① $-x_1 + x_2 - x_3 = -1$

$2x_1 + 6x_2 - x_3 = 3$

$6x_1 + 5x_2 + 3x_3 = 8$

(a) $A = \begin{pmatrix} -1 & 1 & -1 \\ 2 & 6 & -1 \\ 6 & 5 & 3 \end{pmatrix}$

$\det(A) = -1(18+5) - 1(6+6) - 1(10-12)$

$= -23 - 12 + 2$

$= -33 \neq 0$

This system has unique solution.

(b)

$Ax = b$

$$\begin{pmatrix} -1 & 1 & -1 \\ 2 & 6 & -1 \\ 6 & 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix}$$

augmented matrix:

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & -1 \\ 2 & 6 & -1 & 3 \\ 6 & 5 & 3 & 8 \end{array}\right)$$

(c) $$\left(\begin{array}{ccc|c} -1 & 1 & -1 & -1 \\ 2 & 6 & -1 & 3 \\ 6 & 5 & 3 & 8 \end{array}\right) = \left(\begin{array}{ccc|c} -1 & 1 & -1 & -1 \\ 0 & 8 & -3 & 1 \\ 0 & 11 & -3 & 2 \end{array}\right) \begin{matrix} [r_2' = r_2 + 2r_1] \\ [r_3' = r_3 + 6r_1] \end{matrix}$$

$$= \left(\begin{array}{ccc|c} -1 & 1 & -1 & -1 \\ 0 & 8 & -3 & 1 \\ 0 & 0 & 9/8 & 5/8 \end{array}\right) [r_3'' = r_3' - \tfrac{11}{8} \times r_2']$$

$x_3 \Rightarrow \frac{9}{\cancel{8}} x_3 = \frac{5}{\cancel{8}}$. $\boxed{x_3 = 5/9}$

$x_2 \Rightarrow 8x_2 - \cancel{3} \times \frac{5}{\cancel{9}_3} = 1$

$\Rightarrow 8x_2 = 1 + \frac{5}{3}$

$\Rightarrow x_2 = \frac{8}{8 \times 3}$ $\boxed{\therefore x_2 = \frac{1}{3}}$

$x_1 \Rightarrow -x_1 + \frac{1}{3} - \frac{5}{9} = 1 \Rightarrow -x_1 = 1 + \frac{2}{9}$

$\boxed{\therefore x_1 = -\frac{11}{9}}$

(2) $P_2(x) = a_0 + a_1 x + a_2 x^2 \Rightarrow \left\{ \begin{array}{l} P_2(x_0) = a_0 + a_1 x_0 + a_2 x_0^2 = f(x_0) \\ P_2(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 = f(x_1) \\ P_2(x_2) = a_0 + a_1 x_2 + a_2 x_2^2 = f(x_2) \\ P_2(x_3) = a_0 + a_1 x_3 + a_2 x_3^2 = f(x_3) \end{array} \right.$

(a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{pmatrix}$,

$b = \begin{pmatrix} 3 \\ 7 \\ -2 \\ -4 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & 1 & 9 \end{pmatrix}$

(b) $A^T \cdot A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & 1 & 9 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{pmatrix}$

$= \begin{pmatrix} 4 & 3 & 11 \\ 3 & 11 & 27 \\ 11 & 27 & 83 \end{pmatrix}$

$A^T \cdot b = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & 1 & 9 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ -2 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ -21 \\ -31 \end{pmatrix}$

(c) $(A^TA)x = (A^Tb)$

$$\Rightarrow \begin{pmatrix} 4 & 3 & 11 \\ 3 & 11 & 27 \\ 11 & 27 & 83 \end{pmatrix} x = \begin{pmatrix} 4 \\ -21 \\ -31 \end{pmatrix}$$

$$\Rightarrow x = \underbrace{\begin{pmatrix} 4 & 3 & 11 \\ 3 & 11 & 27 \\ 11 & 27 & 83 \end{pmatrix}^{-1}}_{A} \underbrace{\begin{pmatrix} 4 \\ -21 \\ -31 \end{pmatrix}}_{b}$$

$$A = \begin{pmatrix} 4 & 3 & 11 \\ 3 & 11 & 27 \\ 11 & 27 & 83 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & 3 & 11 \\ 0 & 35/4 & 75/4 \\ 11 & 27 & 83 \end{pmatrix} \left[R_2' = R_2 - \frac{3R_1}{4}\right]$$

$$= \begin{pmatrix} 4 & 3 & 11 \\ 0 & \frac{35}{4} & \frac{75}{4} \\ 0 & \frac{75}{4} & \frac{211}{4} \end{pmatrix} \left[R_3' = R_3 - \frac{11R_1}{4}\right]$$

$$= \begin{pmatrix} 4 & 3 & 11 \\ 0 & \frac{35}{4} & \frac{75}{4} \\ 0 & 0 & \frac{88}{7} \end{pmatrix} \left[R_3' = R_3' - \frac{15R_2}{7}\right]$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 1 & 0 \\ \frac{11}{4} & \frac{15}{7} & 1 \end{pmatrix}$$

Now, $LY = B$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 1 & 0 \\ \frac{11}{4} & \frac{15}{7} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -21 \\ -31 \end{pmatrix}$$

$r_1 \Rightarrow y_1 = 4$

$r_2 \Rightarrow \frac{3}{4} \times 4 + y_2 = -21 \quad \therefore y_2 = -24$

$r_3 \Rightarrow \frac{11}{\cancel{4}} \times \cancel{4} + \frac{15}{7} \times (-24) + y_3 = -31$

$\Rightarrow 11 - \frac{15 \times 24}{7} + y_3 = -31$

$\therefore y_3 = \frac{66}{7}$

Again, $UX = Y$

$$\Rightarrow \begin{pmatrix} 4 & 3 & 11 \\ 0 & \frac{35}{4} & \frac{75}{4} \\ 0 & 0 & \frac{88}{7} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -24 \\ \frac{66}{7} \end{pmatrix}$$

$r_3 \Rightarrow \frac{\cancel{88}}{\cancel{7}} x_3 = \frac{\cancel{66}}{\cancel{7}}$

$\boxed{\therefore x_3 = \frac{3}{4}}$

$r_2 \Rightarrow \frac{35}{4} x_2 + \frac{75}{4} \times \frac{3}{4} = -24$

$\boxed{\therefore x_2 = -\frac{87}{20}}$

$r_1 \Rightarrow 4x_1 + 3\left(-\frac{87}{20}\right) + 11 \times \left(\frac{3}{4}\right) = 4$

$\boxed{\therefore x_1 = \frac{111}{5}}$