Linear Gaussian Markov Model with Stochastic volatility

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1 The model

1.1 Why adding stochastic volatility

- Replication tentative of the swaption smile for pricing of more complex interest-rate derivatives

1.2 Main equations

Let $f(t,T)$ be the instantaneous forward rate between t and T, we assume without loss of any generality under the risk neutral measure $\mathbb{Q}$ (depending on the currency)

$$\mathrm{d}f(t,T) = \mu_f(t,T)\mathrm{d}t + \sigma_f(t,T) \cdot \left(\sqrt{v}\mathrm{d}W_t\right) \tag{1}$$

where $\cdot$ represents the scalar product and v is a stochastic volatility process that we will define later in the document.

Remark 1.1. *When dealing with multi-currency frameworks, I will write a separate document as it depends highly on the model put on the FX.*

Remark 1.2. *The number of factors N in the rate model is equal to the dimensionality of the considered vectors $v = (v_1, \dots, v_N)$, $\sigma_f = (\sigma_{f,1}, \dots, \sigma_{f,N})$*

Proposition 1.1 (Heath-Jarrow-Morton 1992)**.** *The absence of arbitrage imples that the drift term μ_f is equal to*

$$\mu_f(t,T) = \sum_{i=1}^{N} v_i(t)\sigma_{f,i}(t,T) \int_t^T \sigma_{f,i}(t,u)du \tag{2}$$

Remark 1.3. *For a general specification of $\sigma_{f,i}(t,T)$, the dynamics of the forward rate curve will be path-dependent, which significantly complicates derivatives pricing and the application of standard econometric techniques*

Proposition 1.2 (Ritchken-Sankarasubramanian)**.** *If the volatility vector of functions can be factored as*

$$\sigma_{f,i}(t,T) = \mathcal{P}_n(T-t)e^{-\int_t^T \lambda_u du} \tag{3}$$

with a regular function $u \mapsto \lambda_u$, and $\tau \mapsto \mathcal{P}(\tau)$ a polynomial function of the variable, then the Heath-Jarrow-Morton model is markovian.

Proof. Check if all the conditions are reunited, and check the regularity of the function □

Remark 1.4. *For simplicity in the calibration, we will set the degree of the polynomial to be 1 and we are left with the following formula,*

$$\sigma_{f,i}(t,T) = \left(\alpha_{1,i} + \alpha_{2,i}(T-t)\right) e^{-\lambda_i(T-t)} \tag{4}$$

we can also consider a term-structure of coefficients and have the follozing formula

$$\sigma_{f,i}(t,T) = \alpha_{1,i}(t)e^{-\lambda_i(T-t)} \tag{5}$$

2 Stochastic volatility models

2.1 Log-normal stochastic volatility

2.2 Heston-like stochastic volatility

References

[1] A General Stochastic Volatility Model for the Pricing of Interest Rate Derivatives, published by the Oxford University Press on behalf of The Society for Financial Studies