A Friendly Introduction to Number Theory

Chapter 2: Pythagorean Triples

Solutions

Hunter Matthews

09/03/2022

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1 Chapter 2: Pythagorean Triples

1.1 Exercise 1

Question:
(a) We showed that in any primitive Pythagorean triple (a, b, c), either a or b is even. Use the same sort of argument to show that either a or b must be a multiple of 3.
(b) By examining the above list of primitive Pythagorean triples, make a guess about when a, b, or c is a multiple of 5. Try to show that your guess is correct.

Solution:
(a) *Lemma.* If the remainder of division of x by k is denoted r, then the remainder of division of x^2 by k is denoted by r^2.

If r denotes the remainder, $\exists m \in \mathbb{Z} \mid x = mk + r$. So we have $x^2 = m^2k^2 + 2mkr + r^2$. This shows that r^2 and x^2 have the same remainders. Moreover, from the lemma, we know that the possible remainders of a square when divided by 3 are 0 and 1. This can be shown by a remainder table of division by 3 below.

x	x^2
0	0
1	1
2	1

If we assume that a and b are not divisible by 3, then the remainder of a^2 when divided by 3 is equal to 1. The same works for b^2. However, we know this cannot be possible since c^2 should have a remainder of 2. Therefore, we get a contradiction and at least one of either a or b is divisible by 3.

(b) My guess would be that exactly one number from the PPT should be divisible by 5. To show this, let us build a table of division by 5.

x	x^2
0	0
1	1
2	4
3	4
4	1

Therefore, the only possible triple of remainders would be

a^2	b^2	c^2
0	0	0
0	1	1
0	4	4
1	0	1
1	4	0
4	0	4
4	1	0

With there being exactly one remainder zero in each of them, our assumption is proved.

1.2 Exercise 2

Question: A nonzero integer d is said to divide an integer m if $m = dk$ for some number k. Show that if d divides both m and n, then d also divides $m - n$ and $m + n$.

Solution:
If $m \mid d \; \exists k_1 \in \mathbb{Z} \mid m = k_1 d$. If $n \mid d \; \exists k_2 \in \mathbb{Z} \mid n = k_2 d$. Therefore, we have that $m + n = k_1 d + k_2 d = (k_1 + k_2)d$ and $m - n = k_1 d - k_2 d = (k_1 - k_2)d$. Since we know $k_1 + k_2 \in \mathbb{Z}$ and $k_1 - k_2 \in \mathbb{Z}$, then $m + n \mid d$ and $m - n \mid d$.

1.3 Exercise 3

Question: For each of the following questions, begin by compiling some data; next, examine the data and formulate a conjecture; and finally, try to prove your conjecture is correct.

(a) Which odd numbers a can appear in primitive Pythagorean triple (a, b, c)?

(b) Which even numbers b can appear in a primitive Pythagorean triple (a, b, c)?

(c) Which numbers c can appear in a primitive Pythagorean triple (a, b, c)?

Solution:
(a) Any odd number can exist as the a in a primitive Pythagorean triple. In order to find such a triple, we can just let $t = a$ and $s = 1$ in the Pythagorean Triples Theorem. This gives us the following primitive Pythagorean Triple

$$(a, \frac{(a^2-1)}{2}, (\frac{(a^2+1)}{2})$$

(b) Looking at the table on page 14, it seems as though b is a multiple of 4. We know that $b = \dfrac{(s^2-t^2)}{2}$ with s and t both being odd. This means we can write $s = 2m+1$ and $t = 2n+1$. If we multiply things out, we get

$$\begin{aligned} b &= \frac{(2m+1)^2-(2n+1)^2}{2} \\ &= 2m^2+2m-2n^2-2n \\ &= 2m(m+1)-2n(n+1) \end{aligned}$$

Notice how $m(m+1)$ and $n(n+1)$ must both be even. Hence, we have that $b \mid 4$.

However, if $b \mid 4$, then we can write b as $b = 2^r B$ for B being odd and $r \geq 2$. We can try to find values of s and t such that $b = \dfrac{(s^2-t^2)}{2}$. We can factor this as

$$(s-t)(s+t) = 2^b = 2^{r+1}B$$

Now we know that both $s-t$ and $s+t$ must be even because s and t are odd. So we try

$$\begin{aligned} s-t &= 2^r \\ s+t &= 2B \end{aligned}$$

If we solve for s and t, we get that $s = 2^{r-1} + B$ and $t = -2^{r-1} + B$. Since B is odd and $r \geq 2$, we know that s and t must also be odd.

$$\begin{aligned} a = st &= B^2 - 2^{2r-2} \\ b = \frac{s^2 - t^2}{2} &= 2^r B \\ c = \frac{s^2 + t^2}{2} &= B^2 + 2^{2r-2} \end{aligned}$$

(c) This part was particularly difficult. I did not attempt to solve this.

1.4 Exercise 4

Question: In our list of examples, we have the two primitive Pythagorean triples $33^2 + 56^2 = 65^2$ and $16^2 + 63^2 = 65^2$. Find at least one more example of two primitive Pythagorean triples with the same value of c. Can you find three primitive Pythagorean triples with the same c? Can you find more than three?

Solution:
One example of two primitive Pythagorean triples with the same value for c is $13^2 + 84^2 = 85^2$ and $36^2 + 77^2 = 85^2$. A primitve Pythagorean triple with 4 triples is

$$\begin{aligned} 1105^2 &= 47^2 + 1104^2 \\ &= 264^2 + 1073^2 \\ &= 576^2 + 943^2 \\ &= 744^2 + 817^2 \end{aligned}$$

1.5 Exercise 5

Question: We have seen that the nth triangular number T_n is given by the formula

$$T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

The first few triangular numbers are 1, 3, 6, and 10. In the list of the first few Pythagorean triples (a, b, c), we find (3, 4, 5), (5, 12, 13), (7, 24, 25), and (9, 40, 41). Notice that in each case, the value of b is four times a triangular number

(a) Find a primitive Pythagorean triple (a, b, c) with $b = 4T_5$. Do the same for $b = 4T_6$ and $b = 4T_7$.

(b) Do you think that for every triangular number T_n, there is a primitive Pythagorean triple (a, b, c) with $b = 4T_n$? If you believe that this is true, then prove it. Otherwise, find some triangular number for which it is not true.

Solution:
(a) We will build a table for the first 4 such triples.

n	a	b	c	s	t
1	3	4	5	3	1
2	5	12	13	5	1
3	7	24	25	7	1
4	9	40	41	9	1

Our assumption would be that such triples can be found of the form $(2n+1)$, $(2n(n+1))$, and $2n^2+2n+1)$. We can check with the continuation of the table from above.

n	a	b	c	s	t
1	3	4	5	3	1
2	5	12	13	5	1
3	7	24	25	7	1
4	9	40	41	9	1
5	11	60	61	11	1
6	13	84	89	13	1
7	15	112	113	15	1

(b) Let us show that the triple $((2n+1), (2n(n+1)), 2n^2+2n+1)$ are primitive Pythagorean triples. This triple is PPT since it is triple for $s = 2n+1$ and $t = 1$. We can check this by the following

$$\begin{aligned}(2n+1)^2 + 4n^2(n+1)^2 &= 4n^4 + 8n^3 + 8n^2 + 4n + 1\\ (2n^2+2n+1)^2 &= 4n^4 + 8n^3 + 8n^2 + 4n + 1\\ (2n+1)^2 + 4n^2(n+1)^2 &= (2n^2+2n+1)^2\end{aligned}$$

$2n+1$ is co-prime with $n, n+1$, and 2. Therefore, $2n+1$ and $2n(n+1)$ are co-prime and this triple is a primitive Pythagorean triple.

1.6 Exercise 6

Question: If you look at the table of primitive Pythagorean triples in this chapter, you will see many triples in which c is 2 greater than a. For example, the triples $(3, 4, 5)$, $(15, 8, 17)$, $(35, 12, 37)$, and $(63, 16, 65)$ all have this property

(a) Find two more primitive Pythagorean triples (a, b, c) having $c = a + 2$.

(b) Find a primitive Pythagorean triple (a, b, c) having $c = a + 2$ and $c > 1000$.

(c) Try to find a formula that describes all primitive Pythagorean triples (a, b, c) having $c = a + 2$.

Solution:
(a) We can find values s and t for known triples

n	a	b	c	s	t
1	3	4	5	3	1
2	15	8	17	5	3
3	35	12	37	7	5
4	63	16	65	9	7

We can assume that such triples can be of the form $(4n^2 - 1, 4n, 4n^2 + 1)$

n	a	b	c	s	t
1	3	4	5	3	1
2	15	8	17	5	3
3	35	12	37	7	5
4	63	16	65	9	7
5	99	20	101	11	9
6	143	24	145	13	11

(b)

n	a	b	c	s	t
1	3	4	5	3	1
2	15	8	17	5	3
3	35	12	37	7	5
4	63	16	65	9	7
5	99	20	101	11	9
6	143	24	145	13	11
16	1023	64	1025	33	31
50	9999	200	10001	101	99

(c) If we look at the expressions for $a = st$ and $c = \dfrac{s^2 + t^2}{2}$ and notice that $c - a = 2 = \dfrac{(s-t)^2}{2}$ we can see that $s = t + 2$. Therefore, our formula gives the complete sequence of such numbers.

1.7 Exercise 7

Question: For each primitive Pythagorean tripe (a, b, c) in the table in this chapter, compute the quantity $2c-2a$. Do these values seem to have some special form? Try to prove that your observation is true for all primitive Pythagorean triples.

Solution:
First, let us compute $2c - 2a$ based off the PPT table in Chapter 2.

a	b	c	$2c-2a$
3	4	5	4
5	12	13	16
7	24	25	36
9	40	41	64
15	8	17	4
21	20	29	16
35	12	37	4
45	28	53	16
63	16	65	4

Looking for patterns in the $2c - 2a$ column, it appears that they all are perfect squares. We can prove this using the Pythagorean Triples Theorem, which we know says $a = st$ and $c = \frac{(s^2+t^2)}{2}$. So we have

$$2c - 2a = (s^2 + t^2) - 2st = (s - t)^2$$

Thus proving that $2c - 2a$ is always a perfect square.

1.8 Exercise 8

Question: Let m and n be numbers that differ by 2, and write the sum $\dfrac{1}{m}+\dfrac{1}{n}$ as a fraction in lowest terms. For example, $\dfrac{1}{2}+\dfrac{1}{4}=\dfrac{3}{4}$ and $\dfrac{1}{3}+\dfrac{1}{5}=\dfrac{8}{15}$.

(a) Compute the next three examples.

(b) Examine the numerators and denominators of the fractions in (a) and compare them with the table of Pythagorean triples. Formulate a conjectured about such fractions.

(c) Prove your conjecture is correct.

Solution:
(a)

n	m	$1/n+1/m$	num	den
2	4	3/4	3	4
3	5	8/15	8	15
4	6	5/12	5	12
5	7	12/35	12	35
6	8	7/24	7	24

(b)

n	m	$1/n+1/m$	num	den	a	b	c	s	t
2	4	3/4	3	4	3	4	5	3	1
3	5	8/15	8	15	15	8	17	5	3
4	6	5/12	5	12	5	12	13	5	1
5	7	12/35	12	35	35	12	37	7	5
6	8	7/24	7	24	7	24	25	7	1

Let our conjecture denoted the following. If n is even then $n=2k$, then $s=2k+1$, $t=1$ and

$$\frac{1}{n}+\frac{1}{n+2}=\frac{s}{\frac{s^2-1}{2}}$$

If n is odd then $n=2k+1$, then $s=2k+3$, $t=2k+1$ and

$$\frac{1}{n}+\frac{1}{n+2}=\frac{\frac{s^2-t^2}{2}}{st}$$

Hence, in both cases, we can form a Pythagorean triple from numerator and denominator.

(c) In the even case

$$\frac{1}{2k}+\frac{1}{2k+2}=\frac{2(2k+1)}{4k(k+1)}=\frac{2(2k+1)}{(2k+1)^2-1}=\frac{s}{\frac{s^2-1}{2}}$$

In the odd case

$$\frac{1}{2k+1} + \frac{1}{2k+3} = \frac{4(k+1)}{(2k+3)(2k+1)} = \frac{\frac{(2k+3)^2-(2k+1)^2}{2}}{(2k+3)(2k+1)} = \frac{\frac{s^2-t^2}{2}}{st}$$

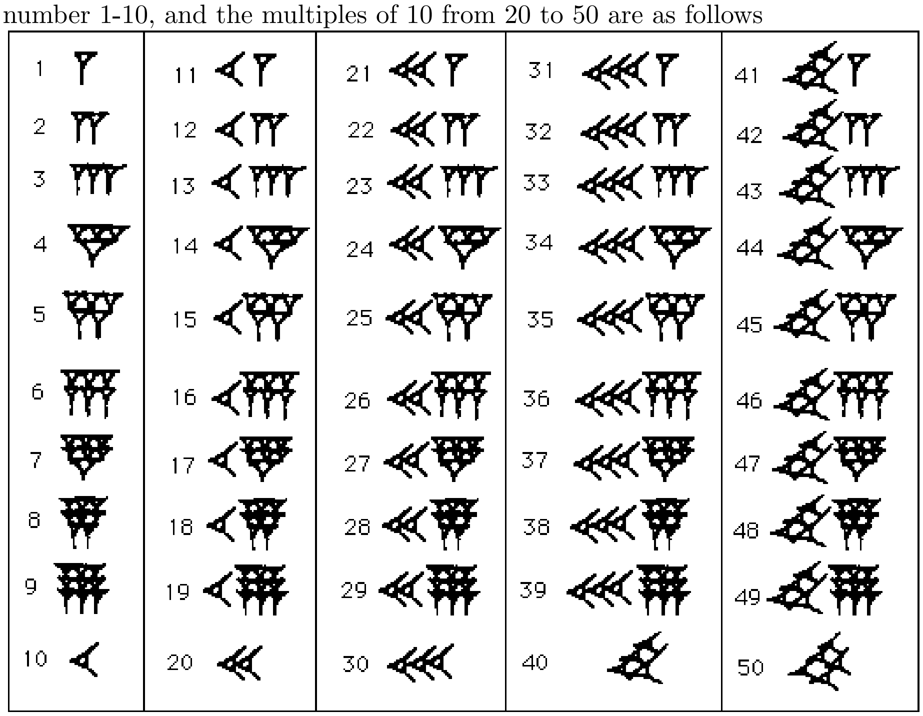
1.9 Exercise 9

Question: Answer the following questions related to the Babylonian number system.
(a) Read about the Babylonian number system and write a short description, including the symbols for the numbers 1 to 10 and the multiples of 10 from 20 to 50.

(b) Read about the Babylonian tablet called Plimpton 322 and write a brief report, including its approximate date of origin.

(c) The second and third columns of Plimpton 322 give pairs of integers (a, c) having the property that $c^2 - a^2$ is a perfect square. Convert some of these pairs from Babylonian numbers to decimal numbers and compute the value of b so that (a, b, c) is a Pythagorean triple.

Solution:
(a) The Babylonians used a sexagesimal positional numerical system. Essentially, this means that they counted in base 60, using 59 different symbols (59 because we are not counting the zero) and that the position of a digit in a number is nothing but the multiplier of the relative power of 60. More information about the Babylonian number system can be found here. The symbols for the number 1-10, and the multiples of 10 from 20 to 50 are as follows



(b,c) You can read about the Babylonian tablet called Plimpton 322 and about the Plimpton 322 tablet containing pairs (a, c) having the property $c^2 - a^2$ letting you compute Pythagorean triples. You can read about it here.