Math 130 04 – A Survey of Calculus

Homework assignment 7

Due: Tuesday, November 1, 2022

Remember: The **chain rule** says that if f and g are real functions, then the derivative of the composite function $(g \circ f)(x) = g(f(x))$ can be calculated as follows.

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

1. Evaluate the derivatives of the following functions.

   (a) $f(x) = (x^2 + 3x + 4)^3$ (b) $f(x) = (x^3 + 5)^{1/4}$ (c) $f(x) = \dfrac{x+3}{(x^2-4)^{2/3}}$

Solution:

(a)

Step 1: We recognise that $f(x)$ can be rewritten as

$$f(x) = (g(x))^3 \qquad \text{where } g(x) = x^2 + 3x + 4.$$

So if we define the function m as $m(x) = x^3$, then f is equal to the composite function $m \circ g$,

$$\text{i.e.} \qquad f(x) = (m \circ g)(x) = m(g(x)) = (g(x))^3 = \left(x^2 + 3x + 4\right)^3$$

Hence, using the chain rule, the derivative of f is

$$f'(x) = (m \circ g)'(x) = m'(g(x)) \cdot g'(x)$$

Step 2: We calculate the derivatives of the functions m and g.

$$\begin{aligned} m'(x) &= \frac{d}{dx}\left(x^3\right) = 3x^2 \\ g'(x) &= \frac{d}{dx}\left(x^2 + 3x + 4\right) = 2x + 3 \end{aligned}$$

Step 3: We calculate the derivative of f.

$$\begin{aligned} f'(x) &= m'(g(x)) \cdot g'(x) \\ &= 3(g(x))^2 \cdot (2x+3) && (\text{since } m'(x) = 3x^2 \text{ and } g'(x) = 2x + 3) \\ &= \boxed{3(x^2 + 3x + 4)^2 \cdot (2x+3)} \end{aligned}$$

(b)

Step 1: We recognise that $f(x)$ can be rewritten as

$$f(x) = (g(x))^{1/4} = m(g(x)) = (m \circ g)(x) \qquad \text{where } g(x) = x^3 + 5 \text{ and } m(x) = x^{1/4}$$

Hence, using the chain rule, the derivative of f is

$$f'(x) = m'(g(x)) \cdot g'(x)$$

Step 2: We calculate the derivatives of m and g.

$$\begin{aligned} m'(x) &= \frac{d}{dx}\left(x^{1/4}\right) = \frac{1}{4} \cdot x^{-3/4} \\ g'(x) &= \frac{d}{dx}\left(x^3 + 5\right) = 3x^2 \end{aligned}$$

Step 3: We calculate the derivative of f.

$$\begin{aligned} f'(x) &= m'(g(x)) \cdot g'(x) \\ &= \frac{1}{4} \cdot (g(x))^{-3/4} \cdot (3x^2) \qquad \left(\text{since } m'(x) = \frac{1}{4} \cdot x^{-3/4} \text{ and } g'(x) = 3x^2\right) \\ &= \boxed{\frac{1}{4} \cdot (x^3 + 5)^{-3/4} \cdot (3x^2)} \end{aligned}$$

(c) Using the quotient rule, we have e

$$\begin{aligned} f'(x) &= \frac{\left(\frac{d}{dx}(x+3)\right) \cdot (x^2 - 4)^{2/3} - (x+3) \cdot \left(\frac{d}{dx}\left((x^2-4)^{2/3}\right)\right)}{\left((x^2 - 4)^{2/3}\right)^2} \\ &= \frac{x \cdot (x^2 - 4)^{2/3} - (x+3) \cdot \left(\frac{d}{dx}\left((x^2-4)^{2/3}\right)\right)}{(x^2 - 4)^{4/3}} \end{aligned}$$

So we need to find the derivative of the function $g(x) = (x^2 - 4)^{2/3}$. First, we recognise that $g(x)$ can be rewritten as

$$g(x) = (k(x))^{2/3} = (m \circ k)(x) \qquad \text{where } k(x) = x^2 - 4 \text{ and } m(x) = x^{2/3}.$$

Hence, using the chain rule, the derivative of g is $g'(x) = m'(k(x)) \cdot k'(x)$.

Next, we calculate the derivatives of m and k.

$$\begin{aligned} m'(x) &= \frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3} \cdot x^{-1/3} \\ k'(x) &= \frac{d}{dx}\left(x^2 - 4\right) = 2x \end{aligned}$$

This lets us calculate the derivative of g.

$$\begin{aligned} g'(x) &= m'(k(x)) \cdot k'(x) \\ &= \frac{2}{3} \cdot (k(x))^{-1/3} \cdot (2x) \qquad \left(\text{since } m'(x) = \frac{2}{3} \cdot x^{-1/3} \text{ and k'(x)=2x}\right) \\ &= \frac{2}{3} \cdot \left(x^2 - 4\right)^{-1/3} \cdot (2x). \end{aligned}$$

Finally, we can calculate the derivative of f.

$$f'(x) = \frac{x \cdot (x^2-4)^{2/3} - (x+3) \cdot g'(x)}{(x^2-4)^{4/3}}$$
$$= \boxed{\frac{x \cdot (x^2-4)^{2/3} - (x+3) \cdot \frac{2}{3} \cdot \left(x^2-4\right)^{-1/3} \cdot (2x)}{(x^2-4)^{4/3}}}$$

2. (Variable interest rates.)

A lender (e.g. a bank) proposes two 20-year loans of \$1.5 million to a borrower (e.g. a homebuyer). The borrower can either pay *compound* interest at an interest rate of x percent, or *simple* interest at an interest rate of $2x$ percent. So, if the borrower chooses to pay compound interest at a rate of x percent, the amount they have to pay after 20 years is

$$f(x) = 1.5 \cdot \left(1 + \frac{x}{100}\right)^{20} \quad \text{million dollars.}$$

If the borrower chooses to pay simple interest, the amount they have to pay after 20 years is

$$g(x) = 1.5 \cdot \left(1 + 20 \cdot \frac{2x}{100}\right) \quad \text{million dollars.}$$

(a) If the compound interest rate is 4 percent, how much would the borrower pay after 20 years if they chose the loan with:

i. Compound interest.

ii. Simple interest.

Which loan makes more sense for the homebuyer?

(b) If the compound interest rate is 8 percent, how much would the borrower pay after 20 years if they choose the loan with:

i. Compound interest.

ii. Simple interest.

Which loan makes more sense for the homebuyer?

(c) The difference between the two loans is measured by the amount $d(x) = g(x) - f(x)$.

i. Calculate the rate of change of the difference between the two loans at a compound interest rate of 2 percent.

ii. Calculate the rate of change of the difference between the two loans at a compound interest rate of 4 percent.

iii. At what compound interest rate is the difference between the two loans *maximum*?

Solution:

(a) If the compound interest rate is 4 percent, we are in the case where $x = 4$.

i. If the borrower chooses the loan with compound interest, they would pay

$$f(4) = 1.5 \cdot \left(1 + \frac{4}{100}\right)^{20} = 3.287 \quad \text{million dollars.}$$

ii. If the borrower chooses the loan with simple interest, they would pay

$$g(4) = 1.5 \cdot \left(1 + 20 \cdot \frac{2 \cdot 4}{100}\right) = 3.9 \quad \text{million dollars.}$$

Since the amount paid on the loan with compound interest at 4 percent is *less* than the amount paid on the loan with simple interest at 8 percent, the loan with $\boxed{\text{compound interest}}$ makes more sense for the borrower.

(b) If the compound interest rate is 8 percent, we are in the case where $x = 8$.

i. If the borrower chooses the loan with compound interest, they would pay

$$f(8) = 1.5 \cdot \left(1 + \frac{8}{100}\right)^{20} = 6.991 \quad \text{million dollars.}$$

ii. If the borrower chooses the loan with simple interest, they would pay

$$g(4) = 1.5 \cdot \left(1 + 20 \cdot \frac{2 \cdot 8}{100}\right) = 6.3 \quad \text{million dollars.}$$

Since the amount paid on the loan with compound interest at 4 percent is *more* than the amount paid on the loan with simple interest at 8 percent, the loan with $\boxed{\text{simple interest}}$ makes more sense for the borrower.

(c) The rate of change of the difference function d is its derivative, which is $d'(x) = g'(x) - f'(x)$. We calculate the derivative of g as

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(1.5 \cdot \left(1 + 20 \cdot \frac{2x}{100}\right)\right) \\ &= 1.5 \cdot \frac{d}{dx}\left(1 + 20 \cdot \frac{2x}{100}\right) \\ &= 1.5 \cdot \left(0 + 20 \cdot \frac{2}{100}\right) = \frac{6}{10}. \end{aligned}$$

To calculate the derivative of f, we recognise that $f(x)$ can be written as

$$f(x) = 1.5 \cdot \left(1 + \frac{x}{100}\right)^{20} = 1.5 \cdot (k(x))^{20} = m(k(x)) = (m \circ k)(x)$$

where $m(x) = 1.5 \cdot x^{20}$ and $k(x) = 1 + \dfrac{x}{100}$.

Hence we can find the derivative of f using the chain rule:

$$\begin{aligned} f'(x) &= m'(k(x)) \cdot k'(x) \\ &= 1.5 \cdot 20 \cdot (k(x))^{19} \cdot \frac{1}{100} \qquad \text{since } m'(x) = 1.5 \cdot 20 \cdot x^{19} \text{ and } k'(x) = \frac{1}{100} \\ &= 30 \cdot \left(1 + \frac{x}{100}\right)^{19} \cdot \frac{1}{100} \\ &= \frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19} \end{aligned}$$

Finally, we calculate $d'(x)$ as

$$\begin{aligned} d'(x) &= g'(x) - f'(x) \\ &= \frac{6}{10} - \frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19} \\ &= \frac{3}{10} \cdot \left(2 - \left(1 + \frac{x}{100}\right)^{19}\right) \end{aligned}$$

i. We have

$$d'(2) = \frac{3}{10} \cdot \left(2 - \left(1 + \frac{2}{100}\right)^{19}\right) = 0.163 \quad \text{million dollars per percent}$$

ii. We have

$$d'(4) = \frac{3}{10} \cdot \left(2 - \left(1 + \frac{4}{100}\right)^{19}\right) = -0.032 \quad \text{million dollars per percent}$$

iii. Any real number a is a local maximum of the difference function d exactly when $d'(a) = 0$ and $d''(a) < 0$.

Step 1: We find the critical points of the function d by solving the equation $d'(x) = 0$.

$$\begin{aligned} & d'(x) = 0 \\ \text{i.e.} \quad & g'(x) - f'(x) = 0 \\ \text{i.e.} \quad & f'(x) = g'(x) \\ \text{i.e.} \quad & \frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19} = \frac{6}{10} \\ \text{i.e.} \quad & \frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19} = 2 \\ \text{i.e.} \quad & 1 + \frac{x}{100} = 2^{1/19} \\ \text{i.e.} \quad & x = 100 \cdot \left(2^{1/19} - 1\right) \\ \text{i.e.} \quad & x = 3.715 \end{aligned}$$

Step 2: The second derivative of d is the derivative of d', which we calculate as

$$\begin{aligned} d''(x) &= g''(x) - f''(x) \\ &= \frac{d}{dx}\left(\frac{6}{10}\right) - \frac{d}{dx}\left(\frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19}\right) \\ &= 0 - \frac{3}{10}\frac{d}{dx}\left(\left(1 + \frac{x}{100}\right)^{19}\right) \end{aligned}$$

$$\text{Using the chain rule, } \frac{d}{dx}\left(\left(1 + \frac{x}{100}\right)^{19}\right) = 19 \cdot \left(1 + \frac{x}{100}\right)^{18} \cdot \frac{1}{100}$$

$$\text{So } \; d''(x) = -\frac{3}{10} \cdot 19 \cdot \left(1 + \frac{x}{100}\right)^{18} \cdot \frac{1}{100}$$

Therefore, $d''(3.715) < 0$ (since $3.715 > 0$).

Hence we can conclude that $x = 3.715$ is the *only* local maximum of d. Therefore, the difference between the two loans is maximum when the compound interest rate is 3.715 percent.