Brownian Motion calculation

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1 Problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Let $W = (W_t)_{t\geq 0}$ be a Wiener process on this probability space. Let $0 < a < b$ two positive numbers

We would like to compute the following probability

$$\mathbb{P}\left(\exists t \in [a,b], W_t = 0\right) \tag{1}$$

2 Auxiliaries

Definition 2.1. *Let $x \in \mathbb{R}$. Let τ_x be the first time when the first time when the Wiener process reaches x*

$$\tau_x = \min\left(t > 0, W_t = x\right) \tag{2}$$

Proposition 2.1. *The probability density function of τ_x is*

$$f_{\tau_x}(y) = \frac{|x|}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right) \tag{3}$$

Proof.

$$\begin{aligned}
\mathbb{P}\left(\tau_x < t\right) &= \mathbb{P}\left(\sup_{s\leq t} W_s > x\right) \\
&= \mathbb{P}\left(|W_t| > x\right) \\
&= \mathbb{P}\left(W_t^2 > x^2\right) \\
&= \mathbb{P}\left(tW_1^2 > x^2\right) \\
&= \mathbb{P}\left(\frac{x^2}{W_1^2} < t\right)
\end{aligned}$$

□

Corollary 2.2. *τ_x and $\frac{x^2}{W_1^2}$ have same law*

Proposition 2.3.