Stopping times in mean-reverted processes

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1 Motivations

Let us consider a single-dimensional stochastic process X under a probability $\mathbb{P}$. The SDE of X under $\mathbb{P}$ is the following

$$\begin{aligned} dX_t &= \mu\left(t, X_t\right) dt + \sigma\left(t, X_t\right) dW_t \\ X_0 &= x \end{aligned} \tag{1}$$

where W is a Wiener process under $\mathbb{P}$.

Proposition 1.1 (Existence and uniqueness of solution). *Let $T > 0$ and μ and σ be measurable functions for which there exists constants α and β such that*

$$\begin{aligned} &|\mu(t, x| + |\sigma(t, x)| \leq \alpha(1 + |x|) \\ &|\mu(t, x) - \mu(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq \beta\,|x - y| \end{aligned} \tag{2}$$

if x is a random variable independent of the σ-algebra generated by $(W_s)_{s\geq 0}$ with finite second moment then (1) has a $\mathbb{P}$-almost surely unique solution $(t, \omega) \mapsto X_t(\omega)$ such that X is adapted to the filtration generated by $(W_s)_{s\geq 0}$ and Z and

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T X_t^2 dt\right] < \infty \tag{3}$$

Let us now assume in the following that all of this conditions are verified.
Let pose $x \in \mathbb{R}$, we will denote by $\tau_{x,a}$ the hitting time of a for a process beginning in x

$$\tau_{x,a} = \inf\left\{t \leq 0, X_t = a\right\} \tag{4}$$

What is the law of $\tau_{x,a}$?
Let us denote by $u_{x,a}$ the Laplace transform of $\tau_{x,a}$

$$u_{x,a}(y) = \mathbb{E}^{\mathbb{P}}\left[e^{-y\tau_{x,a}}\right] \tag{5}$$

2 Ornstein-Ulhenbeck

Let assume first that X is a Ornstein-Ulhenbeck process. There exist measurable functions $t \mapsto \sigma(t)$ and $t \mapsto \lambda_t$ such that

$$\begin{aligned} dX_t &= -\lambda(t) X_t dt + \sigma(t) dW_t \\ X_0 &= x \in \mathbb{R} \end{aligned} \tag{6}$$

2.1 Constant coefficients

Let us now assume that $t \mapsto \sigma(t)$ and $t \mapsto \lambda(t)$ are constant function.

Then

$$X_t = xe^{-\lambda t} + \sigma e^{-\lambda t}\int_0^t e^{\lambda s} dW_s \tag{7}$$

From the Dubins-Schwartz theorem, there exist a Wiener process $B = (B_t)_{t\leq 0}$ such that

$$
\begin{aligned}
B_{\kappa(t)} &= \int_0^t e^{\lambda s} dW_s \\
\kappa(t) &= \sigma^2 \frac{1 - e^{-2\lambda t}}{2\lambda}
\end{aligned} \tag{8}
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References

[1] L. Alili, P. Patie and J.L. Pedersen *Representations of the First Hitting Time Density of an Ornstein-Ulhenbeck Process*