Math 130 04 – A Survey of Calculus

Homework assignment 3

Due: Tuesday, September 20, 2022

> **Instructions:** Write your answers on a separate sheet of paper. Write your name at the top of each page you use, and number each page. Number your answers correctly.
> **Justify all your answers.**

1. Consider the following definition.

$$f(x) = \begin{cases} \frac{1}{x^2+2x} & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

   (a) Does this define a real function? If so, what is its domain? Justify your answer.

   (b) Do the following limits exist? If so, what are they? Justify your answers.

      i. $\lim_{x \to -2^-} f(x)$ (the left-hand limit of f at -2.)

      ii. $\lim_{x \to -1} f(x)$ (the limit of f at -1.)

      iii. $\lim_{x \to 0^+} f(x)$ (the right-hand limit of f at 0.)

      iv. $\lim_{x \to 0} f(x)$

Solution:

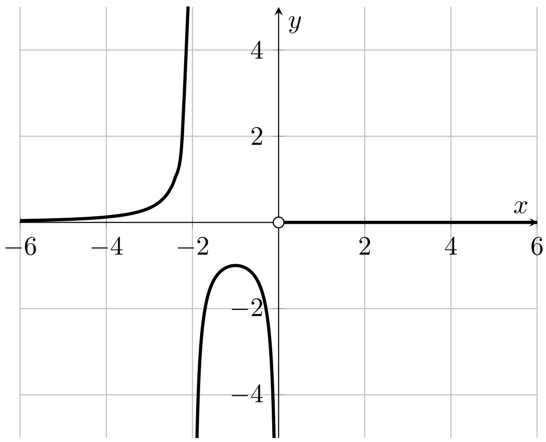
(a) For any real number x, *if* $f(x)$ is defined, then it has a unique value (passes the vertical line test). So f is a real function.

We observe that

- When $x > 0$: $f(x)$ is defined and equals 0.
- When $x = 0$: $f(0)$ is not defined.
- When $x < 0$: $f(x)$ is defined *except* if $x^2 + 2x = 0$. Solving this equation, we find that $f(x)$ is defined whenever $x \neq -2$.

Since $f(x)$ is defined for all real numbers except for $x = 0$ and $x = -2$, the domain of f is $(-\infty, -2) \cup (-2, 0) \cup (0, \infty)$.

We can graph f (for instance, in GeoGebra using the syntax `If(x>0,0,If(x<0,1/(x^2 + 2x)))`) to get a graph that looks like:



(b) i. We observe that as x approaches -2 from the left, $f(x)$ increases indefinitely (goes to ∞). So $\lim_{x\to 2^-} f(x)$ does not exist.

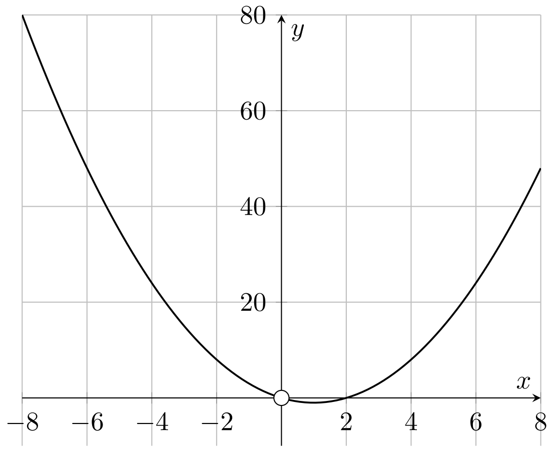
ii. From the graph, we observe that $\lim_{x\to -1} f(x)$ exists. Since $\frac{1}{x^2+2x}$ is a rational function that is defined at -1, we can use the substitution formula to get

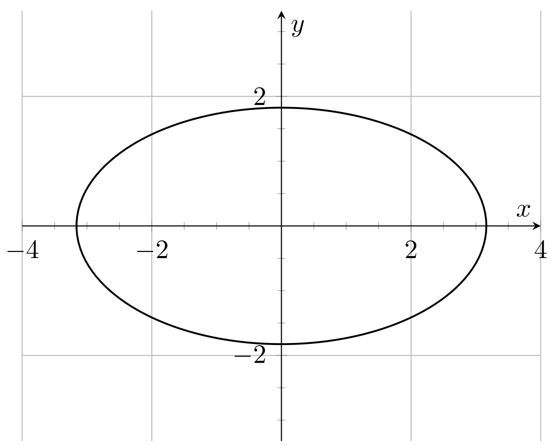
$$\begin{aligned}\lim_{x\to -1} f(x) &= \frac{1}{(-1)^2 + 2(-1)} \\ &= \frac{1}{1-2} \\ &= -1.\end{aligned}$$

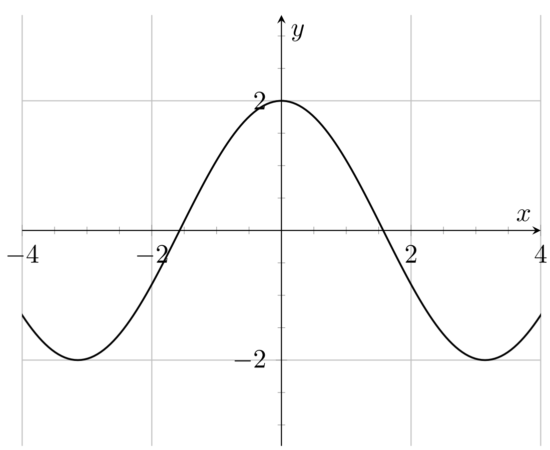
iii. From the graph (or from the definition of f), we observe that as x approaches 0 from the right, $f(x)$ is always equal to 0. So $\lim_{x\to 0^+} = 0$.

iv. We observe that as x approaches 0 from the left, $f(x)$ decreases indefinitely (goes to $-\infty$). This means that the left-hand limit $\lim_{x\to 0^-} f(x)$ does not exist, and so the limit $\lim_{x\to 0} f(x)$ does not exist.

2. For each of the following graphs, justify whether the limit at $x = 0$ exists.

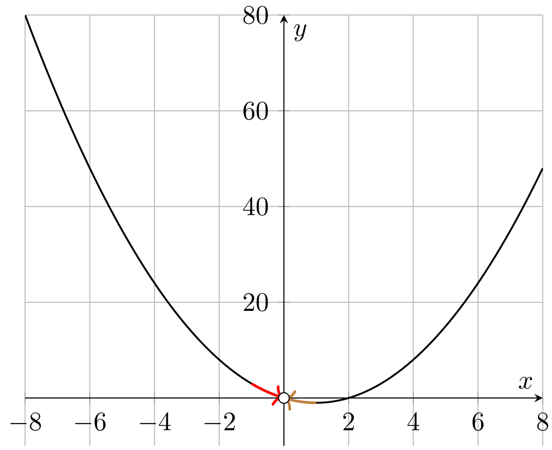
(a)


(b)


(c)


Solution:

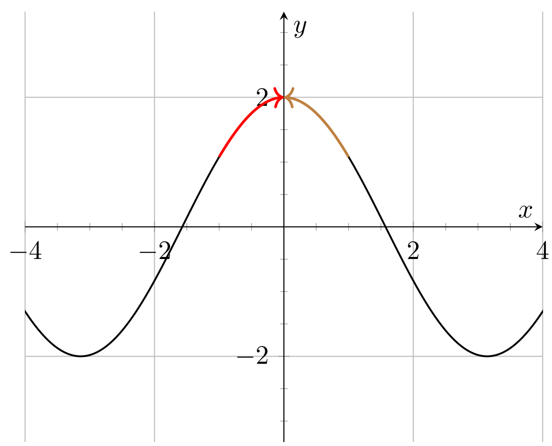
(a)



The graph is that of a real function, since it passes the vertical line test. We observe that as x approaches 0 from the left and the right, the graph approaches the same value. In other words, the left-hand and right-hand limits at $x = 0$ exist and are equal. So the limit at $x = 0$ exists.

(b) The graph is not a real function, since it fails the vertical line test. So it doesn't make sense to talk about its limit at $x = 0$.

(c)



The graph is that of a real function, since it passes the vertical line test. We observe that as x approaches 0 from the left and the right, the graph approaches the same value. In other words, the left-hand and right-hand limits at $x = 0$ exist and are equal. So the limit at $x = 0$ exists.

3. Find the following limits. Justify your answers.

   (a) $\lim_{x \to 0} x^2 + 4$

   (b) $\lim_{x \to \infty} \frac{x^2}{x^2 - 1}$

   (c) $\lim_{x \to 1} \frac{x^2+3x-4}{x^2-1}$

Solution:

(a) Since $x^2 + 4$ is a one-variable polynomial, it always satisfies the substitution formula. So we have

$$\begin{aligned}\lim_{x\to 0} x^2 + 4 &= (0)^2 + 4 \\ &= 4.\end{aligned}$$

(b) If we try to use the substitution formula, we get $\frac{\infty}{\infty}$ which is not a real number (doesn't make sense).

Whenever $x > 0$ (which will be the case when $x \to \infty$), we have

$$\begin{aligned}\frac{x^2}{x^2 - 1} &= \frac{\frac{x^2}{x^2}}{\frac{x^2-1}{x^2}} \\ &= \frac{1}{1 - \frac{1}{x^2}}.\end{aligned}$$

So we can use the algebra of limits to get

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{1 - \frac{1}{x^2}} &= \frac{1}{1 - \lim_{x\to\infty} \frac{1}{x^2}} \\ &= \frac{1}{1 - 0} = 1.\end{aligned}$$

(c) If we try to use the substitution formula we get $\frac{0}{0}$, which doesn't make sense. But this tells us that the polynomials $p(x) = x^2 + 3x - 4$ and $q(x) = x^2 - 1$ both have 1 as a root (that is, $p(1) = 0$ and $q(1) = 0$). Therefore $p(x) = (x-1) \cdot p_1(x)$ and $q(x) = (x-1) \cdot q_1(x)$. We can find $p_1(x)$ and $q_1(x)$ by long division:

$$\begin{array}{r|l} & x + 4 \\ \hline x - 1 & x^2 + 3x - 4 \\ & -x^2 + x \\ \hline & 4x - 4 \\ & -4x + 4 \\ \hline & 0 \end{array} \qquad \begin{array}{r|l} & x + 1 \\ \hline x - 1 & x^2 \quad - 1 \\ & -x^2 + x \\ \hline & x - 1 \\ & -x + 1 \\ \hline & 0 \end{array}$$

So we have

$$\begin{aligned}\lim_{x\to 1} \frac{x^2 + 3x - 4}{x^2 - 1} &= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+4)}{\cancel{(x-1)}(x+1)} \\ &= \lim_{x\to 1} \frac{x+4}{x+1} = \frac{1+4}{1+1} = \frac{5}{2}.\end{aligned}$$