Libor Market Model

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November 16, 2014

1 Derivation

Let us consider a set of dates $(T_i = T_\alpha + (i-\alpha)\delta)_{i=\alpha,\dots,\beta}$ and the Libor associated to the period $[T_i, T_{i+1}]$

$$L_t^i = \frac{1}{\delta}\left(\frac{B\left(t,T_i\right)}{B\left(t,T_{i+1}\right)} - 1\right)$$

The T_i-forward probability measure is denoted by $\mathbb{Q}^i$ (associated to the numeraire $B(t,T_i)$.

Proposition 1.1. *L_t^i is a lognormal martingale of volatiliy $\sigma_i(t) = c_i g(T_i - t)$ under $\mathbb{Q}^{i+1}$*

Which probability can we take to price a product dependent on all the libors ? $\mathbb{Q}^\beta$ the terminal probability.

$$\frac{dL_t^i}{L_t^i} = \gamma^i(t)dW_t + \mathrm{drift}_t^i dt \tag{1}$$

where γ^i is a vectorial volatility function such that $\sigma_i(t) = \|\gamma^i(t)\|$ (function from $\mathbb{R}_+$ to $\mathbb{R}^d$) and W a d-dimensional Wiener process

$$\left.\frac{d\mathbb{Q}^{i+1}}{d\mathbb{Q}^i}\right|_{\mathcal{F}_{T_i}} = \frac{B\left(T_i,T_{i+1}\right)}{B\left(T_i,T_i\right)}\frac{B\left(0,T_i\right)}{B\left(0,T_{i+1}\right)} = \frac{1}{1+\delta L_{T_i}^i}\frac{B\left(0,T_i\right)}{B\left(0,T_{i+1}\right)}$$

$$\xi_t^i = \mathbb{E}^{\mathbb{Q}^i}\left[\frac{d\mathbb{Q}^{i+1}}{d\mathbb{Q}^i}\middle|\mathcal{F}_t\right] = \frac{1}{1+\delta L_t^i}\frac{B\left(0,T_i\right)}{B\left(0,T_{i+1}\right)}$$

Proposition 1.2. *Let X_t be a $\mathbb{Q}^{i+1}$ martingale, then $X_t\xi_t^i$ is a $\mathbb{Q}^i$ martingale*

We know that $L^{\beta-2}$ is a $\mathbb{Q}^{\beta-1}$ martingale, then $L^{\beta-2}\left(1+\delta L^{\beta-1}\right)$ is a $\mathbb{Q}^\beta$ martingale. Applying Itô's formula yields

$$\left(1+\delta L_t^{\beta-1}\right)dL_t^{\beta-2} + \delta L_t^{\beta-2}dL_t^{\beta-1} + \delta d\langle L^{\beta-2}, L^{\beta-1}\rangle_t$$

Taking the drifts,

$$\left(1+\delta L_t^{\beta-1}\right)\mathrm{drift}_t^{\beta-2}L_t^{\beta-2} + \delta\gamma^{\beta-1}(t)\cdot\gamma^{\beta-2}(t)L_t^{\beta-1}L_t^{\beta-2} = 0$$

since $\mathrm{drift}_t^{\beta-1} = 0$. Then

$$\mathrm{drift}_t^{\beta-2} = -\frac{\delta\gamma^{\beta-1}(t)\cdot\gamma^{\beta-2}(t)L_t^{\beta-1}}{\left(1+\delta L_t^{\beta-1}\right)} \tag{2}$$

The general formula is then the following

Theorem 1.3.

$$\begin{aligned}\forall k\in[\alpha,\beta],\ \mathit{drift}_t^k &= -\delta\sum_{i=k+1}^{\beta-1}\frac{L_t^i\gamma^k(t)\cdot\gamma^i(t)}{1+\delta L_t^i}\\ &= -\delta\sum_{i=k+1}^{\beta-1}\frac{L_t^i\rho^{i,k}(t)\sigma^k(t)\sigma^i(t)}{1+\delta L_t^i}\end{aligned} \tag{3}$$

where $\rho^{i,k}(t)$ is the correlation between libors i and k

2 Calibration

Proposition 2.1. *We have the following pricing formula for the caplet of strike K on the i-th Libor*

$$\begin{aligned} C_i(0) &= \delta B\left(0, T_{i+1}\right) \mathbb{E}^{\mathbb{Q}^{i+1}}\left[\left(L^i_{T_i} - K\right)^+\right] \\ &= \delta B\left(0, T_{i+1}\right) B\&S\left(L^i_0, K, \int_0^{T_i} \sigma_i(s)^2 ds\right) \end{aligned} \tag{4}$$

where $(F, K, V) \mapsto B\&S(F, K, V)$ is the Black-Scholes formula

References

[1] A. Brace, D. Gatarek, M. Musiela, *The market model of interest rate dynamics*, 1997, Mathematical Science, 127-155