Math 130 04 – A Survey of Calculus

Homework assignment 1

Due: Tuesday, September 6, 2022

Write your answers on a separate sheet of paper. Write your name at the top of each page you use, and number each page. Remember to number your answers correctly.

General questions

1. What is your major? If you haven't decided yet, what are your options?
2. What other courses are you taking this semester?
3. What do you expect to get from learning calculus?

Algebra

4. Write the following expressions without using exponents.

   (a) 2^4
   (b) x^3
   (c) $x^3 \cdot 3^{-2}$
   (d) $(x^2)^{-2}$
   (e) $x^2 - 1$
   (f) $(3x)^0$

Solution:

(a) $2 \cdot 2 \cdot 2 \cdot 2 = 16$
(b) $x \cdot x \cdot x$
(c) $\frac{x \cdot x \cdot x}{3 \cdot 3}$
(d) $\frac{1}{x \cdot x \cdot x \cdot x}$
(e) $x \cdot x - 1 = (x+1)(x-1)$
(f) 1

5. Multiply the following expressions (and simplify them until no factors remain).

   (a) $(3x - y)(3x + y)$
   (b) $(x - a)^3$
   (c) $xy(x + y)(x - y)$

Solution:

(a) $9x^2 - y^2$
(b) $x^3 - 3x^2a + 3xa^2 - a^3$
(c) $x^3y - xy^3$

6. Factor the following expressions.

(a) $2xy^2 - 50x$ (b) $6x^2 - 23x + 20$

Solution:

(a)

$$\begin{aligned} 2xy^2 - 50x &= 2x(y^2 - 25) \\ &= 2x(y+5)(y-5) \end{aligned}$$

(b)

$$\begin{aligned} 6x^2 - 23x + 20 &= 6x^2 - 15x - 8x + 20 \\ &= (3x-4)(2x-5) \end{aligned}$$

Sets and functions

7. Consider the following sets.

$$A = \{1, 3, 5, 7, 9\} \quad B = \{2, 4, 6, 8\} \quad C = \{1, 2, 3, 7, 8, 9\}$$

(a) For each of the following, say whether it is an element of A, B or C.
   i. 6
   ii. 7
   iii. Anees
   iv. 5

(b) Calculate:
   i. $A \cup B$ (The union of A and B.)
   ii. $A \cap B$ (The intersection of A and B.)
   iii. $A \cup B \cup C$
   iv. $A \cap B \cap C$
   v. $A \cup (B \cap C)$

Solution:

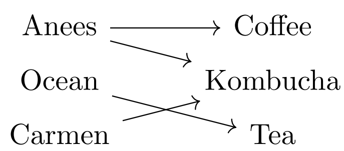
(a) i. $6 \in B$
   ii. $7 \in A$ and $7 \in C$.
   iii. Anees is not an element of A, B or C.
   iv. $5 \in A$.

(b) i. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
   ii. $A \cap B = \emptyset$ (the empty set).

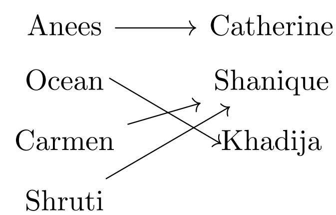
iii. $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
iv. $A \cap B \cap C = \emptyset$
v. $A \cup (B \cap C) = A \cup \{2, 8\} = \{1, 2, 3, 5, 7, 8, 9\}$

8. Write the domain and the range of each of the following correspondences. Which of the following correspondences are functions? Justify your answers.

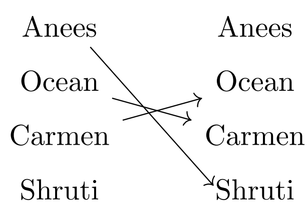
(a)



(c)



(b)



Solution:

(a)

Domain = {Anees, Ocean, Carmen} *Range* = {Coffee, Kombucha, Tea}

The correspondence **is not** a function, because the element "Anees" of the domain maps to more than one element of the range ("Coffee" and "Kombucha").

(b)

Domain = {Anees, Ocean, Carmen, Shruti} *Range* = {Anees, Ocean, Carmen, Shruti}

The correspondence **is not** a function, because the element "Shruti" of the domain does not map to any element of the range.

(c)

Domain = {Anees, Ocean, Carmen, Shruti} *Range* = {Catherine, Shanique, Khadija}

The correspondence **is** a function, because every element of the domain maps to exactly one element of the range.

Math 130 04 – A Survey of Calculus

Homework assignment 2

Due: Tuesday, September 13, 2022

Write your answers on a separate sheet of paper. Write your name at the top of each page you use, and number each page. Remember to number your answers correctly.

Sets and functions

1. Consider the sets

$$A = \{1, 3, 5, 7, 9\} \quad B = \{2, 4, 6, 8\} \quad C = \{1, 2, 3, 7, 8, 9\}.$$

What are the following sets? Justify your answers.

(a) $A \cap B$ (The intersection of A and B.)

(b) $A \cap B \cap C$

(c) $A \cup (B \cap C)$

(d) $(A \cup B) \cap C$

Solution:

(a) $A \cap B = \emptyset$

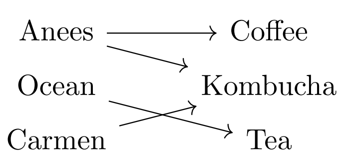
(b) $A \cap B \cap C = \emptyset$

(c) $A \cup (B \cap C) = A \cup \{2, 8\} = \{1, 2, 3, 5, 7, 8, 9\}$

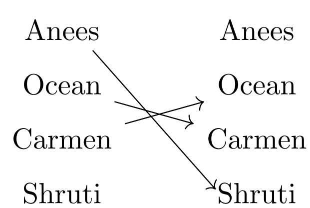
(d) $(A \cup B) \cap C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \cap C = \{1, 2, 3, 7, 8, 9\}$

2. Write the domain and the range of each of the following correspondences. Which of the following correspondences are functions? Justify your answers.

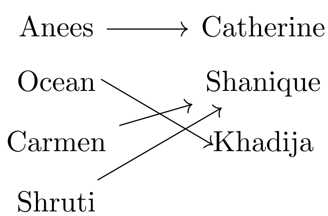
(a)



(b)



(c)



Solution:

(a)

$$\textit{Domain}: \{\text{Anees, Ocean, Carmen}\} \qquad \textit{Range}: \{\text{Coffee, Kombucha, Tea}\}$$

The correspondence **is not** a function, because the element "Anees" of the domain maps to more than one element of the range ("Coffee" and "Kombucha").

(b)

$$\textit{Domain}: \{\text{Anees, Ocean, Carmen, Shruti}\} \qquad \textit{Range}: \{\text{Anees, Ocean, Carmen, Shruti}\}$$

The correspondence **is not** a function, because the element "Shruti" of the domain does not map to any element of the range.

(c)

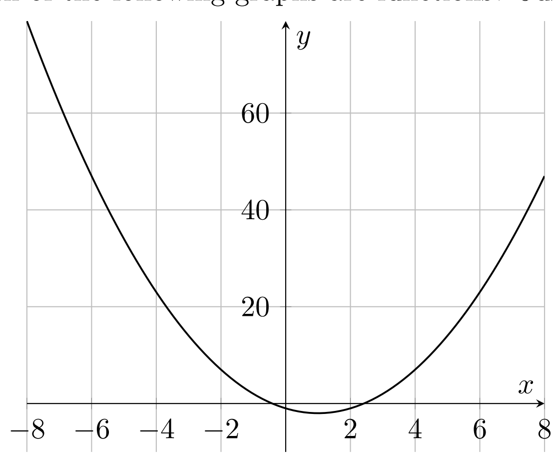
$$\textit{Domain}: \{\text{Anees, Ocean, Carmen, Shruti}\} \qquad \textit{Range}: \{\text{Catherine, Shanique, Khadija}\}$$

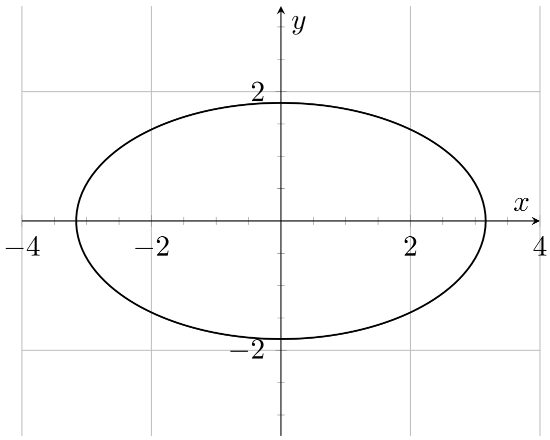
The correspondence **is** a function, because every element of the domain maps to exactly one element of the range.

Graphs

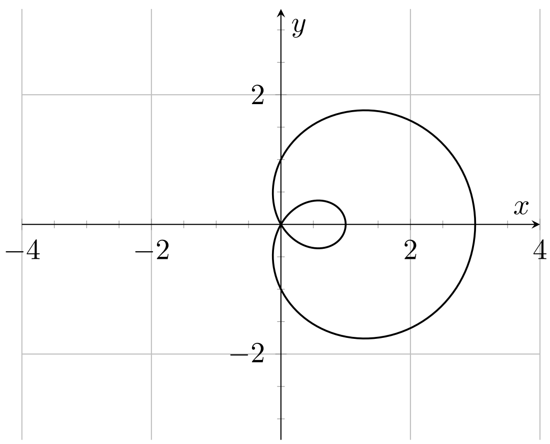
3. Which of the following graphs are functions? Justify your answers.

(a)





(b)

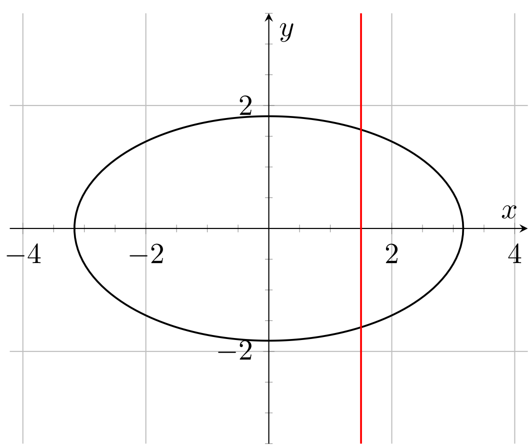


(c)

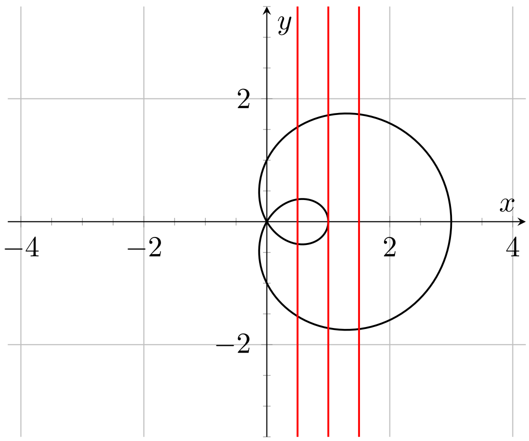
Solution:

(a) The graph is a function, since it satisfies the vertical line test. That is, every vertical line passes through exactly one point of the graph.

(b) The graph is not a function, since it fails the vertical line test. There are vertical lines that pass through two points of the graph, like the red line below.



(c) The graph is not a function, since it fails the vertical line test. There are vertical lines that pass through two, three, and four points of the graph, like the red lines below.



4. A new tech manufacturer, Lemon Inc., released a phone (the piePhone) in 2022 at a price point of \$500. The piePhone was a success, and by September, the 2 million units of the piePhone produced in 2022 had sold out. In 2023, Lemon plans to release the piePhone2 (a minor upgrade to the piePhone). To decide how to price the piePhone2, Lemon has studied the market.

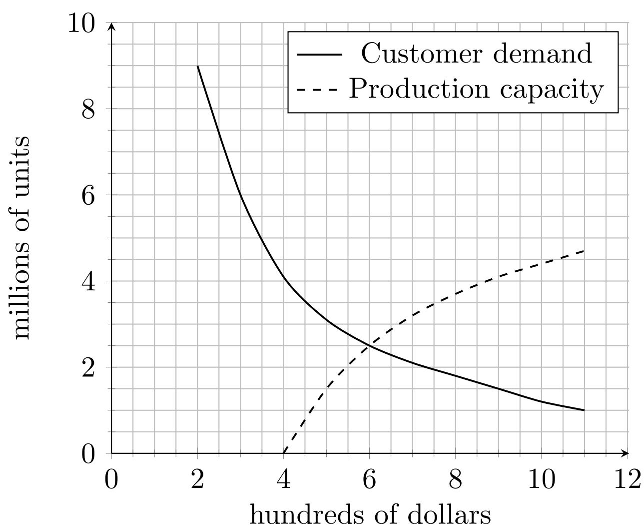


Figure 1: Market study

The result of the study (the "Customer demand" graph above) shows the number of units of the piePhone2 that customers will buy at a given price point. For example, if the piePhone2 is priced at \$300, then Lemon can expect to sell 6 million units, but if it is priced at \$900, then only 1.5 million units will sell. On top of the market study, Lemon has drawn a graph ("Production capacity") showing how many units they can produce at a given price point per unit. For instance, if the piePhone2 is priced at less than \$400, then Lemon can't afford to produce it, but if it is priced at \$1000, then Lemon can produce about 4.5 million units in 2023.

(a) Suppose Lemon produces 2 million units of the piePhone2 in 2023.

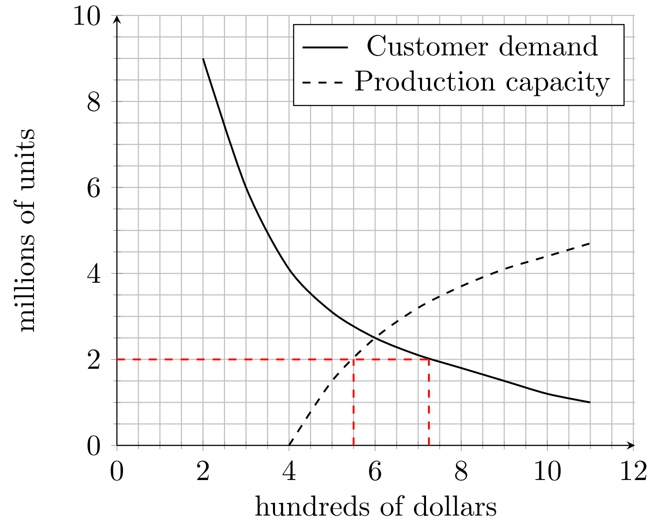
i. What is the lowest price that Lemon can afford to sell the piePhone2 at?
ii. How many units can Lemon expect to sell?
iii. What will the final price per unit be?
iv. What will Lemon's revenue (number of units sold × price per unit) be?

(b) Suppose Lemon produces 4 million units of the piePhone2 in 2023.
i. What is the lowest price that Lemon can afford to sell the piePhone2 at?
ii. How many units can Lemon expect to sell?
iii. What will the final price per unit be?
iv. What will Lemon's revenue (number of units sold × price per unit) be?

(c) How many units of the piePhone2 should Lemon produce in 2023 so that every unit will sell? What will the final price per unit be? What will Lemon's revenue be in 2023?

Solution:

(a) We begin by sketching this situation on the graph. Since Lemon produces 2 million units, we draw a horizontal line passing through the y axis at 2.



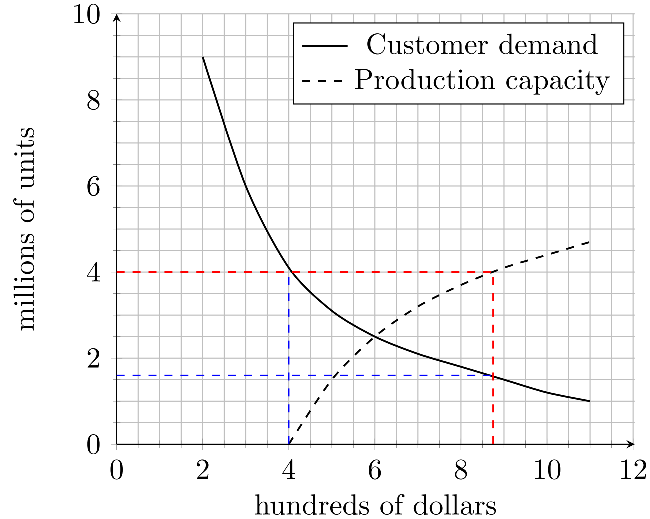
i. Since the horizontal line passes through the "Production capacity" graph at the point $(x = 5.5, y = 2)$, we see that Lemon can afford to produce 2 million units at a minimum price of \$550 per unit.

ii. Since the horizontal line passes through the "Customer demand" graph at $(x = 7.25, y = 2)$, we see that customers will buy 2 million units of the piePhone2 at a price point of \$725. Since this is *higher* than the lowest price at which Lemon can afford to produce 2 million units of the piePhone2, Lemon will sell *all* 2 *million units* of the piePhone2.
Remark: This situation is called a **shortage** in economics.

iii. Since (in this situation) customers are willing to pay \$725 per unit of the piePhone2, the final price per unit will be \$725.

iv. Lemon's revenue will be \$725 × 2 million = 1.45 billion dollars (1450 million dollars).

(b) We begin by sketching this situation on the graph. Since Lemon produces 4 million units, we draw a horizontal line passing through the y axis at 4.



i. Since the horizontal line at $y = 4$ passes through the "Production capacity" graph at the point $(x = 8.75, y = 4)$, we see that Lemon can afford to produce 4 million units at a minimum price of \$875 per unit.

ii. There are two possible outcomes:

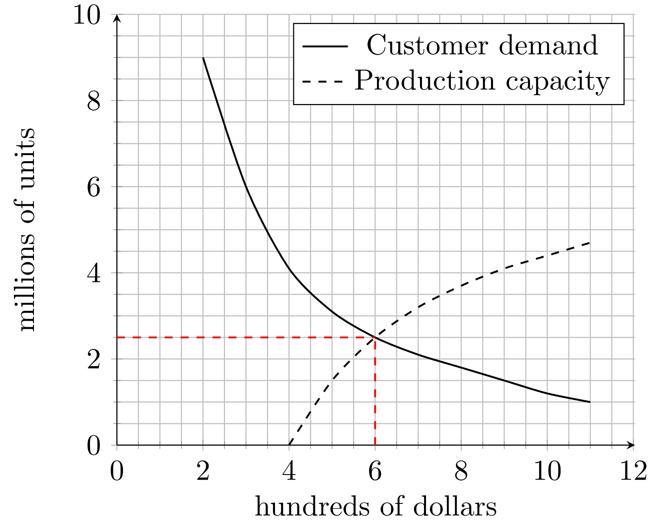
- Since Lemon cannot afford to sell the piePhone2 at less than \$875 per unit, Lemon will sell as many units as customers are willing to buy at this price. Since the vertical line at $x = 8.75$ passes through the "Customer demand" graph at the point $(x = 8.75, y = 1.6)$, in this scenario Lemon will sell 1.6 million units, and hold on to $(4 - 1.6) = 2.4$ million units.
- Alternatively, since the horizontal line at $y = 4$ passes though the "Customer demand" graph at the point $(x = 4, y = 4)$, Lemon can choose to sell all 4 million units at a price of \$400 per unit.

Remark: This situation is called a **surplus** in economics. Lemon will likely choose the first outcome, since it allows them to sell the unsold units (the surplus) later at a discounted price.

iii. If Lemon chooses to sell 1.6 million units, the final price will be \$875 per unit. If they choose to sell all 4 million units, the final price will be \$400 per unit.

iv. If 1.6 million units are sold for \$875 each, Lemon's revenue will be $\$875 \times 1.6$ million $=$ 1.4 billion dollars (1400 million dollars). If 4 million units are sold for \$400 each, Lemon's revenue will be $\$400 \times 4$ million $=$ 1.6 billion dollars (1600 million dollars).

(c) In the ideal situation, there will be no surplus and no shortage. This is at the point where the "Customer demand" graph meets the "Production capacity" graph, i.e. the point $(x = 6, y = 2.5)$. This means that Lemon can produce at most 2.5 million units if they want to sell every unit.



In this situation, the price at which customers will buy the piePhone2 is exactly the price at which Lemon can afford to sell it, namely \$600. Therefore the final price will be \$600.

In this situation, Lemon's revenue will be $\$600 \times 2.5$ million $= 1.5$ billion dollars (1500 million dollars).

Math 130 04 – A Survey of Calculus

Homework assignment 3

Due: Tuesday, September 20, 2022

Instructions: Write your answers on a separate sheet of paper. Write your name at the top of each page you use, and number each page. Number your answers correctly.
Justify all your answers.

1. Consider the following definition.

$$f(x) = \begin{cases} \frac{1}{x^2+2x} & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

   (a) Does this define a real function? If so, what is its domain? Justify your answer.

   (b) Do the following limits exist? If so, what are they? Justify your answers.

      i. $\lim\limits_{x \to -2^-} f(x)$ (the left-hand limit of f at -2.)

      ii. $\lim\limits_{x \to -1} f(x)$ (the limit of f at -1.)

      iii. $\lim\limits_{x \to 0^+} f(x)$ (the right-hand limit of f at 0.)

      iv. $\lim\limits_{x \to 0} f(x)$

Solution:

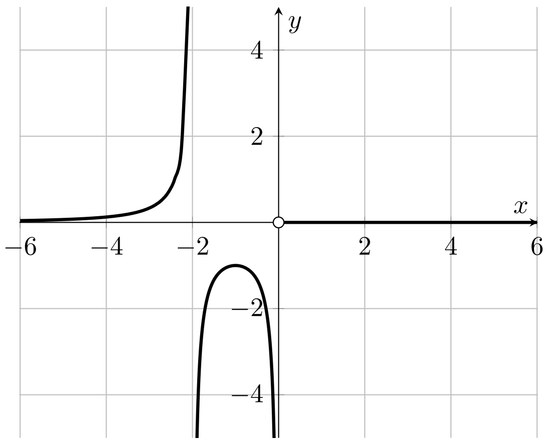
(a) For any real number x, *if* $f(x)$ is defined, then it has a unique value (passes the vertical line test). So f is a real function.

We observe that

- When $x > 0$: $f(x)$ is defined and equals 0.
- When $x = 0$: $f(0)$ is not defined.
- When $x < 0$: $f(x)$ is defined *except* if $x^2 + 2x = 0$. Solving this equation, we find that $f(x)$ is defined whenever $x \neq -2$.

Since $f(x)$ is defined for all real numbers except for $x = 0$ and $x = -2$, the domain of f is $(-\infty, -2) \cup (-2, 0) \cup (0, \infty)$.

We can graph f (for instance, in GeoGebra using the syntax `If(x>0,0,If(x<0,1/(x^2 + 2x)))`) to get a graph that looks like:



(b) i. We observe that as x approaches -2 from the left, $f(x)$ increases indefinitely (goes to ∞). So $\lim_{x\to 2^-} f(x)$ does not exist.

ii. From the graph, we observe that $\lim_{x\to -1} f(x)$ exists. Since $\frac{1}{x^2+2x}$ is a rational function that is defined at -1, we can use the substitution formula to get

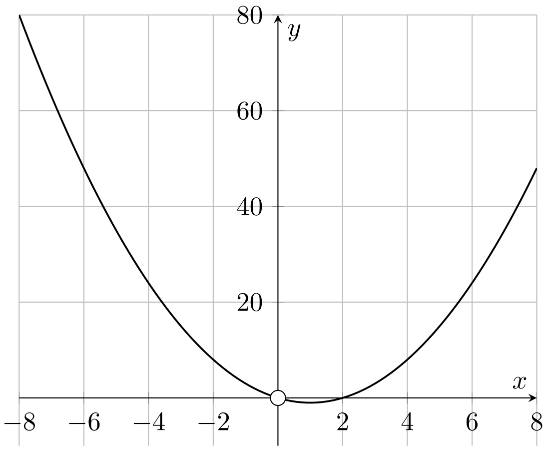
$$\begin{aligned}\lim_{x\to -1} f(x) &= \frac{1}{(-1)^2 + 2(-1)} \\ &= \frac{1}{1-2} \\ &= -1.\end{aligned}$$

iii. From the graph (or from the definition of f), we observe that as x approaches 0 from the right, $f(x)$ is always equal to 0. So $\lim_{x\to 0^+} = 0$.

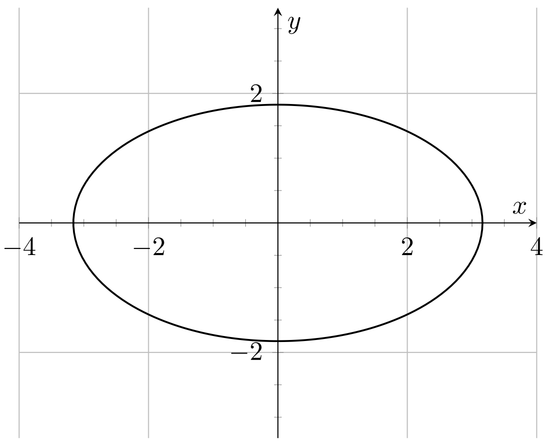
iv. We observe that as x approaches 0 from the left, $f(x)$ decreases indefinitely (goes to $-\infty$). This means that the left-hand limit $\lim_{x\to 0^-} f(x)$ does not exist, and so the limit $\lim_{x\to 0} f(x)$ does not exist.

2. For each of the following graphs, justify whether the limit at $x = 0$ exists.

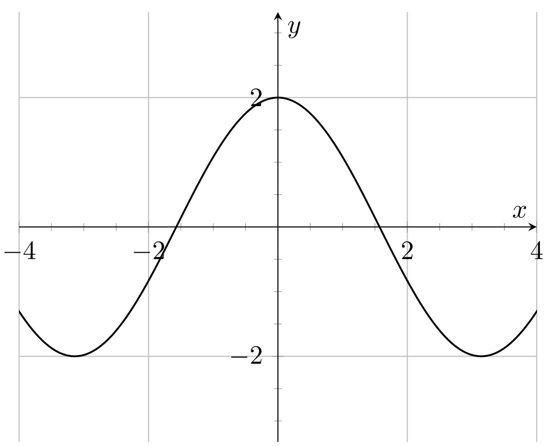
(a)



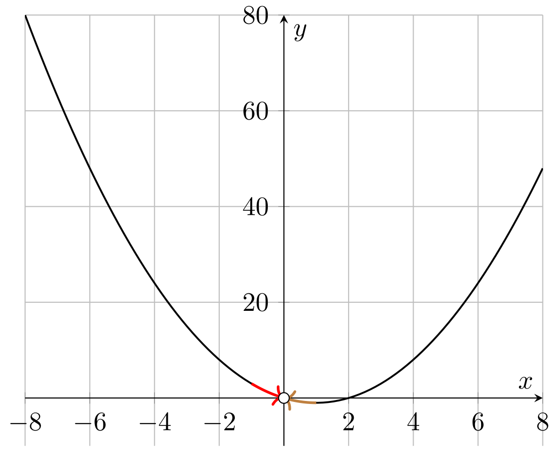
(b)



(c)

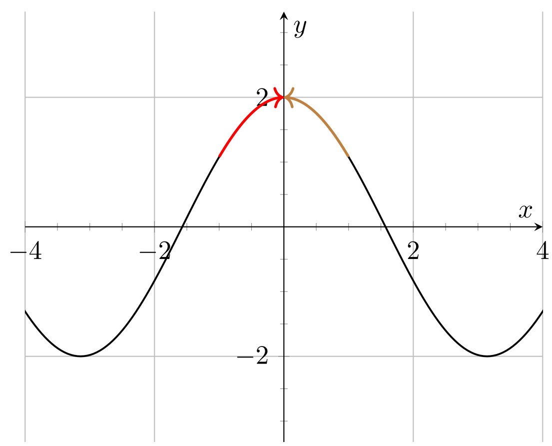


Solution:

(a) 

The graph is that of a real function, since it passes the vertical line test. We observe that as x approaches 0 from the left and the right, the graph approaches the same value. In other words, the left-hand and right-hand limits at $x = 0$ exist and are equal. So the limit at $x = 0$ exists.

(b) The graph is not a real function, since it fails the vertical line test. So it doesn't make sense to talk about its limit at $x = 0$.

(c) 

The graph is that of a real function, since it passes the vertical line test. We observe that as x approaches 0 from the left and the right, the graph approaches the same value. In other words, the left-hand and right-hand limits at $x = 0$ exist and are equal. So the limit at $x = 0$ exists.

3. Find the following limits. Justify your answers.
   (a) $\lim_{x \to 0} x^2 + 4$
   (b) $\lim_{x \to \infty} \frac{x^2}{x^2 - 1}$
   (c) $\lim_{x \to 1} \frac{x^2 + 3x - 4}{x^2 - 1}$

Solution:

(a) Since x^2+4 is a one-variable polynomial, it always satisfies the substitution formula. So we have

$$\begin{aligned}\lim_{x\to 0} x^2+4 &= (0)^2+4\\ &= 4.\end{aligned}$$

(b) If we try to use the substitution formula, we get $\frac{\infty}{\infty}$ which is not a real number (doesn't make sense).

Whenever $x>0$ (which will be the case when $x\to\infty$), we have

$$\begin{aligned}\frac{x^2}{x^2-1} &= \frac{\frac{x^2}{x^2}}{\frac{x^2-1}{x^2}}\\ &= \frac{1}{1-\frac{1}{x^2}}.\end{aligned}$$

So we can use the algebra of limits to get

$$\begin{aligned}\lim_{x\to\infty}\frac{1}{1-\frac{1}{x^2}} &= \frac{1}{1-\lim_{x\to\infty}\frac{1}{x^2}}\\ &= \frac{1}{1-0} = 1.\end{aligned}$$

(c) If we try to use the substitution formula we get $\frac{0}{0}$, which doesn't make sense. But this tells us that the polynomials $p(x) = x^2+3x-4$ and $q(x) = x^2-1$ both have 1 as a root (that is, $p(1) = 0$ and $q(1) = 0$). Therefore $p(x) = (x-1)\cdot p_1(x)$ and $q(x) = (x-1)\cdot q_1(x)$. We can find $p_1(x)$ and $q_1(x)$ by long division:

$$\begin{array}{r|l} & x+4 \\ \hline x-1 & x^2+3x-4 \\ & -x^2+x \\ \hline & 4x-4 \\ & -4x+4 \\ \hline & 0 \end{array} \qquad \begin{array}{r|l} & x+1 \\ \hline x-1 & x^2 \quad -1 \\ & -x^2+x \\ \hline & x-1 \\ & -x+1 \\ \hline & 0 \end{array}$$

So we have

$$\begin{aligned}\lim_{x\to 1}\frac{x^2+3x-4}{x^2-1} &= \lim_{x\to 1}\frac{\cancel{(x-1)}(x+4)}{\cancel{(x-1)}(x+1)}\\ &= \lim_{x\to 1}\frac{x+4}{x+1} = \frac{1+4}{1+1} = \frac{5}{2}.\end{aligned}$$

Math 130 04 – A Survey of Calculus

Homework assignment 4

Due: Tuesday, September 27, 2022

Instructions: Write your answers on a separate sheet of paper. Write your name at the top of each page you use, and number each page. Number your answers correctly.
Justify all your answers.

1. Evaluate the following limits, justifying your answers each time.

   (a) $\lim\limits_{x\to 4} \dfrac{x^3 - 9x^2 + 16}{3x^2 - 9x + 12}$ (b) $\lim\limits_{x\to \infty} \dfrac{x^3 + 25}{x^3 + 13}$ (c) $\lim\limits_{x\to 0} \dfrac{x+3}{x^2}$

Solution:

(a) $\dfrac{x^3 - 9x^2 + 16}{3x^2 - 9x + 12}$ is a rational function, so we can try direct substitution to see if we get an answer.

$$\begin{aligned}
\lim_{x\to 4} \frac{x^3 - 9x^2 + 16}{3x^2 - 9x + 12} &= \frac{4^3 - 9(4^2) + 16}{3(4^2) - 9(4) + 12} \\
&= \frac{64 - 9(16) + 3}{3(16) - 36 + 12} \\
&= \frac{64 - 144 + 16}{48 - 36 + 12} \\
&= \frac{-64}{24} \\
&\boxed{= \frac{-8}{3}}
\end{aligned}$$

(b) If we try to use direct substitution, we get $\dfrac{\infty}{\infty}$, which is a nonsense answer (not a real number). So we have to try another method.

Whenever $x > 0$ (which will be the case as $x \to \infty$) we can divide the numerator and the denominator by x^3 to get:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3+25}{x^3+13} &= \lim_{x\to\infty} \frac{\dfrac{x^3+25}{x^3}}{\dfrac{x^3+13}{x^3}} \\
&= \lim_{x\to\infty} \frac{1+\dfrac{25}{x^3}}{1+\dfrac{13}{x^3}} \\
&= \frac{1+25\cdot\left(\lim\limits_{x\to\infty}\dfrac{1}{x}\right)^3}{1+13\cdot\left(\lim\limits_{x\to\infty}\dfrac{1}{x}\right)^3} && \text{(using the algebra of limits)} \\
&= \frac{1+25\cdot 0}{1+13\cdot 0} && \text{(since we know that } \lim_{x\to\infty}\frac{1}{x}=0\text{)} \\
&= \frac{1}{1} \\
&\boxed{=1}
\end{aligned}$$

(c) If we try to use direct substitution, we get $\dfrac{0}{0}$ which is a nonsense answer (not a real number). So we have to try another method.

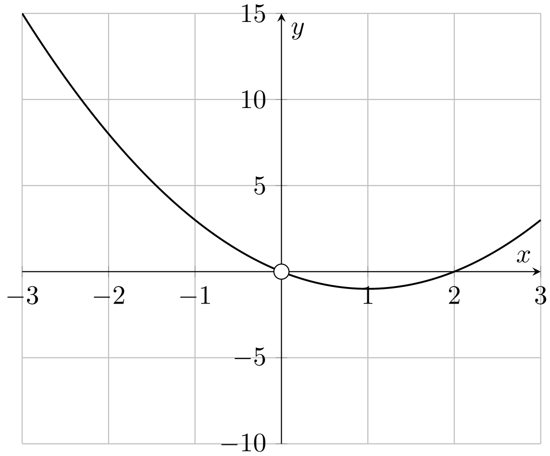
As $x \to 0$, from the left or from the right, it will always be true that $x > 0$ *or* $x < 0$. So we can divide the numerator and denominator by x^2 to get:

$$\begin{aligned}
\lim_{x\to 0} \frac{\dfrac{x+3}{x^2}}{\dfrac{x^2}{x^2}} &= \lim_{x\to 0} \frac{\dfrac{1}{x}+\dfrac{3}{x^2}}{1} \\
&= \left(\lim_{x\to 0}\frac{1}{x}\right) + 3\cdot\left(\lim_{x\to 0}\frac{1}{x}\right)^2 && \text{(using the algebra of limits)}
\end{aligned}$$

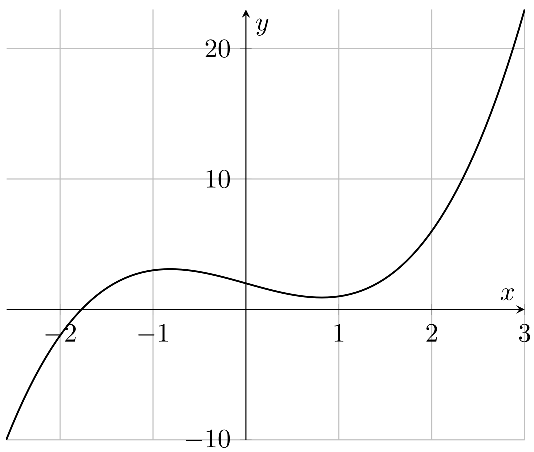
We know that as $x \to 0^+$ (x goes to 0 on the right), $\frac{1}{x}$ increases indefinitely (goes to ∞). Therefore, the same will be true of $\dfrac{1}{x}+\dfrac{3}{x^2}$. So the limit **does not exist**.

Remark: In each of the previous questions, using a graphing calculator to find the limit is a **valid justification** (although it is preferable to know *why* the limit is what it is). However writing things like "$\frac{\infty}{\infty}=1$" is **not** a valid justification since it does not make sense. For example, there are other functions, like $f(x)=\frac{x^2+3}{x+2}$ whose "limit" as $x\to\infty$ by direct substitution gives $\frac{\infty}{\infty}$, but that do not actually have a limit as $x \to \infty$ (the function $f(x)=\frac{x^2+3}{x+2}$ increases indefinitely, i.e. goes to ∞).

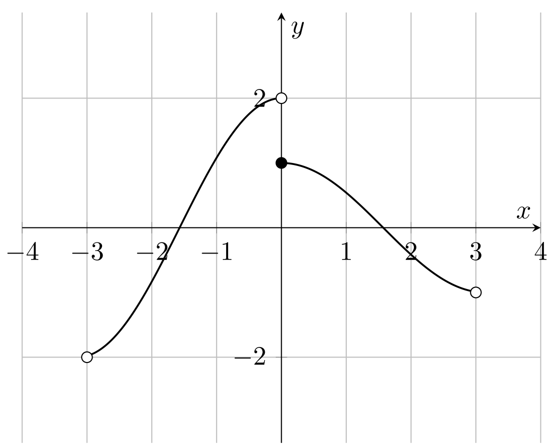
2. Which of the following graphs represent a function continuous over the interval $(-1, 1)$? Justify your answers.



(a)



(b)



(c)

Solution:

(a) The graph is a function, since it satisfies the vertical line test. However, it is **not** continuous over $[-1, 1]$, since it does not satisfy the pen-to-paper test.

More precisely, the function is not defined at $x = 0$, and so although the limit at 0 exists, it is not continuous at 0.

(b) The graph is a continuous function over $[-1, 1]$, since it passes both the vertical line test and the pen-to-paper test.

(c) The graph is a function, since it satisfies the vertical line test. However, it is **not** continuous over $[-1, 1]$, since it does not satisfy the pen-to-paper test.

More precisely, the function is defined for every $x \in [-1, 1]$, but it does not have a limit at 0, since the left- and right-hand limits are not the same.

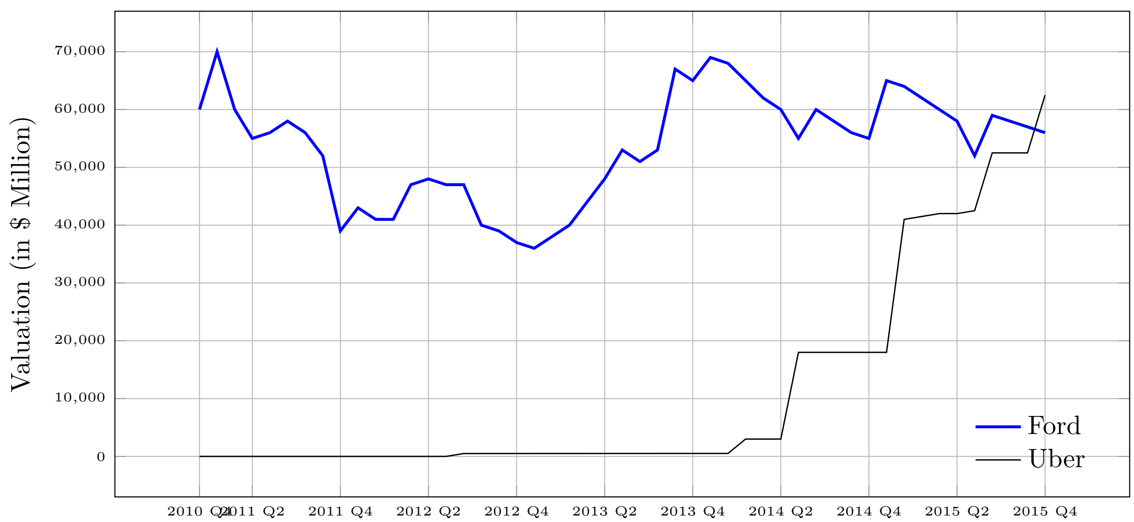


Figure 1: Valuation of Uber and Ford. (Source: Forbes)

3. The figure shows the valuation of the companies Ford and Uber between 2011 and 2015.
   (a) Are the graphs continuous functions in the interval (2014 Q2, 2015 Q4)?
   (b) Is there a point between 2014 Q2 and 2015 Q4 where:
      i. Ford's valuation equals that of Uber?
      ii. Ford's valuation is five times that of Uber?

   Justify all your answers.

Solution:

(a) Both graphs are functions that are continuous over the interval (2014 Q2, 2015 Q4) since they satisfy both the vertical line test and the pen-to-paper test.

(b) i. Yes, since there is a point in the interval (2014 Q2, 2015 Q4) where the graphs intersect.

ii. Let f be the function that is Ford's valuation, and let g be the function that is Uber's valuation. Namely, we have:

$$f(x) = \text{ Ford's valuation at time } x$$
$$g(x) = \text{ Uber's valuation at time } x$$

We would like to find if there is some time x between 2014 Q2 and 2015 Q4, such that $\dfrac{f(x)}{g(x)} = 5$. We know that:

- f and g are continuous over the interval (2013 Q4, 2015 Q4) since they satisfy the pen-to-paper test over that interval. So both functions are continuous over the *closed* interval [2014 Q2, 2015 Q2], since [2014 Q2, 2015 Q2] is contained in the bigger interval (2013 Q4, 2015 Q4).
- Over the closed interval [2014 Q2, 2015 Q2], the function g is never 0. So the function $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$ is continuous over [2014 Q2, 2015 Q2] (by the algebra of continuous functions).
- Since $\dfrac{f}{g}$ is continuous over the *closed* interval [2014 Q2, 2015 Q2], we can apply the intermediate value theorem to it. From the graph, we see that

$$\frac{f(2014\text{ Q2})}{g(2014\text{ Q2})} > \frac{60,000}{10,000} = 6$$

  (since $f(2014\text{ Q2}) = 60,000$ and $g(2014\text{ Q2}) < 10,000$), and that

$$\frac{f(2015\text{ Q2})}{g(2015\text{ Q2})} < \frac{60,000}{40,000} = \frac{3}{2} = 1.5$$

  (since $f(2015\text{ Q2}) < 60,000$ and $g(2015\text{ Q2}) > 40,000$).
- The intermediate value theorem then tells us that over the interval [2014 Q2, 2015 Q2], $\dfrac{f}{g}$ takes all values in the closed interval $[1.5, 6]$.

Since $5 \in [1.5, 6]$, there is some x between 2014 Q2 and 2015 Q2 such that $\dfrac{f(x)}{g(x)} = 5$, which is what we want.

Therefore, the answer is **yes**.

Math 130 04 – A Survey of Calculus

Homework assignment 6

Due: Tuesday, October 18, 2022

1. Evaluate the derivatives of the following functions (over any interval where they are defined).

   (a) $f(x) = x^3 - 9x^2 + 16$ (b) $f(x) = \dfrac{x^3 + 25}{3x - 2}$ (c) $f(x) = \dfrac{x+3}{x^2 - 4}$

Solution:

(a) Using the rules for derivatives, we get:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^3\right) - 9\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(16) && \text{(using the sum and product rules)} \\ &= 3x^2 - 9 \cdot 2x + 0 \\ &= \boxed{3x^2 - 18x} \end{aligned}$$

(b) Using the rules for derivatives, we get:

$$\begin{aligned} f'(x) &= \frac{\left(\frac{d}{dx}\left(x^3 + 25\right)\right) \cdot (3x-2) - (x^3+25) \cdot \frac{d}{dx}(3x-2)}{(3x-2)^2} && \text{(using the quotient rule)} \\ &= \frac{(3x^2+25) \cdot (3x-2) - (x^3+25) \cdot 3}{(3x-2)^2} \\ &= \frac{9x^3 + 75x - 6x^2 - 50 - 3x^3 - 75}{(3x-2)^2} \\ &= \boxed{\frac{6x^3 - 6x^2 + 75x - 125}{(3x-2)^2}} \end{aligned}$$

(c) Using the rules for derivatives, we get:

$$\begin{aligned} f'(x) &= \frac{\left(\frac{d}{dx}(x+3)\right) \cdot (x^2-4) - (x+3) \cdot \frac{d}{dx}\left(x^2-4\right)}{(x^2-4)^2} && \text{(using the quotient rule)} \\ &= \frac{3 \cdot (x^2-4) - (x+3) \cdot 2x}{(x^2-4)^2} \\ &= \frac{3x^2 - 12 - 2x^2 - 6x}{(x^2-4)^2} \\ &= \boxed{\frac{x^2 - 6x - 12}{(x^2-4)^2}} \end{aligned}$$

2. The distance between Los Angeles and San Diego on the I-5 highway is 118 miles.
   (a) What is the average speed (in mph) required to do the trip in 1.5 hours?
   (b) If the speed limit is 70 mph all along the I-5, is it possible to do the trip in 1.5 hours without breaking the law? Explain. What about in 2 hours?
   (c) A car going from L.A. to San Diego on the I-5 travels $f(x)$ miles (measured from L.A.) after x hours, where f is the function:
   $$f(x) = x(91 - 16x)$$
      i. How long does the car take to reach San Diego (i.e. cover 118 miles)?
      ii. Does the car ever break the law? (Is its speed ever more than 70 mph?)
      iii. What is the car's speed when it leaves L.A. (at the starting time)? What is the car's speed when it arrives in San Diego? (That is, at the time calculated in part i.)

   Remember: Speed (or velocity) is the derivative of distance as a function of time.

Solution:

(a) The average speed required to do 118 miles in 1.5 hours is $\frac{118}{1.5} = \boxed{78.67}$ miles per hour (mph).

(b) Since the average speed required to do 118 miles in 1.5 hours is 78.67 mph, it is $\boxed{\text{not possible}}$ to stay below 70 mph and do the trip in 1.5 hours.

The average speed required to do 118 miles in 2 hours is $\frac{118}{2} = 59$ mph, so it is $\boxed{\text{possible}}$ to do the trip in 2 hours while staying below the speed limit.

(c) i. Since the car travels $f(x)$ miles in x hours, to find how many hours the car will take to do 118 miles, we need to solve the equation $f(x) = 118$,

$$\begin{aligned}
\text{i.e.}\quad & x(91 - 16x) = 118\\
\text{i.e.}\quad & 91x - 16x^2 = 118\\
\text{i.e.}\quad & 16x^2 - 91x + 118 = 0\\
\text{i.e.}\quad & 16x^2 - 32x - 59x + 118 = 0\\
\text{i.e.}\quad & 16x(x - 2) - 59(x - 2) = 0\\
\text{i.e.}\quad & (16x - 59)(x - 2) = 0
\end{aligned}$$

The solutions of this equation are $x = 2$ and $x = \frac{59}{16} = 3.69$, so we can conclude that the car will be 118 miles from L.A. on the I-5 after 2 hours and after 3.69 hours, respectively. Since $2 < 3.69$, we can conclude that $\boxed{\text{the car will take 2 hours to do the trip.}}$

ii. The speed of the car at time x hours is the derivative $f'(x)$. We can use the rules for derivatives to calculate

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(x(91 - 16x)\right)\\
&= \left(\frac{d}{dx}(x)\right) \cdot (91 - 16x) + x \cdot \frac{d}{dx}(91 - 16x) \qquad \text{(using the product rule)}\\
&= 1 \cdot (91 - 16x) + x \cdot \left(\frac{d}{dx}(91) - \frac{d}{dx}(16x)\right)\\
&= 91 - 16x + x \cdot (0 - 16)\\
&= 91 - 16x - 16x\\
&= 91 - 32x.
\end{aligned}$$

So, at time 0 hours (the starting time in L.A.), the speed of the car is $f'(0) = 91$ mph, and so the car **does** break the law.

iii. The speed of the car at the starting time ($x = 0$ hours) is $f'(0) = \boxed{91}$ mph. Since the car takes 2 hours to do the trip to San Diego, the speed at the arrival time ($x = 2$ hours) is $f'(2) = 91 - 32 \cdot 2 = 91 - 64 = \boxed{27}$ mph.

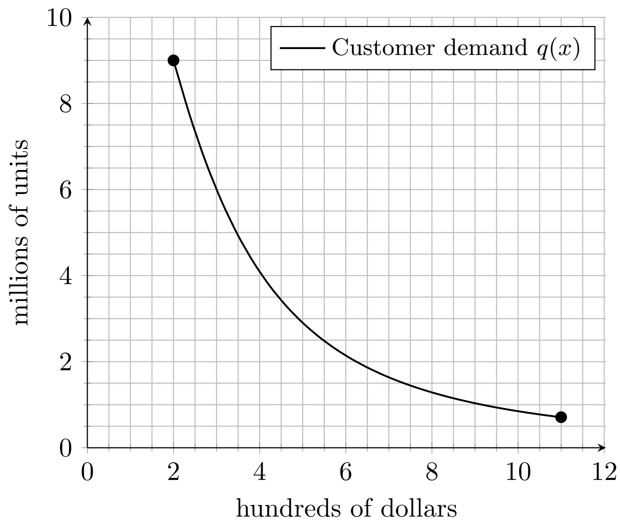


Figure 1: Market demand model

3. (The return of Lemon, Inc. ...) In Homework 2, we saw Lemon Inc.'s market study that estimated the number of units of the piePhone2 that they could expect to sell at a given price point (the "customer demand" graph). Lemon can see that customer demand ($q(x)$ millions of units sold at a price per unit of x hundred dollars) is a continuous function (over the interval $[2, 11]$), since it satisfies the vertical line test and the pen-to-paper test.

   Lemon also knows that their *revenue* from selling $q(x)$ million units at x hundred dollars each is $x \cdot q(x)$ hundred million dollars (Hmn\$). That is, their revenue function is:

   $$r(x) = x \cdot q(x) \quad \text{(in Hmn\$)}$$

   (a) Lemon wants to know the *rate of change* (i.e. the derivative) of revenue in terms of the rate of change (i.e. the derivative) of demand. Show that we can write the rate of change of revenue as:

   $$r'(x) = q(x) \cdot \left(1 + \frac{x}{q(x)} \cdot q'(x)\right)$$

   Remark: The function $E_d(x) = \dfrac{x}{q(x)} \cdot q'(x)$ is called the *"price elasticity of demand"* (Wikipedia article) (that some of you may have seen in your other classes). It is extremely important in economics — it measures how sensitive demand is to changes in price. $E_d(x)$ is almost always a negative real number (i.e. $E_d(x) < 0$). If $E_d(x) = -2$, it means that a 10% increase in price will result in a 20% *decrease* in demand.*

   (b) Lemon hires some pretty solid economists who figure out that the demand function q is given by:

   $$q(x) = \frac{90}{x^2 + 6}$$

*Now that you understand derivatives, you know the exact definition of the price elasticity of demand.

i. Calculate $q'(x)$ and $r'(x)$.
ii. Calculate $r'(2)$. Is revenue increasing or decreasing at a price point of \$200 per unit?
iii. Calculate $r'(6)$. Is revenue increasing or decreasing at a price point of \$600 per unit?
iv. What is $E_d(6)$? At a price point of \$600 per unit, how much (in %) will demand increase or decrease if the price increases by 7%?

Solution:

(a) Lemon's revenue function is $r(x) = x \cdot q(x)$. Using the rules for derivatives, the rate of change of revenue is

$$\begin{aligned} r'(x) &= \frac{d}{dx}\left(x \cdot q(x)\right) \\ &= \left(\frac{d}{dx}(x)\right) \cdot q(x) + x \cdot q'(x) && \text{(using the product rule)} \\ &= q(x) + x \cdot q'(x) \\ &= \boxed{q(x) \cdot \left(1 + \frac{x}{q(x)} \cdot q'(x)\right)} \end{aligned}$$

(b) i. We can use the rules for derivatives to calculate

$$\begin{aligned} q'(x) &= \frac{d}{dx}\left(\frac{90}{x^2+6}\right) \\ &= \frac{\left(\frac{d}{dx}(90)\right)(x^2+6) - 90 \cdot \frac{d}{dx}(x^2+6)}{(x^2+6)^2} && \text{(using the quotient rule)} \\ &= \frac{0 \cdot (x^2+6) - 90 \cdot (2x+0)}{(x^2+6)^2} \\ &= \boxed{-\frac{180x}{(x^2+6)^2}} \end{aligned}$$

Now that we know $q(x)$ and $q'(x)$, we can calculate $r'(x)$ as:

$$\begin{aligned} r'(x) &= q(x) + x \cdot q'(x) \\ &= \frac{90}{x^2+6} + x \cdot \left(-\frac{180x}{(x^2+6)^2}\right) \\ &= \frac{90}{x^2+6} - \frac{180x^2}{(x^2+6)^2} \\ &= \frac{90(x^2+6) - 180x^2}{(x^2+6)^2} \\ &= \frac{90(x^2+6-2x^2)}{(x^2+6)^2} \\ &= \boxed{\frac{90(6-x^2)}{(x^2+6)^2}} \end{aligned}$$

ii. We have

$$r'(2) = \frac{90(6-(2)^2)}{((2)^2+6)^2} = \frac{90(6-4)}{(4+6)^2} = \frac{180}{100} = \boxed{1.8}$$

Since $r'(2) > 0$, Lemon's revenue (the function r) is $\boxed{\text{increasing}}$ at a price point of \$200 per unit.

iii. We have

$$r'(6) = \frac{90(6-(6)^2)}{((6)^2+6)^2} = \frac{90(-30)}{(42^2)} = \boxed{-\frac{75}{49}}$$

Since $r'(6) < 0$, Lemon's revenue is $\boxed{\text{decreasing}}$ at a price point of \$600 per unit.

iv. We can calculate the price elasticity of demand at a price point of x hundred dollars per unit (i.e. the function $E_d(x)$) as follows.

$$\begin{aligned} E_d(x) &= \frac{x}{q(x)} \cdot q'(x) \\ &= x \cdot \frac{(x^2+6)}{90} \cdot \left(-\frac{180x}{(x^2+6)^2}\right) \\ &= x \cdot \left(-\frac{2x}{x^2+6}\right) \\ &= -\frac{2x^2}{x^2+6} \end{aligned}$$

Hence we have

$$E_d(6) = -\frac{2(6)^2}{(6)^2+6} = -\frac{2\cdot 6\cdot 6}{6\cdot 7} = \boxed{-\frac{12}{7}}$$

So, at a price point of \$600 per unit, if the price increases by 7%, the demand will change by

$$E_d(6)\cdot 7 = \left(-\frac{12}{7}\right)\cdot 7 = -12\%$$

In other words, the demand will $\boxed{\text{decrease by }12\%}$.

Math 130 04 – A Survey of Calculus

Homework assignment 7

Due: Tuesday, November 1, 2022

Remember: The **chain rule** says that if f and g are real functions, then the derivative of the composite function $(g \circ f)(x) = g(f(x))$ can be calculated as follows.

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

1. Evaluate the derivatives of the following functions.

   (a) $f(x) = (x^2 + 3x + 4)^3$ (b) $f(x) = (x^3 + 5)^{1/4}$ (c) $f(x) = \dfrac{x+3}{(x^2-4)^{2/3}}$

Solution:

(a)

Step 1: We recognise that $f(x)$ can be rewritten as

$$f(x) = (g(x))^3 \qquad \text{where } g(x) = x^2 + 3x + 4.$$

So if we define the function m as $m(x) = x^3$, then f is equal to the composite function $m \circ g$,

$$\text{i.e.} \qquad f(x) = (m \circ g)(x) = m(g(x)) = (g(x))^3 = \left(x^2 + 3x + 4\right)^3$$

Hence, using the chain rule, the derivative of f is

$$f'(x) = (m \circ g)'(x) = m'(g(x)) \cdot g'(x)$$

Step 2: We calculate the derivatives of the functions m and g.

$$m'(x) = \frac{d}{dx}\left(x^3\right) = 3x^2$$
$$g'(x) = \frac{d}{dx}\left(x^2 + 3x + 4\right) = 2x + 3$$

Step 3: We calculate the derivative of f.

$$\begin{aligned} f'(x) &= m'(g(x)) \cdot g'(x) \\ &= 3(g(x))^2 \cdot (2x+3) && (\text{since } m'(x) = 3x^2 \text{ and } g'(x) = 2x+3) \\ &= \boxed{3(x^2+3x+4)^2 \cdot (2x+3)} \end{aligned}$$

(b)

Step 1: We recognise that $f(x)$ can be rewritten as

$$f(x) = (g(x))^{1/4} = m(g(x)) = (m \circ g)(x) \qquad \text{where } g(x) = x^3 + 5 \text{ and } m(x) = x^{1/4}$$

Hence, using the chain rule, the derivative of f is

$$f'(x) = m'(g(x)) \cdot g'(x)$$

Step 2: We calculate the derivatives of m and g.

$$\begin{aligned} m'(x) &= \frac{d}{dx}\left(x^{1/4}\right) = \frac{1}{4} \cdot x^{-3/4} \\ g'(x) &= \frac{d}{dx}\left(x^3 + 5\right) = 3x^2 \end{aligned}$$

Step 3: We calculate the derivative of f.

$$\begin{aligned} f'(x) &= m'(g(x)) \cdot g'(x) \\ &= \frac{1}{4} \cdot (g(x))^{-3/4} \cdot (3x^2) \qquad \left(\text{since } m'(x) = \frac{1}{4} \cdot x^{-3/4} \text{ and } g'(x) = 3x^2\right) \\ &= \boxed{\frac{1}{4} \cdot (x^3 + 5)^{-3/4} \cdot (3x^2)} \end{aligned}$$

(c) Using the quotient rule, we have e

$$\begin{aligned} f'(x) &= \frac{\left(\frac{d}{dx}(x+3)\right) \cdot (x^2-4)^{2/3} - (x+3) \cdot \left(\frac{d}{dx}\left((x^2-4)^{2/3}\right)\right)}{\left((x^2-4)^{2/3}\right)^2} \\ &= \frac{x \cdot (x^2-4)^{2/3} - (x+3) \cdot \left(\frac{d}{dx}\left((x^2-4)^{2/3}\right)\right)}{(x^2-4)^{4/3}} \end{aligned}$$

So we need to find the derivative of the function $g(x) = (x^2 - 4)^{2/3}$. First, we recognise that $g(x)$ can be rewritten as

$$g(x) = (k(x))^{2/3} = (m \circ k)(x) \qquad \text{where } k(x) = x^2 - 4 \text{ and } m(x) = x^{2/3}.$$

Hence, using the chain rule, the derivative of g is $g'(x) = m'(k(x)) \cdot k'(x)$.

Next, we calculate the derivatives of m and k.

$$\begin{aligned} m'(x) &= \frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3} \cdot x^{-1/3} \\ k'(x) &= \frac{d}{dx}\left(x^2 - 4\right) = 2x \end{aligned}$$

This lets us calculate the derivative of g.

$$\begin{aligned} g'(x) &= m'(k(x)) \cdot k'(x) \\ &= \frac{2}{3} \cdot (k(x))^{-1/3} \cdot (2x) \qquad \left(\text{since } m'(x) = \frac{2}{3} \cdot x^{-1/3} \text{ and k'(x)=2x}\right) \\ &= \frac{2}{3} \cdot \left(x^2 - 4\right)^{-1/3} \cdot (2x). \end{aligned}$$

Finally, we can calculate the derivative of f.

$$f'(x) = \frac{x \cdot (x^2-4)^{2/3} - (x+3) \cdot g'(x)}{(x^2-4)^{4/3}}$$
$$= \boxed{\frac{x \cdot (x^2-4)^{2/3} - (x+3) \cdot \frac{2}{3} \cdot \left(x^2-4\right)^{-1/3} \cdot (2x)}{(x^2-4)^{4/3}}}$$

2. (Variable interest rates.)

A lender (e.g. a bank) proposes two 20-year loans of \$1.5 million to a borrower (e.g. a homebuyer). The borrower can either pay *compound* interest at an interest rate of x percent, or *simple* interest at an interest rate of $2x$ percent. So, if the borrower chooses to pay compound interest at a rate of x percent, the amount they have to pay after 20 years is

$$f(x) = 1.5 \cdot \left(1 + \frac{x}{100}\right)^{20} \quad \text{million dollars.}$$

If the borrower chooses to pay simple interest, the amount they have to pay after 20 years is

$$g(x) = 1.5 \cdot \left(1 + 20 \cdot \frac{2x}{100}\right) \quad \text{million dollars.}$$

(a) If the compound interest rate is 4 percent, how much would the borrower pay after 20 years if they chose the loan with:

i. Compound interest.

ii. Simple interest.

Which loan makes more sense for the homebuyer?

(b) If the compound interest rate is 8 percent, how much would the borrower pay after 20 years if they choose the loan with:

i. Compound interest.

ii. Simple interest.

Which loan makes more sense for the homebuyer?

(c) The difference between the two loans is measured by the amount $d(x) = g(x) - f(x)$.

i. Calculate the rate of change of the difference between the two loans at a compound interest rate of 2 percent.

ii. Calculate the rate of change of the difference between the two loans at a compound interest rate of 4 percent.

iii. At what compound interest rate is the difference between the two loans *maximum*?

Solution:

(a) If the compound interest rate is 4 percent, we are in the case where $x = 4$.

i. If the borrower chooses the loan with compound interest, they would pay

$$f(4) = 1.5 \cdot \left(1 + \frac{4}{100}\right)^{20} = 3.287 \quad \text{million dollars.}$$

ii. If the borrower chooses the loan with simple interest, they would pay

$$g(4) = 1.5 \cdot \left(1 + 20 \cdot \frac{2 \cdot 4}{100}\right) = 3.9 \quad \text{million dollars.}$$

Since the amount paid on the loan with compound interest at 4 percent is *less* than the amount paid on the loan with simple interest at 8 percent, the loan with $\boxed{\text{compound interest}}$ makes more sense for the borrower.

(b) If the compound interest rate is 8 percent, we are in the case where $x = 8$.

i. If the borrower chooses the loan with compound interest, they would pay

$$f(8) = 1.5 \cdot \left(1 + \frac{8}{100}\right)^{20} = 6.991 \quad \text{million dollars.}$$

ii. If the borrower chooses the loan with simple interest, they would pay

$$g(4) = 1.5 \cdot \left(1 + 20 \cdot \frac{2 \cdot 8}{100}\right) = 6.3 \quad \text{million dollars.}$$

Since the amount paid on the loan with compound interest at 4 percent is *more* than the amount paid on the loan with simple interest at 8 percent, the loan with $\boxed{\text{simple interest}}$ makes more sense for the borrower.

(c) The rate of change of the difference function d is its derivative, which is $d'(x) = g'(x) - f'(x)$. We calculate the derivative of g as

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(1.5 \cdot \left(1 + 20 \cdot \frac{2x}{100}\right)\right) \\ &= 1.5 \cdot \frac{d}{dx}\left(1 + 20 \cdot \frac{2x}{100}\right) \\ &= 1.5 \cdot \left(0 + 20 \cdot \frac{2}{100}\right) = \frac{6}{10}. \end{aligned}$$

To calculate the derivative of f, we recognise that $f(x)$ can be written as

$$f(x) = 1.5 \cdot \left(1 + \frac{x}{100}\right)^{20} = 1.5 \cdot (k(x))^{20} = m(k(x)) = (m \circ k)(x)$$

where $m(x) = 1.5 \cdot x^{20}$ and $k(x) = 1 + \dfrac{x}{100}$.

Hence we can find the derivative of f using the chain rule:

$$\begin{aligned} f'(x) &= m'(k(x)) \cdot k'(x) \\ &= 1.5 \cdot 20 \cdot (k(x))^{19} \cdot \frac{1}{100} && \text{since } m'(x) = 1.5 \cdot 20 \cdot x^{19} \text{ and } k'(x) = \frac{1}{100} \\ &= 30 \cdot \left(1 + \frac{x}{100}\right)^{19} \cdot \frac{1}{100} \\ &= \frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19} \end{aligned}$$

Finally, we calculate $d'(x)$ as

$$\begin{aligned} d'(x) &= g'(x) - f'(x) \\ &= \frac{6}{10} - \frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19} \\ &= \frac{3}{10} \cdot \left(2 - \left(1 + \frac{x}{100}\right)^{19}\right) \end{aligned}$$

i. We have

$$d'(2) = \frac{3}{10} \cdot \left(2 - \left(1 + \frac{2}{100}\right)^{19}\right) = 0.163 \quad \text{million dollars per percent}$$

ii. We have

$$d'(4) = \frac{3}{10} \cdot \left(2 - \left(1 + \frac{4}{100}\right)^{19}\right) = -0.032 \quad \text{million dollars per percent}$$

iii. Any real number a is a local maximum of the difference function d exactly when $d'(a) = 0$ and $d''(a) < 0$.

Step 1: We find the critical points of the function d by solving the equation $d'(x) = 0$.

$$\begin{aligned}
& d'(x) = 0 \\
\text{i.e.}\quad & g'(x) - f'(x) = 0 \\
\text{i.e.}\quad & f'(x) = g'(x) \\
\text{i.e.}\quad & \frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19} = \frac{6}{10} \\
\text{i.e.}\quad & \frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19} = 2 \\
\text{i.e.}\quad & 1 + \frac{x}{100} = 2^{1/19} \\
\text{i.e.}\quad & x = 100 \cdot \left(2^{1/19} - 1\right) \\
\text{i.e.}\quad & x = 3.715
\end{aligned}$$

Step 2: The second derivative of d is the derivative of d', which we calculate as

$$\begin{aligned}
d''(x) &= g''(x) - f''(x) \\
&= \frac{d}{dx}\left(\frac{6}{10}\right) - \frac{d}{dx}\left(\frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19}\right) \\
&= 0 - \frac{3}{10}\frac{d}{dx}\left(\left(1 + \frac{x}{100}\right)^{19}\right)
\end{aligned}$$

Using the chain rule, $\frac{d}{dx}\left(\left(1 + \frac{x}{100}\right)^{19}\right) = 19 \cdot \left(1 + \frac{x}{100}\right)^{18} \cdot \frac{1}{100}$

So $d''(x) = -\frac{3}{10} \cdot 19 \cdot \left(1 + \frac{x}{100}\right)^{18} \cdot \frac{1}{100}$

Therefore, $d''(3.715) < 0$ (since $3.715 > 0$).

Hence we can conclude that $x = 3.715$ is the *only* local maximum of d. Therefore, the difference between the two loans is maximum when the compound interest rate is 3.715 percent.

Math 130 04 – A Survey of Calculus

Homework assignment 8

Due: Tuesday, December 6, 2022

Remember: If f is a continuous real function, the *Riemann integral* of f is the function

$$\left(\int f\right)(x) = \int_0^x f(x)\cdot dx = \lim_{h\to 0} S_h f(x)$$

Remember: The *fundamental theorems of calculus* are the following facts.

FTC1 If f is a differentiable real function, then

$$\boxed{\left(\int f'\right)(x) = \int_0^x f'(x)\cdot dx \ = \ f(x) - f(0)}$$

FTC2 If f is a continuous real function, then

$$\boxed{\left(\int f\right)'(x) = f(x)}$$

Remember:

- If $f(x) = x^a$ where a is any constant real number such that $a \neq -1$, then
$$\left(\int f\right)(x) = \int_0^x x^a \cdot dx \ = \ \frac{x^{a+1}}{a+1}$$
- If $f(x) = e^{kx}$ where k is any constant real number such that $k \neq 0$, then
$$\left(\int f\right)(x) = \int_0^x e^{kx} \cdot dx \ = \ \frac{e^{kx}-1}{k}$$

Rules for Riemann integrals

- **Constant rule:** If $f(x) = c\cdot g(x)$, where c is any constant real number, then
$$\left(\int f\right)(x) = c\cdot \left(\int g\right)(x)$$
or using alternate notation,
$$\int_0^x c\cdot g(x)\cdot dx = c\cdot \int_0^x g(x)\cdot dx$$
- **Sum rule:** If f and g are continuous real functions, then
$$\left(\int (f+g)\right)(x) = \left(\int f\right)(x) + \left(\int g\right)(x)$$
or using alternate notation,
$$\int_0^x (f(x)+g(x))\cdot dx = \left(\int_0^x f(x)\cdot dx\right) + \left(\int_0^x g(x)\cdot dx\right)$$

1. Calculate the Riemann integrals of the following functions.

   (a) $f(x) = x^2 + 3x + 4$

   (b) $f(x) = 8e^{-2x} + 4$

   (c) $f(x) = 4x^{-1/3}$

   (d) $f(x) = 3x(4 + x^{2/5})$

Solution:

(a) We can use the rules for Riemann integrals to find

$$\int_0^x (x^2 + 3x + 4) \cdot dx = \left(\int_0^x x^2 \cdot dx\right) + 3 \cdot \left(\int_0^x x \cdot dx\right) + 4 \cdot \left(\int_0^x 1 \cdot dx\right)$$
$$= \boxed{\frac{x^3}{3} + 3\frac{x^2}{2} + 4x}$$

(b) We can use the rules for Riemann integrals to find

$$\int_0^x (8e^{-2x} + 4) \cdot dx = 8 \cdot \left(\int_0^x e^{-2x} \cdot dx\right) + 4 \cdot \left(\int_0^x 1 \cdot dx\right)$$
$$= 8\frac{(e^{-2x} - 1)}{-2} + 4x$$
$$= \boxed{4(1 - e^{-2x}) + 4x}$$

(c) We can use the rules for Riemann integrals to find

$$\int_0^x (4x^{-1/3}) \cdot dx = 4 \cdot \left(\int_0^x x^{-1/3} \cdot dx\right)$$
$$= 4 \cdot \frac{x^{2/3}}{\frac{2}{3}}$$
$$= 4 \cdot \frac{3}{2} \cdot x^{2/3}$$
$$= \boxed{6x^{2/3}}$$

(d) We can first simplify the function f as

$$f(x) = 3x(4 + x^{2/5}) = 12x + 3x^{7/5}$$

Next, we can use the rules for Riemann integrals to find

$$\int_0^x (12x + 3x^{7/5}) \cdot dx = 12 \cdot \left(\int_0^x x \cdot dx\right) + 3 \cdot \left(\int_0^x x^{7/5} \cdot dx\right)$$
$$= 12 \cdot \frac{x^2}{2} + 3 \cdot \frac{x^{12/5}}{\frac{12}{5}}$$
$$= 6x^2 + 3 \cdot \frac{5}{12} \cdot x^{12/5}$$
$$= \boxed{6x^2 + \frac{5}{4}x^{12/5}}$$

Recall: The *marginal cost function* is the derivative of the total cost function. Similarly, the marginal revenue and marginal profit functions are the derivatives of the total revenue and total profit functions respectively.

2. A company calculates its marginal cost function C' as follows: If x thousand units have been produced, the marginal cost (i.e. the cost to produce the next unit) is

$$C'(x) = 2x^{-1/3} \qquad \text{dollars per unit.}$$

   (a) Find the company's total cost function C (i.e. $C(x)$ thousands of dollars to produce x thousand units) if the fixed cost to produce 0 units is 3 thousand dollars (i.e. $C(0) = 3$).

   (b) Suppose the company's marginal revenue function is as follows: If x thousand units have been sold, the marginal revenue (i.e. the revenue from selling the next unit) is

$$R'(x) = 3x^{-1/2} \qquad \text{dollars per unit.}$$

      i. Find the marginal profit function P'. (Remember that the total profit function P is defined as $P(x) = R(x) - C(x)$, where R and C are the total revenue and total cost functions.)

      ii. Does the total profit function P have a maximum in the interval $[0, 20]$? If so, find the value $a \in [0, 20]$ such that P has a maximum at a.

      iii. Calculate the total profit function P, assuming that the revenue from selling 0 units is 0 dollars (i.e. $R(0) = 0$).

Solution:

(a) By the FTC1, we know that

$$\left(\int C'\right)(x) = C(x) - C(0)$$

Since we know that $C(0) = 3$, we can find the total cost as

$$\begin{aligned}
C(x) &= \left(\int C'(x) \cdot dx\right) + C(0) \\
&= \left(\int (2x^{-1/3}) \cdot dx\right) + 3 \\
&= 2 \cdot \frac{x^{2/3}}{\frac{2}{3}} + 3 \\
&= 2 \cdot \frac{3}{2} \cdot x^{2/3} + 3 \\
&= 3x^{2/3} + 3
\end{aligned}$$

(b) i. Since the total profit function is $P(x) = R(x) - C(x)$, the marginal profit function is

$$\begin{aligned}
P'(x) &= \frac{d}{dx}\left(R(x) - C(x)\right) \\
&= R'(x) - C'(x) \\
&= 3x^{-1/2} - 2x^{-1/3}
\end{aligned}$$

ii. Since any maximum a of the function P has to be a critical point (i.e. $P'(a) = 0$), we first solve the equation $P'(x) = 0$ to look for critical points in the interval $[0, 20]$.

$$\begin{aligned}
& P'(x) = 0 \\
\text{i.e.} \quad & R'(x) = C'(x) \\
\text{i.e.} \quad & 3x^{-1/2} = 2x^{-1/3} \\
\text{i.e.} \quad & \frac{3}{x^{1/2}} = \frac{2}{x^{1/3}} \\
\text{i.e.} \quad & 2 \cdot \frac{x^{1/2}}{x^{1/3}} = 3 \\
\text{i.e.} \quad & 2x^{1/2-1/3} = 3 \\
\text{i.e.} \quad & 2x^{1/6} = 3 \\
\text{i.e.} \quad & x = \left(\frac{3}{2}\right)^6 \\
\text{i.e.} \quad & x \approx 11.39
\end{aligned}$$

Therefore P has a critical point at $x = 11.39$, which is in the interval $[0, 20]$. To check whether 11.39 is a local maximum, we use the second derivative test, i.e. we have to check that $P''(11.39) < 0$.
We have

$$\begin{aligned}
P''(x) &= \frac{d}{dx}\left(3x^{-1/2} - 2x^{-1/3}\right) \\
&= -\frac{3}{2}x^{-3/2} - \left(-\frac{2}{3}x^{-4/3}\right) \\
&= \frac{2}{3}x^{-4/3} - \frac{3}{2}x^{-3/2}
\end{aligned}$$

and so we calculate

$$P''(11.39) = \frac{2}{3}(11.39)^{-4/3} - \frac{3}{2}(11.39)^{-3/2} = -0.013$$

Therefore $P''(11.39) < 0$, and so $\boxed{P \text{ has a maximum at } 11.39}$.

iii. Since $R(0) = 0$, we can calculate $P(0) = R(0) - C(0) = -3$. Next, by the FTC1, we know that

$$\left(\int P'\right)(x) = P(x) - P(0)$$

Therefore, we have

$$P(x) = \left(\int_0^x (R'(x) - C'(x)) \cdot dx\right) + P(0)$$

Recall: If f is a continuous function, the value of the Riemann integral of f at x, i.e. the real number $(\int f)(x)$, is the area "under" the graph of f between the points 0 and x on the horizontal axis.

3. Consider the following definition.

$$f(x) = \begin{cases} x^5 + 2x^2 - 2 & \text{if } x \leq 0 \\ \\ x^3 + 4x - 2 & \text{if } x > 0 \end{cases}$$

(a) Is the function f continuous? Namely, is f continuous at every real number x?

(b) What is the value of $(\int f)(x)$ if $x \leq 0$?

(c) What is the value of $(\int f)(x)$ if $x > 0$?

Solution:

(a) When $x < 0$, $f(x) = x^5 + 2x^2 - 2$ is a polynomial, so f is continuous at every real number strictly less than 0.

When $x > 0$, $f(x) = x^3 + 4x - 2$ is a polynomial, so f is continuous at every real number strictly greater than 0.

When $x = 0$, the left hand limit $\lim_{x \to 0^-} f(x)$ is

$$\lim_{x \to 0^-} x^5 + 2x^2 - 2 = (0)^5 + 2(0)^2 - 2 = -2$$

and the right hand limit $\lim_{x \to 0^+} f(x)$ is

$$\lim_{x \to 0^+} (0)^3 + 4(0) - 2 = -2$$

Finally $f(0) = (0)^5 + 2(0)^2 - 2 = -2$. Therefore, since

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = -2$$

f is continuous at 0.

Therefore, since f is continuous at every $x < 0$, at every $x > 0$ and at $x = 0$, $\boxed{f \text{ is continuous at all real numbers}}$.

(b) If $f(x) \leq 0$, then $f(x) = x^5 + 2x^2 - 2$, so

$$\int_0^x f(x) \cdot dx = \int_0^x (x^5 + 2x^2 - 2) \cdot dx$$

$$\left(\int f \right)(x) = \boxed{\frac{x^6}{6} + 2\frac{x^3}{3} - 2x}$$

(c) If $f(x) > 0$, then $f(x) = x^3 + 4x - 2$, so

$$\int_0^x f(x) \cdot dx = \int_0^x (x^3 + 4x - 2) \cdot dx$$

$$\left(\int f \right)(x) = \boxed{\frac{x^4}{4} + 4\frac{x^2}{2} - 2x}$$