Linear Gaussian Markov Model with Stochastic volatility

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1 The model

1.1 Why adding stochastic volatility

- Replication tentative of the swaption smile for pricing of more complex interest-rate derivatives

1.2 Main equations

Let $f(t,T)$ be the instantaneous forward rate between t and T, we assume without loss of any generality under the risk neutral measure $\mathbb{Q}$ (depending on the currency)

$$\mathrm{d}f(t,T) = \mu_f(t,T)\mathrm{d}t + \sigma_f(t,T) \cdot \left(\sqrt{v}\mathrm{d}W_t\right) \tag{1}$$

where $\cdot$ represents the scalar product and v is a stochastic volatility process that we will define later in the document.

Remark 1.1. *When dealing with multi-currency frameworks, I will write a separate document as it depends highly on the model put on the FX.*

Remark 1.2. *The number of factors N in the rate model is equal to the dimensionality of the considered vectors $v = (v_1, \ldots, v_N)$, $\sigma_f = (\sigma_{f,1}, \ldots, \sigma_{f,N})$*

Proposition 1.1 (Heath-Jarrow-Morton 1992)**.** *The absence of arbitrage imples that the drift term μ_f is equal to*

$$\mu_f(t,T) = \sum_{i=1}^{N} v_i(t)\sigma_{f,i}(t,T) \int_t^T \sigma_{f,i}(t,u)du \tag{2}$$

Remark 1.3. *For a general specification of $\sigma_{f,i}(t,T)$, the dynamics of the forward rate curve will be path-dependent, which significantly complicates derivatives pricing and the application of standard econometric techniques*

Proposition 1.2 (Ritchken-Sankarasubramanian)**.** *If the volatility vector of functions can be factored as*

$$\sigma_{f,i}(t,T) = \mathcal{P}_n(T-t)e^{-\int_t^T \lambda_u du} \tag{3}$$

with a regular function $u \mapsto \lambda_u$, and $\tau \mapsto \mathcal{P}(\tau)$ a polynomial function of the variable, then the Heath-Jarrow-Morton model is markovian.

Proof. Check if all the conditions are reunited, and check the regularity of the function □

Remark 1.4. *For simplicity in the calibration, we will set the degree of the polynomial to be 1 and we are left with the following formula,*

$$\sigma_{f,i}(t,T) = \left(\alpha_{1,i} + \alpha_{2,i}(T-t)\right)e^{-\lambda_i(T-t)} \tag{4}$$

we can also consider a term-structure of coefficients and have the follozing formula

$$\sigma_{f,i}(t,T) = \alpha_{1,i}(t)e^{-\lambda_i(T-t)} \tag{5}$$

2 Stochastic volatility models

2.1 Log-normal stochastic volatility

2.2 Heston-like stochastic volatility

2.2.1 Main equations

Let us assume that under the risk-neutral domestic probability measure, the domestic short rate evolves with the following dynamics

$$\begin{aligned} \mathrm{d}r_t &= \lambda(\theta - r_t)\mathrm{d}t + \sigma\sqrt{v_t}\mathrm{d}W_t^r \\ \mathrm{d}v_t &= a(b - v_t)\mathrm{d}t + \sigma_v\sqrt{v_t}dW_t^v \\ \mathrm{d}\langle W^r, W^v\rangle_t &= \rho dt \end{aligned} \tag{6}$$

As $f(t,t) = r_t$ we will be able to derive the dynamic of the domestic instantaneous forward rate under the risk neutral measure.

2.2.2 PDE Approach

Let us derive the price of a zero-coupon bond of maturity T denoted at time t by $B(t,T)$
We have

$$B(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s \mathrm{d}s}\middle|\mathcal{F}_t\right]$$

As we are in an affine model, we can look for an exponentially affine function of the state variables r_t and x_t

$$B(t,T) = e^{\alpha_T(t) + \beta_T(t)r_t + \gamma_T(t)v_t} = f(t, r_t, v_t) \tag{7}$$

with the following boundary conditions

$$\begin{aligned} \alpha_T(T) &= 0 \\ \beta_T(T) &= 0 \\ \gamma_T(T) &= 0 \end{aligned} \tag{8}$$

Proposition 2.1. *The function f satisfies the following 2-dimensional PDE*

$$\frac{\partial f}{\partial t} + \lambda(\theta - r)\frac{\partial f}{\partial r} + \frac{1}{2}\sigma^2 v\frac{\partial^2 f}{\partial r^2} + a(b - v)\frac{\partial f}{\partial v} + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 f}{\partial v} + \rho\sigma\sigma_v v\frac{\partial^2 f}{\partial r\partial v} = rf \tag{9}$$

Proof. Portfolio replication □

Some more calculations will be now helpful to derive ODEs on α, β and γ

$$\begin{aligned} \frac{\partial f}{\partial t}(t,r,v) &= \left[\alpha_T'(t) + \beta_T'(t)r + \gamma_T'(t)v\right] f(t,r,v) \\ \frac{\partial f}{\partial r}(t,r,v) &= \beta_T(t)f(t,r,v) \\ \frac{\partial f}{\partial v}(t,r,v) &= \gamma_T(t)f(t,r,v) \\ \frac{\partial^2 f}{\partial r^2}(t,r,v) &= \beta_T^2(t)f(t,r,v) \\ \frac{\partial^2 f}{\partial v^2}(t,r,v) &= \gamma_T^2(t)f(t,r,v) \\ \frac{\partial^2 f}{\partial v\partial r}(t,r,v) &= \gamma_T(t)\beta_T(t)f(t,r,v) \end{aligned}$$

We are left with the following ODEs

$$\begin{aligned} &\alpha_T'(t) + \lambda\theta\beta_T(t) + ab\gamma_T(t) = 0 \\ &\beta_T'(t) - \lambda\beta_T(t) = 1 \\ &\gamma_T'(t) + \frac{1}{2}\sigma^2\beta_T^2(t) - a\gamma_T(t) + \frac{1}{2}\sigma_v^2\gamma_T^2(t) + \rho\sigma\sigma_v\gamma_T(t)\beta_T(t) = 0 \end{aligned}$$

It yields

$$\beta_T(t) = \frac{1 - e^{-\lambda(T-t)}}{\lambda} \tag{10}$$

Remark 2.1. *This expression can easily extended to term-structure of mean-reversion* λ_t*, in this case the function* β_T *is equal to*

$$\beta_T(t) = \int_t^T e^{-\int_t^s \lambda_u du} ds \tag{11}$$

Let us now solve the Ricatti equation that solves the function γ_T

References

[1] A General Stochastic Volatility Model for the Pricing of Interest Rate Derivatives, published by the Oxford University Press on behalf of The Society for Financial Studies