# Standard $z$-Transform Pair

| $x(n)$ | $X(z)$ | ROC |
|---|---|---|
| $\delta[n]$ | $1$ | All $z$ |
| $\delta[n-m]$ | $z^{-m}$ | All $z$ except $0$ if $m>0$, $\infty$ if $m<0$ |
| $u[n]$ | $\frac{1}{1-z^{-1}}, \frac{z}{z-1}$ | $\|z\|>1$ |
| $-u[-n-1]$ | $\frac{1}{1-z^{-1}}, \frac{z}{z-1}$ | $\|z\|<1$ |
| $a^n u[n]$ | $\frac{1}{1-az^{-1}}, \frac{z}{z-a}$ | $\|z\|>\|a\|$ |
| $-a^n u[-n-1]$ | $\frac{1}{1-az^{-1}}, \frac{z}{z-a}$ | $\|z\|<\|a\|$ |
| $\cos\omega_0 n\, u[n]$ | $\frac{z^2-\cos\omega_0 z}{z^2-2\cos\omega_0 z+1}$ | $\|z\|>1$ |
| $\sin\omega_0 n\, u[n]$ | $\frac{\sin\omega_0 z}{z^2-2\cos\omega_0 z+1}$ | $\|z\|>1$ |
| $r^n\cos\omega_0 n\, u[n]$ | $\frac{z^2-r\cos\omega_0 z}{z^2-2r\cos\omega_0 z+r^2}$ | $\|z\|>\|r\|$ |

$r^n \sin \omega_0 n \, u[n]$ $\quad \dfrac{r \sin \omega_0 z}{z^2 - 2r\cos\omega_0 z + r^2}$ $\quad |z| > |r|$

$x(n) = \begin{cases} a^n, & \text{for } 0 \le n \le N-1 \\ 0, & \text{else} \end{cases}$ $\quad \dfrac{1 - a^N z^{-N}}{1 - a z^{-1}}$ $\quad |z| > 0$

$n a^n u[n]$ $\quad \dfrac{a z}{(z-a)^2}$ $\quad |z| > |a|$

$-n a^n u[-n-1]$ $\quad \dfrac{a z}{(z-a)^2}$ $\quad |z| < |a|$

formlae from property.

$y[n] \longleftrightarrow Y(z)$

$x[n] \longleftrightarrow X(z)$

$x[n-n_0] \longleftrightarrow z^{-n_0} X(z)$

$x[-n] \longleftrightarrow X(z^{-1})$

$a^n x(n) \longleftrightarrow X(z/a)$

$n x[n] \longleftrightarrow -z \dfrac{d}{dz} [X(z)]$

$x^*[n] \longleftrightarrow X^*(z^*)$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$x(n) = z^{-1}(X(z))$

## Inverse z - Transform :-

$$x[n] = z^{-1}\{X(z)\}$$

finding the seq. $x[n]$ from its z transform $X(z)$ is called inverse z-transform

① Inversion formula

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)\, z^{-n-1}\, dz$$

Where C is anticlockwise contour of integration enclosing with origin.

② Using tables

Given $X(z)$,

$$X(z) = X_1(z) + X_2(z) + \cdots\cdots + X_K(z)$$

Where

$X_1(z), X_2(z) \cdots\cdots X_K(z)$ are known z transform pair of seq $x_1[n], x_2[n] \cdots x_K[n]$ respectively

so that,

$$x[n] = x_1[n] + x_2[n] + \cdots\cdots + x_K[n]$$

(iii) Division method (Power Series expansion)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots + x[-2] z^{2} + x[-1] z + x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots$$

In this method coeff. of seq. $x[n]$ is determined by comparing the coeff. of appropriate power of $z^{-1}$ in $x(z)$. This method is useful for finite length seq. This method may not provide closed form sol/n, in such a case long division method can be applied.

(iv) Partial fraction expansion

Given $X(z)$,

$$X(z) = \frac{N(z)}{D(z)}$$

$$X(z) = \frac{(z - z_1)(z - z_2) \cdots (z - z_m)}{(z - p_1)(z - p_2) \cdots (z - p_n)}$$

$m \rightarrow$ zeros

$n \rightarrow$ poles

(a) if $m < n$ and all-poles are simple then partial fraction has the form

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \dots + \frac{c_n}{z-p_n}$$

where,

$$c_0 = X(z)\Big|_{z=0} \qquad c_K = (z-p_K)\frac{X(z)}{z}\Big|_{z=p_K}$$

(b) if $m > n$, improper rational function

First make $X(z)$ proper rational function then go for partial fraction expansion

$$X(z) = \sum_{q=0}^{m-n} b_q z^{-q} + \sum_{K=1}^{n} c_K \cdot \frac{1}{1-p_K \cdot z^{-1}}$$

(c) multiple poles at $(z-p_i)^r$

$r$ poles at $z = p_i$

then term containing multiple poles can be expressed by partial function as

$$\frac{d_1}{z-p_i} + \frac{d_2}{(z-p_i)^2} + \dots + \frac{d_r}{(z-p_i)^r}$$

Where

$$d_{r-K} = \frac{1}{K!}\frac{d^K}{dz^K}(z-p_i)^r \frac{X(z)}{z}\Big|_{z=p_i}$$

## Properties of z-Transform

z transform of $x[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

and

inverse z-transform of $X(z)$ is given by

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz$$

Notation: $x[n] \overset{z}{\longleftrightarrow} X(z)$, ROC: R

(i) linearity

if $x_1[n] \overset{z}{\longleftrightarrow} X_1(z)$, ROC: $R_1$

$x_2[n] \overset{z}{\longleftrightarrow} X_2(z)$, ROC: $R_2$

then,

$$a_1 x_1[n] + a_2 x_2[n] \overset{z}{\longleftrightarrow} a_1 X_1(z) + a_2 X_2(z),$$

ROC: $R_1 \cap R_2$

(ii) Time Reversal

if $x[n] \overset{z}{\longleftrightarrow} X(z)$, ROC: R

then

$$x[-n] \overset{z}{\longleftrightarrow} X(z^{-1}) = X\left(\frac{1}{z}\right), \quad ROC = \frac{1}{R}$$

After time Reversal, a pole or zero in $X(z)$ at $z = z_k$ moves to $z = \frac{1}{z_k}$

Proof:-

$$\mathcal{Z}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n] z^{-n}$$

put

$-n = k$

$$= \sum_{k=-\infty}^{\infty} x[k] z^{k}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \left(\frac{1}{z}\right)^{-k}$$

$$= X\left(\frac{1}{z}\right), \quad ROC: \frac{1}{R}$$

(iii) Time shifting

if $x[n] \xleftrightarrow{z} X(z)$, ROC: R

then $x[n-n_0] \xleftrightarrow{z} z^{-n_0} X(z)$,

ROC: $R \cap |z| > 0$ if $n_0 > 0$

$R \cap |z| < \infty$ if $n_0 < 0$

$x[n-1] \xleftrightarrow{z} z^{-1} X(z)$, ROC: $R \cap |z| > 0$

unit delay

$x[n+1] \xleftrightarrow{z} zX(z)$ , ROC: $R \cap |z| < \infty$

↳ Unit Advance.

Proof :-

$$\mathcal{Z}\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0] z^{-n}$$

put $n - n_0 = k$

$$= \sum_{k=-\infty}^{\infty} x[k] z^{-k-n_0}$$

$$= z^{-n_0} \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

$$= z^{-n_0} X(z),$$

(iv) multiplication by $z_0^n$

if $x[n] \xleftrightarrow{z} X(z)$ , ROC: R

then,

$$z_0^n x[n] \xleftrightarrow{z} X\left(\frac{z}{z_0}\right), \text{ ROC: } |z_0| R$$

After multiplication of sequence $x[n]$ by $z_0^n$, a pole or zero in $x(z)$ at $z = z_k$ moves to $z = z_0 \cdot z_k$ & region of convergence will expand or contract by a factor of abs$(z_0)$

proof :-

$$\mathcal{Z}\{z_0^{\,n} x[n]\} = \sum_{n=-\infty}^{\infty} z_0^{\,n} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{z}{z_0}\right)^{-n}$$

$$= X\left(\frac{z}{z_0}\right)$$

$$e^{j\omega_0 n} x[n] \xleftrightarrow{z} X\left(\frac{z}{e^{j\omega_0}}\right), \quad \text{ROC}: R$$

(v) multiplication by $n$ (differentiation in $z$)

if $x[n] \xleftrightarrow{z} X(z)$, ROC: R

then,

$$n x[n] \xleftrightarrow{z} -z \frac{d}{dz} X(z) \quad \text{ROC}: R$$

proof :-

we have

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] = z^{-n}$$

differentiating on both sides wrt $z$,

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x[n] \left(-n z^{-n-1}\right)$$

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$$-z \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} n\,x[n]\,z^{-n}$$

$$\therefore\ Z\{n\,x[n]\} = -z \frac{d}{dz} X(z)$$

Point 2:

(vi) Complex Conjugate:

Contd. from Red Copy:-

# $X(z) = \dfrac{z}{2z^2-3z+1}$ , $|z|>1$

Using division method.

Since, ROC is $|z|>1$ Seq. $x[n]$ must be Right sided So we must divide ~~so we~~ to obtain the series in power of $z^{-1}$

$$
\begin{array}{r|l}
 & \frac{1}{2}z^{-1} \\
\hline
2z^2-3z+1 & \cancel{z} \\
 & -\cancel{z} - \frac{3}{2} + \frac{1}{2}z^{-1} \\
 & \quad\;\; + \qquad - \\
\hline
 & \cancel{\frac{3}{2}} - \frac{1}{2}z^{-1} \\
 & \cancel{\frac{3}{2}} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2} \\
 & - \qquad + \qquad - \\
\hline
 & \cancel{\frac{7}{4}z^{-1}} - \frac{3}{4}z^{-2} \\
 & \cancel{\frac{7}{4}z^{-1}} - \frac{21}{8}z^{-2} + \frac{7}{8}z^{-1} \\
 & \qquad + \qquad\quad - \\
\hline
 & \frac{15}{8}z^{-2} - \frac{7}{8}z^{-3}
\end{array}
$$

$$X(z) = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \frac{15}{16}z^{-4} + \cdots$$

Computing with $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

We get

$x(1) = \frac{1}{2}$, $x[2] = \frac{3}{4}$ $x[3] = \frac{7}{8}$, $x(4) = \frac{15}{16}$

$x[n] = 0$, for $n \leq 0$

$$x(n) = \left\{ 0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \cdots \right\}$$

or

$$x[n] = \left(1 - \left(\frac{1}{2}\right)^n\right) u(n)$$

(*) $X(z) = \dfrac{z}{2z^2 - 3z + 1}$

(a) $|z| > 1$

(b) $|z| < \frac{1}{2}$

(c) $\frac{1}{2} < |z| < 1$

so/n.

$$\frac{x(z)}{z} = \frac{1}{2z^2 - 3z + 1}$$

$$\frac{X(z)}{z} = \frac{1}{2\left(z - \frac{1}{2}\right)(z - 1)} = \frac{C_1}{z - \frac{1}{2}} + \frac{C_2}{z - 1}$$

Where

$$c_1 = \left(z - \frac{1}{2}\right) \frac{X(z)}{z} \Bigg|_{z=\frac{1}{2}}$$

$$= \left(z - \frac{1}{2}\right) \frac{1}{2\left(z-\frac{1}{2}\right)(z-1)} \Bigg|_{z=\frac{1}{2}}$$

$$= -1$$

$$c_2 = (z-1) \frac{x(z)}{z} \Bigg|_{z=1}$$

$$= \cancel{(z-1)} \times \frac{1}{2\left(z-\frac{1}{2}\right)\cancel{(z-1)}} \Bigg|_{z=1}$$

$$= 1$$

$$X(z) = -\frac{z}{z-\frac{1}{2}} + \frac{z}{z-1}$$

(2) ROC: $|z| > 1$,

We have

$$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a} \quad |z| > |a|$$

Taking Inverse z-transform using table, we get

$$x[n] = -\left(\frac{1}{2}\right)^n u[n] + u[n]$$

$$= \left(1 - \left(\frac{1}{2}\right)^n\right) u[n]$$

(b) ROC: $|z| < \frac{1}{2}$

We have

$$-a^n u[-n-1] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| < |a|$$

Taking inverse z-transform using table, we get

$$x[n] = \left(\frac{1}{2}\right)^n u[-n-1] - u[-n-1]$$

$$= \left(\left(\frac{1}{2}\right)^n - 1\right) u[-n-1]$$

(c). ROC: $\frac{1}{2} < |z| < 1$

We have

$$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| > |a|$$

$$-a^n u[-n-1] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| < |a|$$

Taking inverse z-transform using tables, we get

$$x[n] = -\left(\frac{1}{2}\right)^n u[n] - u[-n-1]$$

Find $x[n]$

H/W

if $X(z) = \frac{z}{z^2 - z - 6}$

(a) $|z| > 3$
(b) $|z| < 2$
(c) $2 < |z| < 3$

eg: Find $x[n]$

$$X(z) = \frac{2z^3 + 5z^2 + z - 1}{2z^2 - 3z + 1}, \quad |z| < \frac{1}{2}$$

Since $X(z)$ is improportional function first make it propotional function then go for partial fraction expansion.

Sol^n,

$$\begin{array}{r|l} & z \\ 2z^2 - 3z + 1 & \cancel{2z^3} + 5z^2 + z - 1 \\ & \cancel{2z^3} - 3z^2 + z \\ & - \quad + \quad - \\ \hline & \cancel{8z^2} + 0 - 1 \\ & \cancel{8z^2} - 12z + 4 \\ & - \quad + \quad - \\ \hline & 12z - 5 \end{array}$$

$$X(z) = z + 4 + \frac{12z - 5}{2z^2 - 3z + 1}$$

let

$$X_1(z) = \frac{12z-5}{2z^2-3z+1}$$

$$\frac{X_1(z)}{z} = \frac{12z-5}{2z\left(z-\frac{1}{2}\right)(z-1)}$$

$$= \frac{c_0}{z} + \frac{c_1}{z-\frac{1}{2}} + \frac{c_2}{z-1}$$

where,

$$c_0 = X_1(z)\Big|_{z=0}$$

$$= \frac{12z-5}{2z^2-3z+1}\Big|_{z=0}$$

$$= -5$$

$$c_1 = \left(z-\frac{1}{2}\right)\frac{X_1(z)}{z}\Big|_{z=\frac{1}{2}}$$

$$= \cancel{\left(z-\frac{1}{2}\right)} \times \frac{12z-5}{2z\cancel{\left(z-\frac{1}{2}\right)}(z-1)}$$

$$= \frac{12/2-5}{\frac{2}{2}\left(\frac{1}{2}-1\right)} = \frac{6-5}{-\frac{1}{2}} = -2$$

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$$C_2 = \left.(z-1)\frac{X_1(z)}{z}\right|_{z=1}$$

$$= \left.\frac{12z-5}{2z(z-\frac{1}{2})}\right|_{z=1}$$

$$= 7$$

$$X_1(z) = -5 - \frac{2z}{z-\frac{1}{2}} + \frac{7z}{z-1}$$

$$X(z) = z+4-5 - \frac{2z}{z-\frac{1}{2}} + \frac{7z}{z-1}$$

$$= z-1 - \frac{2z}{z-\frac{1}{2}} + \frac{7z}{z-1}$$

ROC: $|z| < \frac{1}{2}$

We have,

$\delta[n] \xleftrightarrow{z} 1$, All $z$

$\delta[n-m] \xleftrightarrow{z} z^{-m}$, All $z$ except $0$ if $m>0$
$\infty$ if $m<0$

$-a^n u[-n-1] \xleftrightarrow{z} \frac{z}{z-a}$, $|z|<|a|$

Taking inverse z-transform using Tables

we get

$$x(n) = \delta(n+1) - \delta(n) + 2\left[\left(\tfrac{1}{2}\right)^n u[-n-1]\right]$$
$$- 7u[-n-1]$$

H/W

$$X(z) = \frac{2z^4 + z^2 - 6}{z^2 - 5z + 6}, \quad |z| < 2$$

eg:- find $x[n]$

$$\text{if } X(z) = \frac{z}{(z-1)(z-2)^2}, \quad |z| > 2$$

sol/n

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z-2)^2}$$

$$\frac{X(z)}{z} = \frac{c_1}{z-1} + \frac{d_1}{z-2} + \frac{d_2}{(z-2)^2}$$

where,

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$$C_1 = (z-1)\frac{X(z)}{z}\Big|_{z=1}$$

$$= (z-1)\frac{1}{(z-1)(z-2)^2}\Big|_{z=1}$$

~~= 1~~

$$d_2 = \frac{1}{0!}\frac{\partial^0}{\partial z^0}(z-2)^2\frac{X(z)}{z}\Big|_{z=2}$$

$$= \cancel{(z-2)^2} \times \frac{1}{(z-1)\cancel{(z-2)^2}}\Big|_{z=2}$$

$$= 1$$

$$d_1 = \frac{1}{1!}\frac{d}{dz}(z-2)^2\frac{X(z)}{z}\Big|_{z=2}$$

$$= \frac{d}{dz}(z-2)^2\frac{1}{(z-1)(z-2)^2}\Big|_{z=2}$$

$$= -1(z-1)^{-2}\frac{d}{dz}(z-1)\Big|_{z=2}$$

$$= -(z-1)^{-2}\Big|_{z=2}$$

$$= -1$$

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$$X(z) = \frac{z}{z-1} - \frac{z}{z-2} + \left(\frac{z}{(z-2)^2}\right) \times \frac{1}{2} \times 2$$

ROC: $|z| > 2$

We have

$$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| > |a|$$

$$n a^n u[n] \xleftrightarrow{z} \frac{az}{(z-a)^2}, \quad |z| > |a|$$

Taking inverse z-transform apply tables,

We get

$$x[n] = u[n] - 2^n u[n] + \frac{1}{2} n 2^n u[n]$$

$$= \left(1 - 2^n + n 2^{n-1}\right) u[n]$$

H/W.

find $x[n]$

if $X(z) = \dfrac{z}{(z+2)(z-3)^2}, \quad |z| < 2$

Find o/p of LTI system having I/P sequence $x(n) = \{4, -2, \underset{\uparrow}{3}, 5\}$ and impulse Response $h(n) = \{2, 1, 6, 3\}$ Using z transform.

Soln:

z-transform of $x(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \cdots + x(-2) z^{2} + x(-1) z + x(0) + x(1) z^{-1} + \cdots$$

$$= 4z^{2} - 2z + 3 + 5z^{-1} \qquad \text{ROC: } 0 < |z| < \infty$$

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Similarly z-transform of $h(n)$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$= \cdots + h(0) + h(1) \cdot z^{-1} + h(2) \cdot z^{-2} + h(3) \cdot z^{-3} + \cdots$$

$$= 2 + z^{-1} + 6z^{-2} + 3z^{-3}, \qquad \text{ROC: } |z| > 0$$

as else of 0 is valid.

O/p of LTI system is given by

$$y(n) = x(n) * h(n)$$

In z-domain,

$$Y(z) = X(z) \cdot H(z), \qquad \text{ROC: } R_1 \cap R_2$$

$$= (4z^{2} - 2z + 3 + 5z^{-1})(2 + z^{-1} + 6z^{-2} + 3z^{-3})$$

$$= 8z^{2} - 4z + 6 + 10z^{-1} + 4z - 2 + 3z^{-1} + 5z^{-2}$$
$$+ 24 - 12z^{-1} + 18z^{-2} + 30z^{-3} + 12z^{-1}$$
$$- 6z^{-2} + 9z^{-3} + 15z^{-4}$$

$$= 8z^{2} + 28 + 13z^{-1} + 17z^{-2} + 39z^{-3} + 15z^{-4}$$

ROC: $0 < |z| < \infty$

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$$C_2 = (z-\beta) \cdot \frac{y(z)}{z} \Big|_{z=\beta}$$

$$= (z-\beta) \frac{z}{(z-\alpha)(z-\beta)} \Big|_{z=\beta}$$

$$= \frac{\beta}{\beta-\alpha} = -\frac{\beta}{\alpha-\beta}$$

$$y(z) = \frac{\alpha}{\alpha-\beta} \cdot \frac{z}{z-\alpha} - \frac{\beta}{\alpha-\beta} \cdot \frac{z}{z-\beta}, \quad |z| > \max\{|\alpha| \,\&\, |\beta|\}$$

Taking inverse z-transform using table,

$$y(n) = \frac{\alpha}{\alpha-\beta} \alpha^n u(n) - \frac{\beta}{\alpha-\beta} \beta^n u(n)$$

$$= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha-\beta} u(n)$$

When $\alpha = \beta$,

$$Y(z) = \frac{z}{z-\alpha} \cdot \frac{z}{z-\alpha}, \quad |z| > |\alpha|$$

$$\frac{Y(z)}{z} = \frac{z}{(z-\alpha)^2} = \frac{d_1}{z-\alpha} + \frac{d_2}{(z-\alpha)^2}$$

Where,

$$d_2 = \frac{1}{0!} \frac{d^0}{dz^0} (z-\alpha)^2 \frac{Y(z)}{z} \Big|_{z=\alpha}$$

$$= (z-\alpha)^2 \cdot \frac{z}{(z-\alpha)^2} \Big|_{z=\alpha}$$

$$= \alpha$$

$$c_1 = \frac{1}{1!}\frac{d}{dz}(z-\alpha)^2 \frac{Y(z)}{z}\Bigg|_{z=\alpha}$$

$$= \frac{d}{dz}(z-\alpha)^2 \frac{z}{(z-\alpha)^2}\Bigg|_{z=\alpha}$$

$$= 1$$

$$Y(z) = \frac{z}{z-\alpha} + \frac{\alpha z}{(z-\alpha)^2}, \quad |z|>|\alpha|$$

we have

$$n a^n u(n) \xleftrightarrow{z} \frac{az}{(z-a)^2}, \quad |z|>|a|$$

Taking inverse z transform using table we get,

$$y(n) = \alpha^n u(n) + n\alpha^n u(n)$$
$$= (1+n)\alpha^n u(n)$$

H/w Find $y(n) = x(n) * h(n)$

(i) $x(n) = \left(\frac{1}{2}\right)^n u(n)$
$h(n) = \left(\frac{1}{3}\right)^n u(n)$

(ii) $x(n) = \left(\frac{1}{3}\right)^n u(n)$

$h(n) = 2^n u(n-1)$

## Causality of LTI System in $z$ - domain

To be an LTI System Causal,

$$h(n) = 0, \text{ for } n<0$$

i.e. $h(n)$ is Right sided sequence

if $H(z)$ exists,

then,

Roc has the form At least

$$|z|>0, \ |z| > r_{min}$$

## Stability of LTI System in $z$-domain

To be an LTI System stable,

$$\left|\sum_{n=-\infty}^{\infty} h(n)\right| < \infty$$

if $H(z)$ exists,

$$H(z) = \left|\sum_{n=-\infty}^{\infty} h(n)\right| z^{-n}$$

To get $H(z)$ finite,

Roc must include $|z|=1$ i.e. Unit Circle in $z$-plane

IF LTI System is both stable & Causal, then

then,

poles of LTI System lies inside the Unit Circle in $z$ plane

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(a) $h(n) = 3^n u[-n-1] \overset{z}{\longleftrightarrow} H(z) = \frac{-z}{z-3}, |z| < 3$

↳ non causal (as $z > 3$ for causal), stable (1 included)

(b) $h(n) = 3^n u(n) \overset{z}{\longleftrightarrow} H(z) = \frac{z}{z-3}, |z| > 3$

↳ causal, as $z > 3$ and 1 is not Included so Unstable

(c) $h(n) = \left(\frac{1}{3}\right)^n u(-n-1) \overset{z}{\longleftrightarrow} H(z) = \frac{z}{z-1/3}, |z| < 1/3$

↳ non causal, Unstable

(d) $h(n) = \left(\frac{1}{3}\right)^n u(n) \overset{z}{\longleftrightarrow} H(z) = \frac{z}{z-1/3}, |z| > 1/3$

↳ causal, stable.

→ Recursive System:

There are many system whose present O/P depends upon a no. of past O/P. Such a system is called Recursive System & its O/P can be represented as

$$y[n] = F\{y[n-1], y[n-2], \cdots y(n-N], x[n], x[n-1], \cdots x[n-M]\}$$

where,

$f\{\} \Rightarrow$ function

ie: Feedback System.

→ Non-Recursive System:

A System is said to be non-recursive if its present O/P only depends upon present I/P & past I/P and represented as

$$y[n] = f\{x(n), x[n-1] \ldots, x(n-M)\}$$

where

$f\{\} \rightarrow$ function

ie: non-feedback System.

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→ Infinite Impulse Response (IIR) System

if length of impulse Respone is $\infty$ then it is called IIR.

eg: $h[n] = \left(\frac{1}{2}\right)^n u(n)$

↳ Feedback System
↳ Recursive System

→ Finite impulse Response (FIR) System

If length of impulse Response is finite the it is Called FIR System.

eg: $h[n] = \{3, 2, 5, 6\}$

→ It is non-feedback system
→ It is non-recursive System.

# moving avg. System

$$y[n] = \frac{1}{n+1} \sum_{k=0}^{n} x[k]$$ → non recursive System

$$y[n-1] = \frac{1}{n-1+1} \sum_{k=0}^{n-1} x[k]$$

$$n y[n-1] = \sum_{k=0}^{n-1} x[k]$$

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$$y[n] = \frac{1}{n+1}\left[\sum_{k=0}^{n-1} x[k] + x[n]\right]$$

$$y[n] = \frac{1}{n+1}\left[n\,y[n-1] + x[n]\right]$$

$$y[n] = \frac{n}{n+1}\,y[n-1] + \frac{1}{n+1}\,x[n]$$ → Recursive.

→ Recursivity depends upon implementation of System.

# Difference Equation:-

→ The General form of Const. Coeff. difference equation for LTI System can be represented as:

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

$a_k, b_k$ → Coeff.

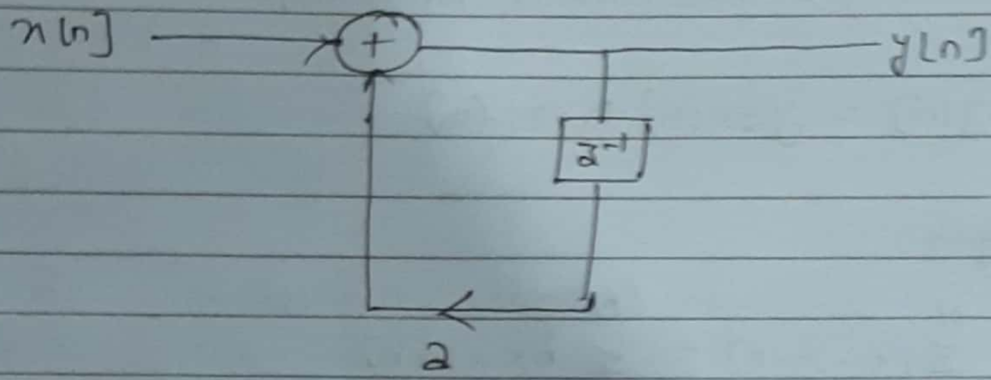
$M, N$ → no. of delay in all I/P & O/P respectively.

$$a_0 y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] = \frac{1}{a_0}\left[\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right]$$

## First order difference Equation

$$y[n] = a\,y[n-1] + x[n]$$




For $n=0$,

$$y[0] = a\,y[-1] + x[0]$$

For $n=1$,

$$y[1] = a\,y[0] + x[1] = a^2 y[-1] + a\,x[0] + x[1]$$

For $n=2$,

$$y[2] = a\,y[1] + x[2] = a^3 y[-1] + a^2 x[0] + a\,x[1] + x[2]$$

$\vdots$

$$y[n] = a^{n+1} y[-1] + a^n x[0] + a^{n-1} x[1] + \cdots + x[n]$$

$$y[n] = a^{n+1} y(-1] + \sum_{K=0}^{n} a^K x[n-K]$$

When $x[n] = 0$, for $n \geq 0$, ie zero ~~initial~~ I/P cond$^n$ then

$$y_{zI}(n) = a^{n+1} y[-1]$$

→ Natural response
→ zero-I/P response

When initial condⁿ zero ie $y[-1] = 0$

then

$$y_{zs}[n] = \sum_{k=0}^{n} a^k x[n-k]$$

→ zero-state or forced response.

$$y[n] = y_{zi}(n) + y_{zs}(n)$$

Again,

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Taking z transform using time shifting property of z-transform

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Transfer function / System function :- Ration of O/p to I/p in frequency domain, laplace domain or in z-domain.

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

$H(z)$ Contains $M$ zeros
$N$ poles

If $N=0$, no poles (non-zero poles)
then FIR System

If $N \geq 1$, at least one non-zero pole then
IIR System