Stochastic Local Volatility

Alexandre Humeau

March 31, 2014

1 The Model

1.1 General description

Let T be a finite time horizon (the final maturity of the deal we want to price, for example) Let S be an asset of drift μ_t under the domestic risk-neutral probability measure. We assume the following dynamics under $\mathbb{Q}_d$

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(S_t, t) e^{X_t} dW_t \tag{1}$$

where $(S, t) \mapsto \sigma(S, t)$ is the local volatility function (deterministic function of spot and time, calibrated to the vanilla option market at time 0. We will develop later a methodology to calibrate that function σ. X is an Ornstein-Ulhenbeck process so that :

$$\forall t \in [0, T], \mathbb{E}\left[e^{2X_t}\right] = 1 \tag{2}$$

The goal of the stochastic volatility would be to calibrate to the forward smile (to be consistent with the

1.2 Stochastic volatility part

Let us assume that X follows the following dynamics under the domestic risk-neutral probability measure

$$\begin{aligned} dX_t &= (\theta(t) - \lambda X_t)dt + \sigma_X dW_t^X \\ X_0 &= 1 \end{aligned} \tag{3}$$

The coefficients are kept constant for now for simplicity.
We have

$$X_t = e^{-\lambda t} + \int_0^t \theta(s) e^{-\lambda(t-s)} ds + \int_0^t \sigma_X e^{-\lambda(t-s)} dW_s^X \tag{4}$$

Then,

$$\begin{aligned} \exp\left(2e^{-\lambda t} + 2\int_0^t \theta(s) e^{-\lambda(t-s)} ds + 2\int_0^t \sigma_X^2 e^{-2\lambda(t-s)} ds\right) &= 1 \\ e^{-\lambda t} + \int_0^t \theta(s) e^{-\lambda(t-s)} ds + \int_0^t \sigma_X^2 e^{-2\lambda(t-s)} ds &= 0 \\ e^{-\lambda t} + \int_0^t \theta(s) e^{\lambda s} ds e^{-\lambda t} + \int_0^t \sigma_X^2 e^{-2\lambda(t-s)} ds &= 0 \\ -\lambda e^{-\lambda t} - \int_0^t \theta(s) e^{\lambda s} ds \lambda e^{-\lambda t} + 2\sigma_X^2 e^{-2\lambda t} + \theta(t) &= 0 \end{aligned}$$

Setting $f(t) = \theta(t) e^{\lambda t}$ it yields

$$\begin{aligned} \lambda \int_0^t f(s) ds + f(t) &= c(t) \\ c(t) &= \lambda - 2\sigma_X^2 e^{-\lambda t} \end{aligned} \tag{5}$$

Then f satisfies the following ODE

$$\begin{aligned} f'(t) + \lambda f(t) &= -2\sigma_X^2 \lambda e^{-\lambda t} \\ f(0) &= \lambda - 2\sigma_X^2 \end{aligned} \tag{6}$$

Then

$$\begin{aligned} f(t) &= Ce^{-\lambda t} - 2\sigma_X^2 \lambda t e^{-\lambda t} \\ &= \left(\lambda - 2\sigma_X^2\right) e^{-\lambda t} - 2\sigma_X^2 \lambda t e^{-\lambda t} \\ f(t) &= \lambda e^{-\lambda t} - 2\sigma_X^2 (1 + \lambda t) e^{-\lambda t} \end{aligned} \tag{7}$$

Then

$$\theta(t) = \lambda e^{-2\lambda t} - 2\sigma_X^2 (1 + \lambda t) e^{-2\lambda t} \tag{8}$$

The stochastic volatility X can be computed quite easily and be simulated exactly

$$X_t \sim \mathcal{N}\left(e^{-\lambda t} + \int_0^t \theta(s) e^{-\lambda(t-s)} ds, \sigma^2 \frac{1 - e^{-2\lambda t}}{2\lambda}\right) \tag{9}$$

1.3 Calibration of the local volatility

Theorem 1.1 (Markovian Projection)**.** *Put here the theorem of the markovian projection of general volatility models into a local volatility*

In our case, we have

$$\begin{aligned} \sigma_D^2(x,t) &= \mathbb{E}\left[e^{2X_t} \sigma^2(S_t, t) \middle| S_t = S\right] \\ &= \mathbb{E}\left[e^{2X_t} \middle| S_t = S\right] \sigma^2(S,t) \end{aligned} \tag{10}$$

1.3.1 Calibration via a PDE

Let us introduce the Green function $G(S_0, X_0, t_0; S, X, t)$. Let us set $b(X) = e^X$. G satisfies

$$\frac{\partial G}{\partial T} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma^2 b^2 S^2 G\right) - \frac{\partial^2}{\partial X^2}\left(\sigma_X^2 G\right) - \frac{\partial^2}{\partial S \partial X}\left(\rho b \sigma S \sigma_X G\right) + \frac{\partial}{\partial X}\left((\theta - \lambda X) G\right) + \frac{\partial}{\partial S}(\mu S G) = 0 \tag{11}$$

Let us assume (in relation of the previous section), that

1. σ_X, θ and λ are not functions of X
2. μ is not a function of S
3. ρ is neither a function of S nor X

The calibration PDE degenerates into

$$\frac{\partial G}{\partial T} - \frac{1}{2} e^{2X} \frac{\partial^2}{\partial S^2}\left(\sigma^2 S^2 G\right) - \sigma_X^2 \frac{\partial^2}{\partial X^2}(G) - \rho \frac{\partial^2}{\partial S \partial X}\left(b \sigma S \sigma_X G\right) + (\theta - \lambda X) \frac{\partial}{\partial X}(G) + \mu \frac{\partial}{\partial S}(SG) = 0 \tag{12}$$

Let us denote by Q the density of S

$$Q(S,t) = \int_{\mathbb{R}} G(S,X,t) dX \tag{13}$$

Then

$$\frac{\partial Q}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma^2 S^2 \int_{\mathbb{R}} e^{2X} G(S,X,t) dX\right) + \mu \frac{\partial}{\partial S}(SG) = 0 \tag{14}$$

Thus,

$$\frac{\partial Q}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma^2 \frac{\int_{\mathbb{R}} e^{2X} G(S,X,t)dX}{\int_{\mathbb{R}} G(X,S,t)dX} S^2 Q\right) + \mu \frac{\partial}{\partial S}(SG) = 0 \tag{15}$$

We recognize the Dupire equation which defines a local volatility function $\sigma_D(S,t)$

$$\sigma_D^2(S,t) = \sigma^2(S,t)\frac{\int_{\mathbb{R}} e^{2X} G(S,X,t)dX}{\int_{\mathbb{R}} G(X,S,t)dX} \tag{16}$$

which permits to re-write the calibration PDE

$$\begin{aligned} &\frac{\partial G}{\partial T} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(S^2 e^{2X}\sigma_D^2 \frac{\int_{\mathbb{R}} G(S,X,t)dX}{\int_{\mathbb{R}} e^{2X} G(S,X,t)dX}\right) + \mu\frac{\partial}{\partial S}(SG) - \frac{\sigma_X^2}{2}\frac{\partial^2 G}{\partial X^2} + \frac{\partial}{\partial X}((\theta - \lambda X)G) \\ &- \rho\sigma_X \frac{\partial^2}{\partial S \partial X}\left(e^X S \sigma_D \sqrt{\frac{\int_{\mathbb{R}} G(S,X,t)dX}{\int_{\mathbb{R}} e^{2X} G(S,X,t)dX}} G\right) = 0 \end{aligned} \tag{17}$$

The only SLV models tractables with a PDE calibration are then mono-asset SLV models. We will develop in the following another method which enable us to calibrate via Monte-Carlo SLV Models. However, a Monte-Carlo will have more noise than a PDE calibration.

1.3.2 Calibration by Monte-Carlo

In this section, we will develop a method to calibrate the SLV models in a higher dimension problem using a Monte-Carlo simulation.

Let us re-write the Dupire equation associated with the SLV model we just described

$$\begin{aligned} &\frac{\partial C}{\partial T} + P_d(0,T)\mathbb{E}^{\mathbb{Q}_T}\left[(Kr_d(T) - S_T r_f(T))\mathbf{1}_{\{S_T>K\}}\right] - \frac{1}{2}\sigma_D^2(K,T)K^2\frac{\partial^2 C}{\partial K^2} = 0 \\ &\frac{\partial C}{\partial T} + P_d(0,T)\mathbb{E}^{\mathbb{Q}_T}\left[(Kr_d(T) - S_T r_f(T))\mathbf{1}_{\{S_T>K\}}\right] - \frac{1}{2}\sigma^2(K,T)\mathbb{E}\left[e^{2X_T}\middle|S_T = K\right]K^2\frac{\partial^2 C}{\partial K^2} = 0 \end{aligned} \tag{18}$$

To write the calibration equation, we are left with, the calculation of the following integral

$$\mathbb{E}\left[e^{2X_T}\middle|S_T = K\right] = \frac{\mathbb{E}\left[e^{2X_T}\mathbf{1}_{S_T=K}\right]}{\mathbb{P}(S_T = K)} \tag{19}$$

Two Monte-Carlo method can then be used for the computation of this conditional expectation

1. Estimation of the integral via a full Monte-Carlo simulation of (S_T, X_T)
2. Approximation of the conditional expectation via a Least-Square approach using well-choosen polynomial functions

Estimation via Least-Squares As we want to compute a S_T-measurable integral, we can approximate

$$\mathbb{E}\left[e^{2X_T}\middle|S_T\right] \approx \sum_{m=1}^{M}\beta_m\zeta_m(S_T) \tag{20}$$

where the coefficients $(\beta_1, \ldots, \beta_M)$ are found by a least-square approach. Let us assume we have N realisations of the random variables (S_T, X_T) denoted by $\left((S_T^1, X_T^1), \ldots, (S_T^N, X_T^N)\right)$
Let us write the minimization problem associated with Least-Square in our case. Let us set $\beta = (\beta_1, \ldots, \beta_M)$

$$\hat{\beta} = \underset{\beta\in\mathbb{R}^M}{\operatorname{argmin}}\left\{\mathbb{E}\left[\left(\mathbb{E}\left[e^{2X_T}\middle|S_T\right] - \sum_{m=1}^{M}\beta_m\zeta_m(S_T)\right)^2\right]\right\} \tag{21}$$

Denoting by $M_{\zeta\zeta}$ and $M_{\zeta X}$ the following quantities, according to [?]

$$\begin{aligned} (M_{\zeta\zeta})_{r,s} &= \mathbb{E}\left[\zeta_r(S_T)\zeta_s(S_T)\right] \\ (M_{\zeta X})_r &= \mathbb{E}\left[\zeta_r(S_T)e^{2X_T}\right] \end{aligned} \tag{22}$$

we have the following estimation for the "regression coefficients" β

$$\beta = (M_{\zeta\zeta})^{-1} M_{\zeta X} \tag{23}$$

Those quantities can be estimated using the Monte-Carlo paths $\big((S_T^1, X_T^1), \ldots, (S_T^N, X_T^N)\big)$

$$\begin{aligned}(\hat{M}_{\zeta\zeta})_{r,s} &= \frac{1}{N}\sum_{n=1}^{N} \zeta_r(S_T^n)\zeta_s(S_T^n) \\ (\hat{M}_{\zeta X})_r &= \frac{1}{N}\sum_{n=1}^{N} \zeta_r(S_T^n) e^{2X_T^n}\end{aligned} \tag{24}$$

However, those estimations can be high-biased because the same paths are used to compute the regression coefficients are used to the conditional expectation.

Estimation via Monte-Carlo

2 Solving the SLV calibration PDE

In this section we will develop numerical methods to solve equation (**??**)

References

[1] Frédéric Abergel, Rémi Tachet, *A Non-linear integro-differential equation from Mathematical Finance*, AIMS Journal, Version 1, 8th Sept. 2011

[2] Alexandre Humeau, *Credit Valuation Adjustment*, Master's thesis Ecole Centrale Paris Version 2, 1st November 2013

[3] Anthonie W. Van Der Stoep, Lech A. Grzelak, Cornelis W. Oosterlee, *The Heston Stochastic-Local volatility Model : Efficient Monte-Carlo simulation*, Version 1, 11th June 2013