Math 130 04 – A Survey of Calculus

Homework assignment 4

Due: Tuesday, September 27, 2022

Instructions: Write your answers on a separate sheet of paper. Write your name at the top of each page you use, and number each page. Number your answers correctly.
Justify all your answers.

1. Evaluate the following limits, justifying your answers each time.

   (a) $\lim\limits_{x\to 4} \dfrac{x^3 - 9x^2 + 16}{3x^2 - 9x + 12}$ (b) $\lim\limits_{x\to\infty} \dfrac{x^3 + 25}{x^3 + 13}$ (c) $\lim\limits_{x\to 0} \dfrac{x + 3}{x^2}$

Solution:

(a) $\dfrac{x^3 - 9x^2 + 16}{3x^2 - 9x + 12}$ is a rational function, so we can try direct substitution to see if we get an answer.

$$\begin{aligned}
\lim_{x\to 4} \frac{x^3 - 9x^2 + 16}{3x^2 - 9x + 12} &= \frac{4^3 - 9(4^2) + 16}{3(4^2) - 9(4) + 12} \\
&= \frac{64 - 9(16) + 3}{3(16) - 36 + 12} \\
&= \frac{64 - 144 + 16}{48 - 36 + 12} \\
&= \frac{-64}{24} \\
&\boxed{= \frac{-8}{3}}
\end{aligned}$$

(b) If we try to use direct substitution, we get $\dfrac{\infty}{\infty}$, which is a nonsense answer (not a real number). So we have to try another method.

Whenever $x > 0$ (which will be the case as $x \to \infty$) we can divide the numerator and the denominator by x^3 to get:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3+25}{x^3+13} &= \lim_{x\to\infty} \frac{\dfrac{x^3+25}{x^3}}{\dfrac{x^3+13}{x^3}} \\
&= \lim_{x\to\infty} \frac{1+\dfrac{25}{x^3}}{1+\dfrac{13}{x^3}} \\
&= \frac{1+25\cdot\left(\lim\limits_{x\to\infty}\dfrac{1}{x}\right)^3}{1+13\cdot\left(\lim\limits_{x\to\infty}\dfrac{1}{x}\right)^3} && \text{(using the algebra of limits)} \\
&= \frac{1+25\cdot 0}{1+13\cdot 0} && \text{(since we know that } \lim_{x\to\infty}\frac{1}{x}=0\text{)} \\
&= \frac{1}{1} \\
&\boxed{=1}
\end{aligned}$$

(c) If we try to use direct substitution, we get $\dfrac{0}{0}$ which is a nonsense answer (not a real number). So we have to try another method.

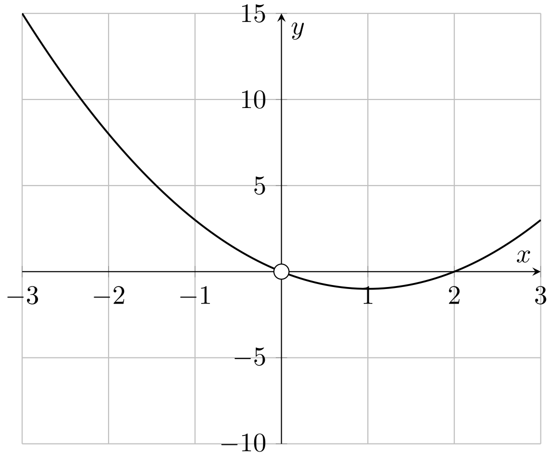
As $x \to 0$, from the left or from the right, it will always be true that $x > 0$ *or* $x < 0$. So we can divide the numerator and denominator by x^2 to get:

$$\begin{aligned}
\lim_{x\to 0} \frac{\dfrac{x+3}{x^2}}{\dfrac{x^2}{x^2}} &= \lim_{x\to 0} \frac{\dfrac{1}{x}+\dfrac{3}{x^2}}{1} \\
&= \left(\lim_{x\to 0}\frac{1}{x}\right) + 3\cdot\left(\lim_{x\to 0}\frac{1}{x}\right)^2 && \text{(using the algebra of limits)}
\end{aligned}$$

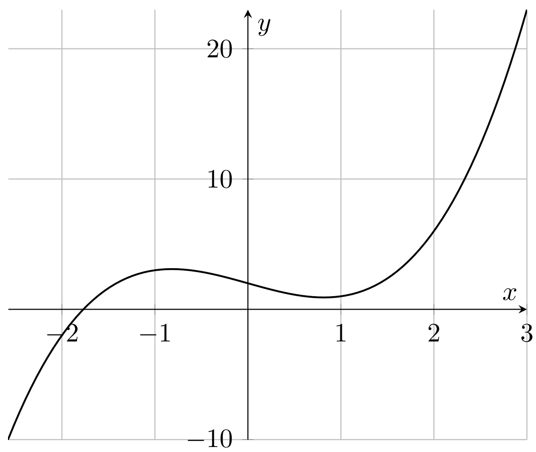
We know that as $x \to 0^+$ (x goes to 0 on the right), $\frac{1}{x}$ increases indefinitely (goes to ∞). Therefore, the same will be true of $\dfrac{1}{x}+\dfrac{3}{x^2}$. So the limit **does not exist**.

Remark: In each of the previous questions, using a graphing calculator to find the limit is a **valid justification** (although it is preferable to know *why* the limit is what it is). However writing things like "$\frac{\infty}{\infty}=1$" is **not** a valid justification since it does not make sense. For example, there are other functions, like $f(x)=\frac{x^2+3}{x+2}$ whose "limit" as $x\to\infty$ by direct substitution gives $\frac{\infty}{\infty}$, but that do not actually have a limit as $x\to\infty$ (the function $f(x)=\frac{x^2+3}{x+2}$ increases indefinitely, i.e. goes to ∞).

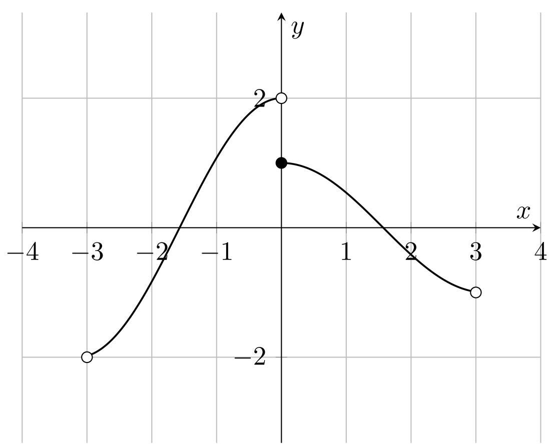
2. Which of the following graphs represent a function continuous over the interval $(-1,1)$? Justify your answers.



(a)



(b)



(c)

Solution:

(a) The graph is a function, since it satisfies the vertical line test. However, it is **not** continuous over $[-1, 1]$, since it does not satisfy the pen-to-paper test.

More precisely, the function is not defined at $x = 0$, and so although the limit at 0 exists, it is not continuous at 0.

(b) The graph is a continuous function over $[-1,1]$, since it passes both the vertical line test and the pen-to-paper test.

(c) The graph is a function, since it satisfies the vertical line test. However, it is **not** continuous over $[-1,1]$, since it does not satisfy the pen-to-paper test.

More precisely, the function is defined for every $x \in [-1,1]$, but it does not have a limit at 0, since the left- and right-hand limits are not the same.

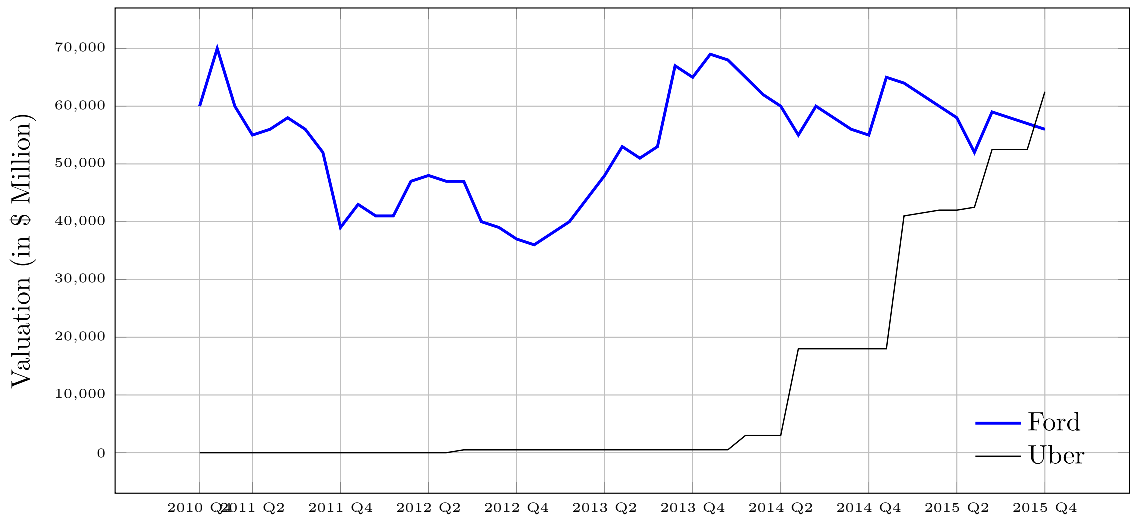


Figure 1: Valuation of Uber and Ford. (Source: Forbes)

3. The figure shows the valuation of the companies Ford and Uber between 2011 and 2015.
   (a) Are the graphs continuous functions in the interval (2014 Q2, 2015 Q4)?
   (b) Is there a point between 2014 Q2 and 2015 Q4 where:
      i. Ford's valuation equals that of Uber?
      ii. Ford's valuation is five times that of Uber?

   Justify all your answers.

Solution:

(a) Both graphs are functions that are continuous over the interval (2014 Q2, 2015 Q4) since they satisfy both the vertical line test and the pen-to-paper test.

(b) i. Yes, since there is a point in the interval (2014 Q2, 2015 Q4) where the graphs intersect.

ii. Let f be the function that is Ford's valuation, and let g be the function that is Uber's valuation. Namely, we have:

$$\begin{aligned} f(x) &= \text{ Ford's valuation at time } x \\ g(x) &= \text{ Uber's valuation at time } x \end{aligned}$$

We would like to find if there is some time x between 2014 Q2 and 2015 Q4, such that $\dfrac{f(x)}{g(x)} = 5$. We know that:

- f and g are continuous over the interval (2013 Q4, 2015 Q4) since they satisfy the pen-to-paper test over that interval. So both functions are continuous over the *closed* interval [2014 Q2, 2015 Q2], since [2014 Q2, 2015 Q2] is contained in the bigger interval (2013 Q4, 2015 Q4).
- Over the closed interval [2014 Q2, 2015 Q2], the function g is never 0. So the function $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$ is continuous over [2014 Q2, 2015 Q2] (by the algebra of continuous functions).
- Since $\dfrac{f}{g}$ is continuous over the *closed* interval [2014 Q2, 2015 Q2], we can apply the intermediate value theorem to it. From the graph, we see that

$$\frac{f(2014\text{ Q2})}{g(2014\text{ Q2})} > \frac{60,000}{10,000} = 6$$

(since $f(2014\text{ Q2}) = 60,000$ and $g(2014\text{ Q2}) < 10,000$), and that

$$\frac{f(2015\text{ Q2})}{g(2015\text{ Q2})} < \frac{60,000}{40,000} = \frac{3}{2} = 1.5$$

(since $f(2015\text{ Q2}) < 60,000$ and $g(2015\text{ Q2}) > 40,000$).
- The intermediate value theorem then tells us that over the interval [2014 Q2, 2015 Q2], $\dfrac{f}{g}$ takes all values in the closed interval $[1.5, 6]$.

Since $5 \in [1.5, 6]$, there is some x between 2014 Q2 and 2015 Q2 such that $\dfrac{f(x)}{g(x)} = 5$, which is what we want.

Therefore, the answer is **yes**.