Ramsey theory

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Lecture notes:

www.chaitanyals.site/teaching/Ramsey-lecture.pdf

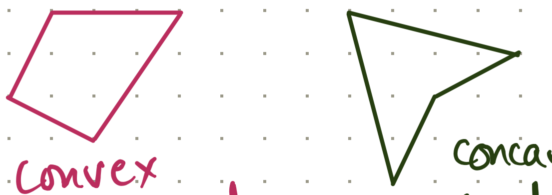
What is Ramsey Theory?

"Any large enough collection of objects has regular patterns in it"

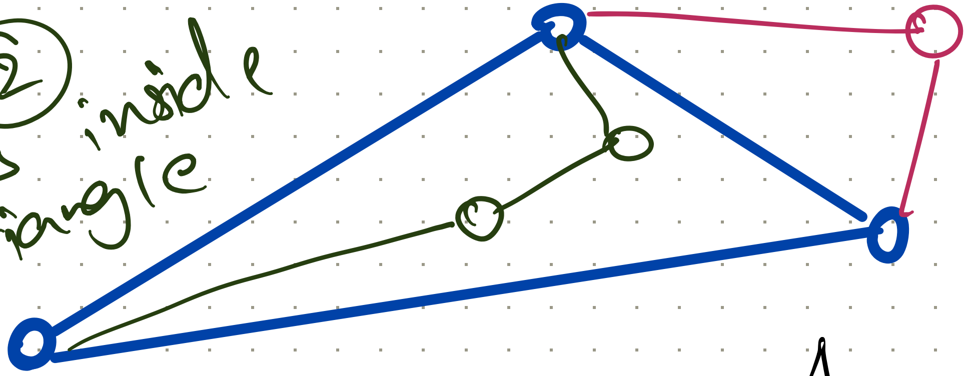
"Complete disorder is impossible."

Example: ("Big Dipper theorem").

Choose any 5 points in the plane, such that no 3 of them lie on the same line. Then 4 of them always form a convex quadrilateral.



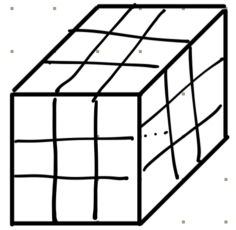
Proof: By contradiction. Assume the thm is false.



But then wherever we place the other two points, we have a convex quadrilateral

Example: (3D Tic-tac-toe)

A game of 3D tic-tac-toe (played on a $3 \times 3 \times 3$ grid) can never end in a tie, no matter how poorly the players play.



Example: Any way of coloring the numbers 1 through 9 in red or blue must contain a monochromatic arithmetic progression of length 3.

1 2 3 4 5 6 7 8 9

Party-goers problem: How many randomly chosen people do you need to invite to a party to guarantee that

(1) there are 3 people who all know each other

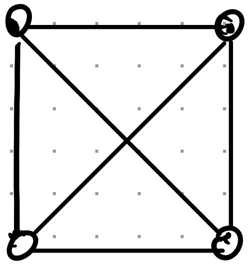
OR

(2) there are 3 people who are complete strangers to each other.

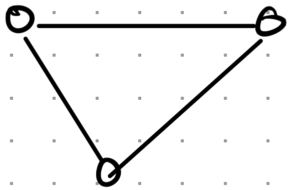
Modelling the problem

Def: The *complete graph* on n vertices is the graph with n vertices such that every pair of vertices is connected by an edge. We call this graph K_n

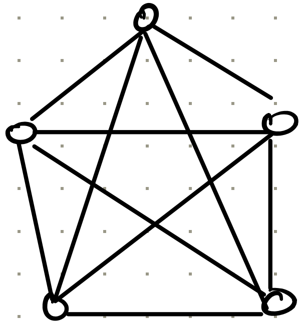
e.g.



K_4

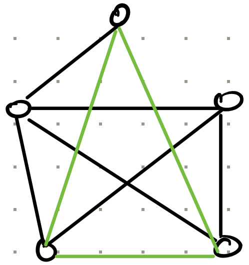


K_3



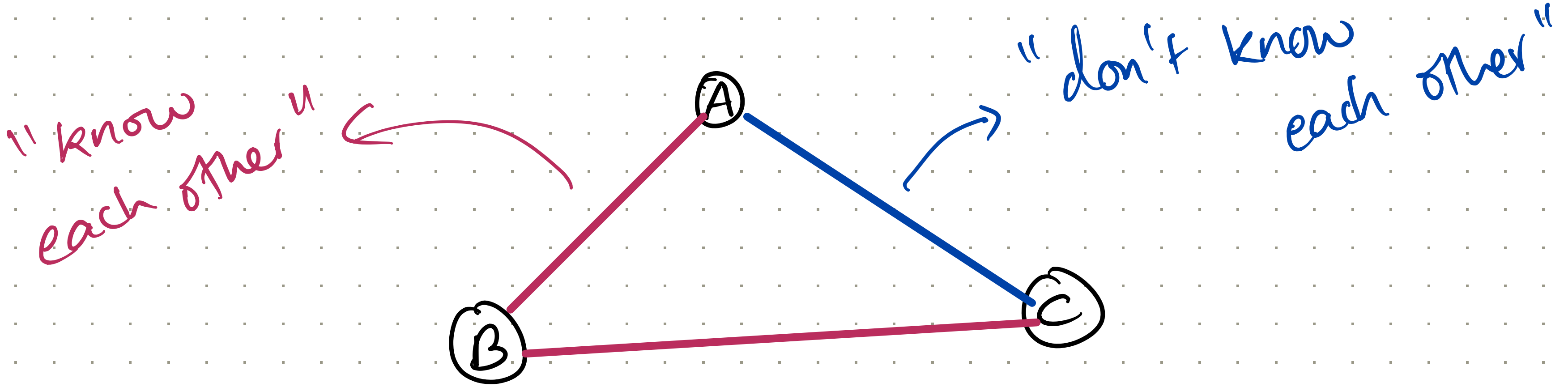
K_5

Def: An *m-clique* inside a graph G is a subgraph of G that is of the form K_m



(a 3-clique inside K_5)

Suppose we have 3 invitees: $\underbrace{A, B \text{ and } C}_{\text{vertices of our graph}}$

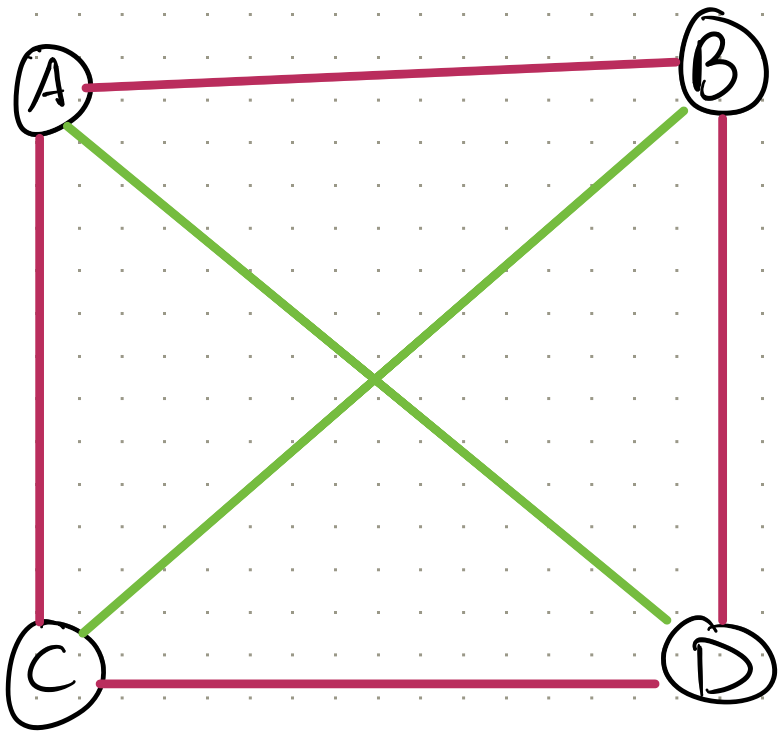


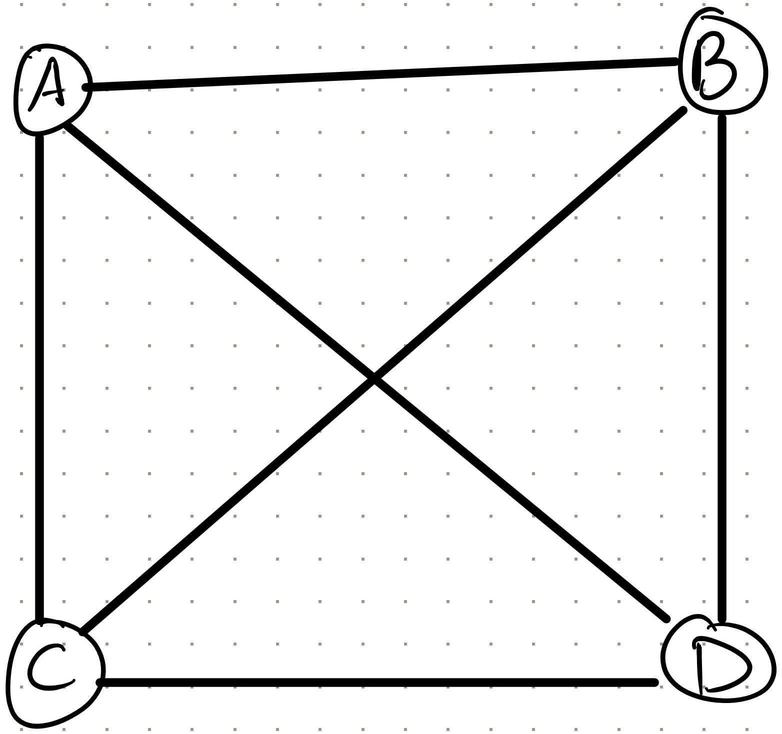
Is it possible to find a 2-coloring of K_3 that does <u>not</u> contain a monochromatic 3-clique?

In-class activity

Is it possible to find a 2-coloring of K_4 that does <u>not</u> contain a monochromatic 3-clique?

<u>Yes</u>.

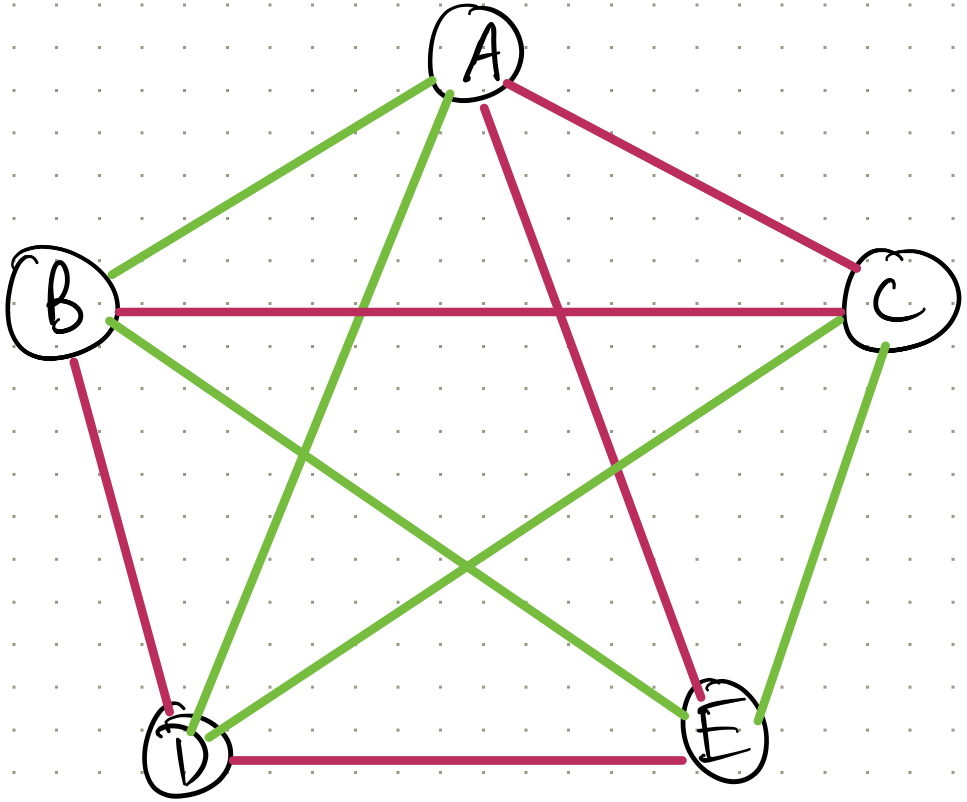




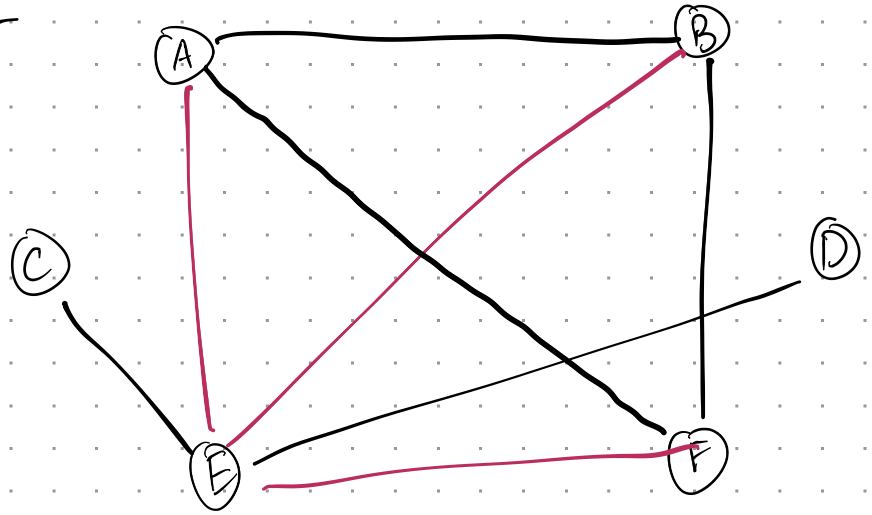
In-class activity

Is it possible to find a 2-coloring of K_5 that does not contain a monochromatic 3-clique?

Yes.



Q: What about K_6? **Ans**: No.



Proof. By contradiction, assume that some 2-coloring of K_6 has no blue or red triangle.

By the pigeonhole principle at least 3 of the 5 edges leaving E must be the same color (say, red). Then consider the triangle formed by those 3 vertices. However we color this triangle, we end up with a contradiction.

So we say that $R(3,3) = 6$

i.e. " The minimum number of people you need in a room to ensure that either

1. there are 3 people who all know each other

OR

2. there are 3 people who are complete strangers to each other

is 6. "

What about $R(k, l)$ for any positive numbers k, l?

i.e. " The minimum number of people you need in a room to ensure that <u>either</u>

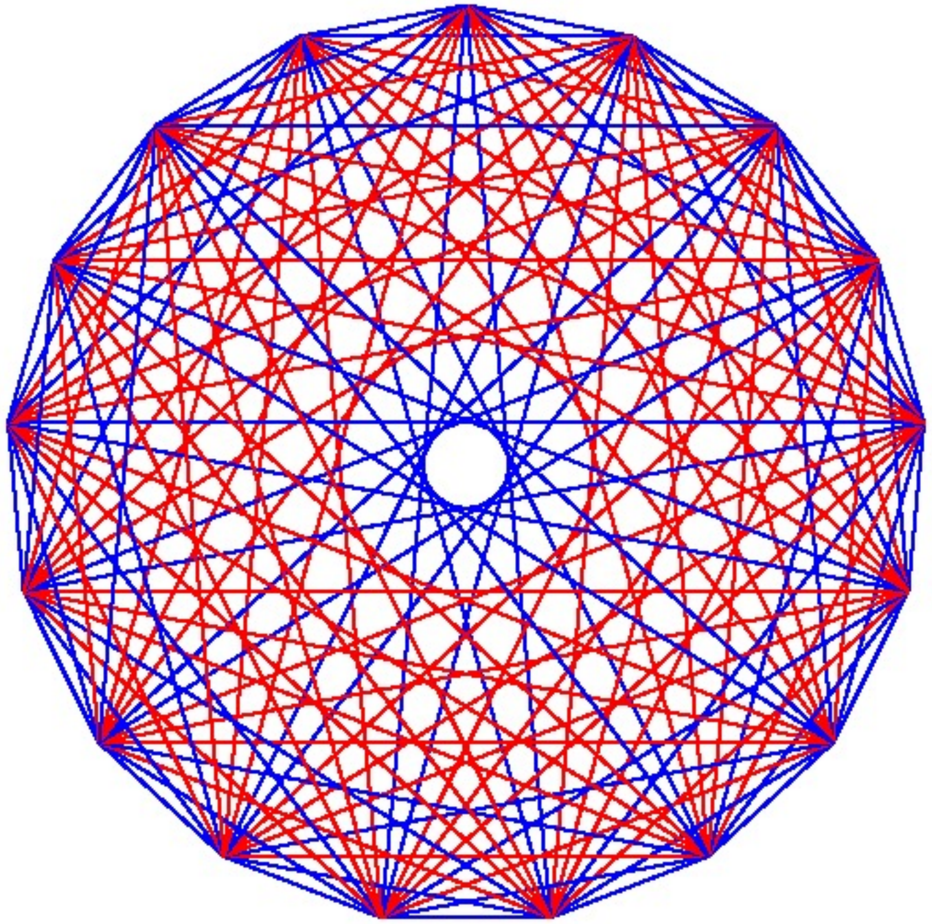
(1) there are k people who all know each other

<u>OR</u>

(2) there are l people who are complete strangers to each other "

is ?

$R(4,4) = 18$



$R(5,5) = ?$

$43 \leq R(5,5) \leq 46$

Theorem: $R(k, l)$ is finite.

Proof: (1) We've seen that $R(k, l)$ is finite when $k, l \leq 3$

(2) Mathematical induction:

Hyp) We assume that $R(m, n)$ is finite whenever $(m < k)$ or $(m = k$ and $n < l)$

and we show that $R(k, l) \leq R(k-1, l) + R(k, l-1)$

under this new hypothesis (Hyp).

Consider the complete graph G with $R(k-1, l) + R(k, l-1)$ vertices. Assume an arbitrary 2-coloring of G.

Pick some vertex x of G.

The remaining vertices of G are divided ("partitioned") into two sets:

$$M = \{ y \in G \mid (x,y) \text{ is blue} \}$$

$$N = \{ y \in G \mid (x,y) \text{ is red} \}$$

Let $|M|$ be the number of elements of M

Let $|N|$ ——— " ——— N

So $|M| + |N| + 1 = R(k-1, l) + R(k, l-1)$

So either $|M|$ or $|N|$ must be $\geq$

either $R(k-1, l)$ or $R(k, l-1)$

Suppose $|M| \geq R(k-1, l)$.

Then by (Hyp) the complete subgraph on M either has a red l-clique or a blue $(k-1)$-clique

If it has a red l-clique, we're done.

If it has a blue $(k-1)$-clique $K_{(k-1)}$, then by the definition of M, $K_{(k-1)} + \{x\}$ is a blue k-clique of G. $\square$