COMP 9602: Assignment 2

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Problem 1

(a).

Proof.

Necessity. Suppose there exists a $v \neq 0$ with $Av \preceq 0$. Since $\mathbf{dom} f_0$ is nonempty, some $x_0 \in \mathbf{dom} f_0$ exists such that $Ax_0 \prec b$. Hence for any $\lambda > 0$,

$$A(x_0 + \lambda v) \prec b \implies x_0 + \lambda v \in \mathbf{dom} f_0$$

With the fact that λ can be arbitrarily large, and $v \neq 0$, $\mathbf{dom} f_0$ is unbounded.

Sufficiency. If $\mathbf{dom} f_0$ is unbounded, there is a sequence of vectors $\{x^k\}$ with $\|x^k\|_2 \to \infty$ in $\mathbf{dom} f_0$. With each vector normalized we get a new sequence $\{v^k\}$, namely

$$v^k = \frac{x^k}{\|x^k\|_2} \quad (\forall k.\ \|v^k\|_2 = 1)$$

Since it's bounded by unit vectors, by **Bolzano-Weierstrass theorem**, $\{v^k\}$ has a convergent subsequence. Denote v as the limit.

Since each $x^k \in \mathbf{dom} f_0$, $Ax^k \prec b$. So for all i, $a_i^T x^k < b_i$. Hence

$$a_i^T v^k < \frac{b_i}{\|x^k\|_2} \to 0$$

Therefore the limit v satisfies $Av \preceq 0$, and sure $v \neq 0$.

□

(b).

Proof.

Necessity. Suppose there exists v such that $Av \preceq 0$, $Av \neq 0$. For any x in the domain of f_0, and for any $\lambda > 0$,

$$f_0(x + \lambda v) = -\sum_{i=1}^{m} \log(b_i - a_i^T x - \lambda a_i^T v)$$

By the assumption, there exist some is with $a_i^T v < 0$, while for others $a_i^T v = 0$. So in the equation above, each component is nondecreasing, and some can be arbitrarily large with respect to λ. Hence the result $f_0(x + \lambda v)$ is unbounded below.

Sufficiency. If f_0 is unbounded below, there is a sequence of vectors $\{x^k\}$ with $f_0(x^k) \to -\infty$. Since f_0 is convex, by the 1st-order condition,

$$f_0(x^k) \geq f_0(x^0) + (x^k - x^0) \sum_{i=1}^{k} \frac{a_i^T}{b_i - a_i^T x^0}$$

And since $b_i - a_i^T x^0 > 0$, we conclude that $\min_i a_i^T x^k \to -\infty$.

Assume there exists some $z \succ 0$ with $A^T z = 0$. On the one hand, since each x^k satisfies $b - Ax^k \succ 0$,

$$z^T(b - Ax^k) = \sum_i z_i^T(b_i - a_i^T x^k) \geq \max_i z_i^T(b_i - a_i^T x^k) \to \infty$$

Note that it's because $z_i^T > 0$, and $\min_i a_i^T x^k \to -\infty$.

Therefore $z^T(b - Ax^k) \to \infty$. On the other hand $z^T Ax^k = 0^T x^k = 0$, so $z^T b \to \infty$, a contradiction derives.

Thus no such z exists. Hence there exists a v with $Av \preceq 0$, $Av \neq 0$.

□

(c). Honestly I don't think this proposition is correct. With the assumption that f_0 is bounded below, if $\mathbf{dom} f_0$ is bounded, surely the minimum can be attained. But by the conclusion of **(a)** and **(b)**, if there exists a $v \neq 0$ with $Av = 0$, potentially we can also have the f_0 bounded below while its domain is unbounded. In that case I don't know why the minimum is attained.

(d).

Proof.

Sufficiency. Since we have

$$f_0(x^* + v) = -\sum_{i=1}^{m} \log(b_i - a_i^T x^* - a_i^T v)$$

If $Av = 0$, it is obvious that $f_0(x^* + v) = f_0(x^*)$. So $Av = 0$ is a sufficient condition for optimal set.

Necessity. I don't know how to prove this direction.

□

Problem 2

Problem 3

Problem 4

The primal problem is

$$
\begin{array}{ll}
\text{minimize} & \frac{1}{2}\|x-a\|_2^2 \\
\text{subject to:} & \|x\|_1 \leq 1
\end{array}
$$

The Lagrangian function $L(x, \lambda)$ is: $(\lambda \geq 0)$

$$
\begin{aligned}
L(x,\lambda) &= \frac{1}{2}\|x-a\|_2^2 + \lambda(\|x\|_1 - 1) \\
&= \sum_{i=1}^{n}\left(\frac{1}{2}(x_i - a_i)^2 + \lambda|x_i|\right) - \lambda
\end{aligned}
$$

Since the Lagrange dual function

$$
g(\lambda) = \inf_x L(x,\lambda) = \sum_i h_i(\lambda) - \lambda
$$

where the function $h_i(\lambda)$ is as follows:

$$
h_i(\lambda) = \inf_{x_i} \begin{cases} \frac{1}{2}(x_i - a_i)^2 - \lambda x_i & , \ x_i \leq 0 \\ \frac{1}{2}(x_i - a_i)^2 + \lambda x_i & , \ x_i > 0 \end{cases}
$$

Two local optimal points $x_-^* = a_i + \lambda$, $x_+^* = a_i - \lambda$.

(1) If $a_i \leq -\lambda$, both x_-^* and x_+^* are non-positive, $x_i^* = x_-^*$, in which case

$$
h_i(\lambda) = \frac{1}{2}a_i^2 - \frac{1}{2}(\lambda + a_i)^2 = \lambda(-a_i - \frac{1}{2}\lambda)
$$

(2) If $|a_i| < \lambda$, x_-^* is positive but x_+^* is negative, hence $x_i^* = 0$, in which case

$$
h_i(\lambda) = \frac{1}{2}a_i^2
$$

(3) If $a_i \geq \lambda$, both x_-^* and x_+^* are non-negative, $x_i^* = x_+^*$, in which case

$$
h_i(\lambda) = \frac{1}{2}a_i^2 - \frac{1}{2}(\lambda - a_i)^2 = \lambda(a_i - \frac{1}{2}\lambda)
$$

Therefore,

$$
h_i(\lambda) = \begin{cases} \lambda(|a_i| - \frac{1}{2}\lambda) & , \ \lambda \leq |a_i| \\ \frac{1}{2}a_i^2 & , \ \lambda > |a_i| \end{cases}
$$

Problem 4 continued on next page...

Hence the dual problem is

$$\begin{array}{ll} \text{maximize} & \sum_i h_i(\lambda) - \lambda \\ \text{subject to:} & \lambda \geq 0 \end{array}$$

Now it comes to a solution for the dual problem. The objective $g(\lambda) = \sum_i h_i(\lambda) - \lambda$ is a unary function, so we can calculate its derivative. And the optimal point λ^* satisfies $g'(\lambda^*) = 0$ if it is attained. Specifically, the derivative of $h_i(\lambda)$ is

$$h_i'(\lambda) = \begin{cases} |a_i| - \lambda & , \ \lambda \leq |a_i| \\ 0 & , \ \lambda > |a_i| \end{cases}$$

And $g'(\lambda) = \sum_i h_i'(\lambda) - 1$. Note that every $h_i'(\lambda)$ is non-increasing, so $g'(\lambda)$ is also non-increasing. Also, $g'(0) = \|a\|_1 - 1$, and for $\lambda \geq \max_i |a_i|$, $g'(\lambda) = -1$. To find the solution to $g'(\lambda) = 0$, we need to discuss the cases.

(1) $\|a\|_1 \leq 1$. In the primal problem, $x^* = a$ can optimize the primal objective and $p^* = 0$.

(2) $\|a\|_1 > 1$. As $g'(\lambda)$ is monotonic, the optimal point $\lambda^* \in [0, \max_i |a_i|]$. To be concise, we can assume that the sequence $\{|a_1|, |a_2|, \cdots, |a_n|\}$ is non-decreasing, namely $|a_1| \leq |a_2| \leq \cdots \leq |a_n|$.

- $\lambda \in [0, |a_1|)$: $g'(\lambda) = |a_1| + |a_2| + \cdots + |a_n| - \lambda n - 1$.
- $\cdots$
- $\lambda \in [|a_k|, |a_{k+1}|)$: $g'(\lambda) = |a_{k+1}| + \cdots + |a_n| - \lambda(n-k) - 1$.
- $\cdots$

Hence to get the optimal point λ^*, we firstly find which interval it belongs to. We calculate each $g'(|a_i|)$ one by one.

$$\begin{aligned} g'(0) &= \|a\|_1 - 1 \\ g'(|a_1|) &= g'(0) - n(|a_1| - 0) \\ g'(|a_2|) &= g'(|a_1|) - (n-1)(|a_2| - |a_1|) \\ &\cdots \\ g'(|a_k|) &= g'(|a_{k-1}|) - (n+1-k)(|a_k| - |a_{k-1}|) \\ &\cdots \end{aligned}$$

If $a_k \leq \lambda \leq a_{k+1}$ for some k, we have

$$g'(\lambda^*) = |a_{k+1}| + \cdots + |a_n| - \lambda^*(n-k) - 1 = 0$$

$$\lambda^* = \frac{\sum_{i=k+1}^n |a_i| - 1}{n-k}$$

Hence

$$d^* = g(\lambda^*) = \frac{1}{2}\sum_{i=1}^k a_i^2 + \lambda^* \sum_{i=k+1}^n |a_i| - \frac{1}{2}{\lambda^*}^2(n-k) - \lambda^*$$

And hence we obtain the optimal value of the primal problem, namely $p^* = d^*$, since by Slater's condition, we can easily find a strictly feasible point x with $\|x\|_1 < 1$, and hence strict duality holds.

Problem 5

(a). The primal problem over $\mathcal{D} = \{(x, y) \mid y > 0\}$ is

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to:} & x^2/y \leq 0 \end{array}$$

The domain is a convex set, and e^{-x} is convex, for sure. And the function $g(x, y) = x^2/y$ is the perspective of $f(x) = x^2$, so it's also convex. Therefore it is a convex optimization problem.

The constraints are satisfied only when $x = 0$, in which case the value is optimal value, namely $p^* = 1$.

(b). The Lagrangian

$$L(x, y, \lambda) = e^{-x} + \frac{\lambda x^2}{y}$$

The Lagrange dual function

$$g(\lambda) = \inf_{y>0,x} \left(e^{-x} + \frac{\lambda x^2}{y} \right)$$

Firstly with $y \gg \lambda x^2$, $\frac{\lambda x^2}{y} \to 0$, and then let $x \to \infty$, the dual problem is

$$\begin{array}{ll} \text{maximize} & 0 \\ \text{subject to:} & \lambda \geq 0 \end{array}$$

Hence any $\lambda \geq 0$ is an optimal solution. $d^* = 0$. The optimal duality gap is $p^* - d^* = 1$.

(c). Slater's condition doesn't hold for this problem, since there is no strictly feasible point (x, y) with $y > 0$ satisfying that $x^2/y < 0$.

(d). Now the perturbed problem is

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to:} & x^2/y \leq u \end{array}$$

(1) If $u > 0$, by setting $x \to \infty$, $y \to \infty$ and $x^2/y \to 0$, the optimal value is $p^*(u) = 0$.

(2) If $u = 0$, $p^*(0) = p^* = 1$, from **(a)**.

The Lagrange dual function of the perturbed problem is

$$g(\lambda) = \inf_{x,y} \left(e^{-x} + \frac{\lambda x^2}{y} - \lambda u \right) = -\lambda u$$

The constraint is $\lambda \geq 0$. When $u > 0$, to maximize $g(\lambda)$, the optimal point is $\lambda^* = 0$.

Hence when $u > 0$, the global sensitivity inequality

$$p^*(u) \geq p^*(0) - \lambda^* u$$

becomes $0 \geq 1 - 0$, which doesn't hold for sure.

Problem 6