Stochastic Local Volatility

Alexandre Humeau

March 24, 2014

1 The Model

1.1 General description

Let T be a finite time horizon (the final maturity of the deal we want to price, for example) Let S be an asset of drift μ_t under the domestic risk-neutral probability measure. We assume the following dynamics under $\mathbb{Q}_d$

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(S_t, t)e^{X_t}dW_t \tag{1}$$

where $(S, t) \mapsto \sigma(S, t)$ is the local volatility function (deterministic function of spot and time, calibrated to the vanilla option market at time 0. We will develop later a methodology to calibrate that function σ. X is an Ornstein-Ulhenbeck process so that :

$$\forall t \in [0, T], \mathbb{E}\left[e^{X_t}\right] = 1 \tag{2}$$

The goal of the stochastic volatility would be to calibrate to the forward smile (to be consistent with the

1.2 Stochastic volatility part

Let us assume that X follows the following dynamics under the domestic risk-neutral probability measure

$$dX_t = (\theta(t) - \lambda X_t)dt + \sigma_X dW_t^X$$
$$X_0 = 1 \tag{3}$$

The coefficients are kept constant for now for simplicity.
We have

$$X_t = e^{-\lambda t} + \int_0^t \theta(s)e^{-\lambda(t-s)}ds + \int_0^t \sigma_X e^{-\lambda(t-s)}dW_s^X \tag{4}$$

Then,

$$\exp\left(e^{-\lambda t} + \int_0^t \theta(s)e^{-\lambda(t-s)}ds + \frac{1}{2}\int_0^t \sigma_X^2 e^{-2\lambda(t-s)}ds\right) = 1$$

$$e^{-\lambda t} + \int_0^t \theta(s)e^{-\lambda(t-s)}ds + \frac{1}{2}\int_0^t \sigma_X^2 e^{-2\lambda(t-s)}ds = 0$$

$$e^{-\lambda t} + \int_0^t \theta(s)e^{\lambda s}ds e^{-\lambda t} + \frac{1}{2}\int_0^t \sigma_X^2 e^{-2\lambda(t-s)}ds = 0$$

$$-\lambda e^{-\lambda t} - \int_0^t \theta(s)e^{\lambda s}ds\lambda e^{-\lambda t} + \sigma_X^2 e^{-2\lambda t} + \theta(t) = 0$$

Setting $f(t) = \theta(t)e^{\lambda t}$ it yields

$$\lambda \int_0^t f(s)ds + f(t) = c(t)$$

$$c(t) = \lambda - \frac{\sigma_X^2}{2}e^{-\lambda t} \tag{5}$$

Then f satisfies the following ODE

$$f'(t) + \lambda f(t) = -\frac{\sigma_X^2 \lambda}{2} e^{-\lambda t}$$
$$f(0) = \lambda - \frac{\sigma_X^2}{2}$$

(6)

Then

$$f(t) = C e^{-\lambda t} - \frac{\sigma_X^2 \lambda}{2} t e^{-\lambda t}$$
$$= \left(\lambda - \frac{\sigma_X^2}{2}\right) e^{-\lambda t} - \frac{\sigma_X^2 \lambda}{2} t e^{-\lambda t}$$
$$f(t) = \lambda e^{-\lambda t} - \frac{\sigma_X^2}{2}(1 + \lambda t) e^{-\lambda t}$$

(7)

Then

$$\theta(t) = \lambda e^{-2\lambda t} - \frac{\sigma_X^2}{2}(1 + \lambda t) e^{-2\lambda t}$$

(8)

The stochastic volatility X can be computed quite easily and be simulated exactly

$$X_t \sim \mathcal{N}\left(e^{-\lambda t} + \int_0^t \theta(s) e^{-\lambda(t-s)} ds, \sigma^2 \frac{1 - e^{-2\lambda t}}{2\lambda}\right)$$

(9)

1.3 Calibration of the local volatility

Theorem 1.1 (Markovian Projection)**.** *Put here the theorem of the markovian projection of general volatility models into a local volatility*

In our case, we have

$$\sigma_D^2(x,t) = \mathbb{E}\left[e^{2X_t} \sigma^2(S_t, t) \big| S_t = S\right]$$
$$= \mathbb{E}\left[e^{2X_t} \big| S_t = S\right] \sigma^2(S, t)$$

(10)

1.3.1 Calibration via a PDE

Let us introduce the Green function $G(S_0, X_0, t_0; S, X, t)$. Let us set $b(X) = e^X$. G satisfies

$$\frac{\partial G}{\partial T} - \frac{1}{2} \frac{\partial^2}{\partial S^2}\left(\sigma^2 b^2 S^2 G\right) - \frac{\partial^2}{\partial X^2}\left(\sigma_X^2 G\right) - \frac{\partial^2}{\partial S \partial X}\left(\rho b \sigma S \sigma_X G\right) + \frac{\partial}{\partial X}\left((\theta - \lambda X)G\right) + \frac{\partial}{\partial S}(\mu S G) = 0$$

(11)

Let us assume (in relation of the previous section), that

1. σ_X, θ and λ are not functions of X

2. μ is not a function of S

3. ρ is neither a function of S nor X

The calibration PDE degenerates into

$$\frac{\partial G}{\partial T} - \frac{1}{2} e^{2X} \frac{\partial^2}{\partial S^2}\left(\sigma^2 S^2 G\right) - \sigma_X^2 \frac{\partial^2}{\partial X^2}(G) - \rho \frac{\partial^2}{\partial S \partial X}\left(b \sigma S \sigma_X G\right) + (\theta - \lambda X)\frac{\partial}{\partial X}(G) + \mu \frac{\partial}{\partial S}(SG) = 0$$

(12)

Let us denote by Q the density of S

$$Q(S,t) = \int_{\mathbb{R}} G(S, X, t) dX$$

(13)

Then

$$\frac{\partial Q}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma^2 S^2 \int_{\mathbb{R}} e^{2X} G(S,X,t)dX\right) + \mu\frac{\partial}{\partial S}(SG) = 0 \tag{14}$$

Thus,

$$\frac{\partial Q}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma^2 \frac{\int_{\mathbb{R}} e^{2X} G(S,X,t)dX}{\int_{\mathbb{R}} G(X,S,t)dX} S^2 Q\right) + \mu\frac{\partial}{\partial S}(SG) = 0 \tag{15}$$

We recognize the Dupire equation which defines a local volatility function $\sigma_D(S,t)$

$$\sigma_D^2(S,t) = \sigma^2(S,t)\frac{\int_{\mathbb{R}} e^{2X} G(S,X,t)dX}{\int_{\mathbb{R}} G(X,S,t)dX} \tag{16}$$

which permits to re-write the calibration PDE

$$\frac{\partial G}{\partial T} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(S^2 e^{2X}\sigma_D^2 \frac{\int_{\mathbb{R}} G(S,X,t)dX}{\int_{\mathbb{R}} e^{2X} G(S,X,t)dX}\right) + \mu\frac{\partial}{\partial S}(SG) - \frac{\sigma_X^2}{2}\frac{\partial^2 G}{\partial X^2} + \frac{\partial}{\partial X}((\theta - \lambda X)G)$$
$$- \rho\sigma_X\frac{\partial^2}{\partial S\partial X}\left(e^X S\sigma_D\sqrt{\frac{\int_{\mathbb{R}} G(S,X,t)dX}{\int_{\mathbb{R}} e^{2X} G(S,X,t)dX}}G\right) = 0 \tag{17}$$

The only SLV models tractables with a PDE calibration are then mono-asset SLV models. We will develop in the following another method which enable us to calibrate via Monte-Carlo SLV Models. However, a Monte-Carlo will have more noise than a PDE calibration.

1.3.2 Calibration by Monte-Carlo

In this section, we will develop a method to calibrate the SLV models in a higher dimension problem using a Monte-Carlo simulation.

2 Solving the SLV calibration PDE

In this section we will develop numerical methods to solve equation (17)

References

[1] Frédéric Abergel, Rémi Tachet, *A Non-linear integro-differential equation from Mathematical Finance*. AIMS Journal, Version 1, 8 sept 2011