CSCI-4470 HW-6

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Homework 6

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Upload a soft copy of your answers (single pdf file) to the submission link on elc before the due date. The answers to the homework assignment should be your own individual work. Make sure to show all the work/steps in your answer to get full credit. Answers without steps or explanation will be given zero.

Extra credit: There is 5 percentage extra credit if you don't submit hand written homework including the figures. You can use latex or any other tool to write your homework. For figures you can use any drawing tool and include the figure as a jpeg or a png file in your latex file.

1. (10 points) Consider the problem below.
Input: A set (or a multiset – i.e., values may be repeated) of numbers $S = \{x_1, x_2, ..., x_n\}$, where $0 < x_i \leq 1$ for all x_i, and a positive integer k.
Accept if S can be partitioned into k subsets, where each subset totals to a value ≤ 1.

The idea here is to see if a set of items of size $x_1, ..., x_n$ can be put into k bins, where each bin can hold a capacity of at most 1.

Show that this problem is in NP by providing a polynomial-time verifier deciding the problem. Be sure to describe your certificate and do the runtime analysis for your verifier.

Step 1: Problem Restatement

- Determine if a set of numbers $S = \{x_1, x_2, ..., x_n\}$, with each $0 < x_i \leq 1$, can be split into k groups where the sum in each group is no more than 1.

Step 2: Definition of NP

- A problem is in NP if its solutions can be verified in polynomial time, given a certificate.

Step 3: Find the Certificate

- If $S = \{0.2, 0.4, 0.3, 0.1\}$ and $k = 2$, a **valid certificate** could be the partition $\{\{0.2, 0.3\}, \{0.4, 0.1\}\}$.
- Subset (certificate) $\{0.2, 0.3\}$: Total $= 0.5 \leq 1$.

- Subset (certificate) $\{0.4, 0.1\}$: Total $= 0.5 \leq 1.$

Step 4: Criteria and Verifier

4.1. Algorithm:

- Receive original set $S = \{0.2, 0.4, 0.3, 0.1\}$, and integer $k = 2$
- For each subset in the certificate, to calculate the total sum of the elements in the subset and verify it is ≤ 1.
- check subsets (certificate) $\{0.2, 0.3\}$ and$\{0.4, 0.1\}$, if the sum is ≤ 1.
  - Total sum $= 0.2 + 0.3 = 0.5$, which is $= 0.5 \leq 1$.
  - Mark elements 0.2 and 0.3 as used.
  - Total sum $= 0.4 + 0.1 = 0.5$, which is $= 0.5 \leq 1$.
  - Mark elements 0.4 and 0.1 as used.
- Verify that all elements of S are included in the subsets, with no duplicates.
  - Confirm all elements in S are used without duplication.

4.2. Verifier Correctness:

- Accept the certificate if each subset's sum is ≤ 1 and every element in S is included once. Reject otherwise.

Step 5: Runtime Analysis

- The verifier runs in polynomial time, specifically $O(nk)$, as it checks each of the k subsets and sums up to n elements in each.

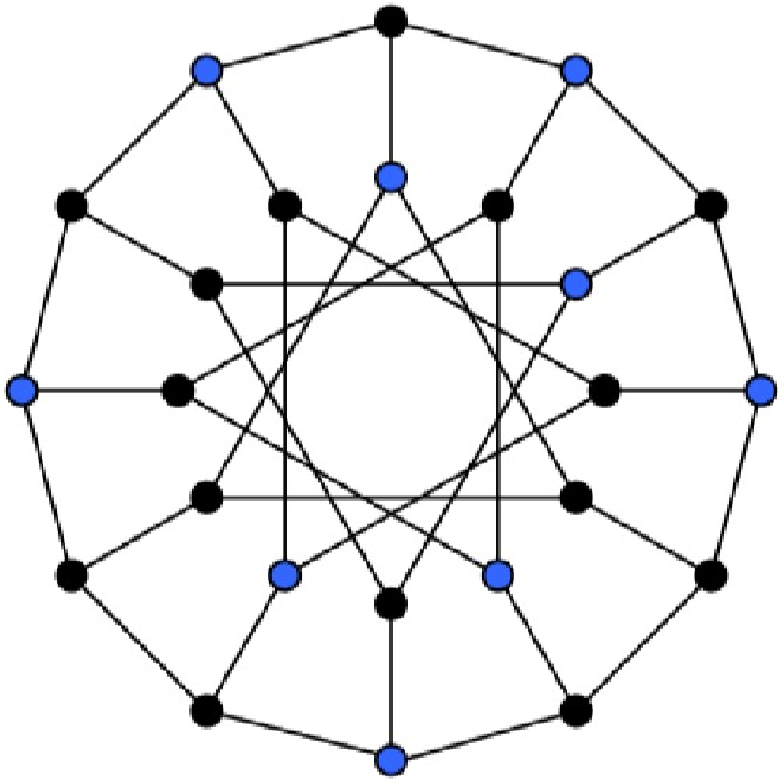
Step 6: Conclusion

- The problem is in NP, as it has a polynomial-time verifier that efficiently confirms the correctness of solutions using certificates. This aligns with the NP class criteria in computational complexity theory.

2. (10 points) The Independent-Set problem is defined as follows:

Given a graph G and a positive integer k, accept $\langle G, k \rangle$ if G contains a subset of vertices V' such that $|V'| = k$ and V' is an independent set – i.e., for any two vertices $u, v \in V'$, $(u, v) \notin E$.

In the graph below nine blue vertices form an Independent-Set.



- Image source: Wikipedia

Prove that Independent-Set is in NP.

Step 1: Certificate, Define the certificate

- Certificate: A subset $V' \subseteq V$ with $|V'| = k$ that forms an independent set

Step 2: Verifier Steps To check if V' is indeed an independent set of size k in the graph G:

2.1. Check Size of V':

- Verify that the size of the subset V' is equal to k.
- If $|V'| \neq k$, the certificate is rejected.

2.2. Check Independence of V':

- For every pair of vertices (u, v) in V', check if an edge (u, v) exists in the edge set E of G.
- If any such edge is found, the certificate is rejected, as it violates the independence criteria.
- If no edges are found between any pairs of vertices in V', the certificate is accepted.

2.3. Runtime Analysis:

- The verification of the independence of V' can be done in $O(k^2)$ time, which is polynomial with respect to the size of the certificate.

Step 3: Determining the Size of the Independent Set

- Count the number of blue vertices, which is $m = 9$, then $m = k$.

Step 4: Solution Statement

Since both of these conditions can be checked in polynomial time. Therefore, Independent-Set is in NP.

- $|V'| = k$
- For all pairs of vertices $(u, v) \in V'$, $(u, v) \notin E$

3. (15 points) Prove that Independent-Set is NP-complete by reducing from Clique and using the solution to problem 2.

Objective: Prove that the Independent-Set problem is NP-complete by using reductions from the Clique and Vertex-Cover problems.

1. **Reduction from CLIQUE to VERTEX-COVER**:
   - Given a graph G and a positive integer k, assume $(G, k) \in$ CLIQUE.
   - Let V' be a clique in G where $|V'| = k$.
   - Define $V'' = V - V'$ as the complement of V' in G.
   - **Claim**: V'' is a vertex cover in G.
     - For any edge $(u, v) \in E$, at least one of u or v must be in V'', because if both were in V', they would form part of the clique, which contradicts the definition of an edge in E.
2. **Relating VERTEX-COVER to INDEPENDENT-SET**:
   - The set V'' is a vertex cover in G if and only if its complement V' is an independent set in G.
   - Therefore, if $(G, k) \in$ CLIQUE, then $(G, |V| - k) \in$ VERTEX-COVER, and consequently, $(G, k) \in$ INDEPENDENT-SET.
3. **Conclusion**:
   - Since we can reduce CLIQUE to VERTEX-COVER, and VERTEX-COVER is closely related to INDEPENDENT-SET, it follows that if CLIQUE is NP-complete, then so is INDEPENDENT-SET.
   - This reduction can be performed in polynomial time, as it involves basic set operations and complementation.