Stochastic Local Volatility

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1 The Model

1.1 General description

Let T be a finite time horizon (the final maturity of the deal we want to price, for example) Let S be an asset of drift μ_t under the domestic risk-neutral probability measure. We assume the following dynamics under $\mathbb{Q}_d$

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(S_t, t) e^{X_t} dW_t \tag{1}$$

where $(S,t) \mapsto \sigma(S,t)$ is the local volatility function (deterministic function of spot and time, calibrated to the vanilla option market at time 0. We will develop later a methodology to calibrate that function σ. X is an Ornstein-Ulhenbeck process so that :

$$\forall t \in [0,T], \mathbb{E}\left[e^{2X_t}\right] = 1 \tag{2}$$

The goal of the stochastic volatility would be to calibrate to the forward smile (to be consistent with the

1.2 Stochastic volatility part

Let us assume that X follows the following dynamics under the domestic risk-neutral probability measure

$$\begin{aligned} dX_t &= (\theta(t) - \lambda X_t)dt + \sigma_X dW_t^X \\ X_0 &= 1 \end{aligned} \tag{3}$$

The coefficients are kept constant for now for simplicity.
We have

$$X_t = e^{-\lambda t} + \int_0^t \theta(s) e^{-\lambda(t-s)} ds + \int_0^t \sigma_X e^{-\lambda(t-s)} dW_s^X \tag{4}$$

Then,

$$\begin{aligned} \exp\left(2e^{-\lambda t} + 2\int_0^t \theta(s) e^{-\lambda(t-s)} ds + 2\int_0^t \sigma_X^2 e^{-2\lambda(t-s)} ds\right) &= 1 \\ e^{-\lambda t} + \int_0^t \theta(s) e^{-\lambda(t-s)} ds + \int_0^t \sigma_X^2 e^{-2\lambda(t-s)} ds &= 0 \\ e^{-\lambda t} + \int_0^t \theta(s) e^{\lambda s} ds e^{-\lambda t} + \int_0^t \sigma_X^2 e^{-2\lambda(t-s)} ds &= 0 \\ -\lambda e^{-\lambda t} - \int_0^t \theta(s) e^{\lambda s} ds \lambda e^{-\lambda t} + 2\sigma_X^2 e^{-2\lambda t} + \theta(t) &= 0 \end{aligned}$$

Setting $f(t) = \theta(t) e^{\lambda t}$ it yields

$$\begin{aligned} \lambda \int_0^t f(s) ds + f(t) &= c(t) \\ c(t) &= \lambda - 2\sigma_X^2 e^{-\lambda t} \end{aligned} \tag{5}$$

Then f satisfies the following ODE

$$
\begin{aligned}
&f'(t) + \lambda f(t) = -2\sigma_X^2 \lambda e^{-\lambda t} \\
&f(0) = \lambda - 2\sigma_X^2
\end{aligned}
\tag{6}
$$

Then

$$
\begin{aligned}
f(t) &= Ce^{-\lambda t} - 2\sigma_X^2 \lambda t e^{-\lambda t} \\
&= \left(\lambda - 2\sigma_X^2\right) e^{-\lambda t} - 2\sigma_X^2 \lambda t e^{-\lambda t} \\
f(t) &= \lambda e^{-\lambda t} - 2\sigma_X^2 (1 + \lambda t) e^{-\lambda t}
\end{aligned}
\tag{7}
$$

Then

$$
\theta(t) = \lambda e^{-2\lambda t} - 2\sigma_X^2 (1 + \lambda t) e^{-2\lambda t} \tag{8}
$$

The stochastic volatility X can be computed quite easily and be simulated exactly

$$
X_t \sim \mathcal{N}\left(e^{-\lambda t} + \int_0^t \theta(s) e^{-\lambda(t-s)} ds, \sigma^2 \frac{1 - e^{-2\lambda t}}{2\lambda}\right) \tag{9}
$$

1.3 Calibration of the local volatility

Theorem 1.1 (Markovian Projection)**.** *Put here the theorem of the markovian projection of general volatility models into a local volatility*

In our case, we have

$$
\begin{aligned}
\sigma_D^2(x,t) &= \mathbb{E}\left[e^{2X_t}\sigma^2(S_t,t) \middle| S_t = S\right] \\
&= \mathbb{E}\left[e^{2X_t} \middle| S_t = S\right] \sigma^2(S,t)
\end{aligned}
\tag{10}
$$

1.3.1 Calibration via a PDE

Let us introduce the Green function $G(S_0, X_0, t_0; S, X, t)$. Let us set $b(X) = e^X$. G satisfies

$$
\frac{\partial G}{\partial T} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma^2 b^2 S^2 G\right) - \frac{\partial^2}{\partial X^2}\left(\sigma_X^2 G\right) - \frac{\partial^2}{\partial S \partial X}\left(\rho b \sigma S \sigma_X G\right) + \frac{\partial}{\partial X}\left((\theta - \lambda X) G\right) + \frac{\partial}{\partial S}(\mu S G) = 0 \tag{11}
$$

Let us assume (in relation of the previous section), that

1. σ_X, θ and λ are not functions of X
2. μ is not a function of S
3. ρ is neither a function of S nor X

The calibration PDE degenerates into

$$
\frac{\partial G}{\partial T} - \frac{1}{2}e^{2X}\frac{\partial^2}{\partial S^2}\left(\sigma^2 S^2 G\right) - \sigma_X^2 \frac{\partial^2}{\partial X^2}(G) - \rho \frac{\partial^2}{\partial S \partial X}\left(b \sigma S \sigma_X G\right) + (\theta - \lambda X)\frac{\partial}{\partial X}(G) + \mu \frac{\partial}{\partial S}(SG) = 0 \tag{12}
$$

Let us denote by Q the density of S

$$
Q(S,t) = \int_{\mathbb{R}} G(S, X, t) dX \tag{13}
$$

Then

$$
\frac{\partial Q}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma^2 S^2 \int_{\mathbb{R}} e^{2X} G(S, X, t) dX\right) + \mu \frac{\partial}{\partial S}(SG) = 0 \tag{14}
$$

Thus,

$$\frac{\partial Q}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma^2 \frac{\int_{\mathbb{R}} e^{2X} G(S,X,t)dX}{\int_{\mathbb{R}} G(X,S,t)dX} S^2 Q\right) + \mu \frac{\partial}{\partial S}(SG) = 0 \tag{15}$$

We recognize the Dupire equation which defines a local volatility function $\sigma_D(S,t)$

$$\sigma_D^2(S,t) = \sigma^2(S,t)\frac{\int_{\mathbb{R}} e^{2X} G(S,X,t)dX}{\int_{\mathbb{R}} G(X,S,t)dX} \tag{16}$$

which permits to re-write the calibration PDE

$$\begin{aligned}
&\frac{\partial G}{\partial T} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(S^2 e^{2X}\sigma_D^2 \frac{\int_{\mathbb{R}} G(S,X,t)dX}{\int_{\mathbb{R}} e^{2X} G(S,X,t)dX}\right) + \mu \frac{\partial}{\partial S}(SG) - \frac{\sigma_X^2}{2}\frac{\partial^2 G}{\partial X^2} + \frac{\partial}{\partial X}((\theta - \lambda X)G) \\
&- \rho\sigma_X \frac{\partial^2}{\partial S \partial X}\left(e^X S \sigma_D \sqrt{\frac{\int_{\mathbb{R}} G(S,X,t)dX}{\int_{\mathbb{R}} e^{2X} G(S,X,t)dX}} G\right) = 0
\end{aligned} \tag{17}$$

The only SLV models tractables with a PDE calibration are then mono-asset SLV models. We will develop in the following another method which enable us to calibrate via Monte-Carlo SLV Models. However, a Monte-Carlo will have more noise than a PDE calibration.

1.3.2 Calibration by Monte-Carlo

In this section, we will develop a method to calibrate the SLV models in a higher dimension problem using a Monte-Carlo simulation.

Let us re-write the Dupire equation associated with the SLV model we just described

$$\begin{aligned}
&\frac{\partial C}{\partial T} + P_d(0,T)\mathbb{E}^{\mathbb{Q}_T}\left[(Kr_d(T) - S_T r_f(T))\mathbf{1}_{\{S_T>K\}}\right] - \frac{1}{2}\sigma_D^2(K,T)K^2\frac{\partial^2 C}{\partial K^2} = 0 \\
&\frac{\partial C}{\partial T} + P_d(0,T)\mathbb{E}^{\mathbb{Q}_T}\left[(Kr_d(T) - S_T r_f(T))\mathbf{1}_{\{S_T>K\}}\right] - \frac{1}{2}\sigma^2(K,T)\mathbb{E}\left[e^{2X_T}\middle|S_T = K\right]K^2\frac{\partial^2 C}{\partial K^2} = 0
\end{aligned} \tag{18}$$

To write the calibration equation, we are left with, the calculation of the following integral

$$\mathbb{E}\left[e^{2X_T}\middle|S_T = K\right] = \frac{\mathbb{E}\left[e^{2X_T}\mathbf{1}_{S_T=K}\right]}{\mathbb{P}(S_T = K)} \tag{19}$$

Re-writting the above equation as a function of log-spot $y_t = \log(S_t)$ yields

$$\mathbb{E}\left[e^{2X_T}\middle|y_T = k\right] = \frac{\mathbb{E}\left[e^{2X_T}\mathbf{1}_{y_T=k}\right]}{\mathbb{P}(y_T = k)} \tag{20}$$

where $k = \log(K)$.

Let us assume that we have an expression for the forward transition probabilities $(y,X,T) \mapsto p(y,X,T) = G(e^y,X,T)$, the two expectations that come into play in equation (20) can be refactored such as

$$\sigma(k,T) = \sigma_D(e^k,T)\sqrt{\frac{I_0(k,T)}{I_1(k,T)}} \tag{21}$$

where

$$I_n(k,T) = \int e^{nx} p(y,x,T)dx, \text{ for } n \in \{0,1\} \tag{22}$$

Two Monte-Carlo method can then be used for the computation of this conditional expectation

1. Estimation of the integral via a full Monte-Carlo simulation of (S_T, X_T)
2. Approximation of the conditional expectation via a Least-Square approach using well-choosen polynomial functions

Estimation via Least-Squares As we want to compute a S_T-measurable integral, we can approximate

$$\mathbb{E}\left[e^{2X_T}\middle|S_T\right] \approx \sum_{m=1}^{M} \beta_m \zeta_m\left(S_T\right) \tag{23}$$

where the coefficients $(\beta_1,\dots,\beta_M)$ are found by a least-square approach. Let us assume we have N realisations of the random variables (S_T,X_T) denoted by $\left((S_T^1,X_T^1),\dots,(S_T^N,X_T^N)\right)$
Let us write the minimization problem associated with Least-Square in our case. Let us set $\beta=(\beta_1,\dots,\beta_M)$

$$\hat{\beta} = \underset{\beta\in\mathbb{R}^M}{\operatorname{argmin}} \left\{ \mathbb{E}\left[\left(\mathbb{E}\left[e^{2X_T}\middle|S_T\right] - \sum_{m=1}^{M} \beta_m \zeta_m\left(S_T\right) \right)^2 \right] \right\} \tag{24}$$

Denoting by $M_{\zeta\zeta}$ and $M_{\zeta X}$ the following quantities, according to [2]

$$\begin{aligned} (M_{\zeta\zeta})_{r,s} &= \mathbb{E}\left[\zeta_r(S_T)\zeta_s(S_T)\right] \\ (M_{\zeta X})_r &= \mathbb{E}\left[\zeta_r(S_T)e^{2X_T}\right] \end{aligned} \tag{25}$$

we have the following estimation for the "regression coefficients" β

$$\beta = (M_{\zeta\zeta})^{-1} M_{\zeta X} \tag{26}$$

Those quantities can be estimated using the Monte-Carlo paths $\left((S_T^1,X_T^1),\dots,(S_T^N,X_T^N)\right)$

$$\begin{aligned} (\hat{M}_{\zeta\zeta})_{r,s} &= \frac{1}{N}\sum_{n=1}^{N} \zeta_r(S_T^n)\zeta_s(S_T^n) \\ (\hat{M}_{\zeta X})_r &= \frac{1}{N}\sum_{n=1}^{N} \zeta_r(S_T^n)e^{2X_T^n} \end{aligned} \tag{27}$$

However, those estimations can be high-biased because the same paths are used to compute the regression coefficients are used to the conditional expectation. [2] treats the analytical removal of the bias and the use of Bessel inequality (to prove the under-estimation of variance of e^{2X_t}

Estimation via Monte-Carlo

2 Overview of the methodology proposed in [4]

2.1 Main equations

We have the following dynamics for the FX Spot S

$$\begin{aligned} &\frac{dS_t}{S_t} = \mu_t^d dt + \sqrt{v_t}A(S_t,t)dW_t^{(1)} \\ &v_t \text{ is a stochastic process : Heston or expoonential-OU,...} \\ &dv_t = \kappa(m(t)-v_t)dt + \alpha\sqrt{v_t}dW_t^{(2)} \text{ for Heston} \\ &d\langle W^{(1)},W^{(2)}\rangle_t = \rho dt \end{aligned} \tag{28}$$

2.1.1 PDE for volatility as Heston process

Let $p(y,v,t)$ be the joint probability distribution for log-spot y and stochastic variance v at time t.

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2\left[vA^2(y,t)p\right]}{\partial y^2} + \rho\alpha\frac{\partial^2\left[vA(y,t)p\right]}{\partial y\partial v} + \frac{1}{2}\alpha^2\frac{\partial^2\left[v^2p\right]}{\partial v^2} + \frac{\partial}{\partial y}\left[\left(\frac{1}{2}vA^2(y,t)-\mu_t^d\right)p\right] + \kappa\frac{\partial\left[\left(v-m_t\right)p\right]}{\partial v} \tag{29}$$

It then reduces to

$$
\begin{aligned}
\frac{\partial p}{\partial t} =& \frac{A^2 v}{2}\frac{\partial^2 p}{\partial y^2} + \rho\alpha A v \frac{\partial^2 p}{\partial y \partial v} + \frac{\alpha^2 v}{2}\frac{\partial^2 p}{\partial v^2} + \left(\frac{1}{2}A^2 v + 2AA'v + \rho\alpha A - \mu_t^d\right)\frac{\partial p}{\partial y} \\
&+ \left(\kappa\left[v - m_t\right] + \alpha^2 + \rho\alpha A' v\right)\frac{\partial p}{\partial v} + \Big[AA'v + \rho\alpha A' + AA''v + \left[A'\right]^2 + \kappa\Big] p
\end{aligned} \tag{30}
$$

where

$$
\begin{aligned}
A &= A(y,t) \\
A' &= \frac{\partial A}{\partial y} \\
A'' &= \frac{\partial^2 A}{\partial y^2}
\end{aligned} \tag{31}
$$

2.1.2 PDE for volatility as exponential of Ornstein-Ulhenbeck process

2.2 Forward induction for local volatility calibration on SLV - PDE Calibration

Let us take the methodology developed in [4].
Let us assume of sequence of timepoints $\{t_0 = 0, t_1, t_2, \dots, t_N = T_{cal}\}$. The approach is somewhat like boostrapping a yield curve but we are trying to infer a surface $A(y,t)$

1. Start at time $t_0 = 0$ with an initial local volatility correction $\forall y, A(y,0) = 1$ and an initial condition

$$
p(y,v,0) = \delta_{\{y-y_0\}}\delta_{\{v-v_0\}} \tag{32}
$$

2. Construct a forward timestepping scheme for (30) using a finite scheme for the PDE such as either explicit finite differencing or ADI timestepping. Spatial derivatives A' and A'' can be estimated at time t by setting a function $f(y) = A(y,t)$ equal to a cubic spline and extracting first and second-order derivative by symbolic differentiation of $f(y)$

3. Refine $A(y,t)$ by the use of (21) by calculating I_0 and I_1 integrals defined in (22) at each required level of y, taking numerical integral in the variance dimension. Then update the diffusion, convection and force terms in (30)

3 Solving the SLV calibration PDE

In this section we will develop numerical methods to solve equation (17)

References

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