Analytical formula for an Asian option

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March 2, 2014

1 Payoff

Let us consider an Asian european option (call or put). The payoff of such option in T is

$$\left(\omega\left(\bar{S}-K\right)\right)^{+} \tag{1}$$

where

$$\bar{S}=\frac{1}{n}\sum_{i=1}^{n}S_{t_i} \tag{2}$$

with $0=t_0<t_1<t_2<\cdots<t_n\leq T$.

2 Model

Let us assume that the stock price S follows a GBM under the risk neutral probability

$$\frac{dS_t}{S_t}=rdt+\sigma dW_t \tag{3}$$

Then

$$\forall i\in[1,n], S_{t_i}=S_0e^{(r-\frac{\sigma^2}{2})t_i+\sigma W_{t_i}} \tag{4}$$

2.1 Moment matching method

2.1.1 Motivations

As we want to find a formula in the Black-Scholes world for the Asian option, let us try to fit a Black-Scholes lognormal underlying for $\bar{S}$.

$$\frac{d\bar{S}_t}{\bar{S}_t}=\bar{\mu}dt+\bar{\sigma}dW_t \tag{5}$$

2.1.2 Calculation

$$\begin{aligned}\mathbb{E}\left[\bar{S}\right]&=S_0e^{\bar{\mu}T}\\&=\frac{1}{n}\sum_{i=1}^{n}S_0\mathbb{E}\left[S_{t_i}\right]\\&=\frac{1}{n}\sum_{i=1}^{n}S_0e^{rt_i}\end{aligned} \tag{6}$$

Then

$$\bar{\mu}=\frac{1}{T}\ln\left(\frac{1}{n}\sum_{i=1}^{n}e^{rt_i}\right) \tag{7}$$

$$
\begin{aligned}
Var\left(\bar{S}\right) &= S_0^2 e^{2\bar{\mu}T}\left(e^{\bar{\sigma}^2 T} - 1\right) \\
&= \frac{1}{n}\left(\sum_{i=1}^{n} Var\left(S_{t_i}\right) + 2\sum_{i<j=1}^{n} Cov\left(S_{t_i}, S_{t_j}\right)\right) \\
&= \frac{1}{n}\left(\sum_{i=1}^{n} S_0 e^{\left(2r+\sigma^2\right)t_i} + 2\sum_{i<j=1}^{n} Cov\left(S_{t_i}, S_{t_j}\right)\right)
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
Cov\left(S_{t_i}, S_{t_j}\right) &= \mathbb{E}\left[S_{t_i} S_{t_j}\right] - \mathbb{E}\left[S_{t_i}\right]\mathbb{E}\left[S_{t_j}\right] \\
&= \mathbb{E}\left[S_{t_i}\mathbb{E}\left[S_{t_j}\middle|S_{t_i}\right]\right] - \mathbb{E}\left[S_{t_i}\right]\mathbb{E}\left[S_{t_j}\right] \\
&= \mathbb{E}\left[S_{t_i}^2\mathbb{E}\left[e^{\left(r-\frac{\sigma^2}{2}\right)(t_j-t_i)+\sigma\left(W_{t_j}-W_{t_i}\right)}\middle|S_{t_i}\right]\right] - \mathbb{E}\left[S_{t_i}\right]\mathbb{E}\left[S_{t_j}\right] \\
&= e^{r(t_j-t_i)}\mathbb{E}\left[S_{t_i}^2\right] - S_0^2 e^{r(t_i+t_j)} \\
&= e^{r(t_j-t_i)} S_0^2 e^{\left(2r+\sigma^2\right)t_i} - S_0^2 e^{r(t_i+t_j)} \\
&= S_0^2 e^{r(t_i+t_j)}\left(e^{\sigma^2 t_i} - 1\right)
\end{aligned} \tag{9}
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