Weber-Davis Wind Model

Steady axisymmetric wind models with frozen-in magnetic fields.

“A **steady-state** model of the solar-wind flow in the equatorial plane including **the effects of pressure gradients** $-\nabla p$, **gravitation** $\rho \mathbf{g}$, and **magnetic forces** $\mathbf{j} \times \mathbf{B}$ is developed and solved for both the **radial** and **azimuthal** motions.”

$$\begin{gathered}\eta = 0 \\ viscosity = 0 \\ scalar\ pressure\end{gathered}$$

the magnetic force is small compared to other forces in the radial equation, however, it is dominant in the rotational motion of the solar wind.

Ideal MHD equations:

$$\begin{gathered}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g} \\ \frac{d}{dt}(\frac{p}{\rho^\gamma}) = 0 \\ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\eta c}{4\pi} \nabla^2 \mathbf{B}\end{gathered}$$

Basic Postulates and Assumptions

1. **MHD equations** for a fluid with and **infinite conductivity**, **zero viscosity** and a **scalar pressure**.
2. The Sun is assumed to have **a general magnetic field that depends on latitude.**
3. The local irregularities in the field, the polarity reversals and wind velocity fluctuations of the sector structure, and the waves superimposed on the smooth field in interplanetary space are all unessential in treating the basic spiral magnetic pattern and the average angular momentum in the solar wind.
4. **Steady-state, complete axial symmetry** in which **in the equatorial plane** of the Sun the field is combed out by the solar wind and has **no component normal to this plane, no** ϕ **- dependence.**
5. In the steady-state solar wind **the velocities** and **magnetic fields** as well as their derivatives are **continuous, smooth functions of position**, i.e., **no shocks** exist anywhere.

Solar-Wind Equations

Solar wind has a velocity (on equatorial plane)

$$\mathbf{v} = u \mathbf{e}_r + v_\phi \mathbf{e}_\phi$$

a magnetic field

$$\mathbf{B} = B_r \mathbf{e}_r + B_\phi \mathbf{e}_\phi$$

Conservation of mass: (steady state: $\partial/\partial t = 0$) Since we only consider the equatorial plane (no $\theta-$component):

$$\frac{1}{r^2}\frac{d}{dr}(r^2\rho u) = 0$$

$$\rho u r^2 = const$$

Since the solar wind is perfect conductor (conductivity → ∞, $\eta = 0$):

$$\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c} = \eta\mathbf{j} = 0$$

$$\begin{aligned} c(\nabla\times\mathbf{E})_\phi &= -\mathbf{v}(\nabla\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{v}) \\ &= \mathbf{B}(\nabla\cdot\mathbf{v}) \\ \frac{1}{r}\frac{d}{dr}[r(uB_\phi - v_\phi B_r)] &= 0 \end{aligned}$$

In a perfectly conducting fluid, $\mathbf{v}$ is parallel to $\mathbf{B}$ in a frame that rotates with the Sun (because of flux freezing), thus we obtain

$$r(uB_\phi - v_\phi B_r) = const = -\Omega r^2 B_r$$

Since

$$\nabla\cdot\mathbf{B} = 0$$

we have

$$\begin{aligned} \frac{d}{dr}(r^2 B_r) &= 0 \\ r^2 B_r &= const = r_0^2 B_0 \end{aligned}$$

where subscript 0 refers to an arbitrary reference level $r = r_0$.

The model has ϕ-symmetry, so the momentum and the magnetic-force term are the only terms that enter the steady state ϕ-equation of motion:

$$\rho\mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{j}\times\mathbf{B}/c$$

$$\rho\frac{u}{r}\frac{d}{dr}(rv_\phi) = \frac{1}{c}(\mathbf{J}\times\mathbf{B})_\phi = \frac{1}{4\pi}[(\nabla\times\mathbf{B})\times\mathbf{B}]_\phi = \frac{B_r}{4\pi r}\frac{d}{dr}(rB_r)$$

($\nabla\times\mathbf{B} = 4\pi\mathbf{j}/c$?)

However,

Since

$$\begin{aligned} \rho u r^2 &= const \\ r^2 B_r &= const \end{aligned}$$

$$\frac{B_r}{4\pi\rho u} = \frac{B_r r^2}{4\pi\rho u r^2} = const$$

Then we have

$$\frac{d}{dr}(rv_\phi) = \frac{B_r}{4\pi\rho u}\frac{d}{dr}(rB_\phi)$$

So we can integrate the azimuthal equation of motion :

$$rv_\phi - \frac{B_r}{4\pi\rho u}rB_\phi = const = L$$

rv_ϕ: **ordinary angular momentum per unit mass.**

$-\frac{B_r}{4\pi\rho\mu} rB_\phi$: **torque associated with the magnetic stresses (magnetic pressure $\nabla_\perp(\frac{B^2}{8\pi})$ and magnetic tension $\kappa\frac{B^2}{4\pi}$).**

Their sum must be a constant, the **total angular momentum carried away from the Sun per unit mass loss L**.

Introduce **radial Alfvenic Mach number**:

$$M_A^2 = \frac{4\pi\rho u^2}{B_r^2}$$

Solve for the azimuthal velocity:

$$r(uB_\phi - v_\phi B_r) = const = -\Omega r^2 B_r$$
$$rv_\phi - \frac{B_r}{4\pi\rho\mu} rB_\phi = const = L$$

we get

$$v_\phi = \Omega r \frac{M_A^2 L r^{-2}\Omega^{-1} - 1}{M_A^2 - 1}$$

M_A is much smaller than 1 near the surface of the Sun, but at 1 a.u. it is approximately 10. So there exists a point between Sun and earth where $M_A = 1$, and let the radial velocity and radius at this point be called u_a, r_a. It is called the **Alfvenic critical point**.

Since at the Alfvenic critical point, $M_A^2 - 1 \to 0$, in order to keep v_ϕ finite, we require the numerator to vanish at the same point, so at this point the total angular momentum should have the value

$$L = \Omega r_a^2$$

From previous equations $\frac{B_r}{4\pi\rho u} = \frac{B_r r^2}{4\pi\rho u r^2} = const$ we have

$$\frac{M_A^2}{ur^2} = const$$

which may be evaluated at the critical point to give

$$M_A^2 = \frac{ur^2}{u_a r_a^2} = \frac{\rho_a}{\rho}$$

and the azimuthal velocity reduces to

$$v_\phi = \frac{\Omega r}{u_a}\frac{u_a - u}{1 - M_A^2}$$

the azimuthal magnetic field is given by

$$B_\phi = -B_r \frac{\Omega r}{u_a}\frac{r_a^2 - r^2}{r_a^2(1 - M_A^2)}$$

Note that L is determined only from the conditions at the Alfvenic critical point $L = \Omega r_a^2$.

For $r \gg r_a$:

the radial velocity u , in the usual solutions is almost a constant and $M_A \propto r, v_\phi \propto \frac{1}{r}, B_\phi \propto \frac{1}{r}$.

For $r \ll r_a$ where $u \ll u_a, M_A^2 \ll 1$:

$$B_\phi = -B_r \frac{\Omega r}{u_a}[1 - \frac{r^2}{r_a^2}(1 - \frac{u}{u_a}) + \cdots]$$

whereas

$$v_\phi = \Omega r[1 - \frac{u}{u_a}(1 - \frac{r^2}{r_a^2}) + \cdots]$$

Near the surface of the Sun where r is really small according to $L = rv_\phi - \frac{B_r}{4\pi\rho u} rB_\phi$, **most of the angular momentum loss is due to the torque exerted by the magnetic fields**. As r increases, the azimuthal fluid velocity v_ϕ increases and the magnetic stress decreases (B_ϕ decreases) until **at large distances the relative contributions to the angular momentum loss are $[1-(u_a/u_\infty)]$ and u_a/u_∞, respectively**.

Take the **radial** momentum equation

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}$$

→

$$\rho u \frac{du}{dr} = -\frac{d}{dr} p - \rho \frac{GM_\odot}{r^2} + \frac{1}{c}(\mathbf{J} \times \mathbf{B})_r + \rho \frac{v_\phi^2}{r^2}$$

Since in a fully ionized gas of pure hydrogen the effective particle mass is only half the hydrogen mass, m, **the equation of state** is:

$$p = \frac{2kT}{m}\rho$$

Assume temperature is equal for ions and electrons. We use the **polytrope law** :

$$\frac{p}{p_0} = (\frac{\rho}{\rho_0})^\gamma$$
$$\frac{p}{p_a} = (\frac{\rho}{\rho_a})^\gamma$$

$$p = p_0(\frac{\rho}{\rho_0})^\gamma = p_a(\frac{\rho}{\rho_a})^\gamma$$

the magnetic force :

$$\frac{1}{c}(\mathbf{J} \times \mathbf{B})_r = -\frac{1}{4\pi r} B_\phi \frac{d}{dr}(rB_\phi)$$

substitute it into the momentum equation we get

$$\frac{d}{dr}\{\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{p_a}{\rho_a}(\frac{\rho}{\rho_a})^{\gamma-1} - \frac{GM_\odot}{r}\} = \frac{v_\phi^2}{r} - \frac{1}{8\pi\rho r^2}\frac{d}{dr}(rB_\phi)^2$$

with the R.H.S equals to zero, we will have the **Parker's equation of motion** for the solar wind:

$$\frac{d}{dr}\{\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{p_a}{\rho_a}(\frac{\rho}{\rho_a})^{\gamma-1} - \frac{GM_\odot}{r}\} = 0$$
$$\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{p_a}{\rho_a}(\frac{\rho}{\rho_a})^{\gamma-1} - \frac{GM_\odot}{r} = const$$

The presence of the R.H.S. is due to the inclusion of the **magnetic force** and **the azimuthal velocity**

Now express ρ, v_ϕ, B_ϕ in terms of u, r.

Using :

$$M_A^2 = \frac{ur^2}{u_a r_a^2} = \frac{\rho_a}{\rho}$$
$$v_\phi = \frac{\Omega r}{u_a}\frac{u_0 - u}{1 - M_A^2}$$
$$B_\phi = -B_r\frac{\Omega r}{u_a}\frac{r_a^2 - r^2}{r_a^2(1 - M_A^2)}$$

We have:

$$\frac{du}{dr} = \frac{u}{r}\{(\frac{2\gamma p_a}{\rho_a M_A^{2(\gamma-1)}} - \frac{GM_\odot}{r})(M_A^2 - 1)^3 + \Omega^2 r^2(\frac{u}{u_a} - 1)[(M_A^2 + 1)\frac{u}{u_a} - 3M_A^2 + 1]\}$$
$$\times[(u^2 - \frac{\gamma p_a}{\rho_a M_A^{2(\gamma-1)}})(M_A^2 - 1)^3 - \Omega^2 r^2(\frac{r_a^2}{r^2} - 1)^2]^{-1}$$

where

$$M_A^2 = \frac{ur^2}{u_a r_a^2} = \frac{\rho_a}{\rho}$$

Then this equation is in only two variables u and r.

And the Alfvenic critical point $r = r_a$ is also a critical point for radial equation.

Integrate the radial equation of motion and give the total energy flux per steradian which is a constant for our solution:

$$F = \rho u r^2\{\frac{u^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_a}{\rho_a}M_A^{-2(\gamma-1)} - \frac{GM_\odot}{r} + \frac{\Omega^2 r_a^2}{2r^2}[1 + \frac{(2M_A^2 - 1)(r^2 - r_a^2)^2}{r_a^4(M_A^2 - 1)^2}]\}$$
$$= const$$

Firs term : **kinetic energy associated with the radial velocity.**

Second term: **the sum of the enthalpy and the energy transported by thermal conduction**, magnetic heating etc.

Third term : **gravitational energy**.

Fourth term: **sum of the magnetic and rotational energies**.

Using

$$v_\phi = \frac{\Omega r}{u_a}\frac{u_0 - u}{1 - M_A^2}$$
$$B_\phi = -B_r\frac{\Omega r}{u_a}\frac{r_a^2 - r^2}{r_a^2(1 - M_A^2)}$$

we can rewrite the fourth term:

$$F_{rot+mag} = \rho u r^2(\frac{v_\phi^2}{2} - \frac{B_\phi B_r}{4\pi\rho}\frac{\Omega r}{u})$$

First term : **kinetic energy associated with the azimuthal velocity**.

Second term : **the energy transported out by the magnetic field, the Poynting energy flux**.

So the model is characterized by six equations:

Polytropic Relation:

$$p = K\rho^\gamma$$

Conservation of Mass:

$$\rho u r^2 = f$$

Conservation of Magnetic Flux:

$$r^2 B_r = \Phi$$

Frozen-in condition (Flux freezing)

$$(v_\phi - \Omega r)B_r = uB_\phi$$

Conservation of Angular Momentum:

$$r(v_\phi - \frac{B_r B_\phi}{4\pi\rho u}) = \Omega r_A^2$$

Bernoulli Integral:

(In rest frame:)

$$F = \rho u r^2 \{\frac{u^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_a}{\rho_a}M_A^{-2(\gamma-1)} - \frac{GM_\odot}{r} + \frac{\Omega^2 r_a^2}{2r^2}[1 + \frac{(2M_A^2 - 1)(r^2 - r_a^2)^2}{r_a^4(M_A^2 - 1)^2}]\}$$

(In rotating frame:)

$$\frac{u^2}{2} + \frac{1}{2}(v_\phi - \Omega r)^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} - \frac{GM_\odot}{r} - \frac{\Omega r^2}{2} = E$$

Topology of the solution

In the neighborhood of the origin, the asymptotic forms for the radial velocity are (Substitute the asymptotic form into the equation and get the coefficients.) :

$$u = a_0 r^{3-2\gamma/(\gamma-1)}(1 + a_1 r - a_2 r^3 + \cdots)$$

$$u = \frac{b_0}{r^{1/2}}[1 + b_1 r - b_2 r^{(5-3\gamma)/2} + \cdots]$$

The model above as well as the Parker’s model shows that there is no solution in which u approaches 0 for small r when $\gamma > 3/2$.

As $r \to 0$, the density increases as $r^{-1/(\gamma-1)}$ for the first expansion of u, and $r^{-3/2}$ for the second.

The coefficients are determined by the initial conditions and the parameters entering function of the total energy flux. If the solutions are required to run through the critical points, these constants depend on the properties there.

The asymptotic behaviors at large distances are :

$$u_\alpha = a_0[1 + a_1\frac{1}{r^{2(\gamma-1)}} + a_2\frac{1}{r} + a_3\frac{1}{r^2} + \cdots]$$

$$u_\beta = \beta_0/r^2[1 - \frac{1}{r^2}(\beta_1 - \beta_2/r) + \cdots]$$

$u_{\alpha 1}$: supersonic,super-Alfvenic wind at infinity

$u_{\alpha 2}$: remain super-Alfvenic, but becomes subsonic after passing the critical points.

For both $u_{\alpha 1}$ and $u_{\alpha 2}$, the pressure tend to zero as r becomes very large.

$u_{\beta 1}, u_{\beta 2}$ yield non-zero pressures an infinity.

The figure below is the family of solutions of :

$$\frac{du}{dr} = \frac{u}{r}\{(\frac{2\gamma p_a}{\rho_a M_A^{2(\gamma-1)}} - \frac{GM_\odot}{r})(M_A^2-1)^3 + \Omega^2 r^2(\frac{u}{u_a}-1)[(M_A^2+1)\frac{u}{u_a} - 3M_A^2+1]\} \times [(u^2 - \frac{\gamma p_a}{\rho_a M_A^{2(\gamma-1)}})(M_A^2 -$$

for a given γ, r_a.

> 💡 The Weber-Davis wind is primarily driven by thermal pressure, centrifugal force, magnetic pressure and wave pressure. Within the slow mode, the solar wind is influenced by both pressure and magnetic fields. After crossing the slow mode critical point r_s, it is primarily dominated by magnetic fields. Upon passing the Alfvén point r_A, the solar wind is mainly driven by its kinetic energy. Finally, after crossing the fast mode critical point r_f, the solar wind reaches supersonic and superalfvénic, and its motion is no longer affected by disturbances from the Sun.

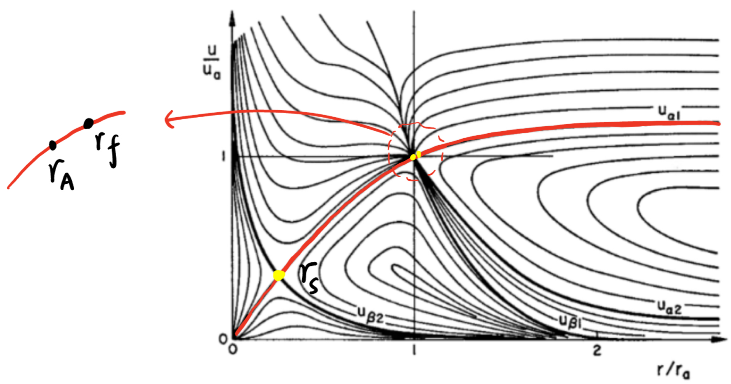

FIG. 1.—Family of solutions of eq. (23) for a given γ and r_a. The solutions passing through the critical points are designated as $u_{\alpha 1}$ and $u_{\alpha 2}$ (with zero pressure at infinity) and $u_{\beta 1}$ and $u_{\beta 2}$ (with non-zero pressure at infinity).

2 critical points :

$$r_c, r_a$$

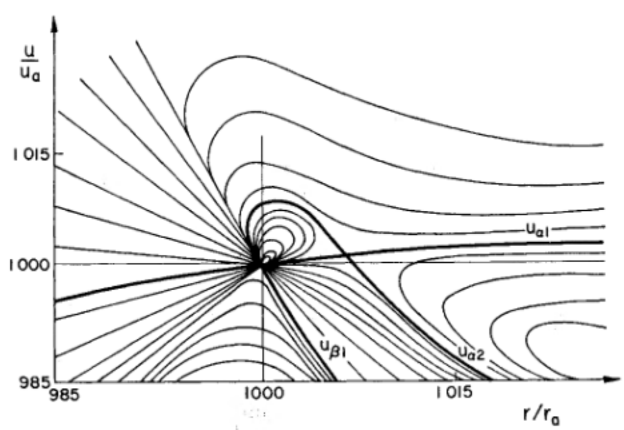

FIG. 2.—Enlargement of part of Fig. 1 near the Alfvénic critical point $r/r_a = 1$

r_f and r_a nearly coincide.

All these singularities are found at points where the flow velocity equals the velocity of a characteristic wave disturbance in the fluid.

r_a: u is equal to the **radial Alfven velocity**.

r_c: u is slightly less than the **pure sound speed**. This is just **Parker's critical point displaced slightly**, because the sound wave for this model is **magneto-acoustic wave**.

r_f : u is very nearly equal to the **Alfven velocity** $[(\mathbf{B}\cdot\mathbf{B}/4\pi\rho)]^{1/2}$ which is slightly larger than radial Alfven velocity.

Propagation of Disturbances and Stability

If magnetic fields are present,

disturbances may travel both in **Sound waves** and **Alfven waves**.

The direction of the magnetic field establishes a preferred axis and thus introduces anisotropy into the fluid. All possible speeds of the longitudinal wavefronts can be determined from the characteristic condition. The characteristic condition is

$$c^2[c^4 - (v_{AT}^2 + v_s^2)c^2 + v_s^2 v_{AN}^2] = 0$$

c : the velocity of the disturbance relative to the fluid.

$v_s = (2\gamma kT/m)^{1/2}$: the local sound velocity.

$v_{AT} = [(B_r^2 + B_\phi^2)/4\pi\rho]^{1/2}$: the local Alfven velocity.

$v_{AN} = (B_r^2/4\pi\rho)^{1/2}$: the local Alfven velocity along the component of the magnetic field normal to the wavefront of interest.

All these quantities vary with r.

In the region between the surface of the Sun and the critical radius r_a, the angle $\delta(r)$ between the magnetic-field vector and the radius vector ranges from a very small value to approximately $\frac{1}{7}$ radian. Thus $B_\phi \approx -\delta B_r$. and to lowest order in δ we solve the characteristic condition for the characteristic disturbances :

$$c = 0, \pm v_s[1 - \frac{v_{AN}^2}{2(v_{AN}^2 - v_s^2)}\delta^2], \pm v_{AN}[1 + \frac{v_{AN}^2}{2(v_{AN}^2 - v_s^2)}\delta^2]$$

This is valid as long as v_{AN} is not close to v_s, which is the case for the Sun.

There are two separate wavefronts traveling with two different possible velocities, a "slow" wave and a "fast" wave.

They are a first-order system of hyperbolic equations, the characteristics can be obtained as :

$$\frac{dr}{dt} = u \pm c$$

It is along these characteristics that all small-amplitude disturbances travel in the solar wind, except those which travel with the solar wind, i.e., which have c = 0.

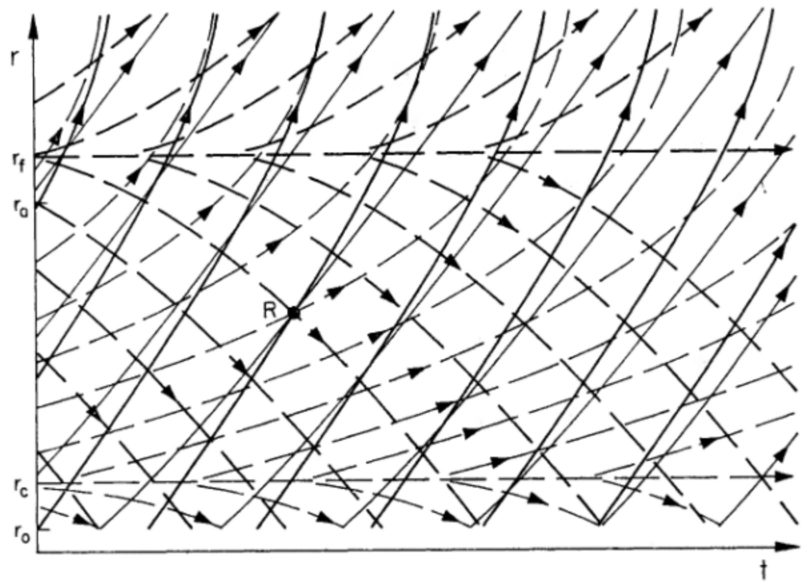

FIG. 3 —A sketch showing the characteristics in the r–t plane The characteristics for the "fast" wave are shown in heavy lines, and for the "slow" wave in light lines The solid lines refer to the solution of eq. (32) with the plus sign and the dashed lines refer to the minus sign; r_0 refers to the base of the corona.

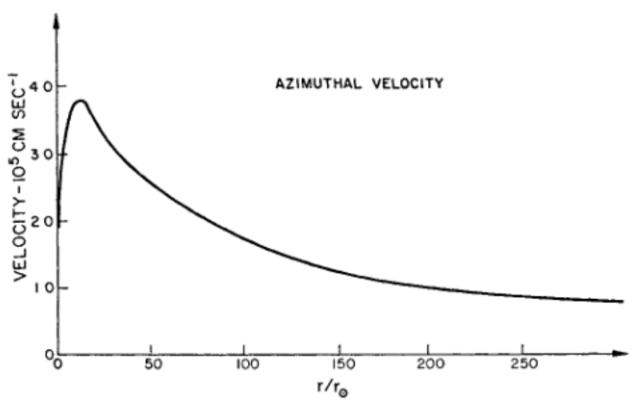

FIG. 4.—Azimuthal velocity of the solar wind

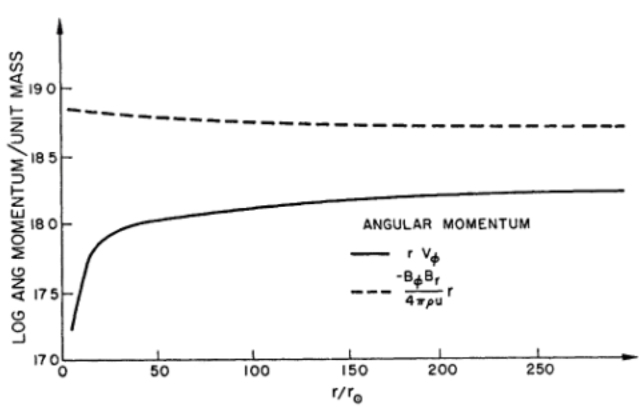

FIG. 5.—Angular momentum and magnetic torque in the solar wind