Math 130 04 – A Survey of Calculus

Homework assignment 6

Due: Tuesday, October 18, 2022

1. Evaluate the derivatives of the following functions (over any interval where they are defined).

   (a) $f(x) = x^3 - 9x^2 + 16$ (b) $f(x) = \dfrac{x^3 + 25}{3x - 2}$ (c) $f(x) = \dfrac{x + 3}{x^2 - 4}$

Solution:

(a) Using the rules for derivatives, we get:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^3\right) - 9\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(16) \qquad \text{(using the sum and product rules)} \\ &= 3x^2 - 9 \cdot 2x + 0 \\ &= \boxed{3x^2 - 18x} \end{aligned}$$

(b) Using the rules for derivatives, we get:

$$\begin{aligned} f'(x) &= \frac{\left(\frac{d}{dx}\left(x^3 + 25\right)\right) \cdot (3x - 2) - (x^3 + 25) \cdot \frac{d}{dx}(3x - 2)}{(3x - 2)^2} \qquad \text{(using the quotient rule)} \\ &= \frac{(3x^2 + 25) \cdot (3x - 2) - (x^3 + 25) \cdot 3}{(3x - 2)^2} \\ &= \frac{9x^3 + 75x - 6x^2 - 50 - 3x^3 - 75}{(3x - 2)^2} \\ &= \boxed{\frac{6x^3 - 6x^2 + 75x - 125}{(3x - 2)^2}} \end{aligned}$$

(c) Using the rules for derivatives, we get:

$$\begin{aligned} f'(x) &= \frac{\left(\frac{d}{dx}(x + 3)\right) \cdot (x^2 - 4) - (x + 3) \cdot \frac{d}{dx}\left(x^2 - 4\right)}{(x^2 - 4)^2} \qquad \text{(using the quotient rule)} \\ &= \frac{3 \cdot (x^2 - 4) - (x + 3) \cdot 2x}{(x^2 - 4)^2} \\ &= \frac{3x^2 - 12 - 2x^2 - 6x}{(x^2 - 4)^2} \\ &= \boxed{\frac{x^2 - 6x - 12}{(x^2 - 4)^2}} \end{aligned}$$

2. The distance between Los Angeles and San Diego on the I-5 highway is 118 miles.
   (a) What is the average speed (in mph) required to do the trip in 1.5 hours?
   (b) If the speed limit is 70 mph all along the I-5, is it possible to do the trip in 1.5 hours without breaking the law? Explain. What about in 2 hours?
   (c) A car going from L.A. to San Diego on the I-5 travels $f(x)$ miles (measured from L.A.) after x hours, where f is the function:
   $$f(x) = x(91 - 16x)$$
      i. How long does the car take to reach San Diego (i.e. cover 118 miles)?
      ii. Does the car ever break the law? (Is its speed ever more than 70 mph?)
      iii. What is the car's speed when it leaves L.A. (at the starting time)? What is the car's speed when it arrives in San Diego? (That is, at the time calculated in part i.)

   Remember: Speed (or velocity) is the derivative of distance as a function of time.

Solution:

(a) The average speed required to do 118 miles in 1.5 hours is $\frac{118}{1.5} = \boxed{78.67}$ miles per hour (mph).

(b) Since the average speed required to do 118 miles in 1.5 hours is 78.67 mph, it is $\boxed{\text{not possible}}$ to stay below 70 mph and do the trip in 1.5 hours.

The average speed required to do 118 miles in 2 hours is $\frac{118}{2} = 59$ mph, so it is $\boxed{\text{possible}}$ to do the trip in 2 hours while staying below the speed limit.

(c) i. Since the car travels $f(x)$ miles in x hours, to find how many hours the car will take to do 118 miles, we need to solve the equation $f(x) = 118$,

$$\begin{aligned}
\text{i.e.} \quad & x(91 - 16x) = 118 \\
\text{i.e.} \quad & 91x - 16x^2 = 118 \\
\text{i.e.} \quad & 16x^2 - 91x + 118 = 0 \\
\text{i.e.} \quad & 16x^2 - 32x - 59x + 118 = 0 \\
\text{i.e.} \quad & 16x(x - 2) - 59(x - 2) = 0 \\
\text{i.e.} \quad & (16x - 59)(x - 2) = 0
\end{aligned}$$

The solutions of this equation are $x = 2$ and $x = \frac{59}{16} = 3.69$, so we can conclude that the car will be 118 miles from L.A. on the I-5 after 2 hours and after 3.69 hours, respectively. Since $2 < 3.69$, we can conclude that $\boxed{\text{the car will take 2 hours to do the trip.}}$

ii. The speed of the car at time x hours is the derivative $f'(x)$. We can use the rules for derivatives to calculate

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(x(91 - 16x)\right) \\
&= \left(\frac{d}{dx}(x)\right) \cdot (91 - 16x) + x \cdot \frac{d}{dx}(91 - 16x) \qquad \text{(using the product rule)} \\
&= 1 \cdot (91 - 16x) + x \cdot \left(\frac{d}{dx}(91) - \frac{d}{dx}(16x)\right) \\
&= 91 - 16x + x \cdot (0 - 16) \\
&= 91 - 16x - 16x \\
&= 91 - 32x.
\end{aligned}$$

So, at time 0 hours (the starting time in L.A.), the speed of the car is $f'(0) = 91$ mph, and so the car **does** break the law.

iii. The speed of the car at the starting time ($x = 0$ hours) is $f'(0) = \boxed{91}$ mph. Since the car takes 2 hours to do the trip to San Diego, the speed at the arrival time ($x = 2$ hours) is $f'(2) = 91 - 32 \cdot 2 = 91 - 64 = \boxed{27}$ mph.

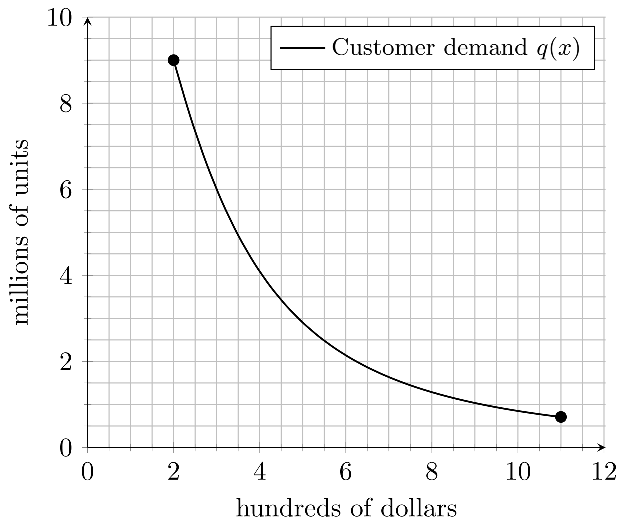

Figure 1: Market demand model

3. (The return of Lemon, Inc. ...) In Homework 2, we saw Lemon Inc.'s market study that estimated the number of units of the piePhone2 that they could expect to sell at a given price point (the "customer demand" graph). Lemon can see that customer demand ($q(x)$ millions of units sold at a price per unit of x hundred dollars) is a continuous function (over the interval $[2, 11]$), since it satisfies the vertical line test and the pen-to-paper test.

   Lemon also knows that their *revenue* from selling $q(x)$ million units at x hundred dollars each is $x \cdot q(x)$ hundred million dollars (Hmn$). That is, their revenue function is:

$$r(x) = x \cdot q(x) \quad \text{(in Hmn\$)}$$

   (a) Lemon wants to know the *rate of change* (i.e. the derivative) of revenue in terms of the rate of change (i.e. the derivative) of demand. Show that we can write the rate of change of revenue as:

$$r'(x) = q(x) \cdot \left(1 + \frac{x}{q(x)} \cdot q'(x)\right)$$

   Remark: The function $E_d(x) = \dfrac{x}{q(x)} \cdot q'(x)$ is called the *"price elasticity of demand"* (Wikipedia article) (that some of you may have seen in your other classes). It is extremely important in economics — it measures how sensitive demand is to changes in price. $E_d(x)$ is almost always a negative real number (i.e. $E_d(x) < 0$). If $E_d(x) = -2$, it means that a 10% increase in price will result in a 20% *decrease* in demand.*

   (b) Lemon hires some pretty solid economists who figure out that the demand function q is given by:

$$q(x) = \frac{90}{x^2 + 6}$$

*Now that you understand derivatives, you know the exact definition of the price elasticity of demand.

i. Calculate $q'(x)$ and $r'(x)$.
ii. Calculate $r'(2)$. Is revenue increasing or decreasing at a price point of \$200 per unit?
iii. Calculate $r'(6)$. Is revenue increasing or decreasing at a price point of \$600 per unit?
iv. What is $E_d(6)$? At a price point of \$600 per unit, how much (in %) will demand increase or decrease if the price increases by 7%?

Solution:

(a) Lemon's revenue function is $r(x) = x \cdot q(x)$. Using the rules for derivatives, the rate of change of revenue is

$$
\begin{aligned}
r'(x) &= \frac{d}{dx}(x \cdot q(x)) \\
&= \left(\frac{d}{dx}(x)\right) \cdot q(x) + x \cdot q'(x) && \text{(using the product rule)} \\
&= q(x) + x \cdot q'(x) \\
&= \boxed{q(x) \cdot \left(1 + \frac{x}{q(x)} \cdot q'(x)\right)}
\end{aligned}
$$

(b) i. We can use the rules for derivatives to calculate

$$
\begin{aligned}
q'(x) &= \frac{d}{dx}\left(\frac{90}{x^2+6}\right) \\
&= \frac{\left(\frac{d}{dx}(90)\right)(x^2+6) - 90 \cdot \frac{d}{dx}(x^2+6)}{(x^2+6)^2} && \text{(using the quotient rule)} \\
&= \frac{0 \cdot (x^2+6) - 90 \cdot (2x+0)}{(x^2+6)^2} \\
&= \boxed{-\frac{180x}{(x^2+6)^2}}
\end{aligned}
$$

Now that we know $q(x)$ and $q'(x)$, we can calculate $r'(x)$ as:

$$
\begin{aligned}
r'(x) &= q(x) + x \cdot q'(x) \\
&= \frac{90}{x^2+6} + x \cdot \left(-\frac{180x}{(x^2+6)^2}\right) \\
&= \frac{90}{x^2+6} - \frac{180x^2}{(x^2+6)^2} \\
&= \frac{90(x^2+6) - 180x^2}{(x^2+6)^2} \\
&= \frac{90(x^2+6-2x^2)}{(x^2+6)^2} \\
&= \boxed{\frac{90(6-x^2)}{(x^2+6)^2}}
\end{aligned}
$$

ii. We have

$$
r'(2) = \frac{90(6-(2)^2)}{((2)^2+6)^2} = \frac{90(6-4)}{(4+6)^2} = \frac{180}{100} = \boxed{1.8}
$$

Since $r'(2) > 0$, Lemon's revenue (the function r) is $\boxed{\text{increasing}}$ at a price point of \$200 per unit.

iii. We have

$$r'(6) = \frac{90(6-(6)^2)}{((6)^2+6)^2} = \frac{90(-30)}{(42^2)} = \boxed{-\frac{75}{49}}$$

Since $r'(6) < 0$, Lemon's revenue is $\boxed{\text{decreasing}}$ at a price point of \$600 per unit.

iv. We can calculate the price elasticity of demand at a price point of x hundred dollars per unit (i.e. the function $E_d(x)$) as follows.

$$\begin{aligned}
E_d(x) &= \frac{x}{q(x)} \cdot q'(x) \\
&= x \cdot \frac{(x^2+6)}{90} \cdot \left(-\frac{180x}{(x^2+6)^2}\right) \\
&= x \cdot \left(-\frac{2x}{x^2+6}\right) \\
&= -\frac{2x^2}{x^2+6}
\end{aligned}$$

Hence we have

$$E_d(6) = -\frac{2(6)^2}{(6)^2+6} = -\frac{2\cdot 6\cdot 6}{6\cdot 7} = \boxed{-\frac{12}{7}}$$

So, at a price point of \$600 per unit, if the price increases by 7%, the demand will change by

$$E_d(6)\cdot 7 = \left(-\frac{12}{7}\right)\cdot 7 = -12\%$$

In other words, the demand will $\boxed{\text{decrease by } 12\%}$.