CSCI 4470 Algorithms

Part VI Graph Algorithms Notes

Chapter 24: Maximum Flow

Before Flow Networks

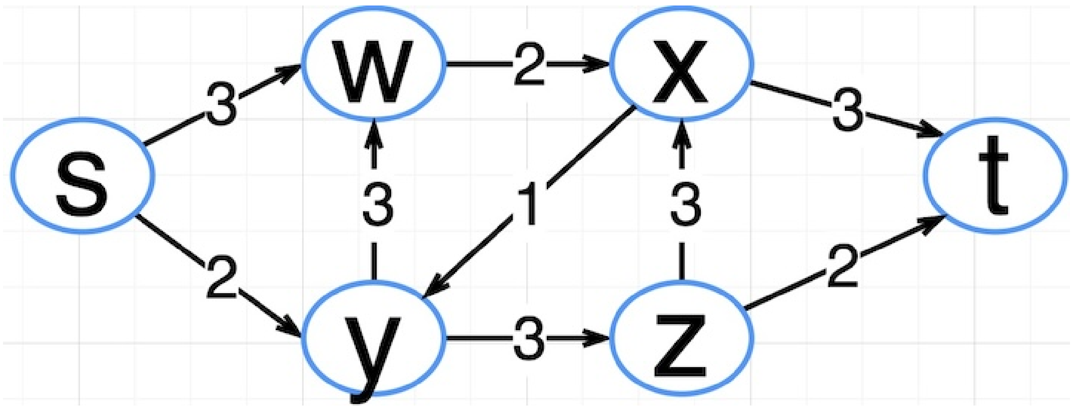
1. Flow Network (Graph): A directed graph where each edge has a capacity and a flow.
2. Maximum Flow: The greatest possible flow from the source to the sink without violating the capacities of the edges.
3. Residual Networks: Derived from the original network, representing the potential for additional flow.
4. Source and Sink: The starting point (source) and ending point (sink) in a flow network.
5. Finding Maximum Flow: Repeatedly find augmenting paths and update flows until no more augmenting paths can be found.
6. Total Flow: The sum of flows exiting the source (or entering the sink).
7. Capacity $c(u, v)$: The maximum amount of flow that can pass through an edge from node u to node v.
8. Flow $f(u, v)$: The actual amount of flow passing through an edge from node u to node v.
9. Capacity Constraints: The flow on an edge cannot exceed its capacity: $0 \leq f(u, v) \leq c(u, v)$.
10. Flow Conservation: The total flow into a node equals the total flow out, except for source and sink.
11. Residual Capacity $c_f(u, v)$: The remaining capacity on an edge considering the current flow: $c_f(u, v) = c(u, v) - f(u, v)$.
12. Ford-Fulkerson Method: An algorithm to compute the maximum flow in a flow network.
13. Augmenting Path: A path from the source to the sink in the residual network where the residual capacity is positive.
14. Update Residual Network: Adjusting the residual network after increasing flow along an augmenting path.

Flow Networks

Flow Network: A directed graph $G = (V, E)$ with a source s and a sink t. Each edge $(u, v) \in E$ has a non-negative capacity $c(u, v)$. If $(u, v) \notin E$, then $c(u, v) = 0$.

Example: $V = \{S, W, T, X, Y, Z\}$, and
$E = \{(s, w, 3), (s, y, 2), (y, w, 3), (w, x, 2), (x, y, 1), (y, z, 3), (z, x, 3), (x, t, 3), (z, t, 2)\}$

- A **Capacity Graph** with flow = 0



Edge Capacities: $c : E \to \mathbb{R}$, $c(u, v) \geq 0$. Edges not in E have zero capacity.

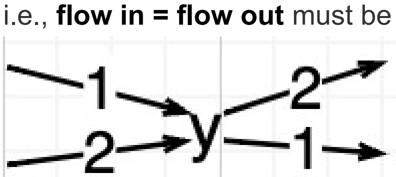
- Which the Edge Weights can't be negative, If Edge $(u, v) \notin E$, then $c(u, v) = 0$, and If Edge $(u, v) \in E$ exists, then $(v, u) \notin E$

Capacity Constraint: $0 \leq f(u, v) \leq c(u, v)$ for all edges $(u, v) \in V$.

- **Source node** s**, Sink node** t, and $\forall\, v \in V, s \rightsquigarrow v \rightsquigarrow t$

Flow Conservation: $\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$ for all $u \in V - \{s, t\}$.
$\sum_{v \in V} f(v, u) - \sum_{v \in V} f(u, v) = 0$。

- i.e., **flow in = flow out** must be satisfied.



Skew Symmetry: for all $(u, v) \in V$, $f(u, v) = -f(v, u)$

Interpretation of Edge Weights: Edges represent conduits (tubes) with capacities, e.g., $c(s, y) = 2$, $c(s, w) = 3$.

Practical Scenarios: Utility grids, supply chains, scheduling problems.

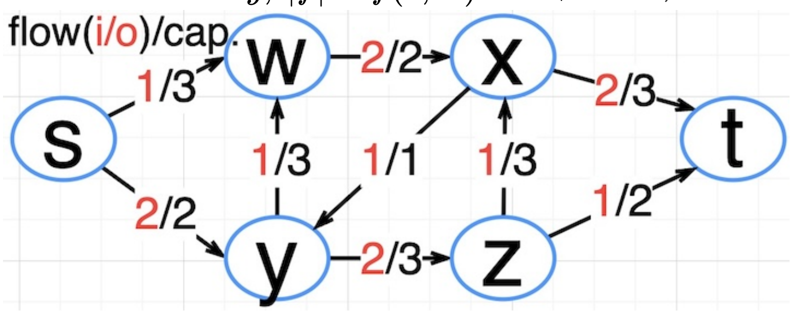
Goal: Find the maximum flow from s to t.

Total Flow:

- **Value of Flow**: $|f| = \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s)$

This formula calculates the net flow leaving the source s in a flow network. It sums up all the flow leaving the source and subtracts any flow entering the source. Typically, flow into the source is zero, so the total flow is just the sum of flow leaving the source.

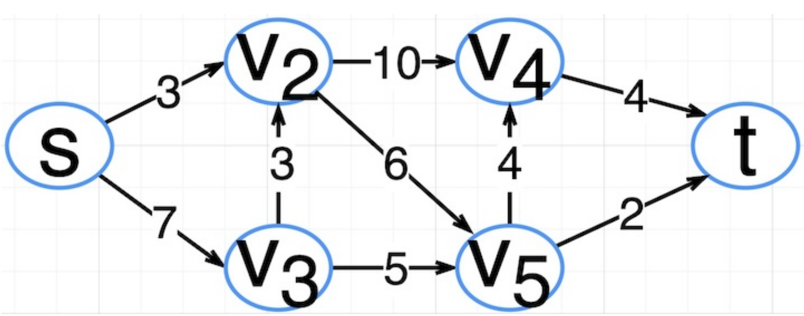
- The sum of flows exiting the source (or entering the sink).
- A **Net Flow** on G is a function $f : V \times V \to \mathbb{R}$ such that: $f(u,v) \leq c(u,v) \forall (u,v) \in E$
    - View Flow as a rate, not a quantity
- There is a flow into y, $|f| = f(s,V) = 2+1 = 3$, the value of the **Maximum Flow** out is $f(V,t) = 3$



Maximum Flow

- The greatest possible flow from the source to the sink without violating the capacities of the edges.
- To find the maximum flow (and min-cut as a by product), the Ford-Fulkerson method repeatedly finds **augmenting paths** (p) through the **Residual Network (Graph)** G_f and **augments the flow** until no more you mean by augment the flow alright let found.

Example of Max Flow: Given the Graph $G(V,E)$, $V = \{s, v_2, v_3, v_4, v_5, t\}$, $E = \{(s,v_2,3),(s,v_3,7),(v_3,v_2,3),(v_2,v_4,10),(v_2,v_5,6),(v_3,v_5,5),(v_5,v_4,4),(v_4,t,4),(v_5,t,2)\}$

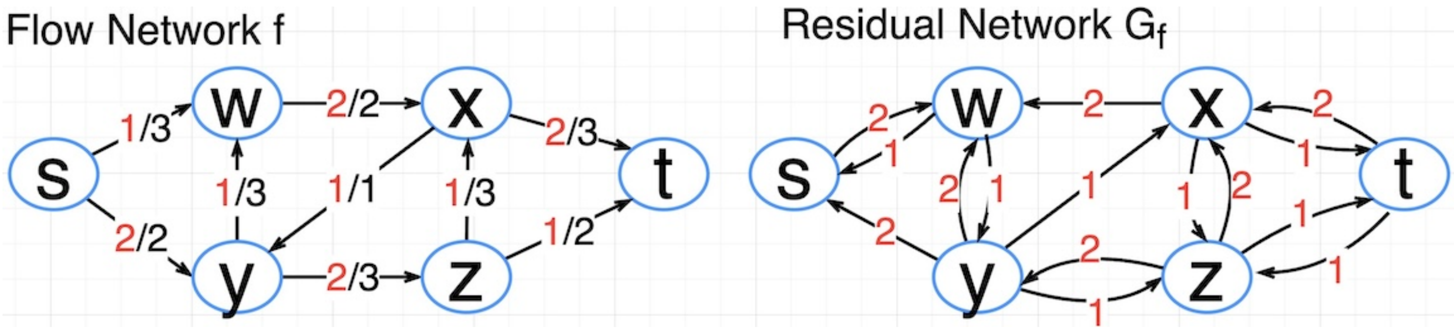


$c(s,v_2) = 3$
$f(s, v_3 = 0)$, if $(s,v_5) \notin E$, $|f| = \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s)$, $|f| = 3$

Residual Networks

Definition: Given a flow network $G = (V, E)$ and a flow f, the **Residual Network** $G_f(V, E_f)$ represents the potential for additional flow in the network. It consists of edges with strictly positive **Residual Capacities**.

- Flow Network = Flow Graph, Residual Network = Residual Graph



Residual Capacity: For an edge (u, v) in G, the residual capacity $c_f(u, v)$ is the capacity for additional flow along that edge, considering the current flow $f(u, v)$.

Calculation:

- If $(u, v) \in E$, the residual capacity can exist in both directions:
  - $c_f(u, v) = c(u, v) - f(u, v)$ // Maximum additional flow possible.
  - $c_f(v, u) = f(u, v)$ // Reversible flow, allowing flow to be sent back.

Significance: The residual network is essential for identifying augmenting paths in flow network optimization algorithms, such as the Ford-Fulkerson method.

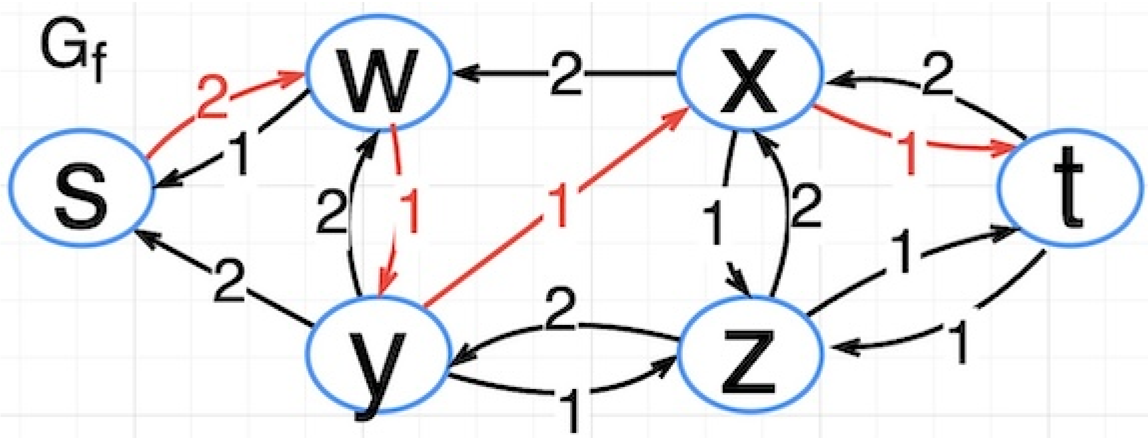
Network Flow Proof

Augmentation of a Flow $(f \uparrow f')$

- This represents how the current flow in the network can be increased by an augmenting flow found in the residual network.

Augmenting Paths

Any path from s to t in G_f is an **Augmenting Path** in G with respect to f. The flow value can be increased along an augmenting path p by $c_f(p) = \min_{(u,v) \in p} \{c_f(u, v)\}$



An augmenting path is a path of edges in the residual graph with unused capacity greater than zero from the source s to the sink t.
$p = \{s, w, y, x, t\}, c_f(p) = 1$

Flow Lemmas

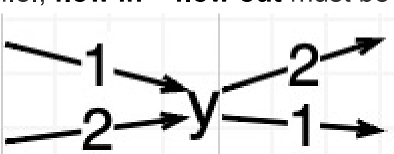
Lemma 1:

Capacity Constraint: $0 \leq f(u, v) \leq c(u, v)$ for all edges $(u, v) \in V$.

- **Source node** s**, Sink node** t, and $\forall\, v \in V, s \rightsquigarrow v \rightsquigarrow t$

Flow Conservation: $\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$ for all $u \in V - \{s, t\}$.
$\sum_{v \in V} f(v, u) - \sum_{v \in V} f(u, v) = 0$。

- i.e., **flow in = flow out** must be satisfied.



Total Flow:

- The Total Flow $|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$ correctly calculates the net flow out of the source s. as $|f|$ represents the value of the flow in the network.

In a flow network G with flow f and a corresponding residual network G_f, when you augment f with another flow f' from G_f, the new total flow in G is the sum of the original flow and the added flow. This is expressed as $|f \uparrow f'| = |f| + |f'|$.

The augmentation $f \uparrow f'$ respects capacity constraints ($0 \leq f(u, v) \leq c(u, v)$) and flow conservation. The new flow $f \uparrow f'$ in any edge (u, v) is the sum of the existing flow $f(u, v)$, additional flow $f'(u, v)$, and reverse flow

$f'(v, u)$, ensuring the flow remains within the edge capacity and the total flow is the sum of the original and additional flows.

Prove of value $|f \uparrow f'| = |f| + |f'|$

Give a flow network G, a flow f in G, and a residual network G_f, let f' be a flow in G_f. Then $f \uparrow$ in G with value $|f \uparrow f'| = |f| + |f'|$

- $f \uparrow f'$ is a flow in G i.e.,$f(u, v) \leq c(u, c)\ \forall (u, v) \in E$, and $\sum\limits_{v \in V} f(v, u) = \sum\limits_{v \in V} f(u, v)$ for all $u \in V - \{s, t\}$
- Is this a proof of capacity Constraint $0 \leq f(u, v) \leq c(u, v)$

if f is the flow in G and f' is flow in G_f(residual network) then augmentation of flow f by f'.

- $0 \leq (f \uparrow f')(u, v) \leq c(u, v)$, $f'(v, u) \leq f(u, v)$, and $f'(u, v) = c(u, v) - f(u, v)$
- $|f_{max}| = |f_{old}| + |f'|$, and $|f \uparrow f'| = |f| + |f'|$

$(f \uparrow f')(u, v) = f(u, v) + f'(u, v) - f'(v, u)$ if $(u, v) \in E$, 0 otherwise
$(f \uparrow f')(u, v) \geq f(u, v) + f'(u, v) - f(u, v)$
$(f \uparrow f')(u, v) \geq f'(u, v)$
$(f \uparrow f')(u, v) \geq 0$

$f'(u, v) = c(u, v) - f(u, v)$
$(f \uparrow f')(u, v) = f(u, v) + f'(u, v) - f'(v, u)$
$(f \uparrow f')(u, v) \leq f(u, v) + f'(u, v)$
$(f \uparrow f')(u, v) = f(u, v) + c(u, v) - f(u, v)$
$(f \uparrow f')(u, v) \leq c(u, v)$

Lemma 2:

If f is the flow in G and f' is flow in G_f(residual network) then augmentation of flow f by f'

- to prove the flow Conservation and the flow in the network $\sum\limits_{v} f(u, v) = \sum\limits_{v} f(v, u)$

$$\sum_{v}(f \uparrow f')(u, v) = \sum_{v}(f \uparrow f')(v, u)$$

$$\sum_{v}(f \uparrow f')(u, v) = \sum_{v} f(u, v) + f'(u, v) - f'(v, u)$$

$$\sum_{v}(f \uparrow f')(u, v) = \sum_{v} f(u, v) + \sum_{v} f'(u, v) - \sum_{v} f'(v, u)$$

$$\sum_{v}(f \uparrow f')(u, v) = \sum_{v} f(v, u) + \sum_{v} f'(v, u) - \sum_{v} f'(u, v)$$

$$\sum_v (f \uparrow f')(u,v) = \sum_v \left(f(v,u) + f'(v,u) - f'(u,v)\right)$$

$$\sum_v (f \uparrow f')(u,v) = \sum_v (f \uparrow f')(v,u)$$

Lemma 3: V_1: Set of vertices that have edges coming from s, V_2: set of vertices that have edges going to s, $V = V_1 \bigcup V_2$

$$|f \uparrow f'| = \sum_v (f \uparrow f')(s,v) - \sum_v (f \uparrow f')(v,s)$$

$$|f \uparrow f'| = \sum_{v_1} (f \uparrow f')(s,v) - \sum_{v_2} (f \uparrow f')(v,s)$$

$$|f \uparrow f'| = \sum_{v_1} [f(s,v) + f'(s,v) - f'(v,s)] - \sum_{v_2} [f(v,s) + f'(v,s) - f'(s,v)]$$

$$|f \uparrow f'| = \sum_{v_1} f(s,v) - \sum_{v_2} f(v,s) + \sum_{v_1} f'(s,v) + \sum_{v_2} f'(s,v) - \sum_{v_1} f'(v,s) - \sum_{v_2} f'(v,s)$$

$$|f \uparrow f'| = \sum_v f(s,v) - \sum_v f(v,s) + \sum_v f'(s,v) - \sum_v f'(v,s)$$

$$= |f| + |f'|$$

FORD-FULKERSON LOOSELY DESCRIBED

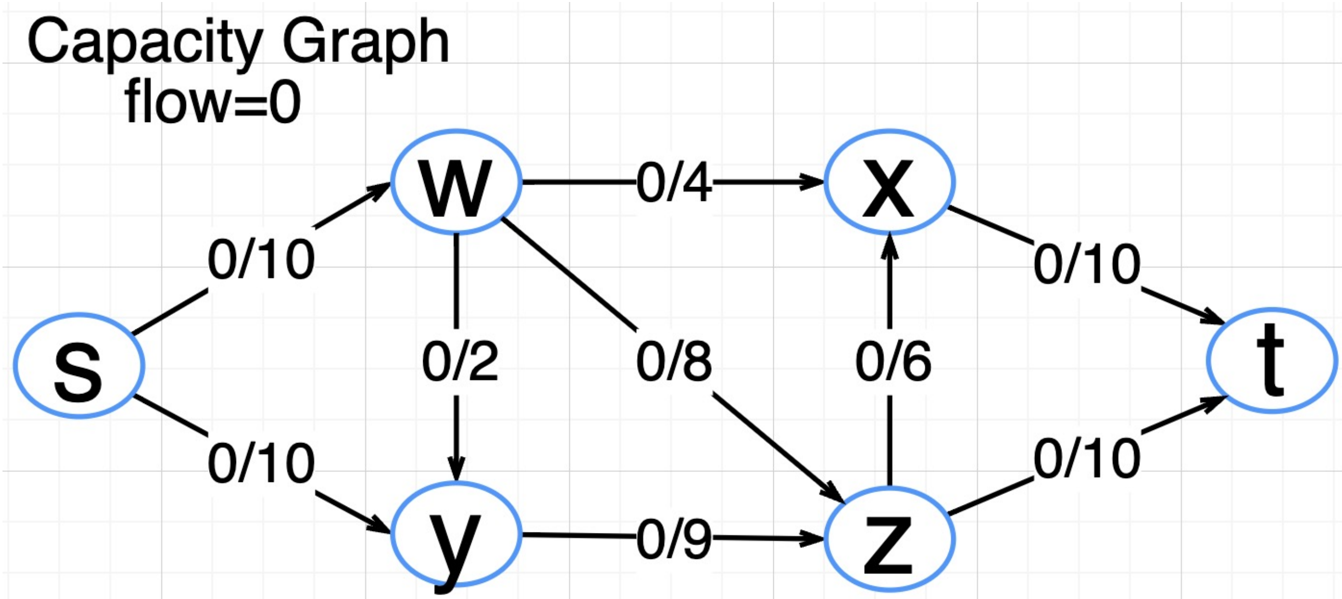
Given the Flow Graph G with flow f, a residual network G_f represents how we can change the flow on edges of G.

An augmenting path p in G_f is a path that increases the flow from s to t.

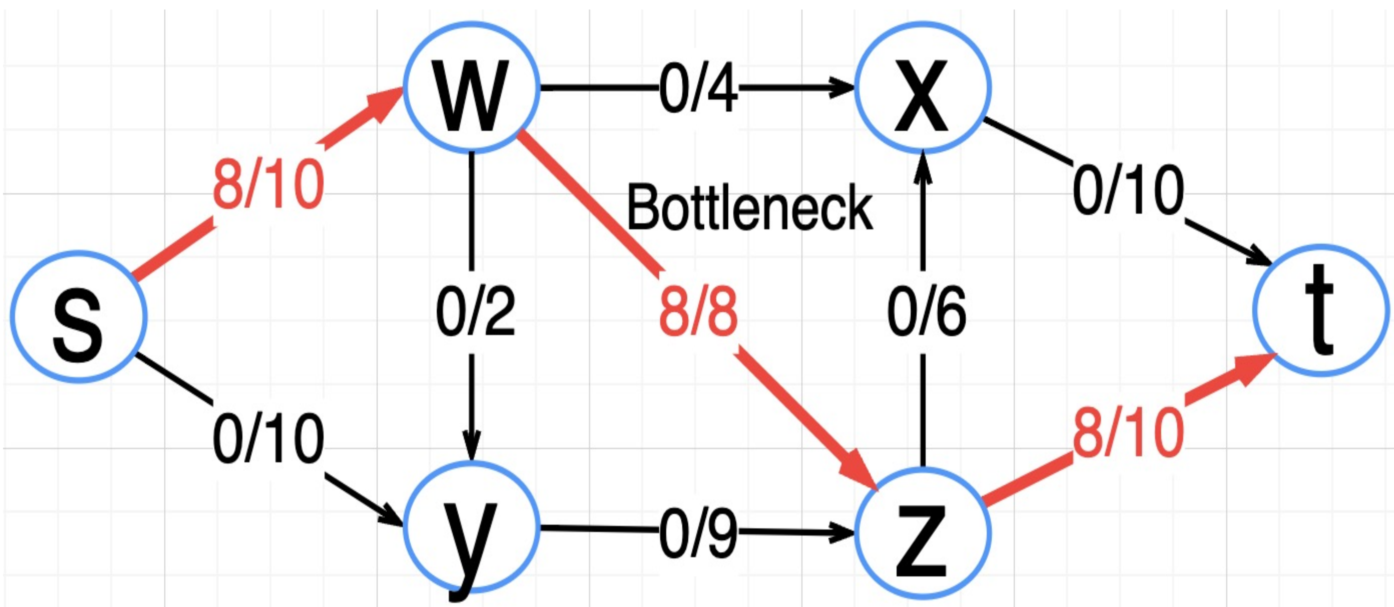
1. initialize flow f to 0
2. while $\exists$ an augmenting path p in the residual network G_f,
3. augment flow f along path p
4. return f

Need to determine how to find residual networks and augmenting paths

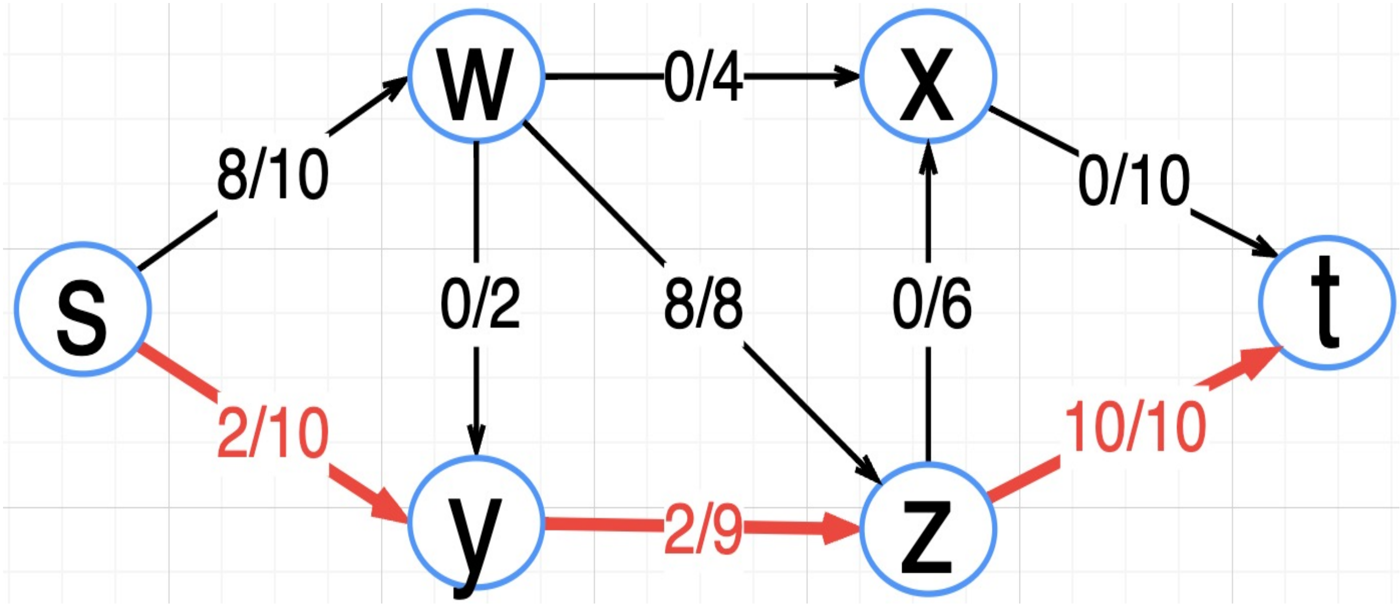
FORD-FULKERSON Example, to find maximum flow in the graph G



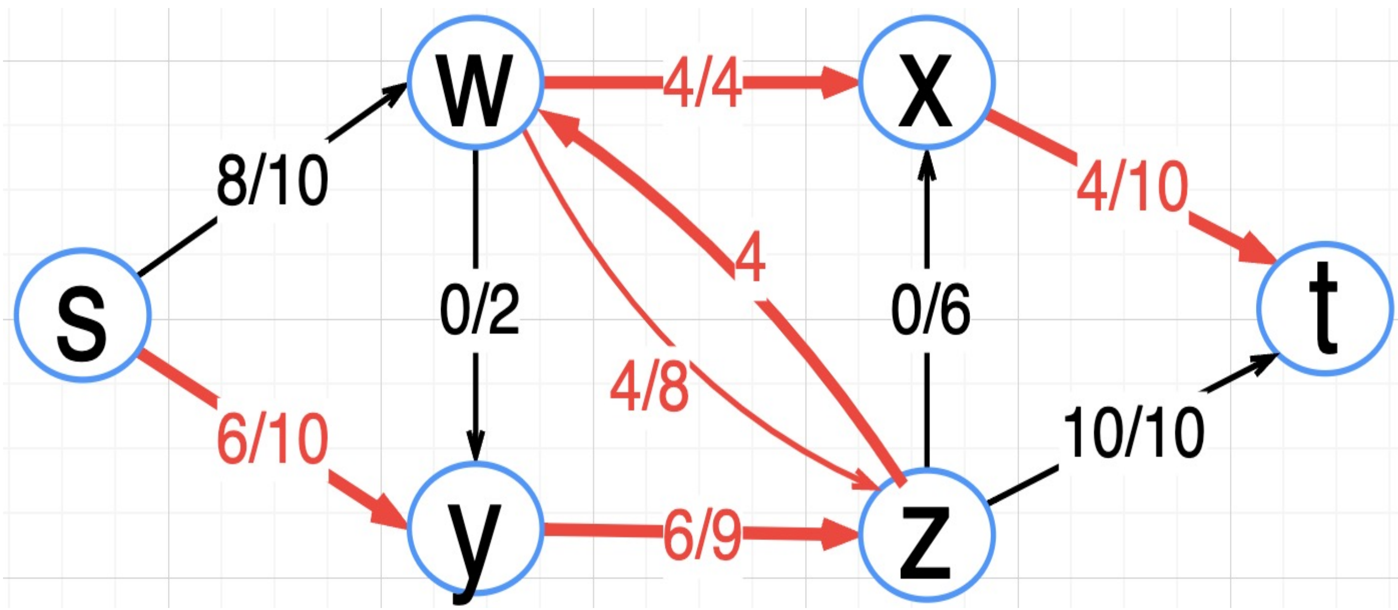
Find the First Augmenting Path $\{s, w, z, t\}$, the bottleneck capacity is 8

- 

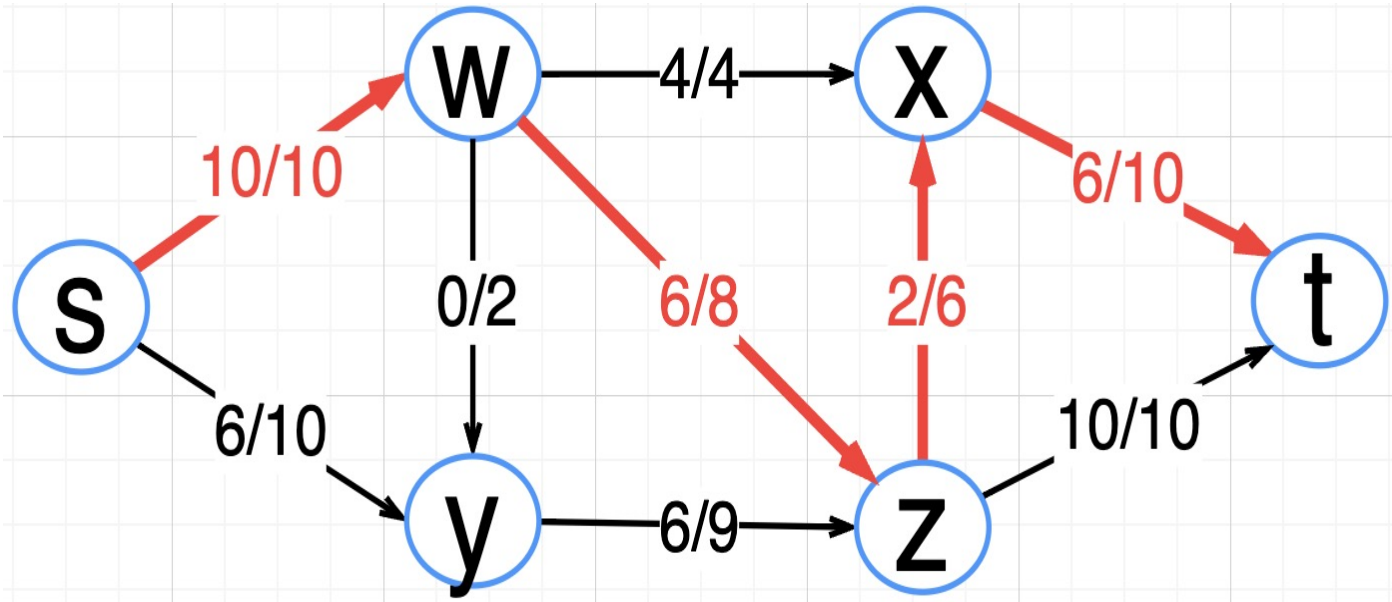
Find the Second Augmenting Path $\{s, y, z, t\}$, the bottleneck capacity is 2

- 

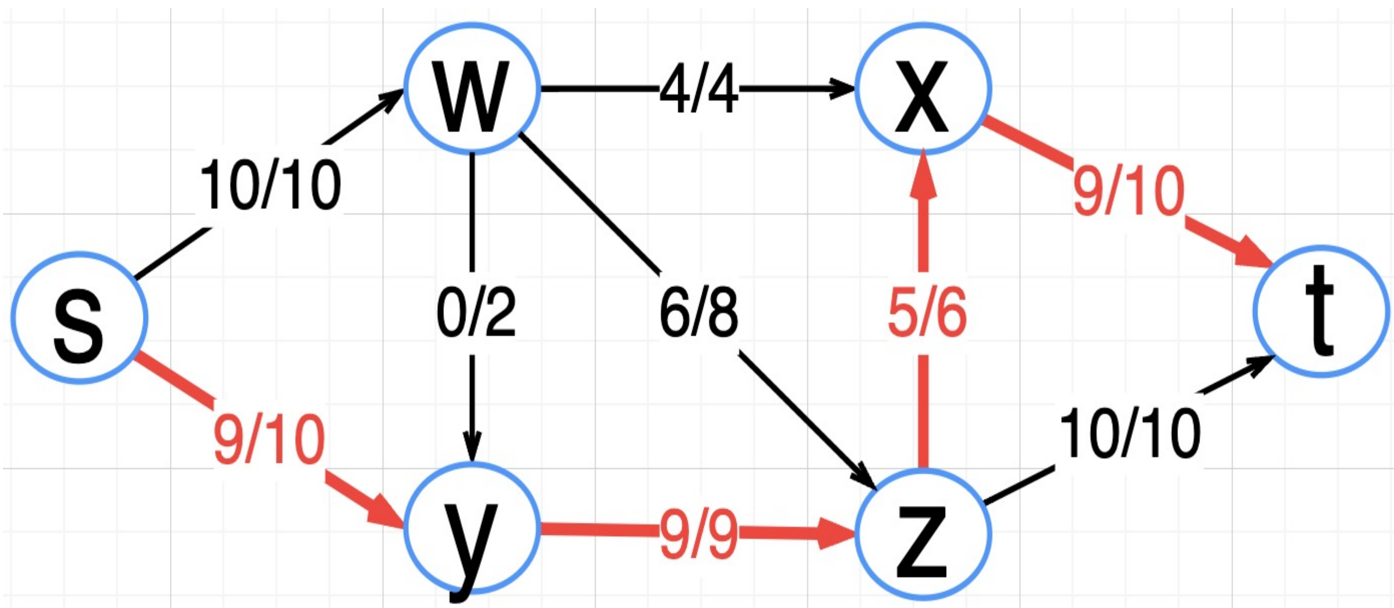
Find the Third Augmenting Path $\{s, y, z, w, x, t\}$, the bottleneck capacity is 4

- 

Find the Fourth Augmenting Path $\{s, w, z, x, t\}$, the bottleneck capacity is 2

- 

Find the Fifth Augmenting Path $\{s, y, z, x, t\}$, the bottleneck capacity is 3

- 

The maximum flow is $8 + 2 + 4 + 2 + 3 = 19$

ST-Cut (Minimum Cut)

A $cut(S, T)$ of flow network G is partition of V into S and $T = V - S$, If f is flow on G the net flow across cut(S,T) is $f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u)$, the capacity of $cut(S, T)$ is $c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v)$

Lemma of ST-Cut:

If f is flow in G with source s and sink t. let (S, T) is any cut of G then net flow across (S, T) is

- to prove $f(S,T) = |f|$ by using the cut set S, $V = S \bigcup T$, and $S \bigcap T = \emptyset$, $u \in S$ and $v \in T$, $(u, v) \in E$ in G and $(u, v) \notin E$ in G_f

$f(S,T) = \sum_{u \in S}\sum_{v \in T} f(u,v) - \sum_{u \in S}\sum_{v \in T} f(v,u)$, $c(S,T) = \sum_{u \in S}\sum_{v \in T} c(u,v)$, and $\sum_{u \in S - \{s\}} (\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u)) = 0$

$$|f| = \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{u \in S-\{s\}} (\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u))$$

$$|f| = \sum_{v \in V} f(s,v) + \sum_{u \in S-\{s\}} \sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,s) - \sum_{u \in S-\{s\}} \sum_{v \in V} f(v,u)$$

$$|f| = \sum_{v \in V} \left(f(s,v) + \sum_{u \in S-\{s\}} f(u,v) \right) - \sum_{v \in V} \left(f(v,s) + \sum_{u \in S-\{s\}} f(v,u) \right)$$

$$|f| = \sum_{v \in V} \sum_{u \in S} f(u,v) - \sum_{v \in V} \sum_{u \in S} f(v,u)$$

$$|f| = \sum_{v \in V} \sum_{u \in S} f(u,v) + \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in S} \sum_{u \in S} f(v,u) - \sum_{v \in T} \sum_{u \in S} f(v,u)$$

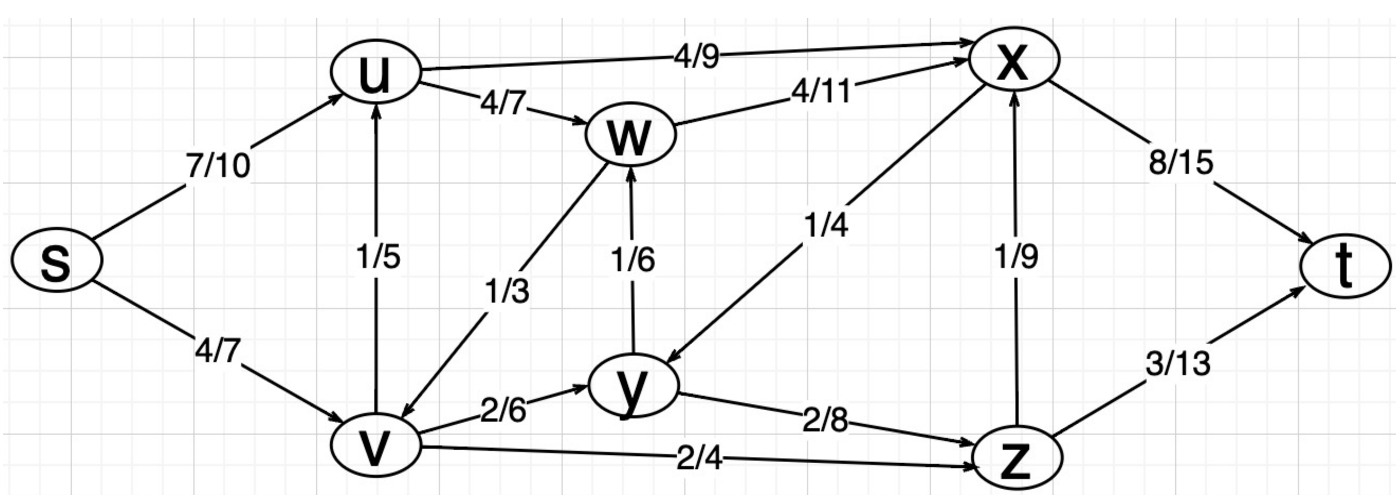
$$|f| = f(S,T)$$

$$|f| = f(S,T) = \sum_{u \in S}\sum_{v \in T} f(u,v) - \sum_{u \in S}\sum_{v \in T} f(v,u)$$

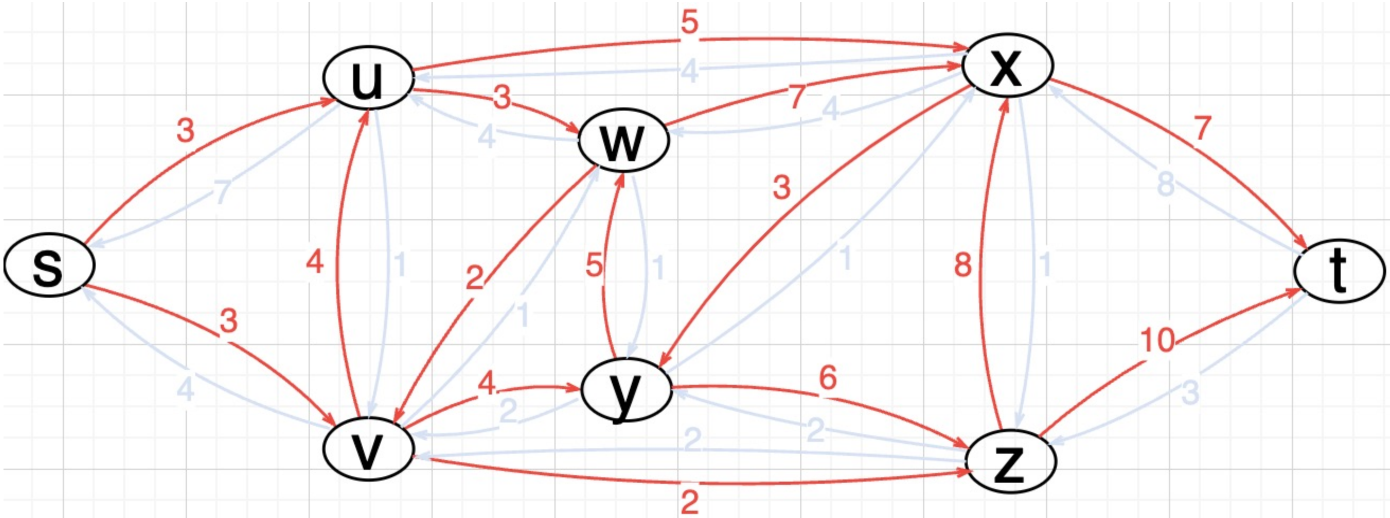
$$|f| = f(S,T) = \sum_{u \in S}\sum_{v \in T} c(u,v) - 0 = c(S,T)$$

- (V, S) which vertices are readable from source s and (V, T) which vertices are not readable from source s
- In the $cut(S,T)$, $f(u,v) = c(u,v)$

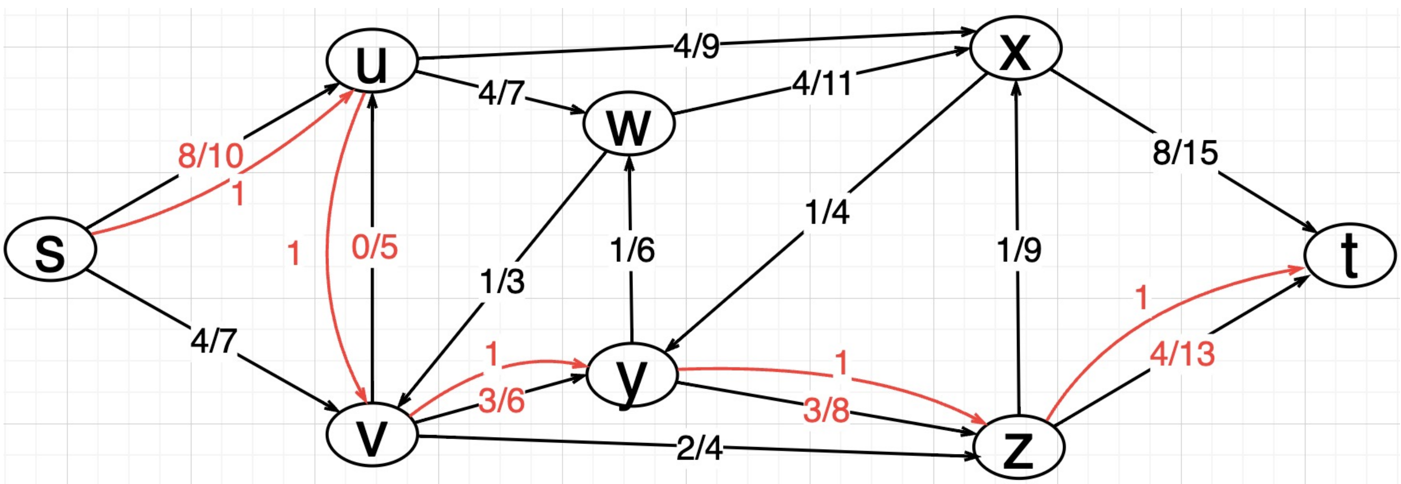
HW5-Q6 Example, Consider the following flow network.

- 

(a) Draw the residual network for this flow.

- 

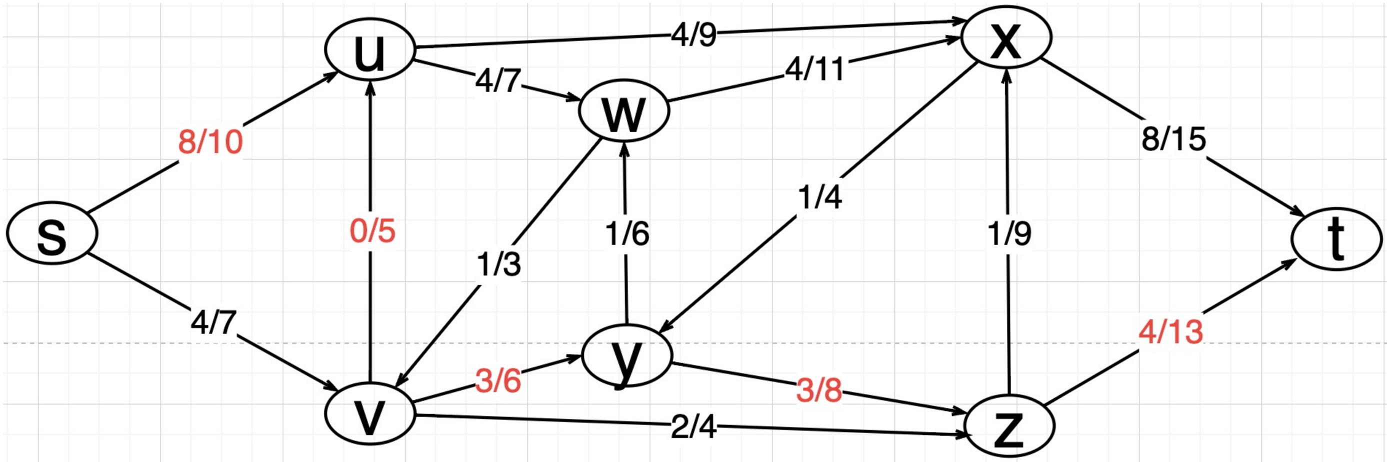
(b) Find an augmenting path that increases at the flow across at least one edge and decreases the flow across another edge. Make the flow be as large as possible for the chosen path.

- 

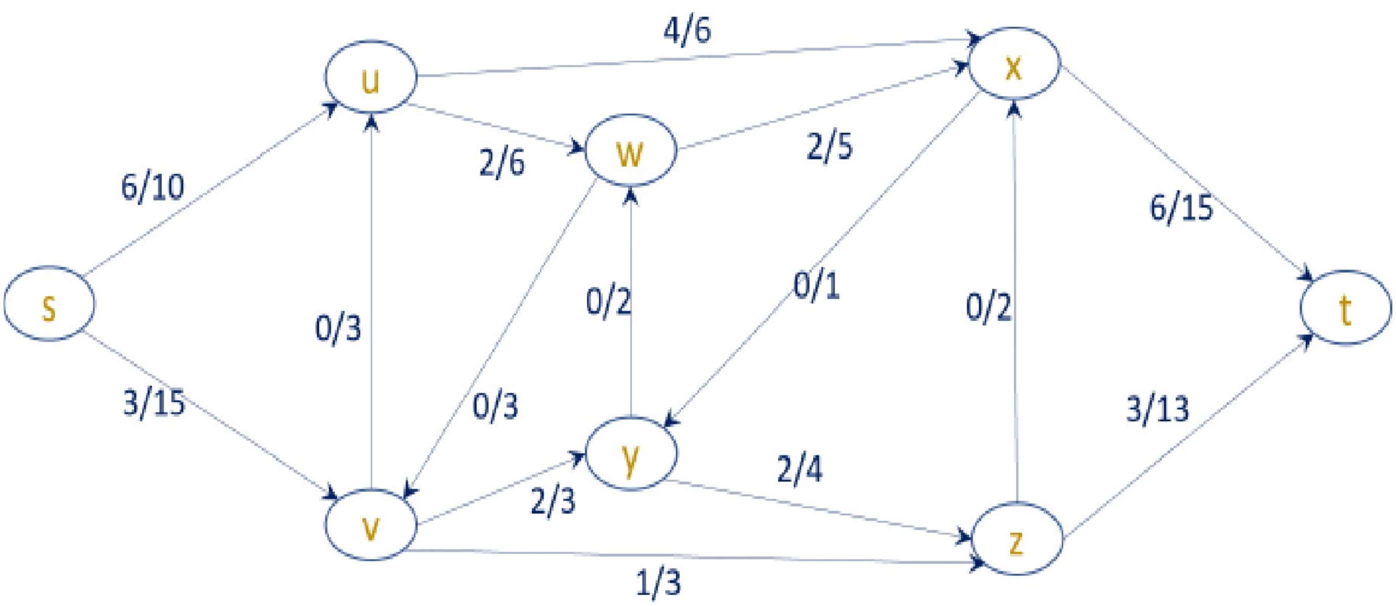
(c) Describe the augmenting path as a ordered list of vertices and state how much flow goes through it.

> An augmenting path that increases flow across some edges and de- creases flow across others could be $\{s, u, v, y, z, t\}$. The flow is decrease across edge (v, u) while it the flow has increases across other edges. There are other choices possible in the graph as well. The capacity of this path is 1.

(d) Finally, draw a new flow graph with the augmented flow.

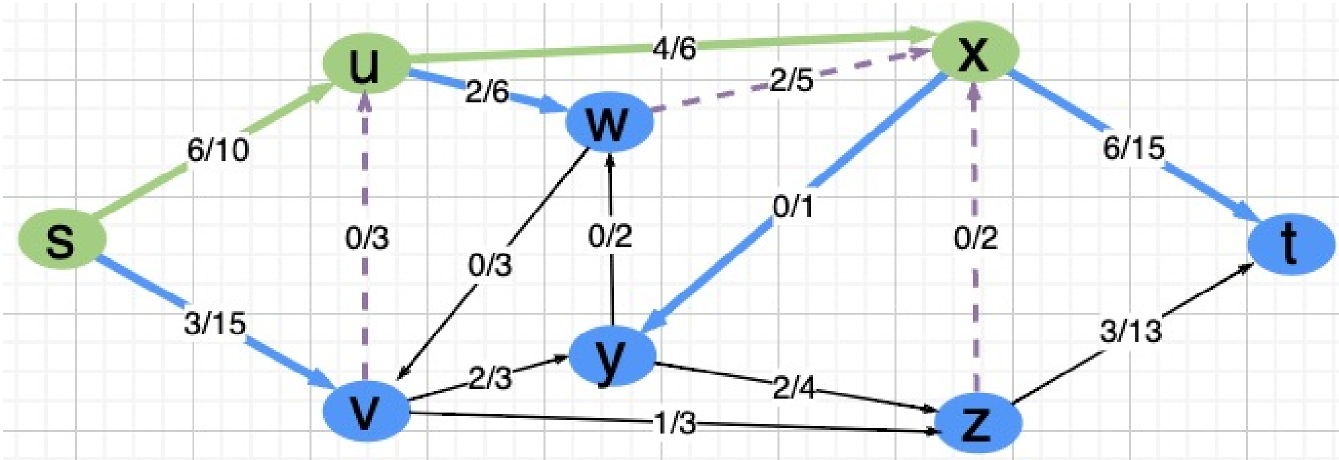
- 

HW5-Q7 Example, Consider the following flow network.

- 

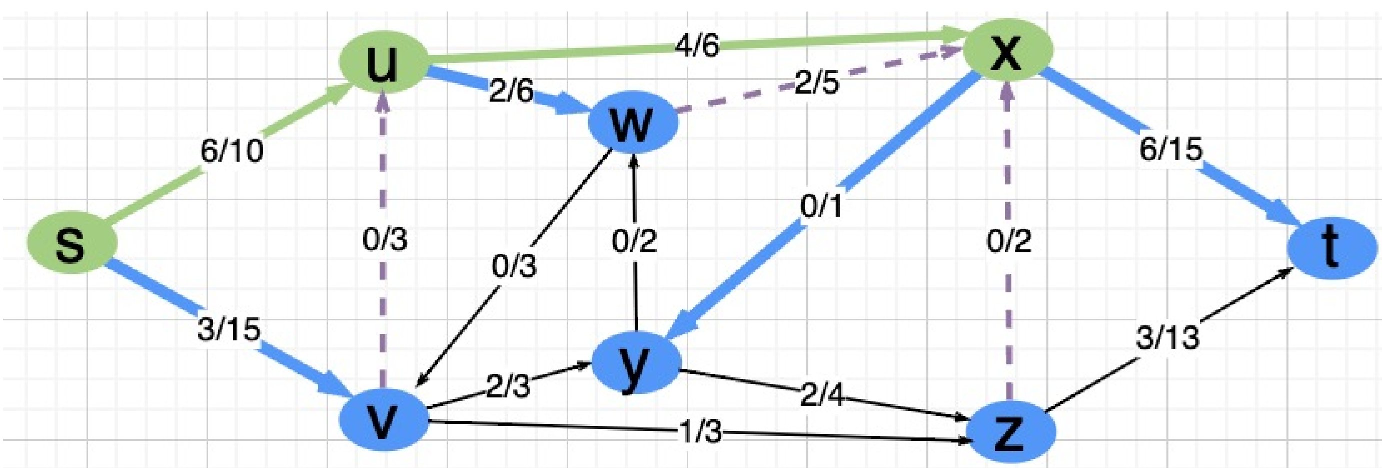
Let $S = \{s, u, x\}$ and $T = \{v, w, y, z, t\}$

(a) What is $f(S, T)$?

- 

- The flow from set S to set S does not count
  - $f(s,u)$, $f(u,x)$
- The flow from the set $S = \{s,u,x\}$ to set $T = \{v,w,y,z,t\}$
  - $f(s,v)=3$, $f(u,w)=2$, $f(x,t)=6$, $f(x,y)=0$
  - $3+2+6+0=11$
- The flow from the **Set T** back to **Set S**
  - $f(v,u)=0$, $f(w,x)=2$, and $f(z,x)=0$
  - $0+2+0=2$
- $f(S,T): 11-2=9$

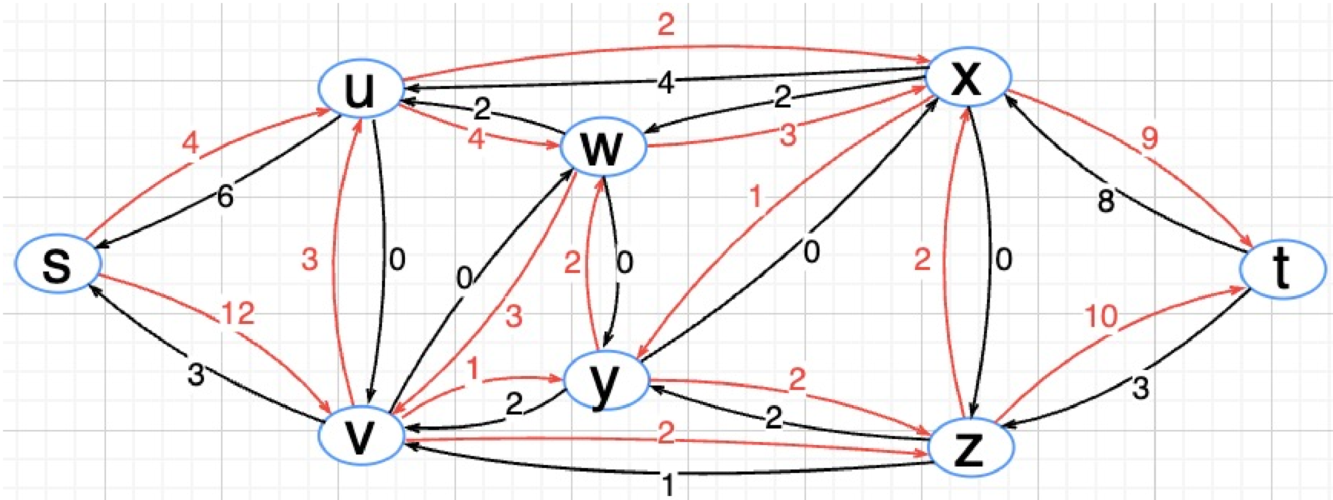
(b) What is $c(S,T)$?

- 

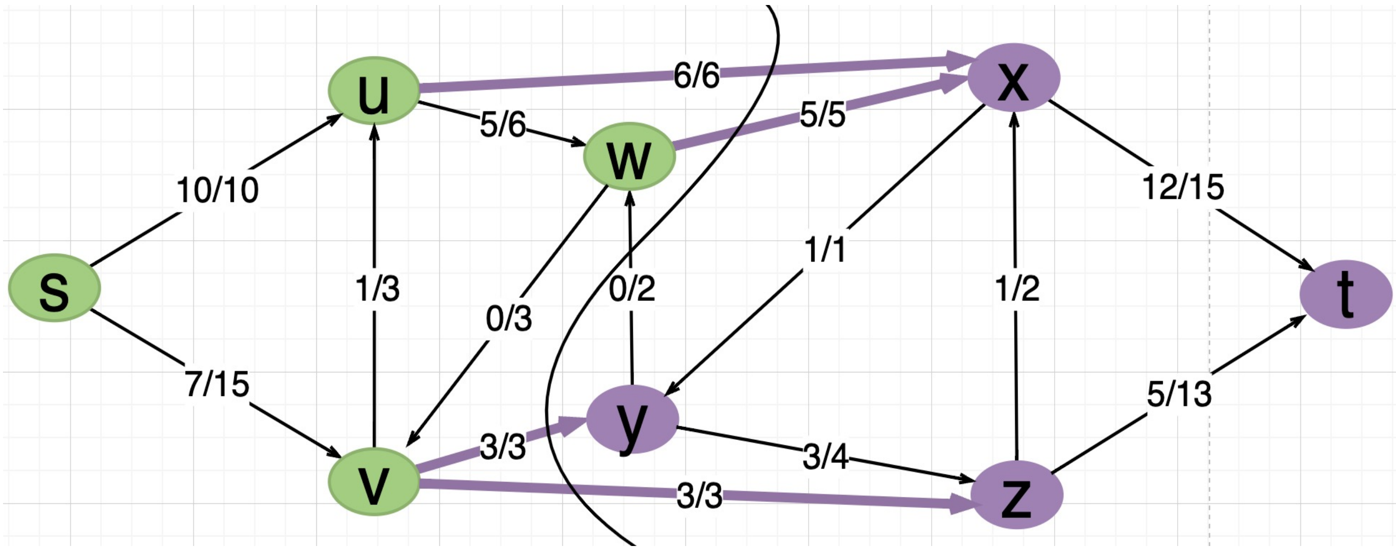
- The capacities from **Set S** to **Set T**
  - $c(s,v)=15$, $c(u,w)=6$, $c(x,y)=1$, $c(x,t)=15$
- $c(S,T): 15+6+1+15=37$

(c) Find a minimum cut for G (Hint: Add augmenting paths until no augmenting path can be found. Which edges are reachable from s in the G_f?)

- **Minimum Cut == Maximum Flow**
- Find out the residual network for the flow network



- Augmenting paths until no augmenting path can be found



  - The cut $S = \{s, u, v, w\}, T = \{x, y, z, t\}$
  - $c(S, T) = 6 + 5 + 3 + 3 = 17$

Before Class

Please use this example to explain those questions step by step concisely and comprehensively with minimal wording, if you are able to show in visualizations that would be more better. please include Taiwan Traditional Chinese after every line of English content please. If possible make it as a detailed note of Network Flow please.

Given a directed graph $G = (V, E)$ with the following assumptions
$V = \{s, w, t, x, y, z\}, E = \{(s, w, 3), (s, y, 2), (y, w, 3), (w, x, 2), (x, y, 1), (y, z, 3), (z, x, 3), (x, t, 3), (z, t, 2)\}$. s is the Source Node, and t is Sink Node.

What is Network Flow, Maximum Flow, Residual Networks, Flow Graph, Source, Sink, and Total Flow?
What is $c(u,v)$ and $f(u,v)$?

- What does $c(u,v) \geq 0$ mean?
- What does $(u,v) \notin E$ then $c(u,v) = 0$ mean?
- What does $(u,v) \in E$ then $c(v,u) \notin E$ mean?

What is capacity $c(u,v)$ and edge (u,v)?
What is the Ford-Fulkerson Method in the Network Flow?
What is the Residual Network G_f??
What is the Residual Capacity $c_f(u,v)$?
What is the Max additional flow $c_f(u,v) = c(u,v) - f(u,v)$
What is the reversible flow? $c_f(v,u) = f(u,v)$
What is the relationship between capacity graph and flow graph??
What is total flow in the network $|f| = \sum_{v \in V} f(s,v) - \sum_{f(v,s)} = 0$?
What is the Capacity Constraint? $0 \leq f(u,v) \leq c(u,v)$
What is the Flow Conservation $u \in v - \{s,t\}$, $\sum_{v \in V} f(v,u) = \sum_{v \in V} f(u,v)$, and $\sum_{v \in V} f(v,u) - \sum_{v \in V} f(u,v) = 0$
What is the Modeling Anti-Parallel Edges? Assume the edge $(u,v) \in E$ then $(v,u) \notin E$
What is the Augmenting Path, Path Finding, Flow Increase, Update Residual Graph
what is the Augmentation of a Flow $(f \uparrow f')$? if Given flow f in G, and flow f' in G_f. what does the “augmentation fo f by f'” mean?
Please explain $(f \uparrow f') = f(u,v) + f'(v,u) - f'(v,u)$ if the edge$(u,v) \in E$, 0 means otherwise.

How to finding the Maximum Flow by using Ford-Fulkerson Method?