Stochastic Local Volatility

Alexandre Humeau

March 22, 2014

1 The Model

1.1 General description

Let T be a finite time horizon (the final maturity of the deal we want to price, for example) Let S be an asset of drift μ_t under the domestic risk-neutral probability measure. We assume the following dynamics under $\mathbb{Q}_d$

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(S_t, t)e^{X_t} dW_t \tag{1}$$

where $(S, t) \mapsto \sigma(S, t)$ is the local volatility function (deterministic function of spot and time, calibrated to the vanilla option market at time 0. We will develop later a methodology to calibrate that function σ. X is an Ornstein-Ulhenbeck process so that :

$$\forall t \in [0, T], \mathbb{E}\left[e^{X_t}\right] = 1 \tag{2}$$

The goal of the stochastic volatility would be to calibrate to the forward smile (to be consistent with the

1.2 Stochastic volatility part

Let us assume that X follows the following dynamics under the domestic risk-neutral probability measure

$$\begin{aligned} dX_t &= (\theta(t) - \lambda X_t)dt + \sigma_X dW_t^X \\ X_0 &= 1 \end{aligned} \tag{3}$$

The coefficients are kept constant for now for simplicity.
We have

$$X_t = e^{-\lambda t} + \int_0^t \theta(s)e^{-\lambda(t-s)}ds + \int_0^t \sigma_X e^{-\lambda(t-s)}dW_s^X \tag{4}$$

Then,

$$\begin{aligned} \exp\left(e^{-\lambda t} + \int_0^t \theta(s)e^{-\lambda(t-s)}ds + \frac{1}{2}\int_0^t \sigma_X^2 e^{-2\lambda(t-s)}ds\right) = 1 \\ e^{-\lambda t} + \int_0^t \theta(s)e^{-\lambda(t-s)}ds + \frac{1}{2}\int_0^t \sigma_X^2 e^{-2\lambda(t-s)}ds = 0 \end{aligned} \tag{5}$$

Setting $f(t) = \theta(t)e^{\lambda t}$ it yields

$$\begin{aligned} &a(t)\int_0^t f(s)ds + f(t) = b(t) \\ &a(t) = \lambda e^{-\lambda t} \\ &b(t) = \lambda e^{-\lambda t} - \frac{\sigma_X^2}{2}e^{-2\lambda t} \end{aligned} \tag{6}$$