COMP 9602: Assignment 2

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Problem 1

(1).

Proof.

Necessity. Suppose there exists a $v \neq 0$ with $Av \preceq 0$. Since $\mathbf{dom} f_0$ is nonempty, some $x_0 \in \mathbf{dom} f_0$ exists such that $Ax_0 \prec b$. Hence for any $\lambda > 0$,

$$A(x_0 + \lambda v) \prec b \implies x_0 + \lambda v \in \mathbf{dom} f_0$$

With the fact that λ can be arbitrarily large, and $v \neq 0$, $\mathbf{dom} f_0$ is unbounded.

Sufficiency. If $\mathbf{dom} f_0$ is unbounded, there is a sequence of vectors $\{x^k\}$ with $\|x^k\|_2 \to \infty$ in $\mathbf{dom} f_0$. With each vector normalized we get a new sequence $\{v^k\}$, namely

$$v^k = \frac{x^k}{\|x^k\|_2} \quad (\forall k.\ \|v^k\|_2 = 1)$$

Since it's bounded by unit vectors, by **Bolzano-Weierstrass theorem**, $\{v^k\}$ has a convergent subsequence. Denote v as the limit.

Since each $x^k \in \mathbf{dom} f_0$, $Ax^k \prec b$. So for all i, $a_i^T x^k < b_i$. Hence

$$a_i^T v^k < \frac{b_i}{\|x^k\|_2} \to 0$$

Therefore the limit v satisfies $Av \preceq 0$, and sure $v \neq 0$.

□

(2).

Proof.

Necessity. Suppose there exists v such that $Av \preceq 0$, $Av \neq 0$. For any x in the domain of f_0, and for any $\lambda > 0$,

$$f_0(x + \lambda v) = -\sum_{i=1}^{m} \log(b_i - a_i^T x - \lambda a_i^T v)$$

By the assumption, there exist some is with $a_i^T v < 0$, while for others $a_i^T v = 0$. So in the equation above, each component is nondecreasing, and some can be arbitrarily large with respect to λ. Hence the result $f_0(x + \lambda v)$ is unbounded below.

Sufficiency. If f_0 is unbounded below, there is a sequence of vectors $\{x^k\}$ with $f_0(x^k) \to -\infty$. Since f_0 is convex, by the 1st-order condition,

$$f_0(x^k) \geq f_0(x^0) + (x^k - x^0) \sum_{i=1}^{k} \frac{a_i^T}{b_i - a_i^T x^0}$$

And since $b_i - a_i^T x^0 > 0$, we conclude that $\min_i a_i^T x^k \to -\infty$.

Assume there exists some $z \succ 0$ with $A^T z = 0$. On the one hand, since each x^k satisfies $b - Ax^k \succ 0$,

$$z^T(b - Ax^k) = \sum_i z_i^T(b_i - a_i^T x^k) \geq \max_i z_i^T(b_i - a_i^T x^k) \to \infty$$

Note that it's because $z_i^T > 0$, and $\min_i a_i^T x^k \to -\infty$.

Therefore $z^T(b - Ax^k) \to \infty$. On the other hand $z^T Ax^k = 0^T x^k = 0$, so $z^T b \to \infty$, a contradiction derives.

Thus no such z exists. Hence there exists a v with $Av \preceq 0$, $Av \neq 0$.

□

(3). ??? (I don't think this is correct if A is not full-rank.)

(4).

Proof.

Sufficiency. Since we have

$$f_0(x^* + v) = -\sum_{i=1}^{m} \log(b_i - a_i^T x^* - a_i^T v)$$

If $Av = 0$, it is obvious that $f_0(x^* + v) = f_0(x^*)$. So $Av = 0$ is a sufficient condition for optimal set.

Necessity. ??? (How to prove it?)

□

Problem 2

Problem 3

Problem 4

The primal problem is

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|x-a\|_2^2 \\ \text{subject to:} & \|x\|_1 \leq 1 \end{array}$$

The Lagrangian function $L(x, \lambda)$ is: $(\lambda \geq 0)$

$$\begin{aligned} L(x, \lambda) &= \frac{1}{2}\|x-a\|_2^2 + \lambda(\|x\|_1 - 1) \\ &= \sum_{i=1}^{n} \left(\frac{1}{2}(x_i - a_i)^2 + \lambda |x_i| \right) - \lambda \end{aligned}$$

Since the Lagrange dual function

$$g(\lambda) = \inf_x L(x, \lambda) = \sum_i h_i(\lambda) - \lambda$$

where the function $h_i(\lambda)$ is as follows:

$$h_i(\lambda) = \inf_{x_i} \left\{ \begin{array}{ll} \frac{1}{2}(x_i - a_i)^2 - \lambda x_i & , \ x_i \leq 0 \\ \frac{1}{2}(x_i - a_i)^2 + \lambda x_i & , \ x_i > 0 \end{array} \right.$$

Two local optimal points $x_-^* = a_i + \lambda$, $x_+^* = a_i - \lambda$.

(1) If $a_i \leq -\lambda$, both x_-^* and x_+^* are non-positive, $x_i^* = x_-^*$, in which case

$$h_i(\lambda) = \frac{1}{2}a_i^2 - \frac{1}{2}(\lambda + a_i)^2 = \lambda(-a_i - \frac{1}{2}\lambda)$$

(2) If $|a_i| < \lambda$, x_-^* is positive but x_+^* is negative, hence $x_i^* = 0$, in which case

$$h_i(\lambda) = \frac{1}{2}a_i^2$$

(3) If $a_i \geq \lambda$, both x_-^* and x_+^* are non-negative, $x_i^* = x_+^*$, in which case

$$h_i(\lambda) = \frac{1}{2}a_i^2 - \frac{1}{2}(\lambda - a_i)^2 = \lambda(a_i - \frac{1}{2}\lambda)$$

Therefore,

$$h_i(\lambda) = \left\{ \begin{array}{ll} \lambda(|a_i| - \frac{1}{2}\lambda) & , \ \lambda \leq |a_i| \\ \frac{1}{2}a_i^2 & , \ \lambda > |a_i| \end{array} \right.$$

Hence the dual problem is

$$\begin{array}{ll} \text{maximize} & \sum_i h_i(\lambda) - \lambda \\ \text{subject to:} & \lambda \geq 0 \end{array}$$

??? (Q: How to solve it?) (A: Use derivative.)

??? (Efficient algorithm?)

Problem 5

(a). The primal problem over $\mathcal{D} = \{(x, y) \mid y > 0\}$ is

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to:} & x^2/y \leq 0 \end{array}$$

The domain is a convex set, and e^{-x} is convex, for sure. And the function $g(x, y) = x^2/y$ is the perspective of $f(x) = x^2$, so it's also convex. Therefore it is a convex optimization problem.

The constraints are satisfied only when $x = 0$, in which case the value is optimal value, namely $p^* = 1$.

(b). The Lagrangian

$$L(x, y, \lambda) = e^{-x} + \frac{\lambda x^2}{y}$$

The Lagrange dual function

$$g(\lambda) = \inf_{y>0,x} \left(e^{-x} + \frac{\lambda x^2}{y} \right)$$

Firstly with $y \gg \lambda x^2$, $\frac{\lambda x^2}{y} \to 0$, and then let $x \to \infty$, the dual problem is

$$\begin{array}{ll} \text{maximize} & 0 \\ \text{subject to:} & \lambda \geq 0 \end{array}$$

Hence any $\lambda \geq 0$ is an optimal solution. $d^* = 0$. The optimal duality gap is $p^* - d^* = 1$.

(c). Slater's condition doesn't hold for this problem, since with the constraint $y > 0$, $x^2/y < 0$ is not feasible. (??? Right?)

(d). ???

Problem 6