ON THE EXTENSION OF A FOURIER DESCRIPTOR BASED METHOD FOR PLANAR FOUR-BAR LINKAGE SYNTHESIS FOR GENERATION OF OPEN AND CLOSED PATHS

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ABSTRACT

In an earlier work, we have presented an efficient method for synthesizing crank-rocker mechanisms that are capable of generating perceptually simple and smooth low-harmonic closed curves. In this paper, we seek to extend this approach to the synthesis of planar four-bar linkages for the generation of open curves. Instead of using Fourier transform that requires a function to be defined over the entire period, we combine finite Fourier series in a curve-fitting scheme for the approximation of periodic as well as non-periodic paths. This yields a general method for planar four-bar path generation that is applicable to both closed and open paths.

1 Introduction

This paper studies the problem of dimensional synthesis of planar four-bar linkages for path generation using Fourier series. Central to this problem is the formulation of an error function that captures the deviation of the generated path from the desired path in terms of nine independent design variables for the problem. When all nine design variables are used simultaneously for comparing the shape, size, location, and orientation of the two curves, as is the case for the so-called *Structural Error Function*, the resulting optimization routine is highly inefficient and sometimes even intractable [1].

In the field of *Computational Shape Analysis* [2], it is routine to process and simplify shapes before comparisons are made. The simplified representation of shapes is called *shape descriptor* or *shape signature*. They include boundary scalar transform such as the centroid-to-boundary distance function, global scalar transform such as moment-based methods, and global space domain methods such as the media axis transform. Hoeltzel and Chieng [3] was one of the early adopters of computational shape analysis techniques for comparing the coupler curves for the dimensional synthesis of four-bar linkages. They used moment invariants to represent and compare coupler curves. Of our interest is a Fourier transform based method for characterizing the shape function. Freudenstein [4] was the first to explore the use of Fourier transform for four-bar linkage analysis and synthesis. This work was followed by Funabashi [5], Farhang et al. [6, 7], Chu and Cao [8–10], McGarva [11], and Nie and Krovi [12]. As shown by Ullah and Kota [1], when reformulated using the Fourier descriptors, the error function that captures the deviation between the synthesized and desired path, leads naturally to a reduced search space that decouples the comparison of shapes from their size, location, and orientation. In addition, they observed that the use of a small set of low-harmonic Fourier coefficients is sufficient for shape comparison. This has been independently observed by cognitive scientists for human judgement of perceptual shape similarity [13]. The combined effect of decoupled search space with small set of Fourier coefficients leads to drastic improvement in optimization routine for four-bar linkage synthesis. Fourier descriptors have seen wide applications in image processing, 2D and 3D shape recognition [14, 15]. Attempts have been made to extend Fourier descriptors from closed curves to open curves [16, 17].

Recently, we found that when the path can be approximated by the first and second harmonics of Fourier series, the search space can be reduced even further and thus leads to an even more efficient method for four-bar linkage synthesis [18]. The Fourier descriptor based approach in its current form, however, is not without limitations. The representation is global in nature and requires a periodic function, which means a closed curve. In many applications for the path synthesis problem, however, only a segment of the specified path needs to be followed either by a non-Grashof or Grashof linkage. The purpose of the current paper is to extend the Fourier descriptor based method for four-bar linkage synthesis from closed curve to open curves and from Grashof linkage to non-Grashof four-bar linkages.

The organization of the paper is as follows. Section 2 presents a least squares approximation of a given curve segment using a finite Fourier series. Section 3 converts the loop closure equations of a four-bar linkage into a periodic function in com-

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plex form and then study the Fourier series approximation of the loop-closure equation. Section 4 presents a general method for kinematic approximation of a curve segment using the coupler curve of a four-bar linkage. Both the curve segment and the loop closure equations are approximated using finite Fourier series. Section 5 presents numerical examples to demonstrate the validity of the proposed method and the developed algorithm.

2 Curve Fitting with Finite Fourier Series

When a closed curve is defined by a sequence of points rather than a continuous curve, a Discrete Fourier Transform (DFT) is used instead of the continuous Fourier Transform. The forward and inverse DFT of a finite-length sequence of N data points of a uniformly sampled, periodic function $\mathbf{z}(t) = x(t) + jy(t)$ are represented as:

$$\mathbf{z}(\frac{2\pi i}{N}) = \sum_{n=0}^{N-1} A_n e^{nj\frac{2\pi i}{N}}, \tag{1}$$

where the Fourier coefficients are given by

$$A_n = \frac{1}{N}\sum_{i=0}^{N-1} \mathbf{z}(\frac{2\pi i}{N}) e^{-nj\frac{2\pi i}{N}}. \tag{2}$$

Now consider the problem of approximating a curve segment $\mathbf{z}(t)$ by a finite series of sinusoidal functions in complex form:

$$\mathbf{z}(t) \approx \sum_{k=-p}^{+p} \alpha_k e^{jk\omega_0 t} \tag{3}$$

where p, is a small positive integer that defines the maximum order of the harmonic terms used in the approximation. Such a curve is also known as a trigonometric polynomial curve of order p in the field of Computer Aided Geometric Design [19, 20]. DFT is in general not applicable, since the function $\mathbf{z}(t)$ represents an open curve segment and is therefore not periodic. In this paper, we use the least-squares curve fitting approach [21]. The resulting coefficients are not Fourier coefficients in general. But when the least-squares curve fitting procedure is applied to a periodic function, the resulting coefficients become Fourier coefficients. For the sake of convenience, we use the term Fourier coefficients to describe coefficients α_k associated with sinusoidal function series, regardless they are "proper" Fourier series or not.

We formulate the least squares problem by defining the error function:

$$\Delta = \sum_{i=1}^{n} ||\mathbf{z}(t_i) - \sum_{k=-p}^{+p} \alpha_k e^{jk\omega_0 t_i}||^2 \tag{4}$$

in which $||\cdot||$ denotes the magnitude of a vector, $\mathbf{z}(t_i)$ ($i = 1,\ldots,n$) denote n points that are sampled from the original curve $\mathbf{z}(t)$. Then the error function, Δ, can be rewritten as

$$\Delta = \sum_{i=1}^{n}(X_i - x_i)^2 + \sum_{k=0}^{n}(Y_i - y_i)^2 \tag{5}$$

where x_i and y_i are the real and the imaginary part of $\mathbf{z}(t_i)$ respectively, and X_i and Y_i are given by

$$X_i = \Re\{\sum_{k=-p}^{+p} \alpha_k e^{ik\theta_i}\} = \sum_{k=-p}^{+p}(a_k \cos k\theta_i - b_k \sin k\theta_i) \tag{6}$$

$$Y_i = \Im\{\sum_{k=-p}^{+p} \alpha_k e^{ik\theta_i}\} = \sum_{k=-p}^{+p}(b_k \cos k\theta_i + a_k \sin k\theta_i) \tag{7}$$

wherein $\alpha_k = a_k + jb_k$ and $\theta_i = \omega_0 t_i$

To minimize the error Δ by selecting the variables a_m and b_m ($m = -p,\cdots,p$), we require:

$$\frac{\partial\Delta}{\partial a_m} = 0, \quad \frac{\partial\Delta}{\partial b_m} = 0 \tag{8}$$

for all $m \in [-p,\cdots,p]$. Rewrite the above necessary conditioins in complex form to obtain

$$\frac{\partial\Delta}{\partial a_m} + j\frac{\partial\Delta}{\partial b_m} = 0 \tag{9}$$

and we can represent the conditions with a complex matrix equation

$$\Omega\mathbb{X} = \mathbb{Y} \tag{10}$$

where

$$\mathbb{X} = \underset{m\rightarrow}{\left[\cdots,\ \alpha_m,\ \cdots\right]^T} \tag{11}$$

$$\Omega = \underset{k\rightarrow}{\begin{bmatrix} & \cdots & \\ \vdots & \sum_{i=0}^{n} e^{j(k-m)\theta_i} & \vdots \\ & \cdots & \end{bmatrix}} m\downarrow \tag{12}$$

$$\mathbb{Y} = \underset{m\rightarrow}{\left[\cdots,\ \sum_{i=0}^{n}\mathbf{z}(t_i)e^{-jm\theta_i},\ \cdots\right]^T} \tag{13}$$

and k and m vary from $-p$ to p, respectively. The solution $\mathbb{X}$ from Eq. (10) yields the best approximation of the curve $\mathbf{z}(t)$. The use of LU decomposition is recommended to avoid potential numerical problems.

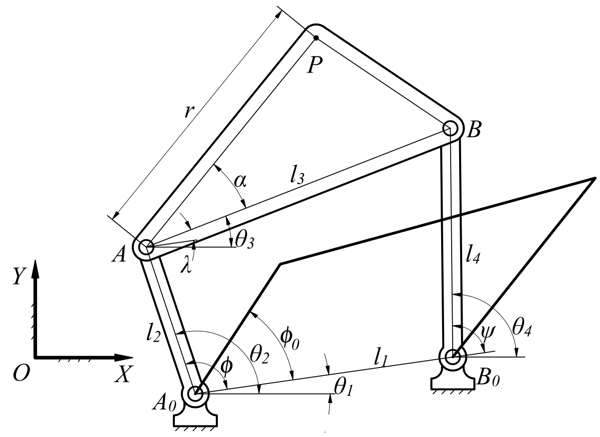


FIGURE 1. A crank-rocker mechanism

3 Fourier Approximation of Loop Closure Equation

Consider a planar four-bar mechanism shown in Figure 1 with XOY being the fixed coordinate frame. The fixed pivot A_0 is located at point (x_0, y_0). A_0B_0 is the ground link, and A_0A is the input link. The ith link has a length l_i and a position angle θ_i measured from the X axis of the fixed frame. Let ϕ, λ and ψ be the angles of link A_0A, AB, B_0B as measured from the ground link A_0B_0, respectively. A point P on the coupler link AB is defined by the distance r and the angle α. Assume that the input link rotates with a constant angular velocity ω, we have

$$\phi = \omega t + \phi_0 \tag{14}$$

where ϕ_0 is the initial input angle.

The loop closure equation for the linkage in complex form is given by

$$l_2 e^{j\phi} + l_3 e^{j\lambda} - l_1 = l_4 e^{j\psi}. \tag{15}$$

The conjugate of (15) is given by

$$l_2 e^{-j\phi} + l_3 e^{-j\lambda} - l_1 = l_4 e^{-j\psi}. \tag{16}$$

Multiplying both sides of Eqs. (15) and (16) to eliminate ψ and after some algebra, we obtain:

$$A(\phi) e^{2j\lambda} + B(\phi) e^{j\lambda} + C(\phi) = 0 \tag{17}$$

where

$$A(\phi) = l_3(l_2 e^{-j\phi} - l_1), \tag{18}$$
$$B(\phi) = l_1^2 + l_2^2 + l_3^2 - l_4^2 - 2l_1 l_2 \cos\phi, \tag{19}$$
$$C(\phi) = l_3(l_2 e^{j\phi} - l_1). \tag{20}$$

The solution to the quadric equation (17) can be obtained as

$$e^{j\lambda} = \frac{-B(\phi) \pm \sqrt{\Delta_1(\phi)\Delta_2(\phi)}}{2A(\phi)}, \tag{21}$$

where

$$\Delta_1(\phi) = l_1^2 + l_2^2 - (l_3 + l_4)^2 - 2l_1 l_2 \cos\phi, \tag{22}$$
$$\Delta_2(\phi) = l_1^2 + l_2^2 - (l_3 - l_4)^2 - 2l_1 l_2 \cos\phi. \tag{23}$$

and the sign $\pm$ correspond to the two configurations of the four-bar linkage.

Eq.(21) yields a meaningful solution for λ when the input angle ϕ satisfies the following well-known feasibility condition (see McCarthy [22]):

$$\Delta_1(\phi)\Delta_2(\phi) \leq 0 \tag{24}$$

If the above inequality is true for all ϕ in the range $[0, 2\pi]$, then the input link is a crank; otherwise, it is a rocker.

From a purely mathematical point of view, the function $e^{j\lambda}$ as defined by (21) is always periodic if we disregard the constraint (24) so that ϕ can take the full range of $[0, 2\pi]$. This allows us to obtain the Fourier series representation of $e^{j\lambda}$ as

$$e^{j\lambda} = \sum_{k=-\infty}^{\infty} C_k e^{jk\phi} \tag{25}$$

where C_k are the proper Fourier coefficients. For the sake of efficiency, only a small group of C_k associated with low harmonics components will be used for approximating $e^{j\lambda}$. When the input link is not a crank, or the path to be approximated is only a portion of the complete coupler curve, the corresponding input angle ϕ rocks in a range that is less than 2π. In this case, better approximation can be achieved if C_k is obtained from the least squares method for open curves as discussed in Section 2.

Now, we seek to obtain the Fourier representation of the coupler curve of the four bar mechanism. Let $\mathbf{A}_0 = x_0 + jy_0$ be the complex number specifying the fixed pivot A_0 and let $\mathbf{z} = re^{j\alpha}$ represent the position of P with respect to the coupler link AB. The position of the coupler point P can be represented as

$$\mathbf{P} = \mathbf{A}_0 + l_2 e^{j\theta_2} + \mathbf{z} e^{j\theta_3} = \mathbf{A}_0 + l_2 e^{j\theta_1} e^{j\phi} + \mathbf{z} e^{j\theta_1} e^{j\lambda}. \tag{26}$$

Substituting Eq. (25) into (26) and in view of Eq. (14), we obtain

$$\mathbf{P} = \sum_{k=-\infty}^{\infty} P_k e^{jk\omega t}, \tag{27}$$

where

$$\begin{cases} P_0 = \mathbf{z}e^{j\theta_1}C_0 + \mathbf{A}_0 \\ P_1 = \mathbf{z}e^{j\theta_1}C_1e^{j\phi_0} + l_2e^{j(\theta_1+\phi_0)} \\ P_k = \mathbf{z}e^{j\theta_1}C_ke^{jk\phi_0}|_{k\neq 0,1} \end{cases} \tag{28}$$

By inspecting Eq.(28), it can be concluded that $\mathbf{A}_0$, l_2 and $\mathbf{z}e^{j\theta_1}$ contribute to the harmonic components of the coupler curve $\mathbf{P}$ in different ways. First, $\mathbf{A}_0$ defines the position of the coupler curve. The variation of the position of $\mathbf{A}_0 = x_0 + jy_0$ results in a translation of the entire coupler curve. Secondly, l_2 is associated with the first harmonic term. The complex number term $\mathbf{z}e^{j\theta_1} = re^{j(\alpha+\theta)}$ acts as a constant multiple of the coefficients, C_k, associated with the loop closure equation as given by (25).

4 Fourier Based Synthesis Method

In this section, we seek to match or approximate as closely as possible a task trajectory with a Fourier approximation of the coupler curve of a four-bar linkage. The task curve is time-prescribed, i.e., it is defined as a function of the input angle, $\phi - \phi_0 = \omega t$, with ϕ_0 as its initial angle. The Fourier representation allows the decoupling of the design variables and thus greatly reduces the dimensions of the search space.

4.1 Decoupling of Design Variables

The finite Fourier series that approximates the task curve $\mathbf{T}$ is given by

$$\mathbf{T} \approx \sum_{k=-p}^{+p} T_k e^{jk\omega t} \tag{29}$$

For either the closed or the open task curve, T_k can be computed by the least squares curve fitting procedure as outlined from Eq.(3) to (13). In addition, the coupler path $\mathbf{P}$ is also approximated by a finite Fourier series:

$$\mathbf{P} \approx \sum_{k=-p}^{+p} P_k e^{jk\omega t}, \tag{30}$$

where P_k are given by Eq. (28).

The task curve $\mathbf{T}$ and the coupler curve $\mathbf{P}$ matches perfectly if

$$T_k = P_k \tag{31}$$

for all $k \in [-p, p]$. In view of Eq. (28), this leads to

$$\text{for } k=0,\ \ T_0 = C_0re^{j(\alpha+\theta_1)} + x_0 + jy_0, \tag{32}$$

$$\text{for } k=1,\ \ T_1 = C_1re^{j(\alpha+\theta_1)}e^{j\phi_0} + l_2e^{j\theta_1}e^{j\phi_0}, \tag{33}$$

$$\text{for } k\neq 0,1,\ \ T_k = C_kre^{j(\alpha+\theta_1)}e^{jk\phi_0}. \tag{34}$$

In general, the problem of path synthesis for a four-bar mechanism involves nine design variables $\{l_1, l_2, l_3, l_4, x_0, y_0, \theta_1, r, \alpha\}$ as well as the initial angle ϕ_0 of the input link. In this paper, we use the following set of ten parameters

$$\mathbf{S} = \{l_2, \frac{l_2}{l_1}, \frac{l_3}{l_1}, \frac{l_4}{l_1}, x_0, y_0, \theta_1, \mathbb{C}, \mathbb{S}, \phi_0\} \tag{35}$$

where

$$\mathbb{C} = r\cos(\alpha+\theta_1),\ \ \mathbb{S} = r\sin(\alpha+\theta_1), \tag{36}$$

for developing the synthesis procedure. Furthermore, we can separate the set $\mathbf{S}$ into two subsets of design variables:

$$\mathbf{S}_1 = \{\frac{l_2}{l_1}, \frac{l_3}{l_1}, \frac{l_4}{l_1}, \phi_0, \mathbb{C}, \mathbb{S}\}, \quad \mathbf{S}_2 = \{l_2, x_0, y_0, \theta_1\}. \tag{37}$$

This is because Eq. (34) includes exclusively the design variables contained in $\mathbf{S}_1$. In addition, it is straightforward to express the four design variables in $\mathbf{S}_2$ in terms of the six design variables in $\mathbf{S}_1$ by solving Eq. (32) and (33), i.e.,

$$x_0 + iy_0 = T_0 - C_0(\mathbb{C} + i\mathbb{S}), \tag{38}$$

$$l_2e^{i\theta_1} = T_1e^{-i\phi_0} - C_1(\mathbb{C} + i\mathbb{S}). \tag{39}$$

Note that C_0, C_1 depend only on link ratios and the initial angle ϕ_0. Thus, the problem of path synthesis is simplified to that of seeking the optimal values for $\mathbf{S}_1$ such that Eq. (34) is satisfied to the extent possible for all $k \in [-p,-1]\bigcup[2,p]$.

To find a least squares solution to Eq. (34), we define the following error function:

$$\begin{aligned} I &= \sum_{k\neq 0,1} |C_kre^{i(\alpha+\theta_1+k\phi_0)} - T_k|^2 \\ &= \sum_{k\neq 0,1} [(A_k\cdot\mathbb{C} - B_k\cdot\mathbb{S} - T_k^x)^2 + (A_k\cdot\mathbb{S} + B_k\cdot\mathbb{C} - T_k^y)^2] \end{aligned} \tag{40}$$

where $T_k = T_k^x + iT_k^y$ and $C_ke^{ik\phi_0} = A_k + iB_k$.

The variables, $\mathbb{C}$ and $\mathbb{S}$, can be used in replacement of $\alpha+\theta_1$ and r and they appear explicitly in Eq.(40). To minimize the error function I, they must satisfy

$$\frac{\partial I}{\partial \mathbb{C}}=0, \quad \frac{\partial I}{\partial \mathbb{S}}=0, \tag{41}$$

which leads to

$$\mathbb{C}+j\mathbb{S}=\frac{\sum_{k\neq 0,1} T_k C_k^* e^{-jk\phi_0}}{\sum_{k\neq 0,1}|C_k|^2} \tag{42}$$

where C_k^* is the conjugate of C_k and depends only on the ratios $l_2/l_1, l_3/l_1, l_4/l_1$ and the initial angle ϕ_0. Thus the least squares solution to Eq. (34) may be obtained by minimizing the error function I involving the following four design variables:

$$\overline{\mathbf{S}}_1=\{\frac{l_2}{l_1},\frac{l_3}{l_1},\frac{l_4}{l_1},\phi_0\}. \tag{43}$$

In summary, the use of Fourier representation enables the decoupling of design variables for the problem of path synthesis and this leads the reduction of search space from ten design variables to four design variables.

4.2 Feasibility Test

In section 3, the loop closure equation (15) has been reduced to (21), which yields $e^{i\lambda}$, as well as the feasibility condition (24), which indicates whether a four-bar linkage can be assembled for a given input angle ϕ. Thus, in addition to (21), it is required that the feasibility condition (24) be satisfied during the path synthesis process. Let the given range of ωt of the target path be $\Delta\phi$ and ϕ_0 be the initial angle. It follows from (22), (23), and (24) that the set of four design parameters $\overline{\mathbf{S}}_1$ must satisfy

$$(\frac{l_3}{l_1}-\frac{l_4}{l_1})^2 \leq \frac{l_1^2}{l_1^2}-2\frac{l_2}{l_1}\cos\phi+1 \leq (\frac{l_3}{l_1}+\frac{l_4}{l_1})^2 \tag{44}$$

for all $\phi=[\phi_0,\phi_0+\Delta\phi]$.

4.3 The Fourier Based Synthesis Algorithm

Now we present a numerical algorithm for path synthesis based on the aforementioned formulation using finite Fourier series. The task curve is specified as a function of the input angle $\phi-\phi_0$. As alluded to earlier, the goal is to find a set of design variables $(\frac{l_2}{l_1},\frac{l_3}{l_1},\frac{l_4}{l_1},\phi_0,\mathbb{C},\mathbb{S},l_2,x_0,y_0,\theta_1)$ such that the error between the task curve and the coupler curve of a four-bar linkage as defined by (40) is minimized. The numerical algorithm outlined below is based on the reduction of the search space from ten design variables to four design variables, $\overline{\mathbf{S}}_1=(\frac{l_2}{l_1},\frac{l_3}{l_1},\frac{l_4}{l_1},\phi_0)$.

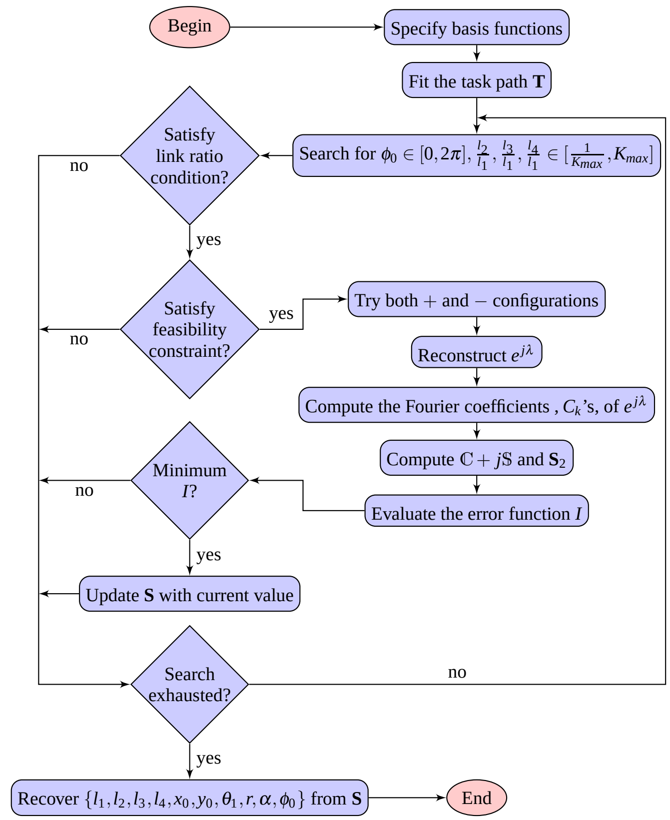


FIGURE 2. The flow chart of the synthesis algorithm

a) The task curve $\mathbf{T}$ is approximated with a user selected finite Fourier series using (29), which include the basis functions to the p^{th} order. Typically, for a smooth curve, a low harmonic curve is sufficient to encode its shape information, and thus p can be small [19,20]. After that, the least squares curve fitting procedure is applied to $\mathbf{T}$ to obtain its coefficients.

b) In this paper, we carry out a numerical search for $\overline{\mathbf{S}}_1$ within a reasonable range as described below. The range of ϕ_0 is allowed to be $[0,2\pi]$. In practical design, the ratio between any two links is not expected to be extremely large or small. So it is reasonable to predefine a max link ratio in the design process, say K_{max}. Correspondingly, the minimum link ratio is $\frac{1}{K_{max}}$. Therefore, we search $\frac{l_2}{l_1},\frac{l_3}{l_1},\frac{l_4}{l_1}$ in $[\frac{1}{K_{max}},K_{max}]$, and we are only interested in the mechanisms that satisfy the link conditions for link ratios:

$$\frac{l_m}{l_n}\in[\frac{1}{K_{max}},K_{max}], \quad \text{for all } m,n\in\{1,2,3,4\}. \tag{45}$$

c) Examine the feasibility constraint for current value of $\overline{\mathbf{S}}_1$ using (44). If the feasibility condition is satisfied, the subsequent steps **(d)** and **(e)** will be processed; otherwise, they will be skipped.

d) For each set of values for $\overline{\mathbf{S}}_1$, use Eq.(21), to obtain $e^{j\lambda}$ as a function of ϕ, and then use the least squares procedure outlined in Section 2 to obtain the Fourier coefficients, C_k.

e) Compute $\mathbb{C}+j\mathbb{S}$ using Eq. (42) and $\mathbf{S}_2$ using (38) and (39). Evaluate the error function I using (40) and store the value of I.

f) Search for the minimum value of the error function I. The corresponding values of the design variables $\mathbf{S}$ yield the optimal design for a four-bar linkage such that its coupler curve best approximates a given task curve $\mathbf{T}$.

Essential to the algorithm (Figure 2) is the search for the optimal value of $\overline{\mathbf{S}}_1$ in a four-dimensional space. For the sake of simplicity, one may adopt the direct search method to obtain the best $\overline{\mathbf{S}}_1$. This is of course not efficient but an optimal solution can always be found. The efficiency of the algorithm will be improved if a more intelligent search method such as the simulated annealing search [23] is used in replacement of the direct search method.

5 Examples

In this section, we present four examples for four-bar linkage synthesis using the Fourier descriptor based algorithm. In the first two examples, the given paths are coupler curves generated from known four-bar linkages demonstrate the validity of the proposed algorithm. In the last two examples, the paths are defined using finite Fourier series whose coefficients are selected without connection to a four-bar linkage. All length units are in inches. All these examples are implemented in MATLAB running Window XP.

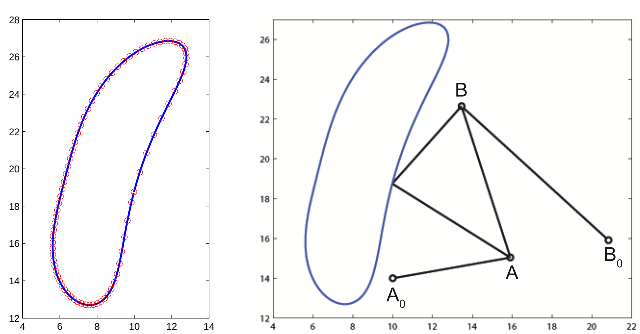


FIGURE 3. (a) The given closed path. (b) The synthesized four-bar linkage and the path it generates.

5.1 Generation of a Closed Path

First presented is an example of synthesizing a crank-rocker mechanism for the generation of a closed path. Known dimensions of a crank-rocker mechanism are used to generate a closed coupler curve as shown in Figure 3(a). This is used as the input to our algorithm to see if we can recover the original link dimensions. To find a global optimum for all choices of the initial angle ϕ_0, we evenly sample ϕ_0 over $[0, 2\pi]$, apply the direct search method to obtain the best choice for $\{\frac{l_2}{l_1}, \frac{l_3}{l_1}, \frac{l_4}{l_1}\}$, and store the resulting minimum structural error. The results show excellent match as indicated in Table 1-2 and Figure 3. The error I vs initial angle ϕ_0 is shown in Figure 4.

TABLE 1. Fourier Coefficients of the given path and two synthesized coupler curves with stepsize $\delta_1 = 0.01$ and $\delta_2 = 0.02$

	Task Path	$\delta_1 = 0.01$	$\delta_2 = 0.02$
C_{-2}	$-0.7574+0.0774i$	$-0.7574+0.0770i$	$-0.7762+0.0809i$
C_{-1}	$-2.3581+0.9571i$	$-2.3585+0.9557i$	$-2.3545+0.9514i$
C_0	$8.5305+19.6441i$	$8.5296+19.6417i$	$8.5289+19.6412i$
C_1	$4.6530-1.3332i$	$4.6516-1.3359i$	$4.6507-1.3361i$
C_2	$0.5513-0.4648i$	$0.5516-0.4645i$	$0.5470-0.4667i$
I		1.5868×10^{-5}	4.0411×10^{-4}

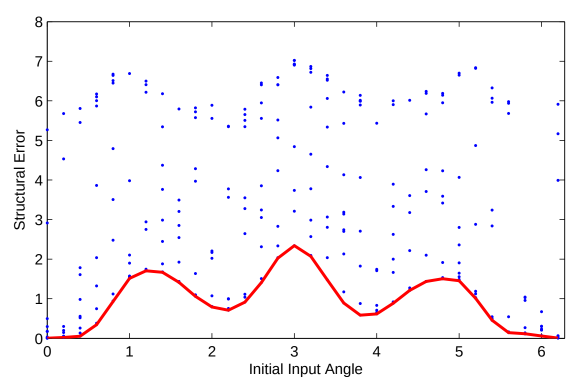


FIGURE 4. Structural error I as a function of the initial input angle

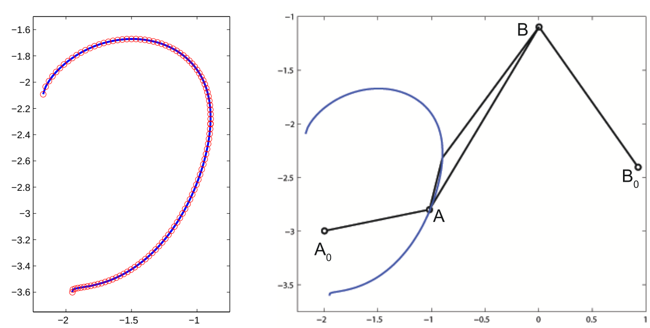
5.2 Generation of an Open Path

In this example, the open path (Figure 5(a)) is generated from a known four-bar mechanism. Again, the results indicate

TABLE 2. The design parameters of the given four-bar linkage and those of two synthesized for two stepsizes. All angles are in rad.

	Task Path	$\delta_1 = 0.01$	$\delta_2 = 0.02$
x_0	10	10.0022	10.0334
y_0	14	13.9988	14.0131
θ_1	0.1745	0.1743	0.1802
l_1	11	10.9943	10.8390
l_2	6	5.9969	5.9716
l_3	8	7.9958	7.9621
l_4	10	9.9996	9.9527
r	7	6.9996	7.0004
α	0.6981	0.6990	0.6867
ϕ_0		0.2991	0.3991

excellent match between the original mechanism and those that have been synthesized (Table 3-4 and Figure 5(a, b)).



(a) (b)

FIGURE 5. (a) The given open path. (b) The synthesized four-bar linkage and the path it generates.

5.3 Generation of Low Harmonic Curve Segments

Now consider the problem of synthesizing a four-bar linkage for generating an arbitrarily selected open path using the proposed algorithm that includes the use of the simulated annealing algorithm instead of direct search on $\{\frac{l_2}{l_1}, \frac{l_3}{l_1}, \frac{l_4}{l_1}\}$.

Simulated annealing is a probabilistic search algorithm for global optimization. The results inherit a degree of randomness. This fact is demonstrated by the following two examples. The first one is a 2-harmonic curve segment as shown as dotted curve

TABLE 3. Fourier Coefficients of the given open path and two synthesized coupler curves with stepsize $\delta_1 = 0.01$ and $\delta_2 = 0.02$

	Task Path	$\delta_1 = 0.01$	$\delta_2 = 0.02$
C_{-2}	$0.0061 + 0.0137i$	$0.0178 - 0.0118i$	$0.0211 - 0.0134i$
C_{-1}	$-0.0590 + 0.1365i$	$-0.0762 + 0.1721i$	$-0.0798 + 0.1757i$
C_0	$-1.6911 - 2.6475i$	$-1.6716 - 2.6864i$	$-1.6676 - 2.6910i$
C_1	$0.8290 + 0.2092i$	$0.8113 + 0.2432i$	$0.8082 + 0.2476i$
C_2	$0.0202 - 0.0364i$	$0.0326 - 0.0600i$	$0.0350 - 0.0631i$
I		1.2340×10^{-7}	9.6908×10^{-7}

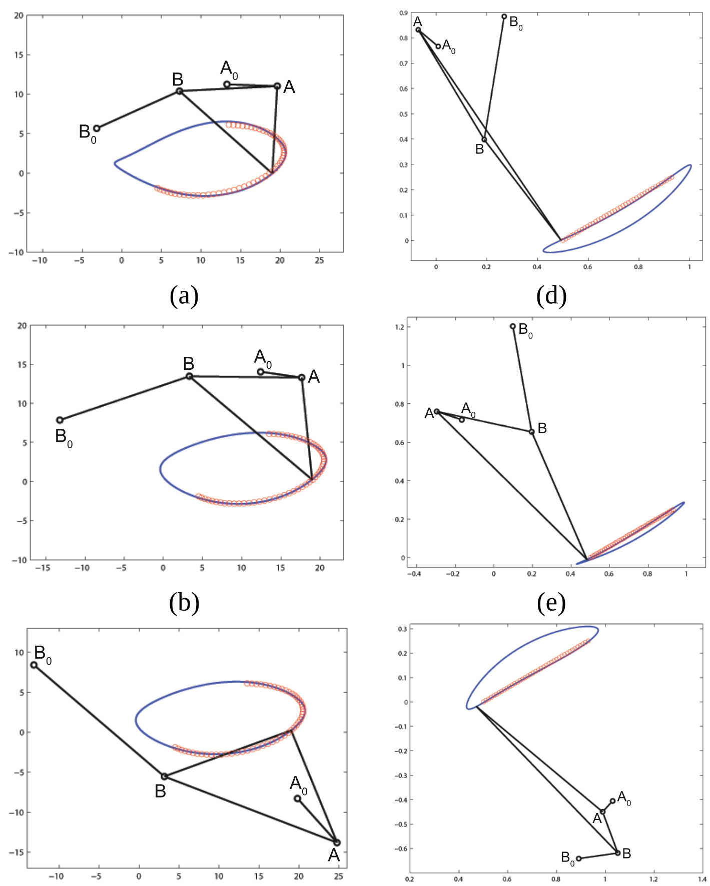
TABLE 4. The design parameters of the given four-bar linkage and those of two synthesized for two different step-sizes. All angles are in rad.

	Task Path	$\delta_1 = 0.01$	$\delta_2 = 0.02$
x_0	-2	-1.9982	-1.9977
y_0	-3	-2.9971	-2.9990
θ_1	0.2	0.1997	0.2015
l_1	3	2.9856	2.9581
l_2	1	0.9998	1.0006
l_3	2	1.9853	1.9868
l_4	1.6	1.6011	1.5705
r	0.5	0.4970	0.4976
α	0.3	0.2963	0.3042
ϕ_0		0	0.0030

in Figure 6(a-c). The equation of the curve is given by

$$\begin{aligned}\mathbf{T} = &(10.0 + 2.0i) + (2.0 + 1.4i)e^{-i\omega t} + (6.0 - 3.0i)e^{i\omega t} \\ &+ (0.2 + 0.4i)e^{-2i\omega t} + (0.9 - 0.5i)e^{2i\omega t}\end{aligned} \tag{46}$$

where ωt varies in the range $[-0.5\pi, 0.5\pi]$. We use the simulated annealing algorithm to search on $\{\frac{l_2}{l_1}, \frac{l_3}{l_1}, \frac{l_4}{l_1}, \phi_0\}$ with the maximum link ratio being 5. Good results are obtained when the algorithm is repeated up to 20 times with an average running time for each search is 24.9375 seconds. This is clocked with MATLAB on a Dell laptop running Window XP with Intel Core Duo CPU T2250 at 1.73GHz and 1 GB of RAM. Three synthesized mechanisms are shown in Figure 6(a-c).



(a) (b) (c) (d) (e) (f)

FIGURE 6. The synthesized mechanisms for the generation of low-harmonic curve segment and line-segment. The given paths and the synthesized curves are represented by circular dots and solid curves, respectively.
(a): The structural error I is 0.0677. The running time is 22.0938 sec.
(b): The structural error I is 0.0049. The running time is 21.8281 sec.
(c): The structural error I is 0.0016. The running time is 28.5000 sec.
(d): The structural error I is 0.0017, and the running time is 52.7813 sec.
(e): The structural error I is 0.0016, and the running time is 58.2656 sec.
(f): The structural error I is 0.0015, and the running time is 178.75 sec.

The second one, as shown in Figure 6(d-f), is a straight-line path from $(0.5, 0)$ to $(0.93, 0.25)$ and the corresponding input angle ϕ varies from ϕ_0 to $\phi_0 + 0.75\pi$. Simulated annealing search is also used with the maximum link ratio $K_{max} = 5$. The straight-line path is approximated by a 3-harmonic curve, a 4-harmonic curve, and a 5-harmonic curve, respectively, and the resulting three optimal mechanisms as illustrated in Figure 6-(e), (f) and (g), respectively.

6 Conclusions

In this paper, we have extended the Fourier descriptor based method for kinematic approximation using a least squares formulation. This leads to a general method for the dimensional synthesis of four-bar linkage for generating not only desired closed curves but also desired open curve segments. The method applies not only to Grashof linkages but also non-Grashof linkages. In addition, the method can be used for approximating a curve segment using only a portion of the coupler curve instead of the complete coupler curve as required in the original Fourier based method.

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