

# Stochastic Local Volatility

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## 1 The Model

### 1.1 General description

Let  $T$  be a finite time horizon (the final maturity of the deal we want to price, for example) Let  $S$  be an asset of drift  $\mu_t$  under the domestic risk-neutral probability measure. We assume the following dynamics under  $\mathbb{Q}_d$

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(S_t, t) e^{X_t} dW_t \quad (1)$$

where  $(S, t) \mapsto \sigma(S, t)$  is the local volatility function (deterministic function of spot and time, calibrated to the vanilla option market at time 0. We will develop later a methodology to calibrate that function  $\sigma$ .  $X$  is an Ornstein-Uhlenbeck process so that :

$$\forall t \in [0, T], \mathbb{E}[e^{2X_t}] = 1 \quad (2)$$

The goal of the stochastic volatility would be to calibrate to the forward smile (to be consistent with the

### 1.2 Stochastic volatility part

Let us assume that  $X$  follows the following dynamics under the domestic risk-neutral probability measure

$$\begin{aligned} dX_t &= (\theta(t) - \lambda X_t) dt + \sigma_X dW_t^X \\ X_0 &= 1 \end{aligned} \quad (3)$$

The coefficients are kept constant for now for simplicity.

We have

$$X_t = e^{-\lambda t} + \int_0^t \theta(s) e^{-\lambda(t-s)} ds + \int_0^t \sigma_X e^{-\lambda(t-s)} dW_s^X \quad (4)$$

Then,

$$\begin{aligned} \exp \left( 2e^{-\lambda t} + 2 \int_0^t \theta(s) e^{-\lambda(t-s)} ds + 2 \int_0^t \sigma_X^2 e^{-2\lambda(t-s)} ds \right) &= 1 \\ e^{-\lambda t} + \int_0^t \theta(s) e^{-\lambda(t-s)} ds + \int_0^t \sigma_X^2 e^{-2\lambda(t-s)} ds &= 0 \\ e^{-\lambda t} + \int_0^t \theta(s) e^{\lambda s} ds e^{-\lambda t} + \int_0^t \sigma_X^2 e^{-2\lambda(t-s)} ds &= 0 \\ -\lambda e^{-\lambda t} - \int_0^t \theta(s) e^{\lambda s} ds \lambda e^{-\lambda t} + 2\sigma_X^2 e^{-2\lambda t} + \theta(t) &= 0 \end{aligned}$$

Setting  $f(t) = \theta(t) e^{\lambda t}$  it yields

$$\begin{aligned} \lambda \int_0^t f(s) ds + f(t) &= c(t) \\ c(t) &= \lambda - 2\sigma_X^2 e^{-\lambda t} \end{aligned} \quad (5)$$

Then  $f$  satisfies the following ODE

$$\begin{aligned} f'(t) + \lambda f(t) &= -2\sigma_X^2 \lambda e^{-\lambda t} \\ f(0) &= \lambda - 2\sigma_X^2 \end{aligned} \quad (6)$$

Then

$$\begin{aligned} f(t) &= C e^{-\lambda t} - 2\sigma_X^2 \lambda t e^{-\lambda t} \\ &= (\lambda - 2\sigma_X^2) e^{-\lambda t} - 2\sigma_X^2 \lambda t e^{-\lambda t} \\ f(t) &= \lambda e^{-\lambda t} - 2\sigma_X^2 (1 + \lambda t) e^{-\lambda t} \end{aligned} \quad (7)$$

Then

$$\theta(t) = \lambda e^{-2\lambda t} - 2\sigma_X^2 (1 + \lambda t) e^{-2\lambda t} \quad (8)$$

The stochastic volatility  $X$  can be computed quite easily and be simulated exactly

$$X_t \sim \mathcal{N} \left( e^{-\lambda t} + \int_0^t \theta(s) e^{-\lambda(t-s)} ds, \sigma^2 \frac{1 - e^{-2\lambda t}}{2\lambda} \right) \quad (9)$$

### 1.3 Calibration of the local volatility

**Theorem 1.1** (Markovian Projection). *Put here the theorem of the markovian projection of general volatility models into a local volatility*

In our case, we have

$$\begin{aligned} \sigma_D^2(x, t) &= \mathbb{E} [e^{2X_t} \sigma^2(S_t, t) | S_t = S] \\ &= \mathbb{E} [e^{2X_t} | S_t = S] \sigma^2(S, t) \end{aligned} \quad (10)$$

#### 1.3.1 Calibration via a PDE

Let us introduce the Green function  $G(S_0, X_0, t_0; S, X, t)$ . Let us set  $b(X) = e^X$ .  $G$  satisfies

$$\frac{\partial G}{\partial T} - \frac{1}{2} \frac{\partial^2}{\partial S^2} (\sigma^2 b^2 S^2 G) - \frac{\partial^2}{\partial X^2} (\sigma_X^2 G) - \frac{\partial^2}{\partial S \partial X} (\rho b \sigma S \sigma_X G) + \frac{\partial}{\partial X} ((\theta - \lambda X) G) + \frac{\partial}{\partial S} (\mu S G) = 0 \quad (11)$$

Let us assume (in relation of the previous section), that

1.  $\sigma_X$ ,  $\theta$  and  $\lambda$  are not functions of  $X$
2.  $\mu$  is not a function of  $S$
3.  $\rho$  is neither a function of  $S$  nor  $X$

The calibration PDE degenerates into

$$\frac{\partial G}{\partial T} - \frac{1}{2} e^{2X} \frac{\partial^2}{\partial S^2} (\sigma^2 S^2 G) - \sigma_X^2 \frac{\partial^2}{\partial X^2} (G) - \rho \frac{\partial^2}{\partial S \partial X} (b \sigma S \sigma_X G) + (\theta - \lambda X) \frac{\partial}{\partial X} (G) + \mu \frac{\partial}{\partial S} (S G) = 0 \quad (12)$$

Let us denote by  $Q$  the density of  $S$

$$Q(S, t) = \int_{\mathbb{R}} G(S, X, t) dX \quad (13)$$

Then

$$\frac{\partial Q}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial S^2} \left( \sigma^2 S^2 \int_{\mathbb{R}} e^{2X} G(S, X, t) dX \right) + \mu \frac{\partial}{\partial S} (S Q) = 0 \quad (14)$$

Thus,

$$\frac{\partial Q}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial S^2} \left( \sigma^2 \frac{\int_{\mathbb{R}} e^{2X} G(S, X, t) dX}{\int_{\mathbb{R}} G(X, S, t) dX} S^2 Q \right) + \mu \frac{\partial}{\partial S} (SG) = 0 \quad (15)$$

We recognize the Dupire equation which defines a local volatility function  $\sigma_D(S, t)$

$$\sigma_D^2(S, t) = \sigma^2(S, t) \frac{\int_{\mathbb{R}} e^{2X} G(S, X, t) dX}{\int_{\mathbb{R}} G(X, S, t) dX} \quad (16)$$

which permits to re-write the calibration PDE

$$\begin{aligned} \frac{\partial G}{\partial T} - \frac{1}{2} \frac{\partial^2}{\partial S^2} \left( S^2 e^{2X} \sigma_D^2 \frac{\int_{\mathbb{R}} G(S, X, t) dX}{\int_{\mathbb{R}} e^{2X} G(S, X, t) dX} \right) + \mu \frac{\partial}{\partial S} (SG) - \frac{\sigma_X^2}{2} \frac{\partial^2 G}{\partial X^2} + \frac{\partial}{\partial X} ((\theta - \lambda X)G) \\ - \rho \sigma_X \frac{\partial^2}{\partial S \partial X} \left( e^X S \sigma_D \sqrt{\frac{\int_{\mathbb{R}} G(S, X, t) dX}{\int_{\mathbb{R}} e^{2X} G(S, X, t) dX}} G \right) = 0 \end{aligned} \quad (17)$$

The only SLV models tractables with a PDE calibration are then mono-asset SLV models. We will develop in the following another method which enable us to calibrate via Monte-Carlo SLV Models. However, a Monte-Carlo will have more noise than a PDE calibration.

### 1.3.2 Calibration by Monte-Carlo

In this section, we will develop a method to calibrate the SLV models in a higher dimension problem using a Monte-Carlo simulation.

Let us re-write the Dupire equation associated with the SLV model we just described

$$\begin{aligned} \frac{\partial C}{\partial T} + P_d(0, T) \mathbb{E}^{\mathbb{Q}_T} [(Kr_d(T) - S_T r_f(T)) \mathbf{1}_{\{S_T > K\}}] - \frac{1}{2} \sigma_D^2(K, T) K^2 \frac{\partial^2 C}{\partial K^2} = 0 \\ \frac{\partial C}{\partial T} + P_d(0, T) \mathbb{E}^{\mathbb{Q}_T} [(Kr_d(T) - S_T r_f(T)) \mathbf{1}_{\{S_T > K\}}] - \frac{1}{2} \sigma^2(K, T) \mathbb{E} [e^{2X_T} | S_T = K] K^2 \frac{\partial^2 C}{\partial K^2} = 0 \end{aligned} \quad (18)$$

To write the calibration equation, we are left with, the calculation of the following integral

$$\mathbb{E} [e^{2X_T} | S_T = K] = \frac{\mathbb{E} [e^{2X_T} \mathbf{1}_{S_T=K}]}{\mathbb{P}(S_T = K)} \quad (19)$$

Re-writting the above equation as a function of log-spot  $y_t = \log(S_t)$  yields

$$\mathbb{E} [e^{2X_T} | y_T = k] = \frac{\mathbb{E} [e^{2X_T} \mathbf{1}_{y_T=k}]}{\mathbb{P}(y_T = k)} \quad (20)$$

where  $k = \log(K)$ .

Let us assume that we have an expression for the forward transition probabilities  $(y, X, T) \mapsto p(y, X, T) = G(e^y, X, T)$ , the two expectations that come into play in equation (20) can be refactored such as

$$\sigma(k, T) = \sigma_D(e^k, T) \sqrt{\frac{I_0(k, T)}{I_1(k, T)}} \quad (21)$$

where

$$I_n(k, T) = \int e^{nx} p(y, x, T) dx, \text{ for } n \in \{0, 1\} \quad (22)$$

Two Monte-Carlo method can then be used for the computation of this conditional expectation

1. Estimation of the integral via a full Monte-Carlo simulation of  $(S_T, X_T)$
2. Approximation of the conditional expectation via a Least-Square approach using well-choosen polynomial functions

**Estimation via Least-Squares** As we want to compute a  $S_T$ -measurable integral, we can approximate

$$\mathbb{E} [e^{2X_T} | S_T] \approx \sum_{m=1}^M \beta_m \zeta_m (S_T) \quad (23)$$

where the coefficients  $(\beta_1, \dots, \beta_M)$  are found by a least-square approach. Let us assume we have  $N$  realisations of the random variables  $(S_T, X_T)$  denoted by  $((S_T^1, X_T^1), \dots, (S_T^N, X_T^N))$

Let us write the minimization problem associated with Least-Square in our case. Let us set  $\beta = (\beta_1, \dots, \beta_M)$

$$\hat{\beta} = \underset{\beta \in \mathbb{R}^M}{\operatorname{argmin}} \left\{ \mathbb{E} \left[ \left( \mathbb{E} [e^{2X_T} | S_T] - \sum_{m=1}^M \beta_m \zeta_m (S_T) \right)^2 \right] \right\} \quad (24)$$

Denoting by  $M_{\zeta\zeta}$  and  $M_{\zeta X}$  the following quantities, according to [2]

$$\begin{aligned} (M_{\zeta\zeta})_{r,s} &= \mathbb{E} [\zeta_r(S_T) \zeta_s(S_T)] \\ (M_{\zeta X})_r &= \mathbb{E} [\zeta_r(S_T) e^{2X_T}] \end{aligned} \quad (25)$$

we have the following estimation for the "regression coefficients"  $\beta$

$$\beta = (M_{\zeta\zeta})^{-1} M_{\zeta X} \quad (26)$$

Those quantities can be estimated using the Monte-Carlo paths  $((S_T^1, X_T^1), \dots, (S_T^N, X_T^N))$

$$\begin{aligned} (\hat{M}_{\zeta\zeta})_{r,s} &= \frac{1}{N} \sum_{n=1}^N \zeta_r(S_T^n) \zeta_s(S_T^n) \\ (\hat{M}_{\zeta X})_r &= \frac{1}{N} \sum_{n=1}^N \zeta_r(S_T^n) e^{2X_T^n} \end{aligned} \quad (27)$$

However, those estimations can be high-biased because the same paths are used to compute the regression coefficients are used to the conditional expectation. [2] treats the analytical removal of the bias and the use of Bessel inequality (to prove the under-estimation of variance of  $e^{2X_t}$ )

**Estimation via Monte-Carlo**

## 2 Overview of the methodology proposed in [4]

### 2.1 Main equations

We have the following dynamics for the FX Spot  $S$

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_t^d dt + \sqrt{v_t} A(S_t, t) dW_t^{(1)} \\ v_t &\text{ is a stochastic process : Heston or exponential-OU, ...} \\ dv_t &= \kappa(m(t) - v_t) dt + \alpha \sqrt{v_t} dW_t^{(2)} \text{ for Heston} \\ d\langle W^{(1)}, W^{(2)} \rangle_t &= \rho dt \end{aligned} \quad (28)$$

#### 2.1.1 PDE for volatility as Heston process

Let  $p(y, v, t)$  be the joint probability distribution for log-spot  $y$  and stochastic variance  $v$  at time  $t$ .

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 [v A^2(y, t) p]}{\partial y^2} + \rho \alpha \frac{\partial^2 [v A(y, t) p]}{\partial y \partial v} + \frac{1}{2} \alpha^2 \frac{\partial^2 [v^2 p]}{\partial v^2} + \frac{\partial}{\partial y} \left[ \left( \frac{1}{2} v A^2(y, t) - \mu_t^d \right) p \right] + \kappa \frac{\partial [(v - m_t) p]}{\partial v} \quad (29)$$

It then reduces to

$$\begin{aligned} \frac{\partial p}{\partial t} = & \frac{A^2 v}{2} \frac{\partial^2 p}{\partial y^2} + \rho \alpha A v \frac{\partial^2 p}{\partial y \partial v} + \frac{\alpha^2 v}{2} \frac{\partial^2 p}{\partial v^2} + \left( \frac{1}{2} A^2 v + 2 A A' v + \rho \alpha A - \mu_t^d \right) \frac{\partial p}{\partial y} \\ & + \left( \kappa [v - m_t] + \alpha^2 + \rho \alpha A' v \right) \frac{\partial p}{\partial v} + \left[ A A' v + \rho \alpha A' + A A'' v + [A']^2 + \kappa \right] p \end{aligned} \quad (30)$$

### 2.1.2 PDE for volatility as exponential of Ornstein-Uhlenbeck process

## 3 Solving the SLV calibration PDE

In this section we will develop numerical methods to solve equation (17)

## References

- [1] Frédéric Abergel, Rémi Tachet, *A Non-linear integro-differential equation from Mathematical Finance*, AIMS Journal, Version 1, 8th Sept. 2011
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- [4] Iain J. Clark, *Foreign Exchange option pricing : A Practitioner's Guide*, Wiley Finance, ,2010