Negation ($\neg$)

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Lecture notes:

www.chaitanyals.site/teaching/Negation-lecture.pdf

What is the logical operator "$\neg$"?

If P is a statement, then $\neg P$ is one too.

$\neg P$ is read "not P"

$\neg P$ is true when P is not true

Truth table of $\neg P$:

P	$\neg P$
$\top$	$\bot$
$\bot$	$\top$

Example ① P = "3 is an even number"

$\neg P$ = "3 is not an even number"

② $x \in \text{Birds}$ = "x is a bird"

$P(x)$ = "x can fly"

(a) $\neg(\forall x \in \text{Birds}, P(x))$ = "not all birds can fly"

(b) $\forall x \in \text{Birds}, \neg P(x)$ = "no bird can fly"

(c) $\exists x \in \text{Birds}, \neg P(x)$ = "there is a bird that cannot fly"

Properties of $\neg$:

① $\neg$ is a unary operator

("whenever P is a statement, $\neg P$ is a statement")

unlike the binary operators $\vee$, $\wedge$, $\Rightarrow$

("whenever P and Q are statements, $P \wedge Q$ is a statement")

Analogy: $\neg P$ is like the negative number -3

Properties of $\neg$:

(2) $\neg$ binds more strongly than all binary operators

$$\neg A \vee B = (\neg A) \vee B$$

Analogy: $-3+5 = (-3)+5$

$$\neg B \wedge \neg C = (\neg B) \wedge (\neg C)$$

Analogy: $-3 \times -5 = (-3) \times (-5)$

$$\neg A \Rightarrow \neg B = (\neg A) \Rightarrow (\neg B)$$

Properties of $\neg$:

(3) $\neg$ is **involutive**: $\neg\neg P = P$

P	$\neg P$	$\neg(\neg P)$
$\top$	$\bot$	$\top$
$\bot$	$\top$	$\bot$

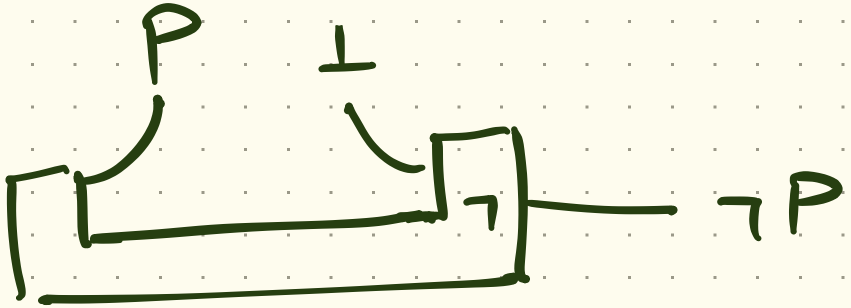
Analogy: $-(-3) = 3$

How to prove $\neg P$

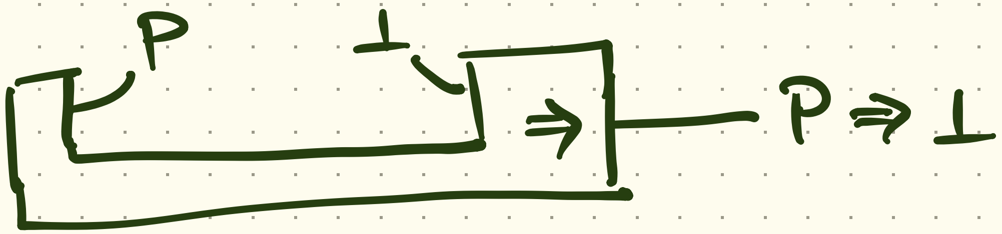
"Assume P and prove $\bot$"

P	$\bot$	$P \Rightarrow \bot$	$\neg P$
$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\bot$	$\top$	$\top$

Why in Olorin, the box



looks exactly like

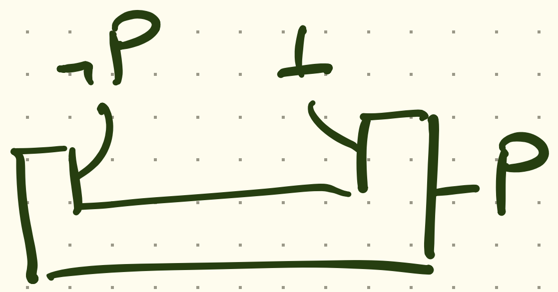


<u>Consequence</u>: So to prove $\neg\neg P$, we assume $\neg P$ and we prove $\perp$.

Since $\neg\neg P = P$, This gives us the mathematical principle of PROOF BY CONTRADICTION

"To prove any mathematical statement P, it's enough to assume $\neg P$ and arrive at a contradiction."

In Latin: "*reductio ad absurdum*"

In Olorin: 

Example: $\sqrt{2}$ is irrational

Proof: Assume that $\sqrt{2}$ is rational.

So $\sqrt{2} = \frac{m}{n}$ where m and n are integers whose greatest common divisor is 1

So $(\sqrt{2})^2 = \frac{m^2}{n^2}$, i.e. $m^2 = 2n^2$

so 2 divides m^2, but this means 2 divides m

i.e. $m = 2a$.

So $(2a)^2 = 2n^2$, i.e. $4a^2 = 2n^2$, i.e. $n^2 = 2a^2$

so 2 divides both m and n $\longrightarrow$ a contradiction!

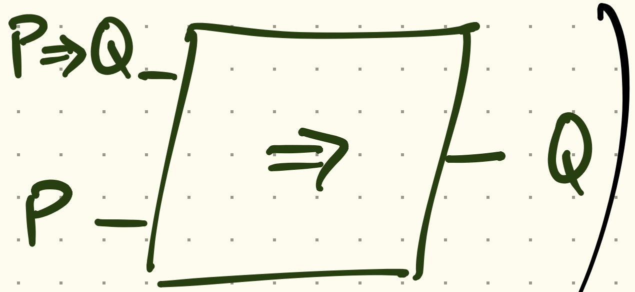
Hence $\sqrt{2}$ is irrational.

How to use $\neg P$

"If we know $\neg P$ and we know P, then we have $\bot$"

In Olorin,

(looks like

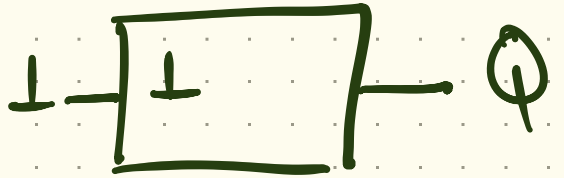


)

Recall: "From $\bot$ we can deduce any Q"

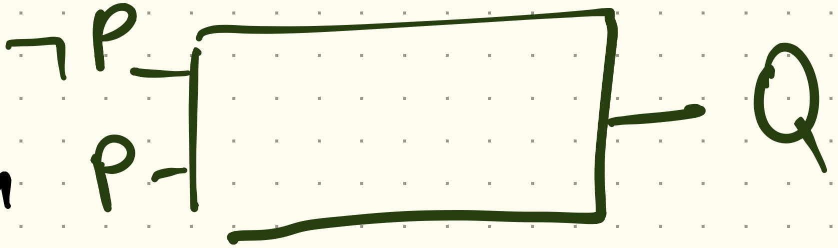
mathematical principle of EXPLOSION

In Latin: *ex falso quodlibet*

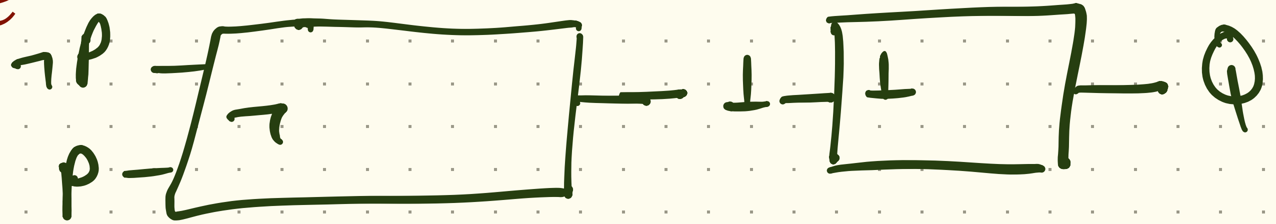
In Olorin 

So by combining, we get

"If we know P and we know $\neg P$, we can deduce any Q"

In Olorin 

In Latin: *ex contradictione quodlibet*



Algebra of negation (De Morgan's Laws)

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$

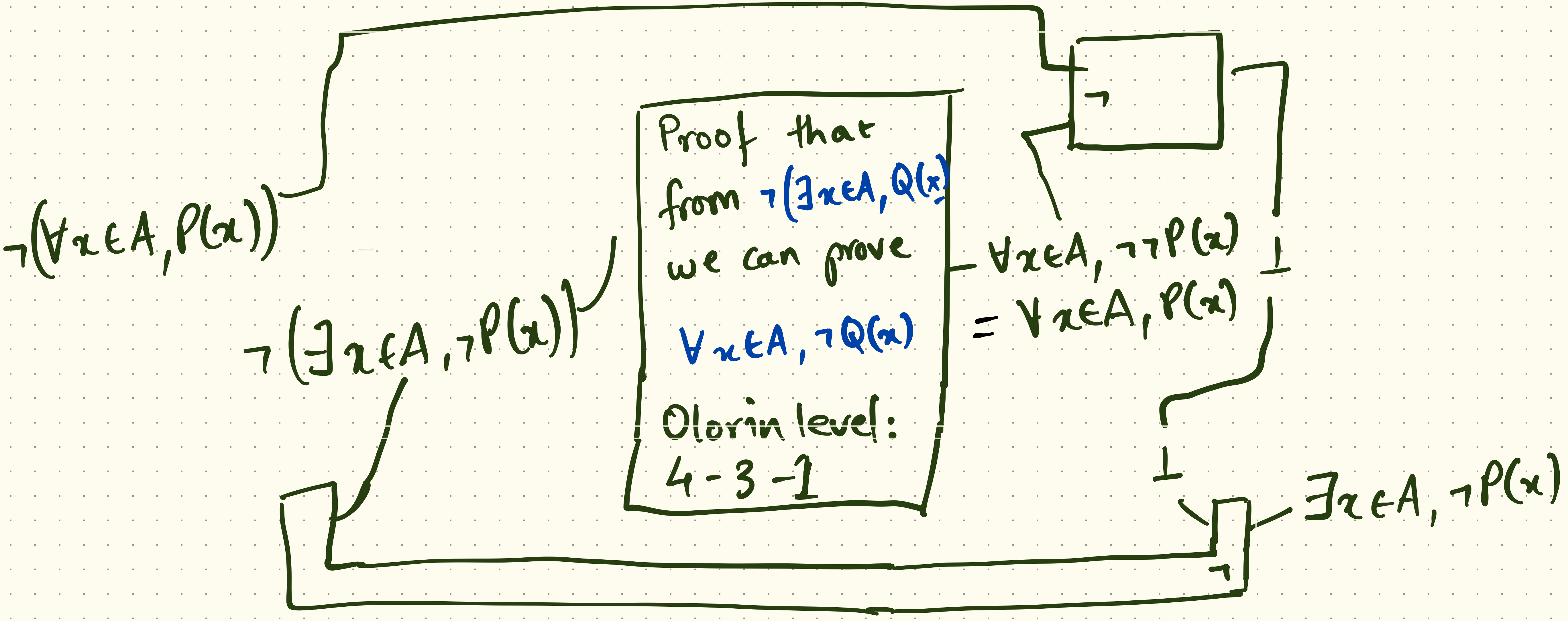
$$\neg(P \wedge Q) = \neg P \vee \neg Q$$

$$\neg(P \Rightarrow Q) = \neg(\neg P \vee Q) = P \wedge \neg Q$$

$$\neg(\forall x \in A,\ P(x)) = \exists x \in A,\ \neg P(x)$$

$$\neg(\exists x \in A,\ P(x)) = \forall x \in A,\ \neg P(x)$$

Example: If we know $\neg(\forall x \in A, P(x))$ we can prove that $\exists x \in A, \neg P(x)$



Example: $\exists$ irrational numbers a and b such that a^b is rational. $\exists x \in A, \exists y \in A, P(x,y)$

Proof. Assume the negation $\forall x \in A, \forall y \in A, \neg P(x,y)$

i.e. for every irrational a and b, a^b is irrational.

Since we know $\sqrt{2}$ is irrational, let $a = b = \sqrt{2}$.

Then $(\sqrt{2})^{\sqrt{2}}$ is irrational. Hence $\underbrace{\left((\sqrt{2})^{\sqrt{2}}\right)^{\sqrt{2}} \text{ is irrational}}_{\neg Q}$.

But $\left((\sqrt{2})^{\sqrt{2}}\right)^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^{\,2} = 2$

$\underbrace{\text{which is a rational number}}_{Q}$.

Thus we have a contradiction.

Hence we conclude $\exists x \in A, \exists y \in A, P(x,y)$