, c$ are the KKT conditions necessary?

(iii) For which values of $\mathbf{Q}, \mathbf{b}, c$ are the KKT conditions sufficient?

(iv) Under the condition of part (ii), find the optimal solution of (P) using the KKT conditions.

Solution.

(i) That is to say, there exists a point $\mathbf{x}_0$ satisfies $\mathbf{x}_0^T\mathbf{Q}\mathbf{x}_0 + 2\mathbf{b}^T\mathbf{x}_0 + c \leq 0$. Since $f_1 = \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$ is a quadratic function with $\mathbf{Q}$ positive definite, the gradient of f_1 is $\nabla f_1 = 2\mathbf{Q}\mathbf{x} + 2\mathbf{b}$. Hence when $\mathbf{x} = -\mathbf{Q}^{-1}\mathbf{b}$, the minimum of f_1 is

$$\min f_1 = -\mathbf{b}^T\mathbf{Q}^{-1}\mathbf{b} + c.$$

Thus when $-\mathbf{b}^T\mathbf{Q}^{-1}\mathbf{b} + c \leq 0$, the problem is feasible.

(ii) For convex problem, the necessity of the KKT conditions holds if the Slater's condition is satisfied. That is to say

$$\exists\ \mathbf{x}_0, \quad \text{s.t.} \quad \mathbf{x}_0^T\mathbf{Q}\mathbf{x}_0 + 2\mathbf{b}^T\mathbf{x}_0 + c < 0.$$

Thus when $-\mathbf{b}^T\mathbf{Q}^{-1}\mathbf{b} + c < 0$, the KKT conditions are necessary.
Note. *You'd better prove that KKT conditions are not necessary when* $-\mathbf{b}^T\mathbf{Q}^{-1}\mathbf{b} + c = 0$, *cause Slater's condition is only a sufficient condition for the necessity of KKT conditions.*

(iii) For convex problem, the sufficiency of the KKT conditions always holds, so the answer is $-\mathbf{b}^T\mathbf{Q}^{-1}\mathbf{b} + c \leq 0$.

(iv) The Lagrangian is

$$L(\mathbf{x}, \lambda) = \mathbf{a}^T\mathbf{x} + \lambda(\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c).$$

The KKT conditions are

$$\begin{aligned} \mathbf{a} + 2\lambda\mathbf{Q}\mathbf{x} + 2\lambda\mathbf{b} &= \mathbf{0} && (1)\\ \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c &\leq 0 && (2)\\ \lambda &\geq 0 && (3)\\ \lambda(\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c) &= 0 && (4) \end{aligned}$$

- If $\lambda = 0$, then we have $\mathbf{a} = \mathbf{0}$ by (1), which is a contradiction.
- If $\lambda \neq 0$, then

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c = 0. \qquad (5)$$

We have $\mathbf{x} = -\frac{1}{2\lambda}\mathbf{Q}^{-1}(\mathbf{a} + 2\lambda\mathbf{b})$ by (1). Plugging $\mathbf{x}$ into (5), we have

$$\lambda = \frac{1}{2}\sqrt{\frac{\mathbf{a}^T\mathbf{Q}^{-1}\mathbf{a}}{\mathbf{b}^T\mathbf{Q}^{-1}\mathbf{b} - c}}. \quad \text{(discarding the negative root)}$$

And thus

$$\mathbf{x} = -\mathbf{Q}^{-1}\left(\sqrt{\frac{\mathbf{b}^T\mathbf{Q}^{-1}\mathbf{b} - c}{\mathbf{a}^T\mathbf{Q}^{-1}\mathbf{a}}}\ \mathbf{a} + \mathbf{b}\right),$$

is the only KKT point. By (ii), we know that it's also the optimal solution of (P).

4. Consider the optimization problem

$$\begin{aligned} \min \quad & x_1^2 - x_2^2 - x_3^2 \\ \text{s.t.} \quad & x_1^4 + x_2^4 + x_3^4 \leq 1. \end{aligned}$$

(i) Is the problem convex?

(ii) Find all the KKT points of the problem.

(iii) Find the optimal solution of the problem.

Solution.

(i) Since

$$\begin{aligned} f_0(x) &= x_1^2 - x_2^2 - x_3^2 \\ &= x^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} x, \end{aligned}$$

where $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ isn't positive definite, it's not a convex problem.

(ii) The Lagrangian is

$$L(x_1, x_2, x_3, \lambda) = x_1^2 - x_2^2 - x_3^2 + \lambda(x_1^4 + x_2^4 + x_3^4 - 1).$$

The KKT conditions are

$$\begin{aligned} x_1(1 + 2\lambda x_1^2) &= 0, && (1)\\ x_2(1 - 2\lambda x_2^2) &= 0, && (2)\\ x_3(1 - 2\lambda x_3^2) &= 0, && (3)\\ x_1^4 + x_2^4 + x_3^4 &\leq 1, && (4)\\ \lambda &\geq 0 && (5)\\ \lambda(x_1^4 + x_2^4 + x_3^4 - 1) &= 0. && (6) \end{aligned}$$

- If $\lambda = 0$, then we have $x_1 = x_2 = x_3 = 0$ by (1)(2)(3). $(x_1, x_2, x_3, \lambda) = (0, 0, 0, 0)$ satisfies the KKT system, hence $(x_1, x_2, x_3) = (0, 0, 0)$ is a KKT point.
- If $\lambda > 0$, by (6) we have

$$x_1^4 + x_2^4 + x_3^4 = 1. \tag{7}$$

  By (1), we have $x_1 = 0$ when $\lambda > 0$.
  - If $x_2 = x_3 = 0$, $(x_1, x_2, x_3) = (0, 0, 0)$ is not a feasible solution by (7).
  - If $x_2 = 0, x_3 \neq 0$, then by (7), $x_3 = \pm 1$, and thus $\lambda = \frac{1}{2}$ by (3). $(x_1, x_2, x_3, \lambda) = (0, 0, \pm 1, \frac{1}{2})$ satisfies the KKT system, hence $(x_1, x_2, x_3) = (0, 0, \pm 1)$ are KKT points.
  - If $x_2 \neq 0, x_3 = 0$, then by (7), $x_2 = \pm 1$, and thus $\lambda = \frac{1}{2}$ by (2). $(x_1, x_2, x_3, \lambda) = (0, \pm 1, 0, \frac{1}{2})$ satisfies the KKT system, hence $(x_1, x_2, x_3) = (0, \pm 1, 0)$ are KKT points.
  - If $x_2, x_3 \neq 0$, then by (2) and (3), $x_2 = \pm \frac{1}{\sqrt{2\lambda}}, x_3 = \pm \frac{1}{\sqrt{2\lambda}}$. Plugging it into (7), we have $\lambda = \frac{\sqrt{2}}{2}$, and thus $x_2 = \pm \frac{1}{\sqrt[4]{2}}, x_3 = \pm \frac{1}{\sqrt[4]{2}}$.
    $(x_1, x_2, x_3, \lambda) = (0, \pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}, \frac{\sqrt{2}}{2})$ satisfies the KKT system, hence $(x_1, x_2, x_3) = (0, \pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}})$ are KKT points.

  To conclude, there are 9 KKT points:

$$(0, 0, 0), \quad (0, 0, \pm 1), \quad (0, \pm 1, 0), \quad (0, \pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}).$$

(iii) The KKT conditions are necessary optimality condition under the regularity condition, so the optimal solutions only exist in KKT points.

- When $(x_1, x_2, x_3) = (0, 0, 0)$, $f_0(x_1, x_2, x_3) = 0$;
- When $(x_1, x_2, x_3) = (0, 0, \pm 1)$ or $(0, \pm 1, 0)$, $f_0(x_1, x_2, x_3) = -1 < 0$;
- When $(x_1, x_2, x_3) = (0, \pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}})$, $f_0(x_1, x_2, x_3) = -\sqrt{2} < -1$.

So the optimal solution of the problem is $(x_1, x_2, x_3) = (0, \pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}})$.