Stochastic Local Volatility

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1 General definitions

1.1 Term-structures

As we work in finance, most of the coefficients exposed in the following models will be piecewise constant.

Definition 1.1. *A term-structure of coefficients (or just term-structure when there is no ambiguity) is a piecewise-constant coefficient.*

Definition 1.2. *If $t \mapsto \gamma_t$ is a term-structure, we will denote by $\Gamma^k(t,s)$, for $t < s$ (or just Γ^k is there is no ambiguity), the following quantity*

$$\Gamma^k(t,s) = \frac{1}{s-t}\int_t^s \gamma_u^k du \tag{1}$$

2 The Model

2.1 General description

Let T be a finite time horizon (the final maturity of the deal we want to price, for example) Let S be an asset of drift μ_t under the domestic risk-neutral probability measure. We assume the following dynamics under $\mathbb{Q}_d$

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(S_t, t).\Sigma_t \cdot dW_t^S \tag{2}$$

where $(S,t) \mapsto \sigma(S,t)$ is the local volatility function (deterministic function of spot and time, calibrated to the vanilla option market at time 0) and W^S is a Wiener process under the domestic risk neutral probability measure.

We will develop later a methodology to calibrate that function σ. $(\Sigma_t)_{t\leq T}$ is an $(\mathcal{F}_t)_{t\leq T}$-measurable process.
This additional constraint equation comes from the Markovian projection theorem. In order to calibrate the function $\sigma(t,S)$ to the vanilla option market, we have to use the Dupire Local volatility function $\sigma_D(t,S)$ equal to

$$\begin{aligned}\sigma_D(T,K)^2 &= \mathbb{E}^{\mathbb{Q}_d^T}\left[\sigma(T,S_T)^2\Sigma_T^2 \middle| S_T = K\right]\\ &= \sigma(T,K)^2\mathbb{E}^{\mathbb{Q}_d^T}\left[\Sigma_T^2 \middle| S_T = K\right]\end{aligned} \tag{3}$$

Then

$$\sigma(T,K)^2 = \frac{\sigma_D(T,K)^2}{\mathbb{E}^{\mathbb{Q}_d^T}\left[\Sigma_t^2 | S_T = K\right]} \tag{4}$$

The calibration of the Dupire Local volatility function equal to

$$\begin{aligned}\sigma_D(T,K)^2 &= \frac{\frac{\partial C}{\partial K} + \mu_T K\frac{\partial C}{\partial K} + (\mu_T - r_d(T))\, C}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}} + A(T)\\ A(T) &= \frac{KB_d(0,T)\left[Cov^{\mathbb{Q}_d^T}\left(-\mu_T, \mathbf{1}_{S_T>K}\right) + \frac{1}{K}Cov^{\mathbb{Q}_d^T}\left(r_d(T) - \mu_T, (S_T-K)^+\right)\right]}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}\end{aligned} \tag{5}$$

where $A(T)$ is the stochastic-rate correction of the Dupire Local volatility function, is already exposed in a large number of papers.

We will additionnally set the following condition for the stochastic volatility process Σ

$$\forall t \in [0,T], \mathbb{E}\left[\Sigma_t^2\right] = 1 \tag{6}$$

The goal of the stochastic volatility would be to calibrate to the forward smile (to be consistent with the forward start option)

2.2 Examples of stochastic volatility models

We will give in this subsection some practical examples of stochastic volatility models.

2.2.1 Heston model

The stochastic volatility process follows under the domestic risk-neutral probability measure

$$\begin{aligned} d\Sigma_t &= \left(\theta_\Sigma(t) - \lambda_\Sigma \Sigma_t\right) dt + \alpha_\Sigma \sqrt{\Sigma_t} dW_t^\Sigma \\ d\langle W^S, W^\Sigma \rangle_t &= \rho_{S,\Sigma} dt \end{aligned} \tag{7}$$

The coefficients λ_Σ, α_Σ and $\rho_{S,\Sigma}$ are kept constant for simplicity. An extension with term-structure of coefficients may be easy to introduce (because we will be working in a Monte-Carlo framework or PDEs most of the time).

$\theta_\Sigma(t)$ is calibrated such as 6 is satisfied.

2.2.2 Lognormal model

The stochastic volatility is lognormally distributed

$$\begin{aligned} \Sigma_t &= e^{X_t} \\ dX_t &= \left(\theta_X(t) - \lambda_X X_t\right) dt + \alpha_X dW_t^X \\ d\langle W^X, W^S \rangle_t &= \rho_{S,X} dt \end{aligned} \tag{8}$$

The coefficients λ_X, α_X and $\rho_{S,X}$ are kept constant for simplicity. An extension with term-structure of coefficients may be easy to introduce (because we will be working in a Monte-Carlo framework or PDEs most of the time).

$\theta_X(t)$ is calibrated such as 6 is satisfied.

2.2.3 Schöbel and Zhu model

The stochastic volatility is normally distributed

$$\begin{aligned} d\Sigma_t &= \left(\theta_\Sigma(t) - \lambda_\Sigma \Sigma_t\right) dt + \alpha_\Sigma dW_t^\Sigma \\ d\langle W^\Sigma, W^S \rangle_t &= \rho_{S,\Sigma} dt \end{aligned} \tag{9}$$

The coefficients λ_Σ, α_Σ and $\rho_{S,\Sigma}$ are kept constant for simplicity. An extension with term-structure of coefficients may be easy to introduce (because we will be working in a Monte-Carlo framework or PDEs most of the time).

$\theta_\Sigma(t)$ is calibrated such as 6 is satisfied.
Under the T-forward domestic measure, the stochastic volatility satisfies

$$\begin{aligned} d\Sigma_t &= \left(\theta_{\Sigma,T,d}(t) - \lambda_\Sigma \Sigma_t\right) dt + \alpha_\Sigma dW_t^{\Sigma,T,d} \\ \theta_{\Sigma,T,d}(t) &= \theta_\Sigma(t) - \int_0^t \rho_{d,\Sigma}(u) b_d(u,T) \sigma_d(u) du \end{aligned} \tag{10}$$

where $\sigma_d(u)$ is the term-structure of volatility of the domestic interest-rate (in a one-factor model), $\rho_{d,\Sigma}(u)$ is the term-structure of correlation between the domestic interest rate and the foreign exchange stochastic volatility, and $b_d(u,T)$ is a function that we will try to find in the next section.

2.3 Interest-rate models

We will focus, in this section, in the derivation of the function $b_d(u, T)$ in simple one-factor models

Definition 2.1. *Let $f(t,T)$ be the instantaneous forward rate between t and T*

$$f(t,T) = -\frac{\partial \ln B(t,T)}{\partial T} \tag{11}$$

Definition 2.2. *The short-rate r_t at time t is equal to*

$$r_t = f(t,t) \tag{12}$$

2.3.1 Linear Gaussian Markov 1-factor Model (LGM1F)

It satisfies under the currency risk-neutral measure

$$df(t,T) = \mu(t,T)dt + \sigma_f(t,T)dW_t \tag{13}$$

It then can be shown that

$$\mu(t,T) = \sigma_f(t,T) \int_t^T \sigma_f(t,s)ds \tag{14}$$

Then r_t satisfies the following SDE

$$\begin{aligned} dr_t &= d(f(t,t)) \\ &= \left.\frac{\partial f(t,T)}{\partial T}\right|_{T=t} + df(t,T)|_{T=t} \\ &= \left.\frac{\partial f(t,T)}{\partial T}\right|_{T=t} + \sigma_f(t,t)dW_t \end{aligned} \tag{15}$$

We will now specify the function $\sigma_f(t,T)$

$$\sigma_f(t,T) = \sigma(t) \exp\left(-\int_t^T \lambda_u du\right) \tag{16}$$

where $\sigma(t)$ is a term-structure of volatility and λ_t is a term-structure of mean-reversion that we will calibrate. The zero-coupon bond price satisfies

$$B(t,T) = \exp\left(a(t,T) + b(t,T)r_t\right) \tag{17}$$

where b is the above function that we wanted to find in the previous section.

2.3.2 Linear Gaussian Markov 1-factor Stochastic volatility model (LGM1FSV)

$$df(t,T) = \mu(t,T)dt + \sigma_f(t,T)\sqrt{v_t}dW_t \tag{18}$$

Let us assume for simplicity that

$$\begin{aligned} dr_t &= \left(\theta_r(t) - \lambda_r r_t\right)dt + \sigma_r(t)\sqrt{v_t}dW_t^r \\ dv_t &= \kappa\left(\theta_v - v_t\right)dt + \sigma_v\sqrt{v_t}dW_t^v \\ d\langle W^v, W^r\rangle_t &= \rho_{r,v}dt \end{aligned} \tag{19}$$

Let us apply Itô's formula to $h(r_t, v_t, t) \in \mathcal{C}^{1,2}\left(\mathbb{R}^2 \times \mathbb{R}_+\right)$

$$dh(r_t,v_t,t) = \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial r}dr_t + \frac{\partial h}{\partial v}dv_t + \frac{1}{2}\frac{\partial^2 h}{\partial r^2}v_t\sigma_r^2(t)dt + \frac{1}{2}\frac{\partial^2 h}{\partial v^2}v_t\sigma_v^2 dt + \frac{\partial^2 h}{\partial v \partial r}\rho_{r,v}\sigma_r(t)\sigma_v v_t dt \tag{20}$$

2.3.3 Calibration of models

2.4 Stochastic volatility part

Let us assume that X follows the following dynamics under the domestic risk-neutral probability measure

$$\begin{aligned} dX_t &= (\theta(t) - \lambda X_t)dt + \sigma_X dW_t^X \\ X_0 &= 1 \end{aligned} \tag{21}$$

The coefficients are kept constant for now for simplicity.
We have

$$X_t = e^{-\lambda t} + \int_0^t \theta(s) e^{-\lambda(t-s)} ds + \int_0^t \sigma_X e^{-\lambda(t-s)} dW_s^X \tag{22}$$

Then,

$$\begin{aligned} \exp\left(2e^{-\lambda t} + 2\int_0^t \theta(s)e^{-\lambda(t-s)}ds + 2\int_0^t \sigma_X^2 e^{-2\lambda(t-s)}ds\right) &= 1 \\ e^{-\lambda t} + \int_0^t \theta(s)e^{-\lambda(t-s)}ds + \int_0^t \sigma_X^2 e^{-2\lambda(t-s)}ds &= 0 \\ e^{-\lambda t} + \int_0^t \theta(s)e^{\lambda s}ds e^{-\lambda t} + \int_0^t \sigma_X^2 e^{-2\lambda(t-s)}ds &= 0 \\ -\lambda e^{-\lambda t} - \int_0^t \theta(s)e^{\lambda s}ds \lambda e^{-\lambda t} + 2\sigma_X^2 e^{-2\lambda t} + \theta(t) &= 0 \end{aligned}$$

Setting $f(t) = \theta(t)e^{\lambda t}$ it yields

$$\begin{aligned} \lambda \int_0^t f(s)ds + f(t) &= c(t) \\ c(t) &= \lambda - 2\sigma_X^2 e^{-\lambda t} \end{aligned} \tag{23}$$

Then f satisfies the following ODE

$$\begin{aligned} f'(t) + \lambda f(t) &= -2\sigma_X^2 \lambda e^{-\lambda t} \\ f(0) &= \lambda - 2\sigma_X^2 \end{aligned} \tag{24}$$

Then

$$\begin{aligned} f(t) &= Ce^{-\lambda t} - 2\sigma_X^2 \lambda t e^{-\lambda t} \\ &= \left(\lambda - 2\sigma_X^2\right) e^{-\lambda t} - 2\sigma_X^2 \lambda t e^{-\lambda t} \\ f(t) &= \lambda e^{-\lambda t} - 2\sigma_X^2 (1 + \lambda t)e^{-\lambda t} \end{aligned} \tag{25}$$

Then

$$\theta(t) = \lambda e^{-2\lambda t} - 2\sigma_X^2(1 + \lambda t)e^{-2\lambda t} \tag{26}$$

The stochastic volatility X can be computed quite easily and be simulated exactly

$$X_t \sim \mathcal{N}\left(e^{-\lambda t} + \int_0^t \theta(s)e^{-\lambda(t-s)}ds, \sigma^2 \frac{1 - e^{-2\lambda t}}{2\lambda}\right) \tag{27}$$

2.5 Calibration of the local volatility

Theorem 2.1 (Markovian Projection)**.** *Put here the theorem of the markovian projection of general volatility models into a local volatility*

In our case, we have

$$\begin{aligned} \sigma_D^2(x,t) &= \mathbb{E}\left[e^{2X_t}\sigma^2(S_t,t)\middle|S_t = S\right] \\ &= \mathbb{E}\left[e^{2X_t}\middle|S_t = S\right]\sigma^2(S,t) \end{aligned} \tag{28}$$

2.5.1 Calibration via a PDE

Let us introduce the Green function $G(S_0, X_0, t_0; S, X, t)$. Let us set $b(X) = e^X$. G satisfies

$$\frac{\partial G}{\partial T} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma^2 b^2 S^2 G\right) - \frac{\partial^2}{\partial X^2}\left(\sigma_X^2 G\right) - \frac{\partial^2}{\partial S \partial X}\left(\rho b \sigma S \sigma_X G\right) + \frac{\partial}{\partial X}\left((\theta - \lambda X)G\right) + \frac{\partial}{\partial S}(\mu S G) = 0 \tag{29}$$

Let us assume (in relation of the previous section), that

1. σ_X, θ and λ are not functions of X
2. μ is not a function of S
3. ρ is neither a function of S nor X

The calibration PDE degenerates into

$$\frac{\partial G}{\partial T} - \frac{1}{2}e^{2X}\frac{\partial^2}{\partial S^2}\left(\sigma^2 S^2 G\right) - \sigma_X^2 \frac{\partial^2}{\partial X^2}(G) - \rho \frac{\partial^2}{\partial S \partial X}\left(b \sigma S \sigma_X G\right) + (\theta - \lambda X)\frac{\partial}{\partial X}(G) + \mu \frac{\partial}{\partial S}(SG) = 0 \tag{30}$$

Let us denote by Q the density of S

$$Q(S,t) = \int_{\mathbb{R}} G(S, X, t) dX \tag{31}$$

Then

$$\frac{\partial Q}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma^2 S^2 \int_{\mathbb{R}} e^{2X} G(S, X, t) dX\right) + \mu \frac{\partial}{\partial S}(SG) = 0 \tag{32}$$

Thus,

$$\frac{\partial Q}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma^2 \frac{\int_{\mathbb{R}} e^{2X} G(S, X, t) dX}{\int_{\mathbb{R}} G(X, S, t) dX} S^2 Q\right) + \mu \frac{\partial}{\partial S}(SG) = 0 \tag{33}$$

We recognize the Dupire equation which defines a local volatility function $\sigma_D(S,t)$

$$\sigma_D^2(S,t) = \sigma^2(S,t)\frac{\int_{\mathbb{R}} e^{2X} G(S, X, t) dX}{\int_{\mathbb{R}} G(X, S, t) dX} \tag{34}$$

which permits to re-write the calibration PDE

$$\begin{aligned}
&\frac{\partial G}{\partial T} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(S^2 e^{2X} \sigma_D^2 \frac{\int_{\mathbb{R}} G(S, X, t) dX}{\int_{\mathbb{R}} e^{2X} G(S, X, t) dX}\right) + \mu \frac{\partial}{\partial S}(SG) - \frac{\sigma_X^2}{2}\frac{\partial^2 G}{\partial X^2} + \frac{\partial}{\partial X}\left((\theta - \lambda X)G\right) \\
&- \rho \sigma_X \frac{\partial^2}{\partial S \partial X}\left(e^X S \sigma_D \sqrt{\frac{\int_{\mathbb{R}} G(S, X, t) dX}{\int_{\mathbb{R}} e^{2X} G(S, X, t) dX}} G\right) = 0
\end{aligned} \tag{35}$$

The only SLV models tractables with a PDE calibration are then mono-asset SLV models. We will develop in the following another method which enable us to calibrate via Monte-Carlo SLV Models. However, a Monte-Carlo will have more noise than a PDE calibration.

2.5.2 Calibration by Monte-Carlo

In this section, we will develop a method to calibrate the SLV models in a higher dimension problem using a Monte-Carlo simulation.

Let us re-write the Dupire equation associated with the SLV model we just described

$$\begin{aligned}
&\frac{\partial C}{\partial T} + P_d(0,T)\mathbb{E}^{\mathbb{Q}_T}\left[\left(K r_d(T) - S_T r_f(T)\right)\mathbf{1}_{\{S_T > K\}}\right] - \frac{1}{2}\sigma_D^2(K,T) K^2 \frac{\partial^2 C}{\partial K^2} = 0 \\
&\frac{\partial C}{\partial T} + P_d(0,T)\mathbb{E}^{\mathbb{Q}_T^d}\left[\left(K r_d(T) - S_T r_f(T)\right)\mathbf{1}_{\{S_T > K\}}\right] - \frac{1}{2}\sigma^2(K,T)\mathbb{E}^{\mathbb{Q}_T}\left[e^{2X_T} \middle| S_T = K\right] K^2 \frac{\partial^2 C}{\partial K^2} = 0
\end{aligned} \tag{36}$$

To write the calibration equation, we are left with, the calculation of the following integral

$$\mathbb{E}^{\mathbb{Q}_T^d}\left[e^{2X_T}\middle|S_T=K\right]=\frac{\mathbb{E}^{\mathbb{Q}_T^d}\left[e^{2X_T}\mathbf{1}_{S_T=K}\right]}{\mathbb{Q}_T^d(S_T=K)} \tag{37}$$

and

$$\mathbb{E}^{\mathbb{Q}_T^d}\left[\left(Kr_d(T)-S_Tr_f(T)\right)\mathbf{1}_{\{S_T>K\}}\right] \tag{38}$$

Re-writting the first equation as a function of log-spot $y_t=\log\left(S_t\right)$ yields

$$\mathbb{E}^{\mathbb{Q}_T^d}\left[e^{2X_T}\middle|y_T=k\right]=\frac{\mathbb{E}^{\mathbb{Q}_T^d}\left[e^{2X_T}\mathbf{1}_{y_T=k}\right]}{\mathbb{Q}_T^d(y_T=k)} \tag{39}$$

where $k=\log(K)$.

Let us assume that we have an expression for the forward transition probabilities under domestic probability $(y,X,T)\mapsto p(y,X,T)=G(e^y,X,T)$, the two expectations that come into play in equation (38) can be refactored such as

$$\sigma(k,T)=\sigma_D(e^k,T)\sqrt{\frac{I_0(k,T)}{I_1(k,T)}} \tag{40}$$

where

$$I_n(k,T)=\int e^{nx}p(y,x,T)dx,\ \text{ for } n\in\{0,1\} \tag{41}$$

Two Monte-Carlo method can then be used for the computation of this conditional expectation

1. Estimation of the integral via a full Monte-Carlo simulation of (S_T,X_T)
2. Approximation of the conditional expectation via a Least-Square approach using well-choosen polynomial functions

Estimation via Least-Squares As we want to compute a S_T-measurable integral, we can approximate

$$\mathbb{E}^{\mathbb{Q}_T^d}\left[e^{2X_T}\middle|S_T\right]\approx\sum_{m=1}^{M}\beta_m\zeta_m\left(S_T\right) \tag{42}$$

where the coefficients $(\beta_1,\ldots,\beta_M)$ are found by a least-square approach. Let us assume we have N realisations of the random variables (S_T,X_T) denoted by $\left((S_T^1,X_T^1),\ldots,(S_T^N,X_T^N)\right)$
Let us write the minimization problem associated with Least-Square in our case. Let us set $\beta=(\beta_1,\ldots,\beta_M)$

$$\hat{\beta}=\underset{\beta\in\mathbb{R}^M}{\operatorname{argmin}}\left\{\mathbb{E}^{\mathbb{Q}_T^d}\left[\left(\mathbb{E}^{\mathbb{Q}_T^d}\left[e^{2X_T}\middle|S_T\right]-\sum_{m=1}^{M}\beta_m\zeta_m\left(S_T\right)\right)^2\right]\right\} \tag{43}$$

Denoting by $M_{\zeta\zeta}$ and $M_{\zeta X}$ the following quantities, according to [2]

$$\begin{aligned}(M_{\zeta\zeta})_{r,s}&=\mathbb{E}^{\mathbb{Q}_T^d}\left[\zeta_r(S_T)\zeta_s(S_T)\right]\\(M_{\zeta X})_r&=\mathbb{E}^{\mathbb{Q}_T^d}\left[\zeta_r(S_T)e^{2X_T}\right]\end{aligned} \tag{44}$$

we have the following estimation for the ”regression coefficients” β

$$\beta=(M_{\zeta\zeta})^{-1}M_{\zeta X} \tag{45}$$

Those quantities can be estimated using the Monte-Carlo paths $\left((S_T^1,X_T^1),\ldots,(S_T^N,X_T^N)\right)$

$$
\begin{aligned}
(\hat{M}_{\zeta\zeta})_{r,s} &= \frac{1}{N}\sum_{n=1}^{N}\zeta_r(S_T^n)\zeta_s(S_T^n) \\
(\hat{M}_{\zeta X})_r &= \frac{1}{N}\sum_{n=1}^{N}\zeta_r(S_T^n)e^{2X_T^n}
\end{aligned} \tag{46}
$$

However, those estimations can be high-biased because the same paths are used to compute the regression coefficients are used to the conditional expectation. [2] treats the analytical removal of the bias and the use of Bessel inequality (to prove the under-estimation of variance of e^{2X_t}

Estimation via Monte-Carlo In this paragraph, we will focus on the estimation of the integral described in equation 37. Thie description is based on [5]. As the equation 37 is based on an expectation under the T-forward neutral domestic probability measure, we need to have the dynamics of all the factors under that probability measure.

$$
\begin{aligned}
\frac{dS_t}{S_t} &= (r_t^d - r_t^f - \text{drift term due to change of measure})dt + \sigma(S_t,t)e^{X_t}dW_t^{S,T,d} \\
dr_t^d &= \left(\theta^d(t) - \lambda^d r_t^d - \left(\sigma_t^d\right)^2\beta^d(t,T)\right)dt + \sigma_t^d dW_t^{d,T,d} \\
dr_t^f &= \left(\theta^f(t) - \lambda^f r_t^f - \rho^{f,S}\sigma^f(t)b^f(t,T)\sigma(t,S_t) - \sigma^d(t)\beta^d(t,T)\sigma^f(t,T)\beta^f(t,T)\rho^{f,d}\right)dt + \sigma_t^f dW_t^{f,T,d}
\end{aligned} \tag{47}
$$

3 Overview of the methodology proposed in [4]

3.1 Main equations

We have the following dynamics for the FX Spot S

$$
\begin{aligned}
&\frac{dS_t}{S_t} = \mu_t^d dt + \sqrt{v_t}A(S_t,t)dW_t^{(1)} \\
&v_t \text{ is a stochastic process : Heston or expoonential-OU,...} \\
&dv_t = \kappa(m(t) - v_t)dt + \alpha\sqrt{v_t}dW_t^{(2)} \text{ for Heston} \\
&d\langle W^{(1)}, W^{(2)}\rangle_t = \rho dt
\end{aligned} \tag{48}
$$

3.1.1 PDE for volatility as Heston process

Let $p(y,v,t)$ be the joint probability distribution for log-spot y and stochastic variance v at time t.

$$
\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2\left[vA^2(y,t)p\right]}{\partial y^2} + \rho\alpha\frac{\partial^2\left[vA(y,t)p\right]}{\partial y\partial v} + \frac{1}{2}\alpha^2\frac{\partial^2\left[v^2p\right]}{\partial v^2} + \frac{\partial}{\partial y}\left[\left(\frac{1}{2}vA^2(y,t) - \mu_t^d\right)p\right] + \kappa\frac{\partial\left[(v - m_t)\,p\right]}{\partial v} \tag{49}
$$

It then reduces to

$$
\begin{aligned}
\frac{\partial p}{\partial t} =& \frac{A^2v}{2}\frac{\partial^2 p}{\partial y^2} + \rho\alpha Av\frac{\partial^2 p}{\partial y\partial v} + \frac{\alpha^2 v}{2}\frac{\partial^2 p}{\partial v^2} + \left(\frac{1}{2}A^2v + 2AA'v + \rho\alpha A - \mu_t^d\right)\frac{\partial p}{\partial y} \\
&+ \left(\kappa\left[v - m_t\right] + \alpha^2 + \rho\alpha A'v\right)\frac{\partial p}{\partial v} + \left[AA'v + \rho\alpha A' + AA''v + \left[A'\right]^2 + \kappa\right]p
\end{aligned} \tag{50}
$$

where

$$
\begin{aligned}
A &= A(y,t) \\
A' &= \frac{\partial A}{\partial y} \\
A'' &= \frac{\partial^2 A}{\partial y^2}
\end{aligned} \tag{51}
$$

3.1.2 PDE for volatility as exponential of Ornstein-Ulhenbeck process

3.2 Forward induction for local volatility calibration on SLV - PDE Calibration

Let us take the methodology developed in [4].
Let us assume of sequence of timepoints $\{t_0 = 0, t_1, t_2, \ldots, t_N = T_{cal}\}$. The approach is somewhat like boostrapping a yield curve but we are trying to infer a surface $A(y,t)$

1. Start at time $t_0 = 0$ with an initial local volatility correction $\forall y, A(y,0) = 1$ and an initial condition

$$p(y,v,0) = \delta_{\{y-y_0\}}\delta_{\{v-v_0\}} \tag{52}$$

2. Construct a forward timestepping scheme for (49) using a finite scheme for the PDE such as either explicit finite differencing or ADI timestepping. Spatial derivatives A' and A'' can be estimated at time t by setting a function $f(y) = A(y,t)$ equal to a cubic spline and extracting first and second-order derivative by symbolic differentiation of $f(y)$

3. Refine $A(y,t)$ by the use of (39) by calculating I_0 and I_1 integrals defined in (40) at each required level of y, taking numerical integral in the variance dimension. Then update the diffusion, convection and force terms in (49)

4 Solving the SLV calibration PDE

In this section we will develop numerical methods to solve equation (34)

5 How to set the correlation between factors

5.1 FX-FX correlations

The correlations between FX spot rates are calibrated on the cross FX volatility market slice if it is quoted and estimated using historical data if it is not. We can distinguish two kind of calibration

- Calibration on the ATM forward volatility
- Calibration on the Full Smile

5.1.1 Calibration on the ATM

Let X_i be the number of domestic currency units corresponding to 1 FORi currency unit. We will call X_{21} the number of FOR2 currency units corresponding to 1 FOR1 currency unit.
Then

$$X_{21} = \frac{X_2}{X_1} \tag{53}$$

Using the volatilities, this equality transposes to

$$\sigma_{12}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2 \tag{54}$$

And then correlation between the FX spots rates X_1 and X_2 is equal to

$$\rho_{12}(T) = \frac{\sigma_1(T)^2 + \sigma_2(T)^2 - \sigma_{12}(T)^2}{2\sigma_1(T)\sigma_2(T)} \tag{55}$$

where $\sigma_i(T)$ is the ATM volatility at time T.

We have to ensure that

$$\forall T, |\rho_{12}(T)| \leq 1 \tag{56}$$

but we will see that the positiveness of the full correlation matrix leads to much straighter conditions on this correlation.

5.1.2 Local correlation - calibration on the full smile

5.2 Rate-FX correlations

5.3 Rate-Rate correlations

5.4 FX-FX stoch vol correlations

5.5 FX stoch vol-IR correlations

5.6 FX stoch vol correlations

6 What needs to be done when the correlation matrix is not semi-definite positive?

References

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