Answer to the question no 1

(5)

x	$f(x)$
$x_0 = 0$	1
$x_1 = 0.5$	1.6487
$x_2 = 1$	2.7183

$$l_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^{n=2} \frac{x - x_j}{x_0 - x_j}$$

$$= \frac{x - x_1}{x_0 - x_1} \times \frac{x - x_2}{x_0 - x_2}$$

$$= \frac{x - 0.5}{0 - 0.5} \times \frac{x - 1}{0 - 1}$$

$$= 2(x - 0.5)(x - 1)$$

6

x	$f(x)$
0	1
0.5	1.6487
1	2.7183

$$l_1(x) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{x - x_j}{x_1 - x_j}$$

$$= \frac{x - x_0}{x_1 - x_0} \times \frac{x_1 - x_2}{x_1 - x_2}$$

$$= \frac{x - 0}{0.5 - 0} \times \frac{x - 1}{1 - 0.5}$$

$$= \boxed{-4x(x-1)}$$

7

x	$f(x)$
0	1
0.5	1.6487
1	2.7183

$$l_2(x) = \prod_{\substack{j=0 \\ j \neq 2}}^{2} \frac{x - x_j}{x_2 - x_j}$$

$$= \frac{x - x_0}{x_2 - x_0} \times \frac{x - x_1}{x_2 - x_1}$$

$$= \frac{x - 0}{1 - 0} \times \frac{x - 0.5}{1 - 0.5}$$

$$= \boxed{2x(x - 0.5)}$$

Using lagrange Method,

$P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$

$= 2(x-0.5)(x-1)(1)$

$- 4x(x-1)(1.6487)$

$+ 2x(x-0.5)(2.7183)$

$\therefore P_2(x=0.2)$

$= 2(0.2-0.5)(0.2-1)(1)$

$- 4\times 0.2(0.2-1)(1.6487)$

$+ 2\times 0.2(0.2-0.5)(2.7183)$

$= \boxed{1.209}$

9)

At $x=0.2$,

Weierstrass approximation says

$$|f(x)-P_n(x)| \le \epsilon$$

$f(x) = f(0.2) = e^{0.2} = 1.2214$

$P_2(x) = 1.209$ [calculated previously]

$\therefore$ Error $= |f(.2) - P_2(0.2)|$

$= |1.2214 - 1.209|$

$\boxed{= 0.0124}$

d) However, when the number of nodes is large, we will have a lot of equations, so our vandermonde matrix will be big and time consuming to compute, so Lagrange will be a better option.

Part B:

Task 1, 2

x_i	$f(x_i)$ $\rightarrow f[x_0]$	$\rightarrow f[x_0, x_1]$	$\rightarrow f[x_0, x_1, x_2]$
0	4		
		$\frac{16-4}{1-0} = 12$	
1	16		$\frac{4-12}{-1-0} = 8$
		$\frac{8-16}{-1-1} = 4$	
-1	8		

$$b_0 = f[x_0] = \boxed{4}$$

$$b_1 = f[x_0, x_1, \cancel{x_2}] = \boxed{12}$$

$$b_2 = f[x_0, x_1, x_2] = \boxed{8}$$

$$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$P_2(0.5) = 4 + 12(0.5 - 0) + 8(0.5 - 0)(0.5 - 1) = \boxed{8}$$

$$P_2(-0.9) = 4 + 12(-0.9 - 0) + 8(-0.9 - 0)(-0.9 - 1) = \boxed{6.88}$$

Task 3

$f(x) = \sin^2(x/2)$ $\quad \{-\pi/3, 0, \pi/3\}$

3 points, degree 2

$\therefore |f(x) - P_2(x)| = \left| \frac{f^3(\xi)}{3!} \omega(x) \right|$

$f'(x) = 2 \sin x/2 \cos x/2 \cdot 1/2$

$= \frac{1}{2} \sin x$

$f''(x) = \frac{1}{2} \cos x$

$f'''(x) = -\frac{1}{2} \sin x$

to find max value of $f'''(x)$

$f^{iv}(x) = 0$

$\therefore -\frac{1}{2} \cos x = 0$

$x = \pi/2$

but $\pi/2 \notin [-1, 1]$

$\therefore f'''(-1) = 0.4207$ (max)

$f'''(1) = -0.4207$

$\omega(x) = (x-0)(x-\pi/3)(x+\pi/3)$

$\omega(x)_{max} = 0.442$

$$w(x) = \left(x - \pi/3\right)(x-0)\left(x + \pi/3\right)$$

$$= x\left(x^2 - \frac{\pi^2}{3^2}\right)$$

$$= x^3 - \frac{\pi^2}{3^2}x$$

to find max of $w(x)$, we need to find x where $w'(x) = 0$

$$w'(x) = 3x^2 - \frac{\pi^2}{3^2} = 0$$

$$\Rightarrow 3x^2 = \frac{\pi^2}{3^2}$$

$$\Rightarrow \sqrt{3}x = \pm \frac{\pi}{3}$$

$$\Rightarrow x = \pm \frac{\pi}{3\sqrt{3}}$$

$w(-\pi/3\sqrt{3}) = 0.4420$ max

$w(\pi/3\sqrt{3}) = -0.4420$

$w(1) = -0.096622$

$w(-1) = 0.0966$

$$\therefore \max \left| f(x) - P_2(x) \right| = \max \left| \frac{f'''(\xi)}{3!} * \omega(x) \right|$$

$$= \frac{.4207}{6} * 0.4420$$

$$= 0.03099$$

b) Because it takes more nodes on the edges and less nodes in the middle.

Answer to the question no 4

$$h_k(x) = (1 - 2(x - x_k)\, l_k'(x_k))\, l_k^2(x)$$

$$\hat{h}_k(x) = (x - x_k)\, l_k^2(x)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 0.2}{0.1 - 0.2}$$

$x_0 = 0.1$
$x_1 = 0.2$

$$= \frac{x - 0.2}{-0.1}$$

$$= -10x + 2$$

$$l_0'(x) = -10$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0.1}{0.2 - 0.1} = \frac{x - 0.1}{0.1}$$

$$= 10x - 1$$

$$l_1'(x) = 10$$

(a) $h_0(x) = \left(1 - 2(x - x_0)\, l_0'(x)\right) l_0^2(x)$

$= \{1 - 2(x - 0.1).(-10)\} (-10x + 2)^2$

$= \left(1 + 20x \cancel{+ 0.2} - 2\right)\left(100x^2 - 40x + 4\right)$

$= (20x - 1)(100x^2 - 40x + 4)$

$= 2000x^3 - 800x^2 + 80x - 100x^2$
$\quad + 40x - 4$

$= 2000x^3 - 900x^2 + 120x - 4$

$h_1(x) = \left(1 - 2(x - x_1)\, l_1'(x)\right) l_1^2(x)$

$= \{1 - 2(x - 0.2).10\} (10x - 1)^2$

$= (1 - 20x + 4)(100x^2 - 20x + 1)$

$= (-20x + 5)(100x^2 - 20x + 1)$

$= -2000x^3 + 400x^2 - 20x$
$\quad + 500x^2 - 100x + 5$

$= -2000x^3 + 900x^2 - 120x + 5$

$$\hat{h}_0(x) = (x - x_0)\, l_0^2(x)$$

$$= (x - 0.1)(-10x + 2)^2$$

$$= (x - 0.1)(100x^2 - 40x + 4)$$

$$= 100x^3 - 40x^2 + 4x - 10x^2 + 4x - 0.4$$

$$= 100x^3 - 50x^2 + 8x - 0.4$$

$$\hat{h}_1(x) = (x - x_1)\, l_1^2(x)$$

$$= (x - 0.2)(10x - 1)^2$$

$$= (x - 0.2)(100x^2 - 20x + 1)$$

$$= 100x^3 - 20x^2 + x - 20x^2 + 4x - 0.2$$

$$= 100x^3 - 40x^2 + 5x - 0.2$$

(b) $P_3(x) = f(x_0)\,h_0(x) + f(x_1)\,h_1(x)$
$+ f'(x_0)\,\hat{h}_0(x) + f'(x_1)\,\hat{h}_1(x)$

$= -0.62050(2000x^3 - 900x^2 + 120x - 4)$
$+ (-0.28340)(-2000x^3 + 900x^2 - 120x + 5)$
$+ 3.58502(100x^3 - 50x^2 + 8x - 0.4)$
$+ 3.14033(100x^3 - 40x^2 + 5x - 0.2)$

$= -1241x^3 + 558.45x^2 - 74.46x - 2.482$
$+ 566.8x^3 - 255.06x^2 + 34.008x - 1.417$
$+ 358.502x^3 - 179.251x^2 + 28.68016x$
$- 1.434008 + 314.033x^3 - 125.6132x^2$
$+ 15.70165x - 0.628066$

$= -1.665x^3 - 1.4742x^2 + 3.92981x$
$- 5.961074$

$\therefore P_3(0.15) = -5.41039137 5$