Smile Interpolation

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1 Motivations

Using a smile is useful to define the distribution of a financial underlying at a given date

Theorem 1.1 (Breeden-Litzenberger formula)**.** *Let S be a financial asset. Let ϕ be the probability density function of S_T. Then*

$$\phi(K) = \frac{d^2 C^T}{dK^2} \tag{1}$$

where C^T is the T-forward price of a call of strike K

Let us assume that we have a smooth parametrisation of the smile $K \mapsto \sigma(K)$ for a given maturity T.

Proposition 1.2. *With the above notations*

$$\phi(K) = \frac{\partial^2 C^T}{\partial K^2} + \sigma''(K)\frac{\partial C^T}{\partial \sigma} + \sigma'(K)^2 \frac{\partial^2 C^T}{\partial \sigma^2} + 2\sigma'(K)\frac{\partial^2 C^T}{\partial \sigma \partial K} \tag{2}$$

If we want the density function to be a $\mathcal{C}^1$ function of the strike K, $K \mapsto \sigma(K)$ has to be at least a $\mathcal{C}^3$ function

2 Interpolation via a polynomial function of degree 5

2.1 Hermite Interpolation

A polynomial function of degree 5 between 0 and 1 can be determined by the values in 0 and 1, the first derivative in 0 and 1 and the second derivative in 0 and 1.
We have

$$f(x) = f(0)H_0^0(x) + f(1)H_1^0(x) + f'(0)H_0^1(x) + f'(1)H_1^1(x) + f''(0)H_0^2(x) + f''(1)H_1^2(x) \tag{3}$$

where

$$\begin{aligned}
H_0^0(x) &= 1 - 10x^3 + 15x^4 - 6x^5 \\
H_0^1(x) &= t - 6x^3 + 8x^4 - 3x^5 \\
H_0^2(x) &= \frac{1}{2}\left(x^2 - 3x^3 + 3x^4 - x^5\right) \\
H_1^0(x) &= 10x^3 - 15x^4 + 6x^5 \\
H_1^1(x) &= -4x^3 + 7x^4 - 3x^5 \\
H_1^2(x) &= \frac{1}{2}\left(x^3 - 2x^4 + x^5\right)
\end{aligned} \tag{4}$$

2.2 Smile interpolation methodology

We assume that the function $K \mapsto \sigma(K)$ is a piecewise Hermite-interpolated polynomial of degree as a function of $\ln\left(\frac{K}{F}\right)$, where is $F = F_{0,T}$ is the forward at time 0 of the underlying asset.

Let us assume that we have the set of quoted volatilities

$$\mathcal{V} = \{(K_i, \sigma_i), i = 1, \ldots, n\}$$