Math 130 04 – A Survey of Calculus

Homework assignment 8

Due: Tuesday, December 6, 2022

Remember: If f is a continuous real function, the *Riemann integral* of f is the function

$$\left(\int f\right)(x) = \int_0^x f(x)\cdot dx = \lim_{h\to 0} S_h f(x)$$

Remember: The *fundamental theorems of calculus* are the following facts.

FTC1 If f is a differentiable real function, then

$$\boxed{\left(\int f'\right)(x) = \int_0^x f'(x)\cdot dx \ = \ f(x) - f(0)}$$

FTC2 If f is a continuous real function, then

$$\boxed{\left(\int f\right)'(x) = f(x)}$$

Remember:

- If $f(x) = x^a$ where a is any constant real number such that $a \neq -1$, then

$$\left(\int f\right)(x) = \int_0^x x^a \cdot dx \ = \ \frac{x^{a+1}}{a+1}$$

- If $f(x) = e^{kx}$ where k is any constant real number such that $k \neq 0$, then

$$\left(\int f\right)(x) = \int_0^x e^{kx} \cdot dx \ = \ \frac{e^{kx}-1}{k}$$

Rules for Riemann integrals

- **Constant rule:** If $f(x) = c\cdot g(x)$, where c is any constant real number, then

$$\left(\int f\right)(x) = c\cdot\left(\int g\right)(x)$$

or using alternate notation,

$$\int_0^x c\cdot g(x)\cdot dx = c\cdot \int_0^x g(x)\cdot dx$$

- **Sum rule:** If f and g are continuous real functions, then

$$\left(\int (f+g)\right)(x) = \left(\int f\right)(x) + \left(\int g\right)(x)$$

or using alternate notation,

$$\int_0^x (f(x)+g(x))\cdot dx = \left(\int_0^x f(x)\cdot dx\right) + \left(\int_0^x g(x)\cdot dx\right)$$

1. Calculate the Riemann integrals of the following functions.
   (a) $f(x) = x^2 + 3x + 4$
   (b) $f(x) = 8e^{-2x} + 4$
   (c) $f(x) = 4x^{-1/3}$
   (d) $f(x) = 3x(4 + x^{2/5})$

Solution:

(a) We can use the rules for Riemann integrals to find

$$\begin{aligned}\int_0^x (x^2 + 3x + 4) \cdot dx &= \left(\int_0^x x^2 \cdot dx\right) + 3 \cdot \left(\int_0^x x \cdot dx\right) + 4 \cdot \left(\int_0^x 1 \cdot dx\right) \\ &= \boxed{\frac{x^3}{3} + 3\frac{x^2}{2} + 4x}\end{aligned}$$

(b) We can use the rules for Riemann integrals to find

$$\begin{aligned}\int_0^x (8e^{-2x} + 4) \cdot dx &= 8 \cdot \left(\int_0^x e^{-2x} \cdot dx\right) + 4 \cdot \left(\int_0^x 1 \cdot dx\right) \\ &= 8\frac{(e^{-2x} - 1)}{-2} + 4x \\ &= \boxed{4(1 - e^{-2x}) + 4x}\end{aligned}$$

(c) We can use the rules for Riemann integrals to find

$$\begin{aligned}\int_0^x (4x^{-1/3}) \cdot dx &= 4 \cdot \left(\int_0^x x^{-1/3} \cdot dx\right) \\ &= 4 \cdot \frac{x^{2/3}}{\frac{2}{3}} \\ &= 4 \cdot \frac{3}{2} \cdot x^{2/3} \\ &= \boxed{6x^{2/3}}\end{aligned}$$

(d) We can first simplify the function f as

$$f(x) = 3x(4 + x^{2/5}) = 12x + 3x^{7/5}$$

Next, we can use the rules for Riemann integrals to find

$$\begin{aligned}\int_0^x (12x + 3x^{7/5}) \cdot dx &= 12 \cdot \left(\int_0^x x \cdot dx\right) + 3 \cdot \left(\int_0^x x^{7/5} \cdot dx\right) \\ &= 12 \cdot \frac{x^2}{2} + 3 \cdot \frac{x^{12/5}}{\frac{12}{5}} \\ &= 6x^2 + 3 \cdot \frac{5}{12} \cdot x^{12/5} \\ &= \boxed{6x^2 + \frac{5}{4}x^{12/5}}\end{aligned}$$

Recall: The *marginal cost function* is the derivative of the total cost function. Similarly, the marginal revenue and marginal profit functions are the derivatives of the total revenue and total profit functions respectively.

2. A company calculates its marginal cost function C' as follows: If x thousand units have been produced, the marginal cost (i.e. the cost to produce the next unit) is

$$C'(x) = 2x^{-1/3} \qquad \text{dollars per unit.}$$

   (a) Find the company's total cost function C (i.e. $C(x)$ thousands of dollars to produce x thousand units) if the fixed cost to produce 0 units is 3 thousand dollars (i.e. $C(0) = 3$).

   (b) Suppose the company's marginal revenue function is as follows: If x thousand units have been sold, the marginal revenue (i.e. the revenue from selling the next unit) is

$$R'(x) = 3x^{-1/2} \qquad \text{dollars per unit.}$$

      i. Find the marginal profit function P'. (Remember that the total profit function P is defined as $P(x) = R(x) - C(x)$, where R and C are the total revenue and total cost functions.)

      ii. Does the total profit function P have a maximum in the interval $[0, 20]$? If so, find the value $a \in [0, 20]$ such that P has a maximum at a.

      iii. Calculate the total profit function P, assuming that the revenue from selling 0 units is 0 dollars (i.e. $R(0) = 0$).

Solution:

(a) By the FTC1, we know that

$$\left(\int C'\right)(x) = C(x) - C(0)$$

Since we know that $C(0) = 3$, we can find the total cost as

$$\begin{aligned} C(x) &= \left(\int C'(x) \cdot dx\right) + C(0) \\ &= \left(\int (2x^{-1/3}) \cdot dx\right) + 3 \\ &= 2 \cdot \frac{x^{2/3}}{\frac{2}{3}} + 3 \\ &= 2 \cdot \frac{3}{2} \cdot x^{2/3} + 3 \\ &= 3x^{2/3} + 3 \end{aligned}$$

(b) i. Since the total profit function is $P(x) = R(x) - C(x)$, the marginal profit function is

$$\begin{aligned} P'(x) &= \frac{d}{dx}\left(R(x) - C(x)\right) \\ &= R'(x) - C'(x) \\ &= 3x^{-1/2} - 2x^{-1/3} \end{aligned}$$

ii. Since any maximum a of the function P has to be a critical point (i.e. $P'(a) = 0$), we first solve the equation $P'(x) = 0$ to look for critical points in the interval $[0, 20]$.

$$\begin{aligned}
& P'(x) = 0 \\
\text{i.e.} \quad & R'(x) = C'(x) \\
\text{i.e.} \quad & 3x^{-1/2} = 2x^{-1/3} \\
\text{i.e.} \quad & \frac{3}{x^{1/2}} = \frac{2}{x^{1/3}} \\
\text{i.e.} \quad & 2 \cdot \frac{x^{1/2}}{x^{1/3}} = 3 \\
\text{i.e.} \quad & 2x^{1/2-1/3} = 3 \\
\text{i.e.} \quad & 2x^{1/6} = 3 \\
\text{i.e.} \quad & x = \left(\frac{3}{2}\right)^6 \\
\text{i.e.} \quad & x \approx 11.39
\end{aligned}$$

Therefore P has a critical point at $x = 11.39$, which is in the interval $[0, 20]$. To check whether 11.39 is a local maximum, we use the second derivative test, i.e. we have to check that $P''(11.39) < 0$.

We have

$$\begin{aligned}
P''(x) &= \frac{d}{dx}\left(3x^{-1/2} - 2x^{-1/3}\right) \\
&= -\frac{3}{2}x^{-3/2} - \left(-\frac{2}{3}x^{-4/3}\right) \\
&= \frac{2}{3}x^{-4/3} - \frac{3}{2}x^{-3/2}
\end{aligned}$$

and so we calculate

$$P''(11.39) = \frac{2}{3}(11.39)^{-4/3} - \frac{3}{2}(11.39)^{-3/2} = -0.013$$

Therefore $P''(11.39) < 0$, and so $\boxed{P \text{ has a maximum at } 11.39}$.

iii. Since $R(0) = 0$, we can calculate $P(0) = R(0) - C(0) = -3$. Next, by the FTC1, we know that

$$\left(\int P'\right)(x) = P(x) - P(0)$$

Therefore, we have

$$P(x) = \left(\int_0^x (R'(x) - C'(x)) \cdot dx\right) + P(0)$$

Recall: If f is a continuous function, the value of the Riemann integral of f at x, i.e. the real number $(\int f)(x)$, is the area "under" the graph of f between the points 0 and x on the horizontal axis.

3. Consider the following definition.

$$f(x) = \begin{cases} x^5 + 2x^2 - 2 & \text{if } x \leq 0 \\ \\ x^3 + 4x - 2 & \text{if } x > 0 \end{cases}$$

(a) Is the function f continuous? Namely, is f continuous at every real number x?

(b) What is the value of $(\int f)(x)$ if $x \leq 0$?

(c) What is the value of $(\int f)(x)$ if $x > 0$?

Solution:

(a) When $x < 0$, $f(x) = x^5 + 2x^2 - 2$ is a polynomial, so f is continuous at every real number strictly less than 0.

When $x > 0$, $f(x) = x^3 + 4x - 2$ is a polynomial, so f is continuous at every real number strictly greater than 0.

When $x = 0$, the left hand limit $\lim\limits_{x \to 0^-} f(x)$ is

$$\lim_{x \to 0^-} x^5 + 2x^2 - 2 = (0)^5 + 2(0)^2 - 2 = -2$$

and the right hand limit $\lim\limits_{x \to 0^+} f(x)$ is

$$\lim_{x \to 0^+} (0)^3 + 4(0) - 2 = -2$$

Finally $f(0) = (0)^5 + 2(0)^2 - 2 = -2$. Therefore, since

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = -2$$

f is continuous at 0.

Therefore, since f is continuous at every $x < 0$, at every $x > 0$ and at $x = 0$, $\boxed{f \text{ is continuous at all real numbers}}$.

(b) If $f(x) \leq 0$, then $f(x) = x^5 + 2x^2 - 2$, so

$$\int_0^x f(x) \cdot dx = \int_0^x (x^5 + 2x^2 - 2) \cdot dx$$

$$\left(\int f\right)(x) = \boxed{\frac{x^6}{6} + 2\frac{x^3}{3} - 2x}$$

(c) If $f(x) > 0$, then $f(x) = x^3 + 4x - 2$, so

$$\int_0^x f(x) \cdot dx = \int_0^x (x^3 + 4x - 2) \cdot dx$$

$$\left(\int f\right)(x) = \boxed{\frac{x^4}{4} + 4\frac{x^2}{2} - 2x}$$