Supplementary Information for "Causal motifs and existence of endogenous cascades in directed networks with application to company defaults"

I. TWO-EDGE VARIANCE CALCULATION NEEDS A CHECK

$$E[C^{(1,2)}] = N \sum_{k_i,k_o} P(k_i,k_o) c^{(1,2)}_{k_i,k_o}(t) \tag{1}$$

$$V[C^{(1,2)}] = N \sum_{k_i,k_o} P(k_i,k_o) \Big[\left(c^{(1,2)}_{k_i,k_o}(t) - \frac{1}{N} E[C^{(1,2)}(t)] \right)^2 + c^{(1,2)COV}_{k_i,k_o}(t) \Big] \tag{2}$$

$$E[\tilde{C}^{(1,2)}(t)] = N \sum_{k_i,k_o} P(k_i,k_o) \tilde{c}^{(1,2)}_{k_i,k_o}(t) \tag{3}$$

$$V[\tilde{C}^{(1,2)}(t)] = N \sum_{k_i,k_o} P(k_i,k_o) \Big[\left(\tilde{c}^{(1,2)}_{k_i,k_o}(t) - \frac{1}{N} E[\tilde{C}^{(1,2)}(t)] \right)^2 + \tilde{c}^{(1,2)COV}_{k_i,k_o}(t) \Big] \tag{4}$$

One-edge:

$$c^{(1)COV}_{k_i,k_o}(t) = 0 \tag{5}$$

$$\tilde{c}^{(1)COV}_{k_i,k_o}(t) = 0 \tag{6}$$

Two-edge:

$$\begin{aligned} c^{(2)COV}_{k_i,k_o}(t) &= 2 \cdot (c^{VI}_{k_i,k_o}(t) + c^{\Lambda I}_{k_i,k_o}(t) + c^{V\Lambda}_{k_i,k_o}(t)) + c^{VV}_{k_i,k_o}(t)) + c^{\Lambda\Lambda}_{k_i,k_o}(t)) + c^{VVI}_{k_i,k_o}(t)) + c^{\Lambda\Lambda I}_{k_i,k_o}(t)) \\ &\quad + c^{I}_{k_i,k_o}(t)) + c^{V}_{k_i,k_o}(t)) + c^{\Lambda}_{k_i,k_o}(t)) \\ &= \sum_{z_i=0}^{k_i} \sum_{z_o=0}^{k_o} \Big[2 \cdot \left(z_i \binom{z_o}{2} + \binom{z_i}{2}\binom{z_o}{2} + \binom{z_i}{2} z_o \right) + \binom{z_i}{3} + \binom{z_i}{4} + \binom{z_o}{3} + \binom{z_o}{4} \\ &\quad + \binom{z_i}{3} z_o + z_i \binom{z_o}{3} + z_i z_o + \binom{z_i}{2} + \binom{z_o}{2} \Big] \times i^{m}_{k_i,k_o,z_i,z_o}(t) \end{aligned} \tag{7}$$

$$\begin{aligned} \tilde{c}^{(2)COV}_{k_i,k_o}(t) &= 2 \cdot (\tilde{c}^{VI}_{k_i,k_o}(t) + \tilde{c}^{\Lambda I}_{k_i,k_o}(t) + \tilde{c}^{V\Lambda}_{k_i,k_o}(t)) + \tilde{c}^{VV}_{k_i,k_o}(t)) + \tilde{c}^{\Lambda\Lambda}_{k_i,k_o}(t)) + \tilde{c}^{VVI}_{k_i,k_o}(t)) + \tilde{c}^{\Lambda\Lambda I}_{k_i,k_o}(t)) \\ &\quad + \tilde{c}^{I}_{k_i,k_o}(t)) + \tilde{c}^{V}_{k_i,k_o}(t)) + \tilde{c}^{\Lambda}_{k_i,k_o}(t)) \\ &= \sum_{m_i=0}^{k_i} \sum_{m_o=0}^{k_o} \Big[2 \cdot \left(\frac{1}{12} m_i \binom{m_o}{2} + \frac{1}{30} \binom{m_i}{2}\binom{m_o}{2} + \frac{1}{12} \binom{m_i}{2} m_o \right) \\ &\quad + \frac{1}{4}\binom{m_i}{3} + \frac{1}{5}\binom{m_i}{4} + \frac{1}{4}\binom{m_o}{3} + \frac{1}{5}\binom{m_o}{4} \\ &\quad + \frac{1}{20}\binom{m_i}{3} m_o + \frac{1}{20} m_i \binom{m_o}{3} + \frac{1}{6} m_i m_o + \frac{1}{3}\binom{m_i}{2} + \frac{1}{3}\binom{m_o}{2} \Big] \times i^{z}_{k_i,k_o,m_i,m_o}(t) \end{aligned} \tag{8}$$

II. TEMPORAL EVOLUTION

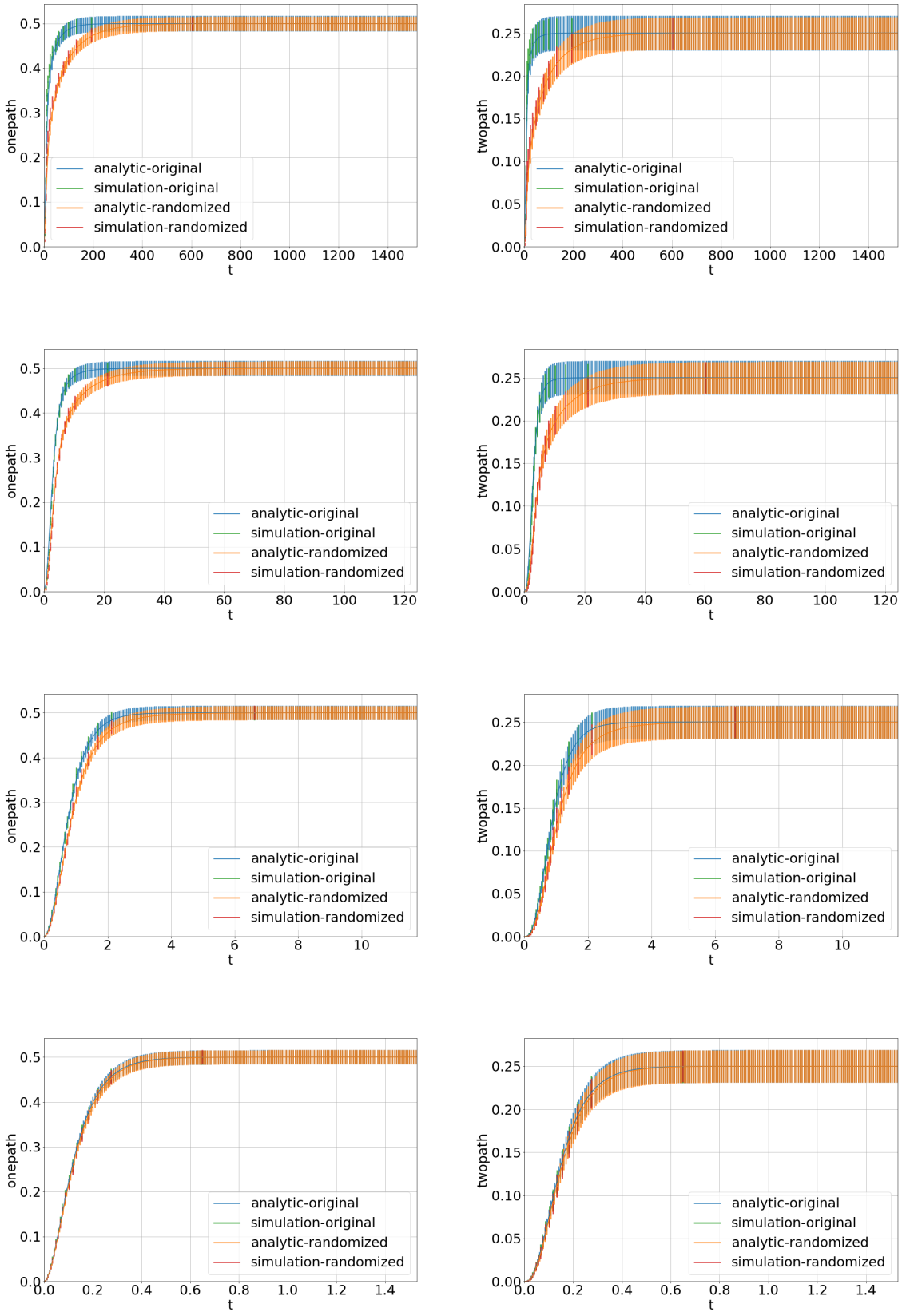


FIG. 1: One-edge and two-edge statistics for $\zeta = 0.01, 0.1, 1, 10$ SI process on an Erdős-Rényi graph with $N = 1000, \langle k \rangle = 2$. x-axis shows the time, while y-axis represents the relative frequency of a test statistic

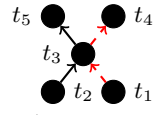
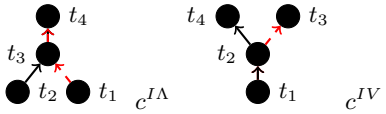
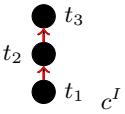
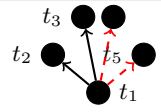
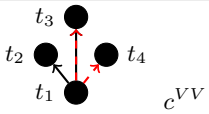
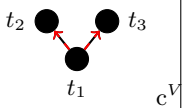
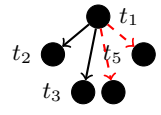
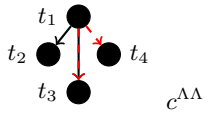
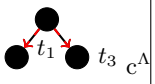
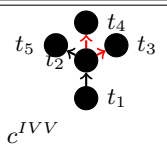
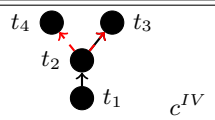
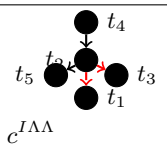
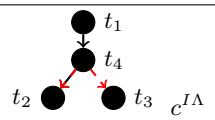
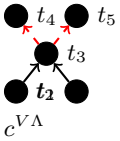
Overlap	One node	Two nodes	Three nodes
I-I			
V-V			
$\Lambda-\Lambda$			
I-V			
I-Λ			
V-Λ			

TABLE I: Table with all possible casual motifs up to order 3.

III. Z-SCORE

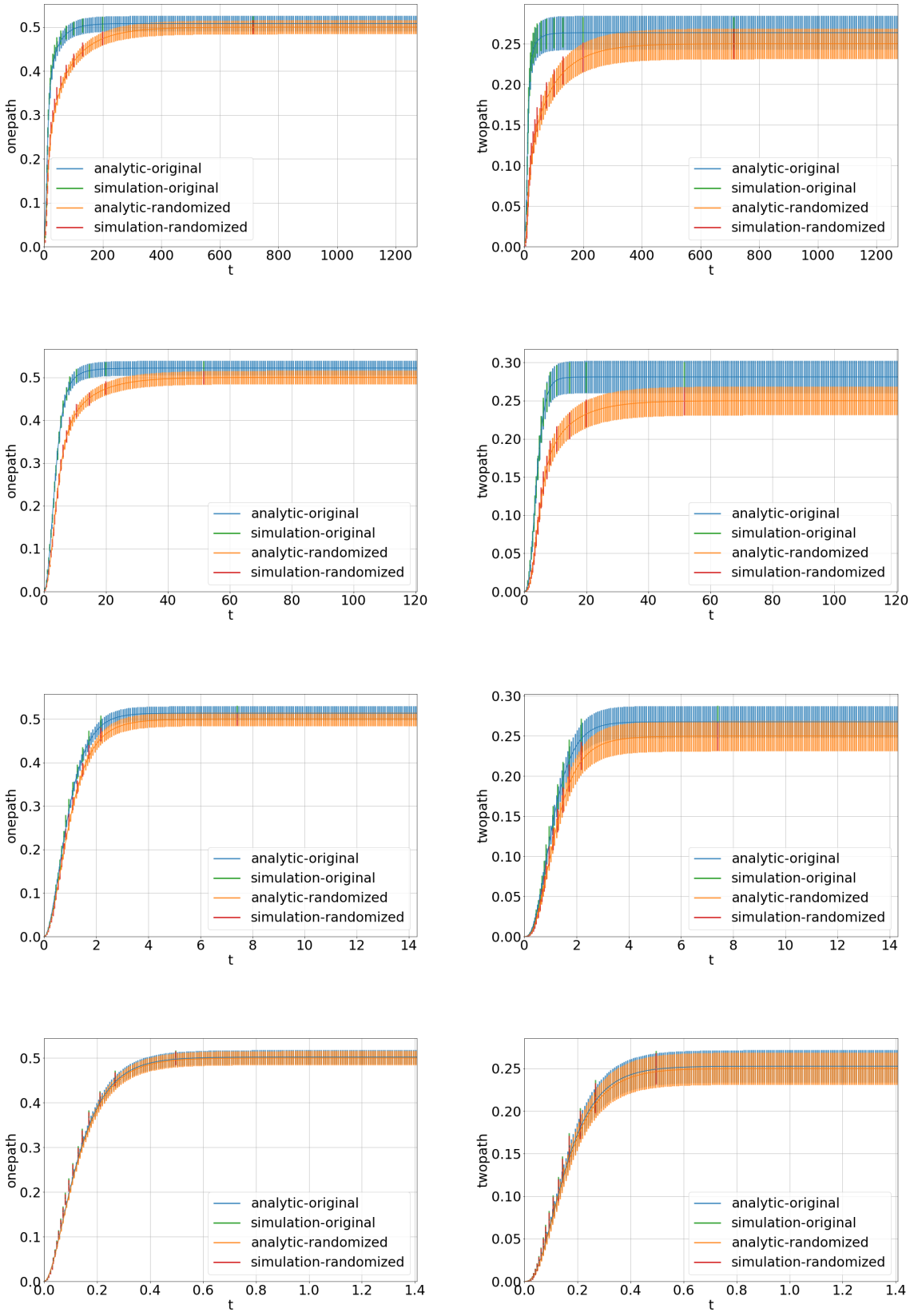


FIG. 2: One-edge and two-edge statistics for $\zeta = 0.01, 0.1, 1, 10$ voter model process on an Erdős-Rényi graph with $N = 1000, \langle k \rangle = 2$. x-axis shows the time, while y-axis represents the relative frequency of a test statistic

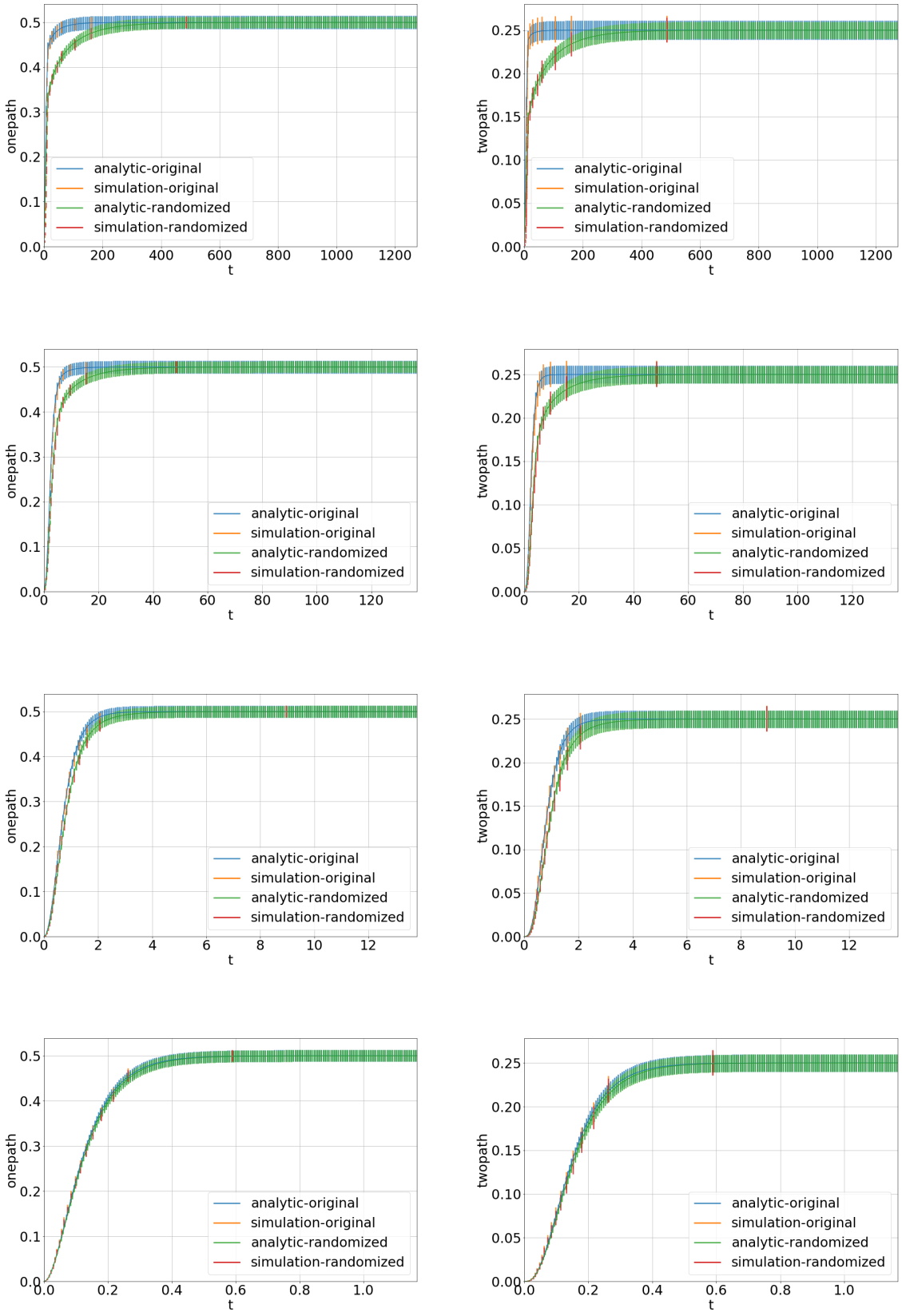


FIG. 3: One-edge and two-edge statistics for $\zeta = 0.01, 0.1, 1, 10$ SI process on an Erdős-Rényi graph with $N = 1000, \langle k \rangle = 3$. x-axis shows the time, while y-axis represents the relative frequency of a test statistic

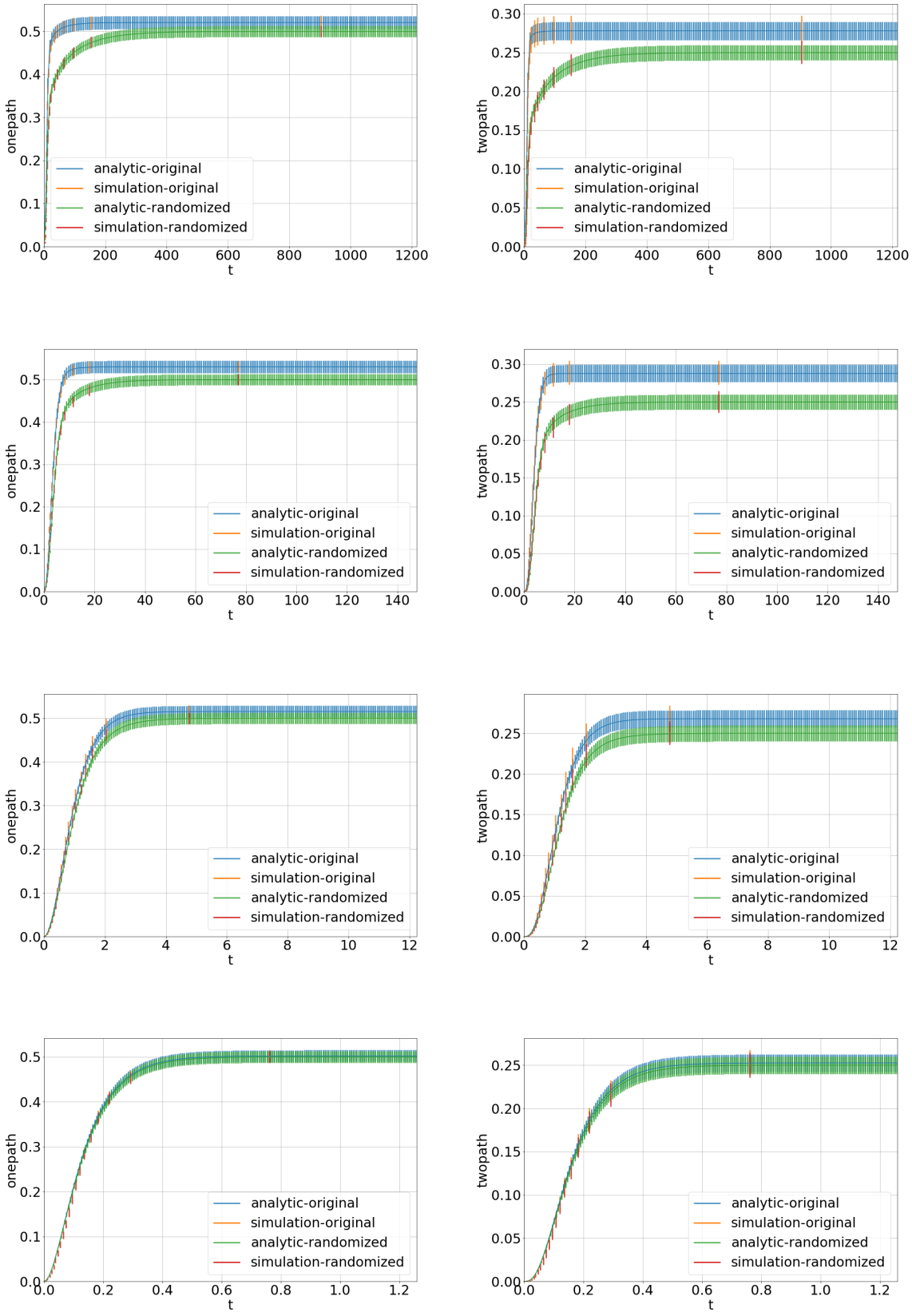


FIG. 4: One-edge and two-edge statistics for $\zeta = 0.01, 0.1, 1, 10$ voter model process on an Erdős-Rényi graph with $N = 1000, \langle k \rangle = 3$. x-axis shows the time, while y-axis represents the relative frequency of a test statistic

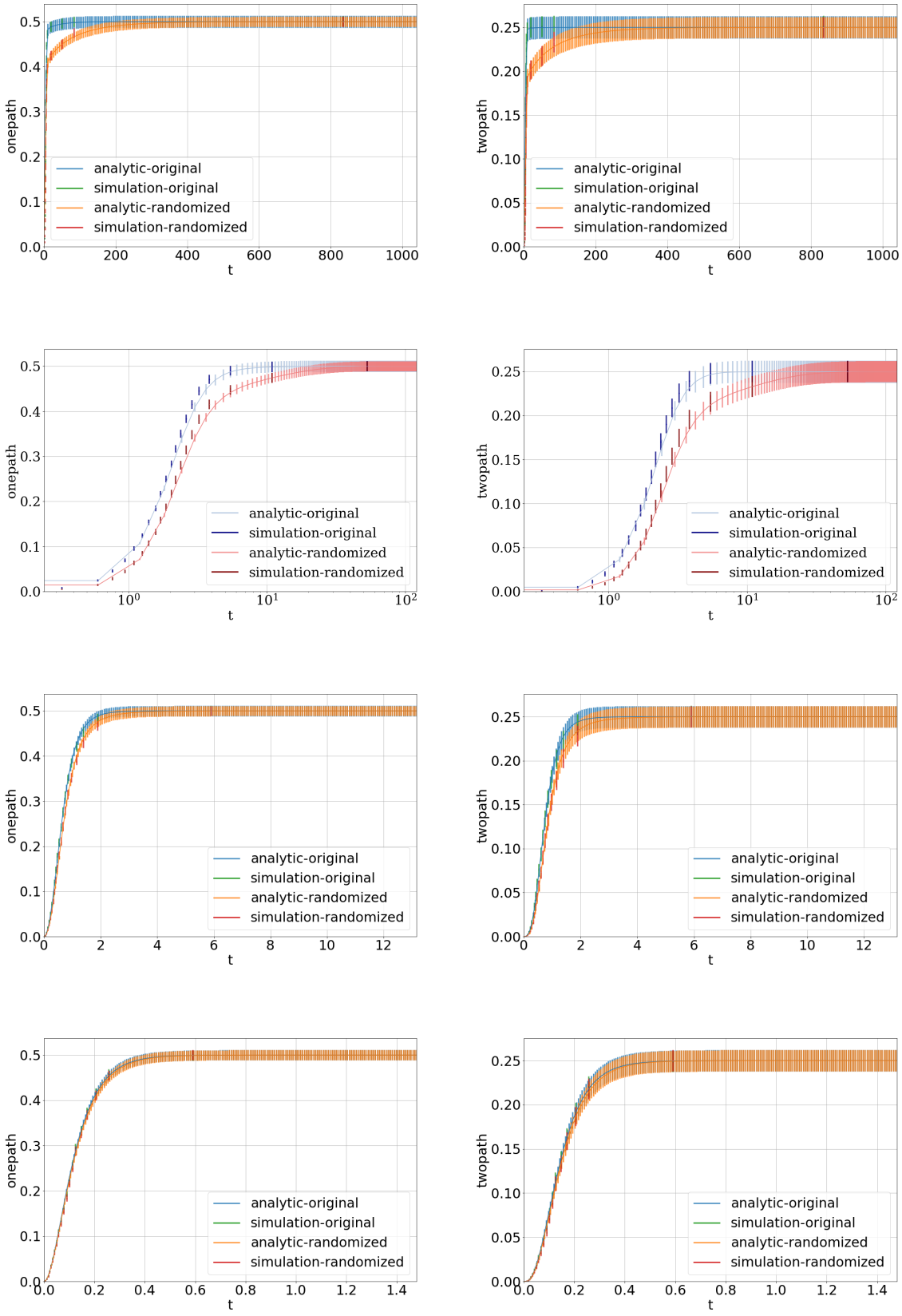


FIG. 5: One-edge and two-edge statistics for $\zeta = 0.01, 0.1, 1, 10$ SI process on an Erdős-Rényi graph with $N = 1000, \langle k \rangle = 4$. x-axis shows the time, while y-axis represents the relative frequency of a test statistic

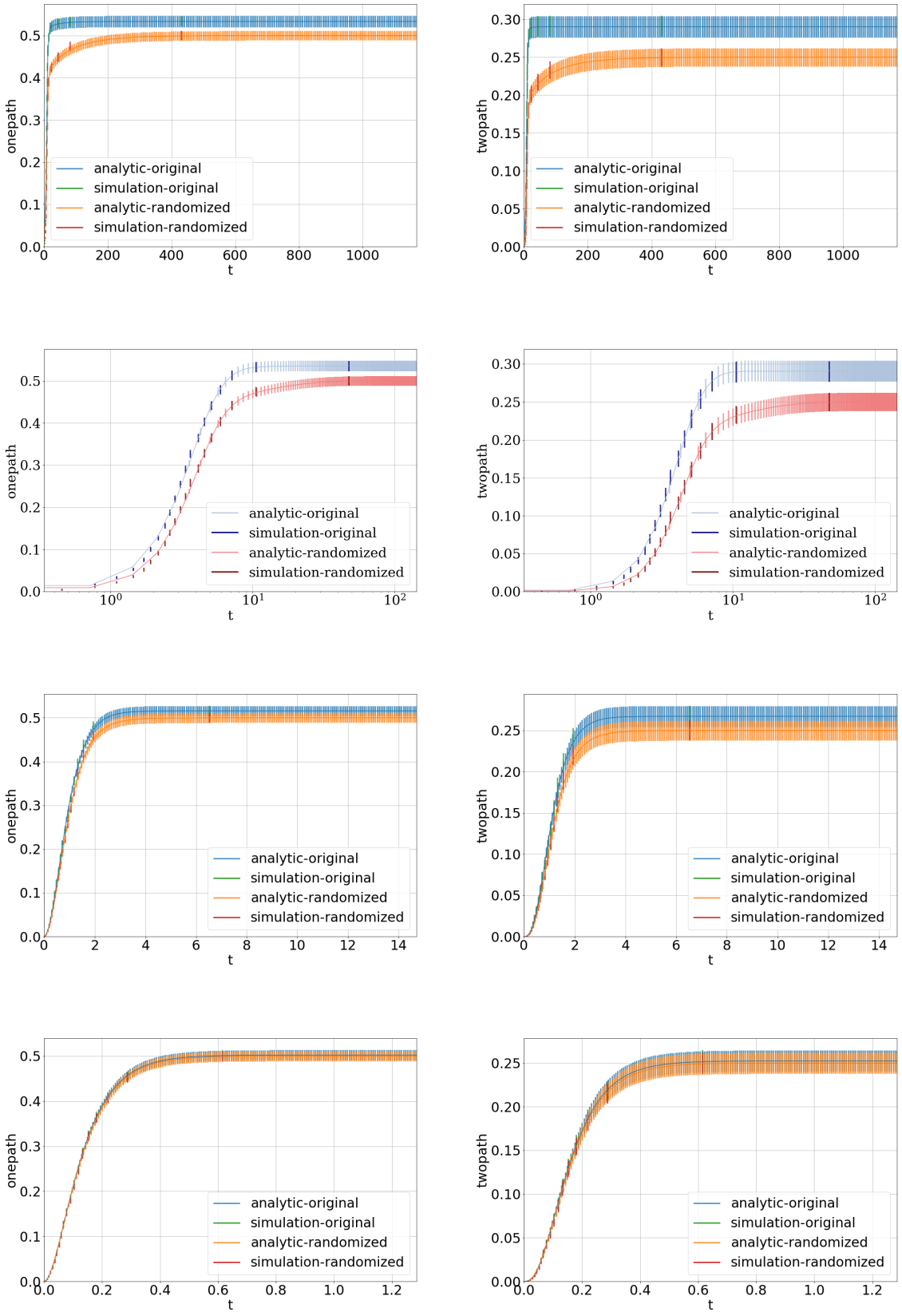


FIG. 6: One-edge and two-edge statistics for $\zeta = 0.01, 0.1, 1, 10$ voter model process on an Erdős-Rényi graph with $N = 1000, \langle k \rangle = 4$. x-axis shows the time, while y-axis represents the relative frequency of a test statistic

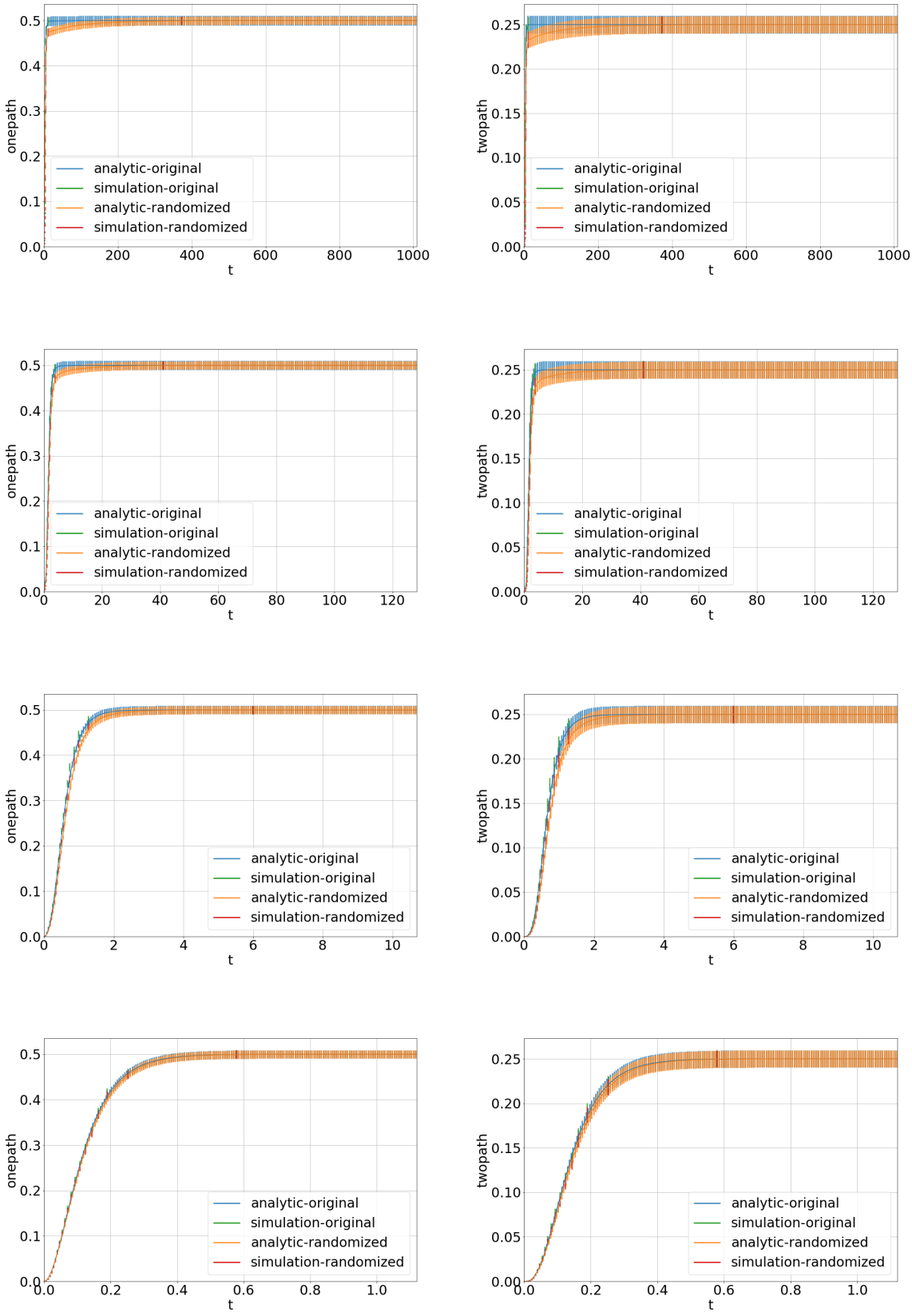


FIG. 7: One-edge and two-edge statistics for $\zeta = 0.01, 0.1, 1, 10$ SI process on an Erdős-Rényi graph with $N = 1000, \langle k \rangle = 6$. x-axis shows the time, while y-axis represents the relative frequency of a test statistic

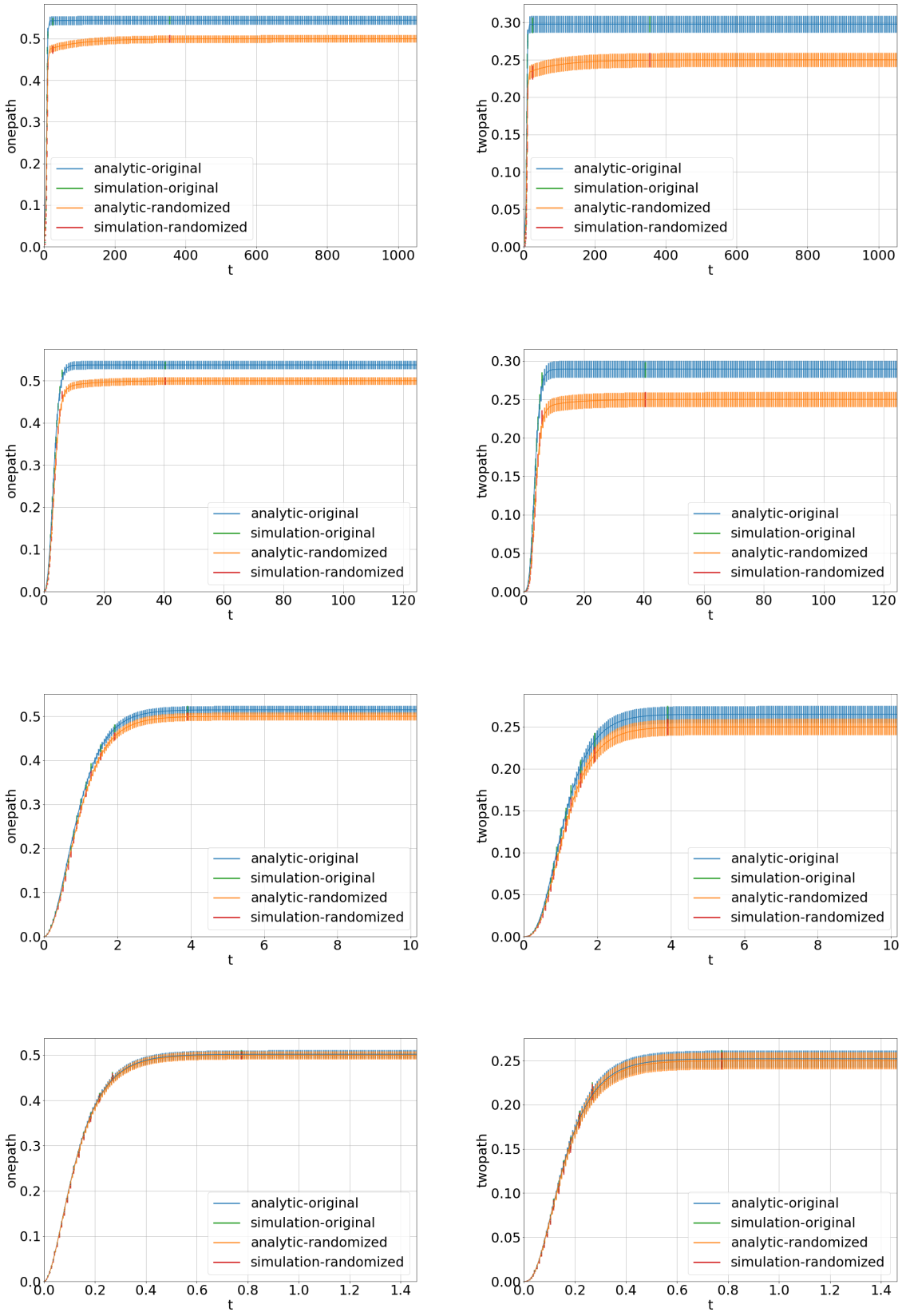


FIG. 8: One-edge and two-edge statistics for $\zeta = 0.01, 0.1, 1, 10$ voter model process on an Erdős-Rényi graph with $N = 1000, \langle k \rangle = 6$. x-axis shows the time, while y-axis represents the relative frequency of a test statistic

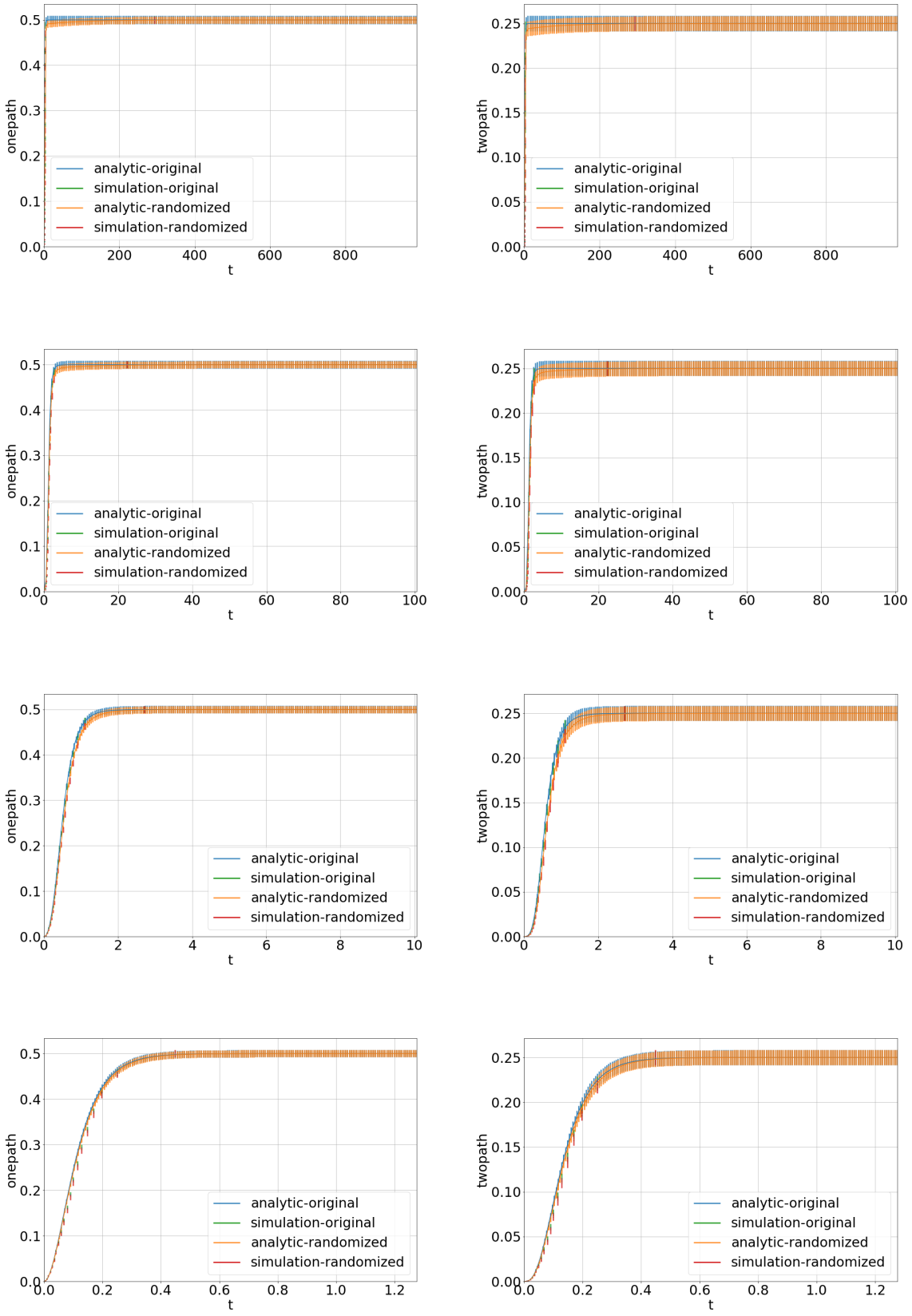


FIG. 9: One-edge and two-edge statistics for $\zeta = 0.01, 0.1, 1, 10$ SI process on an Erdős-Rényi graph with $N = 1000, \langle k \rangle = 8$. x-axis shows the time, while y-axis represents the relative frequency of a test statistic

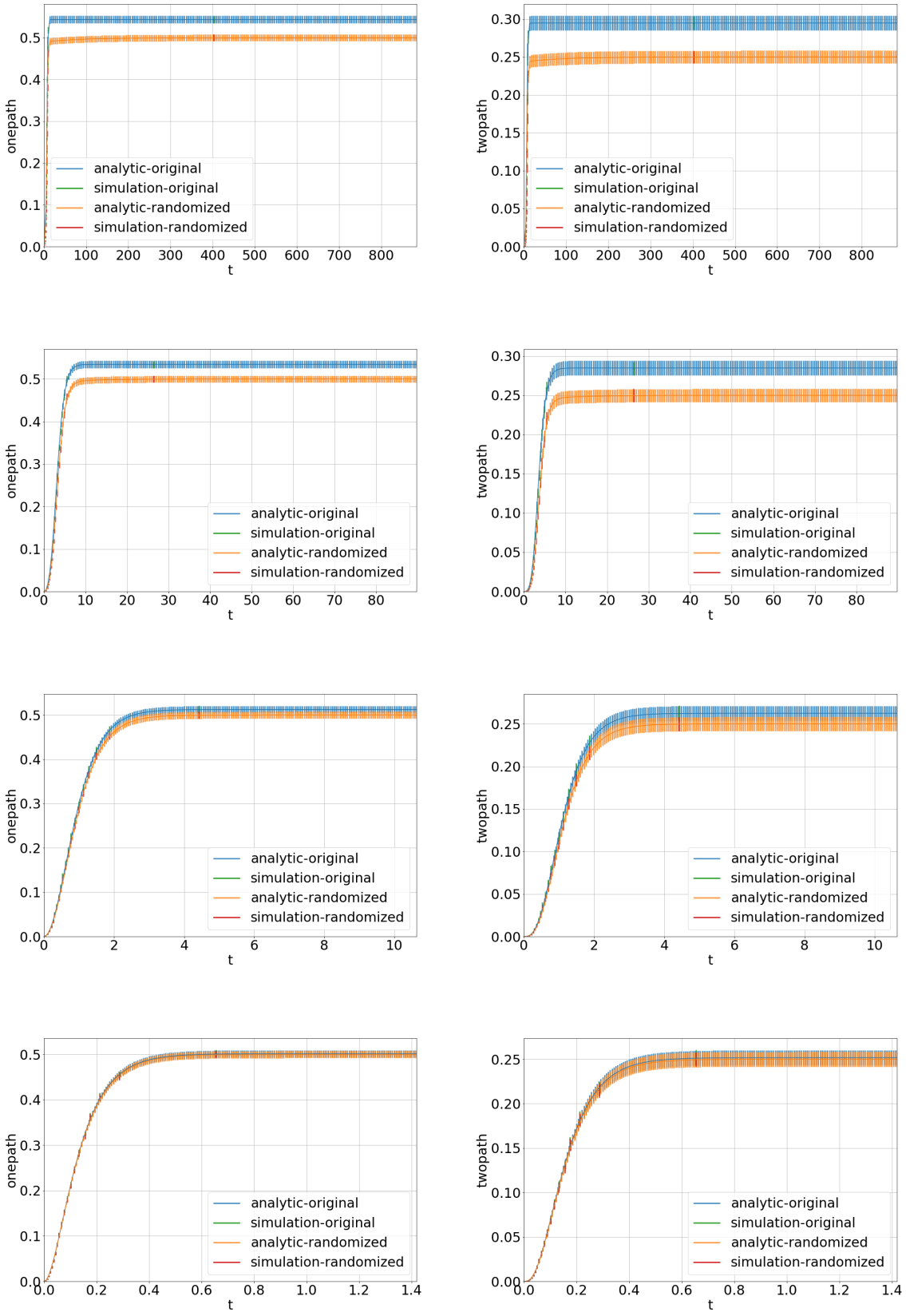


FIG. 10: One-edge and two-edge statistics for $\zeta = 0.01, 0.1, 1, 10$ voter model process on an Erdős-Rényi graph with $N = 1000, \langle k \rangle = 8$. x-axis shows the time, while y-axis represents the relative frequency of a test statistic

⟨ k ⟩ / percentage	2.0	3.0	4.0	6.0	8.0
5.0	4.11 (1.22)	4.66 (1.09)	5.01 (1.04)	5.12 (0.99)	5.14 (0.92)
10.0	6.35 (1.22)	7.0 (1.07)	7.15 (1.04)	7.01 (0.99)	6.9 (0.95)
15.0	7.77 (1.22)	8.5 (1.06)	8.58 (1.05)	8.04 (1.03)	7.63 (1.01)
20.0	8.8 (1.25)	9.45 (1.16)	9.41 (1.12)	8.55 (1.06)	8.14 (1.03)
25.0	9.49 (1.31)	10.11 (1.17)	9.89 (1.12)	8.88 (1.07)	8.25 (1.07)
30.0	9.9 (1.36)	10.6 (1.22)	10.09 (1.15)	9.04 (1.11)	8.39 (1.05)
35.0	10.05 (1.38)	10.95 (1.22)	10.21 (1.13)	9.08 (1.11)	8.37 (1.1)
40.0	9.93 (1.36)	11.12 (1.25)	10.28 (1.15)	9.06 (1.13)	8.23 (1.11)
45.0	9.72 (1.39)	11.15 (1.23)	10.35 (1.18)	9.06 (1.14)	8.12 (1.09)
50.0	9.4 (1.38)	11.16 (1.27)	10.16 (1.13)	8.87 (1.11)	7.83 (1.12)
55.0	9.01 (1.38)	11.09 (1.3)	9.99 (1.16)	8.56 (1.09)	7.53 (1.09)
60.0	8.5 (1.34)	10.61 (1.21)	9.69 (1.18)	8.27 (1.08)	7.27 (1.08)
65.0	7.91 (1.28)	10.12 (1.21)	9.25 (1.13)	7.93 (1.09)	6.92 (1.07)
70.0	7.2 (1.21)	9.26 (1.34)	8.77 (1.15)	7.44 (1.05)	6.5 (1.06)
75.0	6.33 (1.2)	8.05 (1.28)	8.11 (1.09)	6.85 (1.01)	6.02 (1.07)
80.0	5.33 (1.14)	6.71 (1.17)	7.32 (1.09)	6.22 (1.03)	5.46 (1.05)
85.0	4.15 (1.11)	5.29 (1.13)	5.98 (1.2)	5.29 (1.01)	4.69 (1.05)
90.0	2.85 (1.07)	3.72 (1.11)	4.16 (1.16)	4.2 (1.02)	3.7 (1.03)
95.0	1.44 (1.04)	1.94 (1.06)	2.16 (1.14)	2.57 (1.07)	2.38 (1.03)
100.0	-0.08 (1.02)	0.06 (1.04)	-0.03 (1.1)	-0.14 (1.05)	-0.05 (1.04)

TABLE II: z-scores of one-edge statistic for an SI process with $\zeta = 0.01$ on a network of $N = 1000$ vertices.

⟨ k ⟩ / percentage	2.0	3.0	4.0	6.0	8.0
5.0	1.87 (1.31)	2.2 (1.24)	2.49 (1.22)	2.8 (1.17)	2.96 (1.15)
10.0	3.25 (1.27)	3.71 (1.18)	4.1 (1.19)	4.42 (1.07)	4.49 (1.09)
15.0	4.39 (1.23)	4.88 (1.18)	5.24 (1.12)	5.46 (1.09)	5.52 (1.07)
20.0	5.23 (1.24)	5.85 (1.15)	6.06 (1.15)	6.17 (1.11)	6.13 (1.08)
25.0	5.84 (1.22)	6.44 (1.16)	6.59 (1.16)	6.64 (1.1)	6.47 (1.06)
30.0	6.4 (1.22)	6.98 (1.18)	7.08 (1.15)	7.05 (1.14)	6.69 (1.07)
35.0	6.87 (1.16)	7.43 (1.18)	7.52 (1.18)	7.24 (1.14)	6.78 (1.07)
40.0	7.15 (1.19)	7.74 (1.15)	7.71 (1.16)	7.39 (1.12)	6.89 (1.07)
45.0	7.36 (1.18)	7.85 (1.15)	7.8 (1.15)	7.45 (1.11)	6.86 (1.04)
50.0	7.4 (1.17)	7.96 (1.19)	7.82 (1.11)	7.4 (1.13)	6.7 (1.07)
55.0	7.33 (1.16)	7.92 (1.18)	7.74 (1.11)	7.31 (1.05)	6.56 (1.06)
60.0	7.24 (1.14)	7.78 (1.15)	7.64 (1.15)	7.13 (1.04)	6.41 (1.05)
65.0	6.99 (1.2)	7.59 (1.17)	7.36 (1.07)	6.86 (1.04)	6.16 (1.07)
70.0	6.51 (1.15)	7.19 (1.11)	7.03 (1.07)	6.56 (1.04)	5.83 (1.04)
75.0	5.93 (1.17)	6.73 (1.08)	6.64 (1.08)	6.12 (1.08)	5.41 (1.03)
80.0	5.1 (1.12)	6.03 (1.1)	5.99 (1.06)	5.59 (1.03)	4.96 (1.05)
85.0	4.05 (1.1)	5.02 (1.11)	5.18 (1.05)	4.83 (1.0)	4.26 (0.99)
90.0	2.85 (1.05)	3.6 (1.06)	4.0 (1.07)	3.84 (0.99)	3.43 (0.98)
95.0	1.49 (1.02)	1.91 (1.04)	2.22 (1.07)	2.4 (1.0)	2.21 (0.97)
100.0	-0.05 (1.0)	0.04 (1.0)	0.06 (1.05)	-0.05 (1.01)	0.01 (1.01)

TABLE III: z-scores of one-edge statistic for an SI process with $\zeta = 0.1$ on a network of $N = 1000$ vertices.

percentage \ ⟨ k ⟩	2.0	3.0	4.0	6.0	8.0
5.0	0.4 (1.15)	0.55 (1.2)	0.6 (1.15)	0.71 (1.15)	0.82 (1.15)
10.0	0.78 (1.11)	1.02 (1.17)	1.13 (1.12)	1.33 (1.12)	1.46 (1.12)
15.0	1.11 (1.13)	1.43 (1.16)	1.55 (1.11)	1.86 (1.08)	1.99 (1.1)
20.0	1.49 (1.13)	1.75 (1.14)	1.93 (1.09)	2.23 (1.1)	2.43 (1.07)
25.0	1.77 (1.07)	2.05 (1.11)	2.27 (1.1)	2.57 (1.12)	2.83 (1.05)
30.0	2.05 (1.07)	2.34 (1.1)	2.55 (1.1)	2.82 (1.1)	3.1 (1.08)
35.0	2.26 (1.1)	2.61 (1.1)	2.86 (1.12)	3.07 (1.05)	3.36 (1.05)
40.0	2.45 (1.09)	2.79 (1.1)	3.1 (1.12)	3.27 (1.09)	3.52 (1.06)
45.0	2.68 (1.08)	3.0 (1.08)	3.24 (1.08)	3.45 (1.08)	3.62 (1.05)
50.0	2.88 (1.04)	3.13 (1.08)	3.4 (1.08)	3.56 (1.04)	3.72 (1.04)
55.0	2.96 (1.04)	3.26 (1.08)	3.47 (1.07)	3.65 (1.02)	3.75 (1.06)
60.0	3.02 (1.07)	3.3 (1.07)	3.48 (1.07)	3.67 (1.05)	3.7 (1.04)
65.0	3.02 (1.05)	3.32 (1.05)	3.45 (1.05)	3.68 (1.05)	3.73 (1.03)
70.0	2.98 (1.04)	3.26 (1.02)	3.42 (1.04)	3.59 (1.05)	3.61 (1.05)
75.0	2.89 (1.06)	3.15 (1.03)	3.33 (1.06)	3.43 (1.05)	3.48 (1.06)
80.0	2.68 (1.03)	2.96 (1.02)	3.13 (1.05)	3.21 (1.05)	3.23 (1.04)
85.0	2.36 (1.04)	2.61 (1.03)	2.77 (1.04)	2.84 (1.03)	2.87 (1.05)
90.0	1.9 (1.03)	2.1 (1.02)	2.23 (1.02)	2.35 (1.02)	2.35 (1.03)
95.0	1.18 (1.01)	1.33 (1.03)	1.44 (1.03)	1.56 (1.02)	1.58 (1.04)
100.0	0.02 (1.03)	-0.02 (1.04)	-0.04 (1.01)	0.05 (0.99)	0.07 (1.03)

TABLE IV: z-scores of one-edge statistic for an SI process with $\zeta = 1.0$ on a network of $N = 1000$ vertices.

percentage \ ⟨ k ⟩	2.0	3.0	4.0	6.0	8.0
5.0	0.0 (1.03)	0.11 (1.04)	0.03 (1.06)	0.09 (1.01)	0.07 (1.01)
10.0	0.02 (1.03)	0.16 (1.04)	0.13 (1.04)	0.21 (1.07)	0.2 (1.05)
15.0	0.09 (1.06)	0.21 (1.05)	0.17 (1.06)	0.28 (1.05)	0.29 (1.04)
20.0	0.17 (1.04)	0.27 (1.01)	0.26 (1.05)	0.35 (1.05)	0.38 (1.04)
25.0	0.19 (1.07)	0.32 (1.02)	0.31 (1.05)	0.44 (1.01)	0.48 (1.05)
30.0	0.25 (1.06)	0.36 (1.02)	0.39 (1.05)	0.5 (1.0)	0.55 (1.08)
35.0	0.29 (1.05)	0.44 (1.0)	0.45 (1.03)	0.58 (1.02)	0.61 (1.06)
40.0	0.34 (1.06)	0.46 (1.0)	0.5 (1.03)	0.62 (1.05)	0.66 (1.03)
45.0	0.36 (1.06)	0.48 (1.0)	0.54 (1.02)	0.68 (1.04)	0.71 (1.03)
50.0	0.38 (1.09)	0.51 (1.01)	0.6 (1.03)	0.69 (1.03)	0.78 (1.02)
55.0	0.4 (1.07)	0.54 (0.99)	0.64 (1.02)	0.75 (1.03)	0.83 (1.0)
60.0	0.43 (1.06)	0.59 (0.99)	0.67 (1.04)	0.77 (1.01)	0.87 (1.0)
65.0	0.45 (1.04)	0.61 (0.98)	0.67 (1.01)	0.79 (1.03)	0.88 (0.99)
70.0	0.44 (1.01)	0.64 (0.99)	0.7 (1.01)	0.82 (1.04)	0.92 (1.0)
75.0	0.43 (1.0)	0.63 (0.97)	0.7 (0.99)	0.79 (1.02)	0.92 (1.02)
80.0	0.41 (1.0)	0.61 (0.99)	0.67 (1.01)	0.75 (1.02)	0.87 (1.0)
85.0	0.35 (1.0)	0.54 (1.0)	0.63 (1.02)	0.68 (1.02)	0.79 (1.0)
90.0	0.3 (1.0)	0.47 (1.0)	0.55 (1.03)	0.57 (1.02)	0.68 (1.02)
95.0	0.18 (1.01)	0.31 (1.01)	0.39 (1.01)	0.37 (1.03)	0.45 (1.01)
100.0	-0.09 (1.01)	-0.01 (1.01)	0.03 (1.04)	-0.06 (1.04)	-0.02 (1.01)

TABLE V: z-scores of one-edge statistic for an SI process with $\zeta = 10.0$ on a network of $N = 1000$ vertices.

percentage \ ⟨ k ⟩	2.0	3.0	4.0	6.0	8.0
5.0	-0.02 (1.01)	0.04 (1.04)	0.06 (1.01)	0.03 (1.03)	0.02 (1.01)
10.0	-0.0 (1.01)	0.07 (1.04)	0.07 (1.0)	0.03 (1.06)	0.01 (1.04)
15.0	-0.02 (1.05)	0.09 (1.04)	0.07 (1.04)	0.04 (1.02)	0.05 (1.03)
20.0	-0.02 (1.06)	0.1 (1.06)	0.07 (1.08)	0.05 (1.04)	0.08 (1.02)
25.0	-0.03 (1.04)	0.09 (1.05)	0.06 (1.08)	0.05 (1.04)	0.11 (1.02)
30.0	-0.01 (1.02)	0.11 (1.06)	0.07 (1.05)	0.07 (1.03)	0.14 (1.01)
35.0	0.0 (1.01)	0.1 (1.04)	0.08 (1.05)	0.1 (1.0)	0.15 (0.99)
40.0	0.0 (1.0)	0.09 (1.02)	0.07 (1.06)	0.11 (1.02)	0.15 (1.0)
45.0	0.0 (1.0)	0.11 (1.03)	0.05 (1.03)	0.11 (1.05)	0.13 (1.0)
50.0	0.02 (1.02)	0.1 (1.01)	0.07 (1.04)	0.11 (1.02)	0.14 (1.02)
55.0	0.04 (1.02)	0.11 (0.99)	0.08 (1.02)	0.12 (1.0)	0.14 (1.03)
60.0	0.05 (1.03)	0.1 (1.0)	0.08 (1.0)	0.1 (0.99)	0.14 (1.02)
65.0	0.04 (1.03)	0.1 (1.02)	0.08 (1.04)	0.1 (1.01)	0.13 (1.04)
70.0	0.06 (1.05)	0.1 (1.03)	0.08 (1.02)	0.11 (1.03)	0.13 (1.04)
75.0	0.05 (1.05)	0.1 (1.04)	0.07 (1.0)	0.12 (1.02)	0.12 (1.04)
80.0	0.05 (1.07)	0.11 (1.05)	0.06 (1.0)	0.12 (1.01)	0.12 (1.02)
85.0	0.04 (1.06)	0.1 (1.03)	0.05 (1.02)	0.08 (1.02)	0.11 (1.03)
90.0	0.03 (1.05)	0.07 (1.01)	0.05 (1.0)	0.08 (1.03)	0.07 (1.02)
95.0	0.03 (1.05)	0.04 (1.01)	0.01 (1.01)	0.07 (1.02)	0.04 (1.03)
100.0	0.01 (1.04)	0.03 (1.04)	-0.03 (1.01)	0.02 (1.03)	-0.03 (1.01)

TABLE VI: z-scores of one-edge statistic for an SI process with $\zeta = 100.0$on a network of $N = 1000$ vertices.

IV. KS-TEST

⟨ k ⟩ / percentage	2.0	3.0	4.0	6.0	8.0
5.0	4.56 (2.12)	5.14 (1.9)	5.46 (1.86)	5.53 (1.75)	5.52 (1.62)
10.0	7.09 (2.06)	7.75 (1.81)	7.77 (1.76)	7.5 (1.64)	7.43 (1.5)
15.0	8.62 (1.98)	9.33 (1.73)	9.21 (1.64)	8.47 (1.58)	8.1 (1.51)
20.0	9.64 (1.87)	10.19 (1.72)	9.9 (1.6)	9.04 (1.53)	8.53 (1.44)
25.0	10.24 (1.88)	10.7 (1.7)	10.36 (1.56)	9.2 (1.5)	8.61 (1.42)
30.0	10.58 (1.86)	11.06 (1.67)	10.47 (1.57)	9.29 (1.46)	8.68 (1.38)
35.0	10.52 (1.83)	11.25 (1.62)	10.43 (1.49)	9.25 (1.44)	8.51 (1.35)
40.0	10.25 (1.78)	11.17 (1.64)	10.33 (1.48)	9.1 (1.41)	8.3 (1.35)
45.0	9.76 (1.77)	10.95 (1.56)	10.18 (1.47)	8.97 (1.41)	8.11 (1.29)
50.0	9.08 (1.68)	10.63 (1.57)	9.87 (1.38)	8.7 (1.32)	7.74 (1.3)
55.0	8.35 (1.57)	10.3 (1.6)	9.54 (1.41)	8.28 (1.32)	7.37 (1.28)
60.0	7.49 (1.49)	9.5 (1.45)	9.06 (1.39)	7.86 (1.25)	7.04 (1.23)
65.0	6.63 (1.42)	8.66 (1.42)	8.44 (1.33)	7.45 (1.25)	6.63 (1.23)
70.0	5.71 (1.32)	7.61 (1.46)	7.71 (1.35)	6.84 (1.17)	6.14 (1.19)
75.0	4.73 (1.28)	6.34 (1.38)	6.79 (1.26)	6.14 (1.13)	5.58 (1.19)
80.0	3.75 (1.19)	5.02 (1.28)	5.73 (1.24)	5.36 (1.13)	4.93 (1.16)
85.0	2.75 (1.14)	3.76 (1.22)	4.36 (1.28)	4.38 (1.11)	4.11 (1.14)
90.0	1.77 (1.1)	2.52 (1.17)	2.85 (1.22)	3.23 (1.1)	3.08 (1.11)
95.0	0.81 (1.05)	1.27 (1.11)	1.39 (1.16)	1.71 (1.1)	1.79 (1.09)
100.0	-0.11 (1.01)	0.08 (1.07)	-0.02 (1.12)	-0.11 (1.05)	-0.06 (1.06)

TABLE VII: z-scores of two-edge statistic for an SI process with $\zeta = 0.01$ on a network of $N = 1000$ vertices.

⟨ k ⟩ / percentage	2.0	3.0	4.0	6.0	8.0
5.0	1.54 (1.98)	1.89 (1.95)	2.25 (1.89)	2.66 (1.86)	2.85 (1.8)
10.0	2.93 (1.86)	3.47 (1.82)	3.9 (1.78)	4.34 (1.67)	4.47 (1.62)
15.0	4.03 (1.81)	4.67 (1.76)	5.01 (1.67)	5.35 (1.61)	5.47 (1.53)
20.0	4.93 (1.75)	5.63 (1.65)	5.87 (1.57)	6.09 (1.56)	6.1 (1.46)
25.0	5.54 (1.67)	6.2 (1.61)	6.43 (1.57)	6.52 (1.49)	6.43 (1.38)
30.0	6.05 (1.66)	6.67 (1.56)	6.87 (1.53)	6.87 (1.49)	6.62 (1.37)
35.0	6.48 (1.57)	7.05 (1.52)	7.29 (1.52)	7.04 (1.46)	6.68 (1.34)
40.0	6.65 (1.54)	7.24 (1.5)	7.43 (1.48)	7.16 (1.38)	6.75 (1.3)
45.0	6.77 (1.48)	7.29 (1.47)	7.43 (1.4)	7.14 (1.37)	6.69 (1.24)
50.0	6.74 (1.43)	7.28 (1.44)	7.39 (1.33)	7.09 (1.36)	6.48 (1.25)
55.0	6.55 (1.41)	7.18 (1.42)	7.21 (1.3)	6.93 (1.25)	6.3 (1.24)
60.0	6.31 (1.36)	6.92 (1.35)	7.02 (1.34)	6.7 (1.2)	6.1 (1.2)
65.0	5.89 (1.33)	6.61 (1.35)	6.67 (1.23)	6.36 (1.18)	5.79 (1.19)
70.0	5.3 (1.28)	6.11 (1.29)	6.23 (1.21)	5.98 (1.16)	5.42 (1.15)
75.0	4.61 (1.25)	5.51 (1.24)	5.72 (1.19)	5.48 (1.16)	4.97 (1.13)
80.0	3.78 (1.2)	4.72 (1.22)	4.97 (1.14)	4.87 (1.11)	4.45 (1.12)
85.0	2.83 (1.13)	3.69 (1.19)	4.06 (1.12)	4.06 (1.07)	3.74 (1.07)
90.0	1.86 (1.07)	2.49 (1.13)	2.88 (1.09)	3.05 (1.05)	2.87 (1.03)
95.0	0.92 (1.05)	1.24 (1.08)	1.47 (1.05)	1.71 (1.03)	1.7 (1.0)
100.0	-0.02 (1.01)	0.05 (1.04)	0.05 (1.01)	-0.02 (1.0)	-0.01 (1.0)

TABLE VIII: z-scores of two-edge statistic for an SI process with $\zeta = 0.1$ on a network of $N = 1000$ vertices.

⟨ k ⟩ / percentage	2.0	3.0	4.0	6.0	8.0
5.0	0.17 (1.39)	0.32 (1.46)	0.39 (1.46)	0.55 (1.41)	0.64 (1.39)
10.0	0.48 (1.34)	0.69 (1.34)	0.8 (1.35)	1.07 (1.36)	1.25 (1.33)
15.0	0.74 (1.3)	1.09 (1.34)	1.22 (1.33)	1.58 (1.3)	1.76 (1.31)
20.0	1.07 (1.32)	1.37 (1.34)	1.58 (1.29)	1.98 (1.31)	2.19 (1.27)
25.0	1.31 (1.27)	1.66 (1.29)	1.93 (1.27)	2.32 (1.34)	2.57 (1.21)
30.0	1.6 (1.29)	1.94 (1.28)	2.21 (1.28)	2.56 (1.3)	2.87 (1.24)
35.0	1.77 (1.28)	2.21 (1.29)	2.53 (1.28)	2.8 (1.24)	3.12 (1.19)
40.0	1.98 (1.25)	2.4 (1.29)	2.76 (1.29)	2.98 (1.25)	3.3 (1.2)
45.0	2.15 (1.21)	2.61 (1.25)	2.9 (1.25)	3.15 (1.23)	3.4 (1.17)
50.0	2.31 (1.15)	2.72 (1.24)	3.03 (1.23)	3.25 (1.19)	3.5 (1.16)
55.0	2.39 (1.15)	2.83 (1.21)	3.09 (1.2)	3.32 (1.17)	3.54 (1.16)
60.0	2.42 (1.16)	2.85 (1.18)	3.09 (1.18)	3.35 (1.17)	3.49 (1.14)
65.0	2.43 (1.16)	2.85 (1.16)	3.05 (1.15)	3.34 (1.16)	3.51 (1.12)
70.0	2.38 (1.14)	2.79 (1.12)	3.0 (1.14)	3.23 (1.14)	3.36 (1.12)
75.0	2.26 (1.13)	2.67 (1.11)	2.88 (1.14)	3.07 (1.13)	3.22 (1.12)
80.0	2.06 (1.1)	2.47 (1.1)	2.66 (1.11)	2.84 (1.13)	2.94 (1.1)
85.0	1.78 (1.08)	2.15 (1.09)	2.31 (1.09)	2.48 (1.09)	2.59 (1.09)
90.0	1.37 (1.07)	1.68 (1.07)	1.8 (1.05)	2.0 (1.08)	2.07 (1.06)
95.0	0.78 (1.03)	1.01 (1.05)	1.08 (1.03)	1.25 (1.05)	1.33 (1.06)
100.0	-0.01 (1.01)	0.03 (1.03)	-0.03 (0.99)	0.04 (1.01)	0.07 (1.03)

TABLE IX: z-scores of two-edge statistic for an SI process with $\zeta = 1.0$ on a network of $N = 1000$ vertices.

⟨ k ⟩ / percentage	2.0	3.0	4.0	6.0	8.0
5.0	nan (nan)	0.08 (1.21)	0.02 (1.07)	0.09 (1.16)	0.04 (0.99)
10.0	0.04 (1.07)	0.13 (1.14)	0.12 (1.08)	0.18 (1.12)	0.14 (1.05)
15.0	0.08 (1.09)	0.16 (1.11)	0.16 (1.05)	0.26 (1.11)	0.22 (1.09)
20.0	0.1 (1.11)	0.22 (1.09)	0.23 (1.07)	0.31 (1.1)	0.32 (1.06)
25.0	0.13 (1.07)	0.24 (1.07)	0.28 (1.08)	0.4 (1.06)	0.42 (1.09)
30.0	0.18 (1.06)	0.27 (1.05)	0.32 (1.07)	0.45 (1.03)	0.48 (1.1)
35.0	0.23 (1.04)	0.35 (1.04)	0.38 (1.06)	0.53 (1.04)	0.57 (1.09)
40.0	0.28 (1.05)	0.39 (1.06)	0.43 (1.06)	0.56 (1.06)	0.61 (1.06)
45.0	0.29 (1.09)	0.41 (1.04)	0.47 (1.06)	0.62 (1.06)	0.65 (1.05)
50.0	0.31 (1.11)	0.44 (1.04)	0.53 (1.07)	0.63 (1.07)	0.73 (1.04)
55.0	0.33 (1.06)	0.47 (1.01)	0.57 (1.06)	0.69 (1.06)	0.77 (1.04)
60.0	0.37 (1.06)	0.52 (1.01)	0.59 (1.08)	0.72 (1.03)	0.82 (1.02)
65.0	0.39 (1.06)	0.54 (1.0)	0.6 (1.06)	0.73 (1.06)	0.82 (1.0)
70.0	0.38 (1.02)	0.56 (0.99)	0.62 (1.05)	0.77 (1.07)	0.86 (1.01)
75.0	0.36 (1.03)	0.55 (0.98)	0.61 (1.03)	0.73 (1.06)	0.86 (1.04)
80.0	0.35 (1.03)	0.52 (0.99)	0.58 (1.05)	0.7 (1.04)	0.81 (1.02)
85.0	0.31 (1.02)	0.46 (1.0)	0.54 (1.04)	0.64 (1.05)	0.73 (1.0)
90.0	0.25 (1.01)	0.4 (1.0)	0.46 (1.03)	0.53 (1.04)	0.61 (1.02)
95.0	0.14 (1.01)	0.25 (1.0)	0.32 (1.01)	0.35 (1.05)	0.4 (1.01)
100.0	-0.08 (1.01)	-0.03 (0.99)	0.0 (1.02)	-0.03 (1.05)	-0.03 (1.01)

TABLE X: z-scores of two-edge statistic for an SI process with $\zeta = 10.0$ on a network of $N = 1000$ vertices.

percentage \ ⟨ k ⟩	2.0	3.0	4.0	6.0	8.0
5.0	nan (nan)	0.07 (1.17)	-0.04 (0.96)	0.02 (1.08)	0.05 (1.08)
10.0	0.02 (1.03)	0.07 (1.16)	0.04 (1.02)	0.05 (1.03)	0.03 (1.08)
15.0	0.01 (1.06)	0.05 (1.06)	0.06 (1.05)	0.03 (1.02)	0.07 (1.05)
20.0	-0.02 (1.07)	0.08 (1.08)	0.07 (1.1)	0.05 (1.04)	0.09 (1.04)
25.0	-0.02 (1.05)	0.08 (1.06)	0.06 (1.09)	0.05 (1.02)	0.09 (1.02)
30.0	-0.01 (1.04)	0.1 (1.07)	0.07 (1.07)	0.06 (1.02)	0.12 (1.02)
35.0	-0.01 (1.01)	0.1 (1.04)	0.06 (1.05)	0.08 (0.98)	0.13 (1.0)
40.0	-0.0 (1.01)	0.09 (1.03)	0.04 (1.08)	0.1 (1.01)	0.13 (1.0)
45.0	-0.01 (1.03)	0.11 (1.02)	0.01 (1.05)	0.09 (1.04)	0.11 (1.01)
50.0	0.01 (1.06)	0.1 (1.0)	0.04 (1.05)	0.09 (1.01)	0.12 (1.04)
55.0	-0.0 (1.03)	0.11 (0.99)	0.04 (1.03)	0.1 (1.01)	0.12 (1.05)
60.0	0.01 (1.04)	0.1 (1.0)	0.06 (1.02)	0.08 (1.0)	0.13 (1.03)
65.0	0.0 (1.04)	0.08 (1.02)	0.06 (1.07)	0.09 (1.02)	0.12 (1.03)
70.0	0.01 (1.05)	0.09 (1.05)	0.05 (1.04)	0.1 (1.05)	0.12 (1.04)
75.0	0.02 (1.04)	0.08 (1.05)	0.04 (1.01)	0.12 (1.05)	0.12 (1.03)
80.0	0.02 (1.07)	0.09 (1.05)	0.04 (1.0)	0.11 (1.02)	0.11 (1.02)
85.0	0.01 (1.04)	0.08 (1.02)	0.02 (1.02)	0.07 (1.01)	0.1 (1.03)
90.0	0.01 (1.03)	0.04 (1.01)	0.03 (1.01)	0.07 (1.03)	0.06 (1.01)
95.0	0.01 (1.04)	0.02 (1.01)	-0.01 (1.01)	0.07 (1.03)	0.02 (1.02)
100.0	-0.02 (1.04)	0.0 (1.02)	-0.06 (1.0)	0.02 (1.03)	-0.04 (1.01)

TABLE XI: z-scores of two-edge statistic for an SI process with $\zeta = 100.0$ on a network of $N = 1000$ vertices.

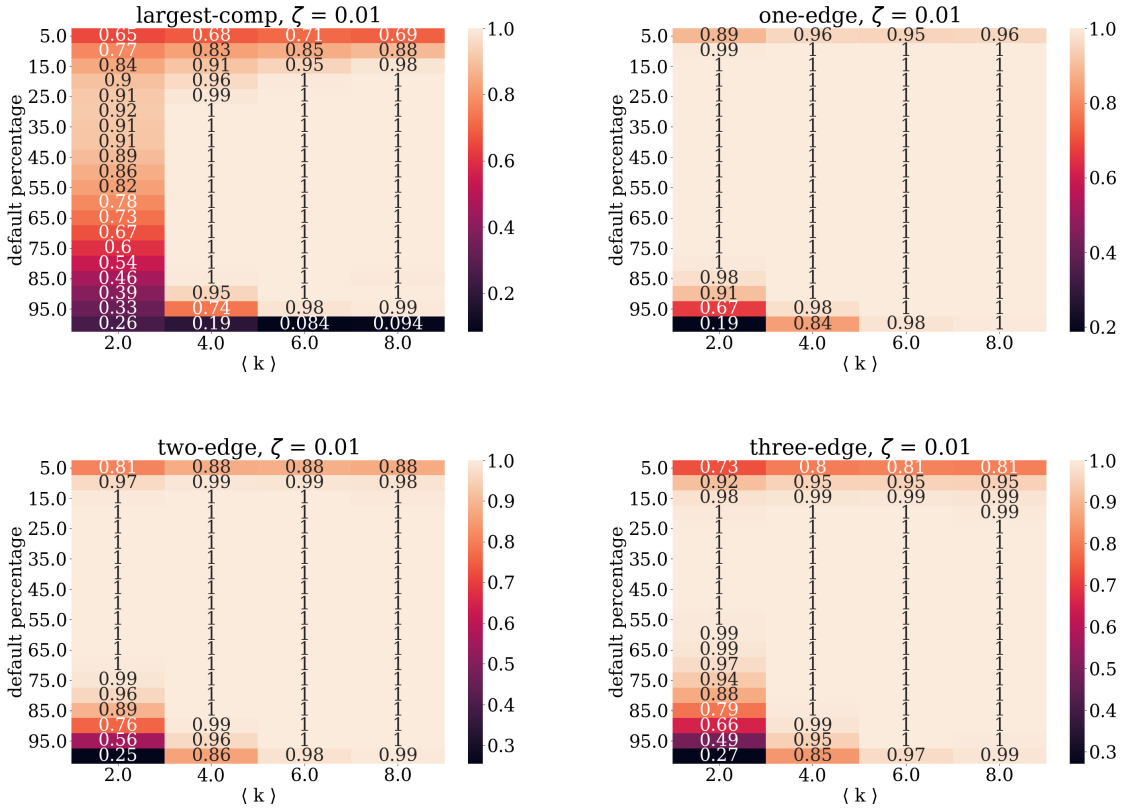


FIG. 11: KS-test of the results for an voter-model process with $\zeta = 0.01$ on a network of $N = 1000$ vertices.

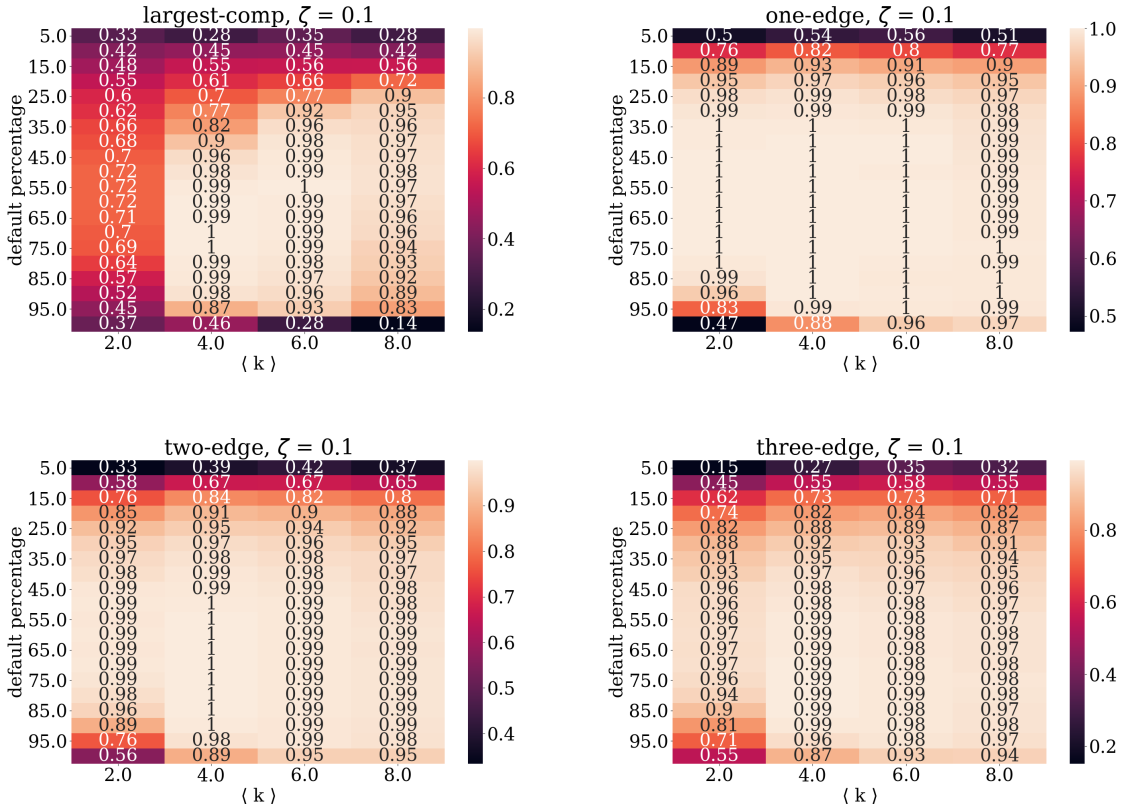


FIG. 12: KS-test of the results for a voter-model process with $\zeta = 0.1$ on a network of $N = 1000$ vertices.

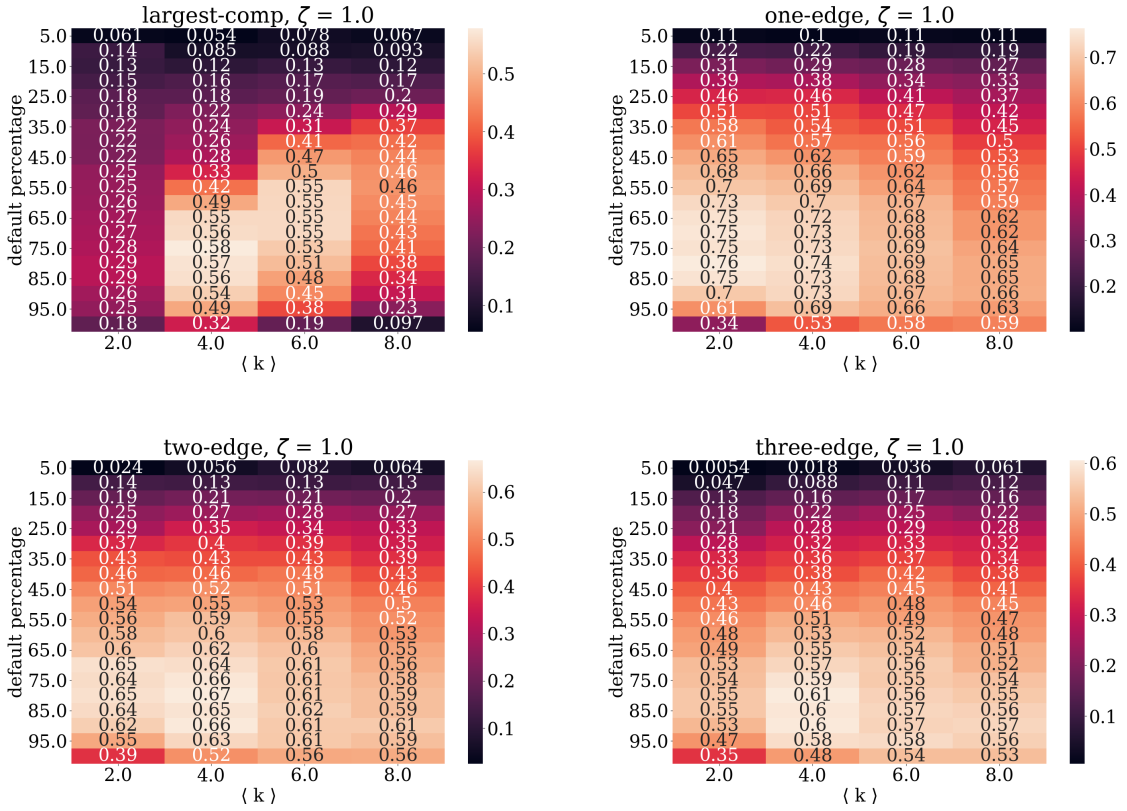


FIG. 13: KS-test of the results for a voter-model process with $\zeta = 1$ on a network of $N = 1000$ vertices.

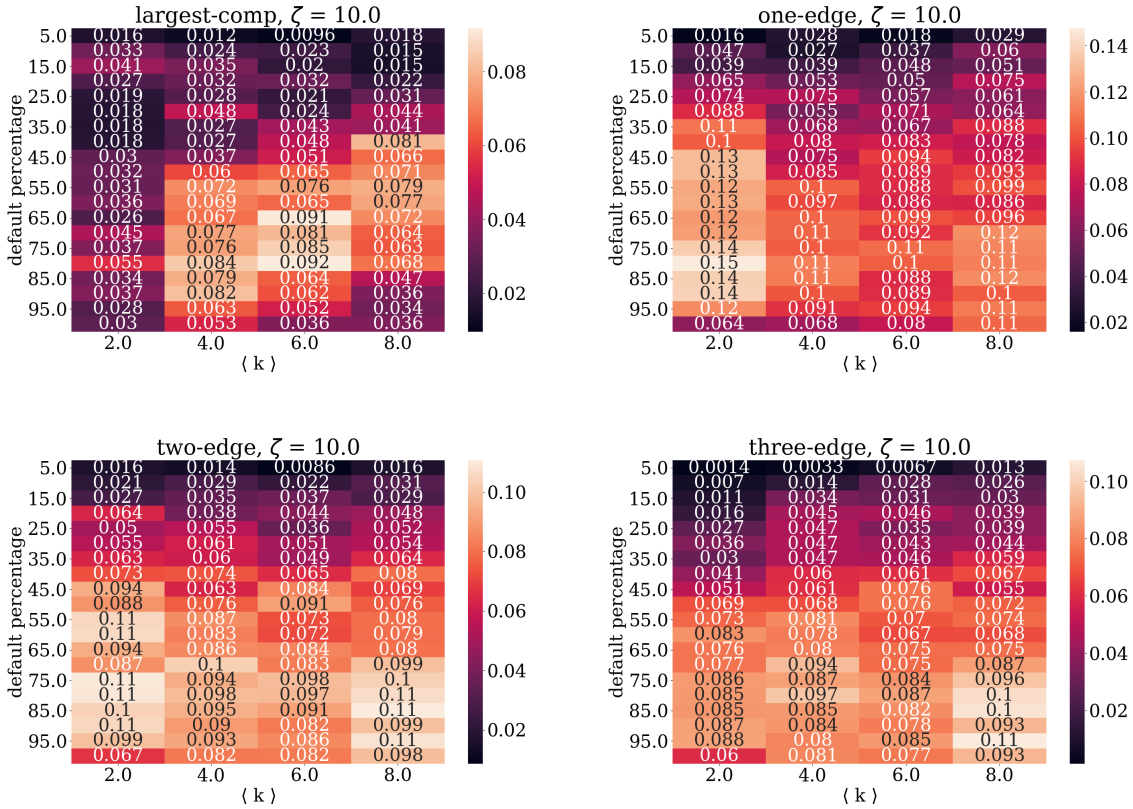


FIG. 14: KS-test of the results for a voter-model process with $\zeta = 10$ on a network of $N = 1000$ vertices.

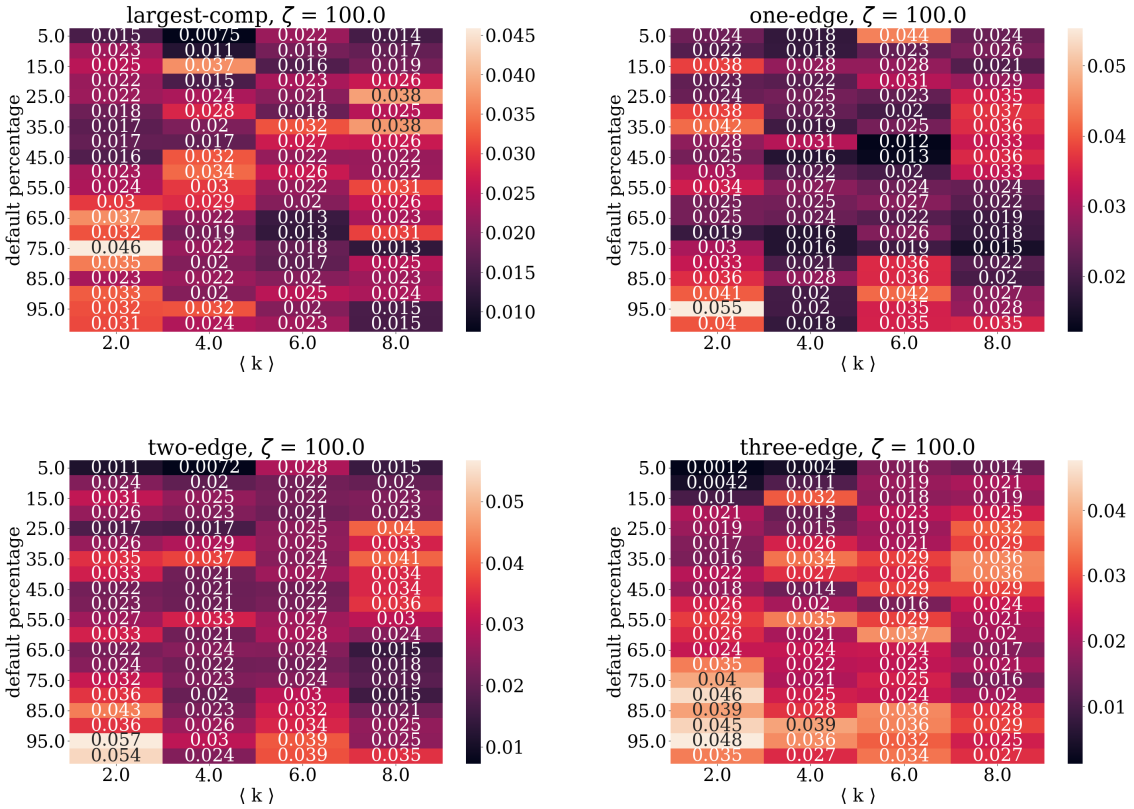


FIG. 15: KS-test of the results for a voter-model process with $\zeta = 100$ on a network of $N = 1000$ vertices.

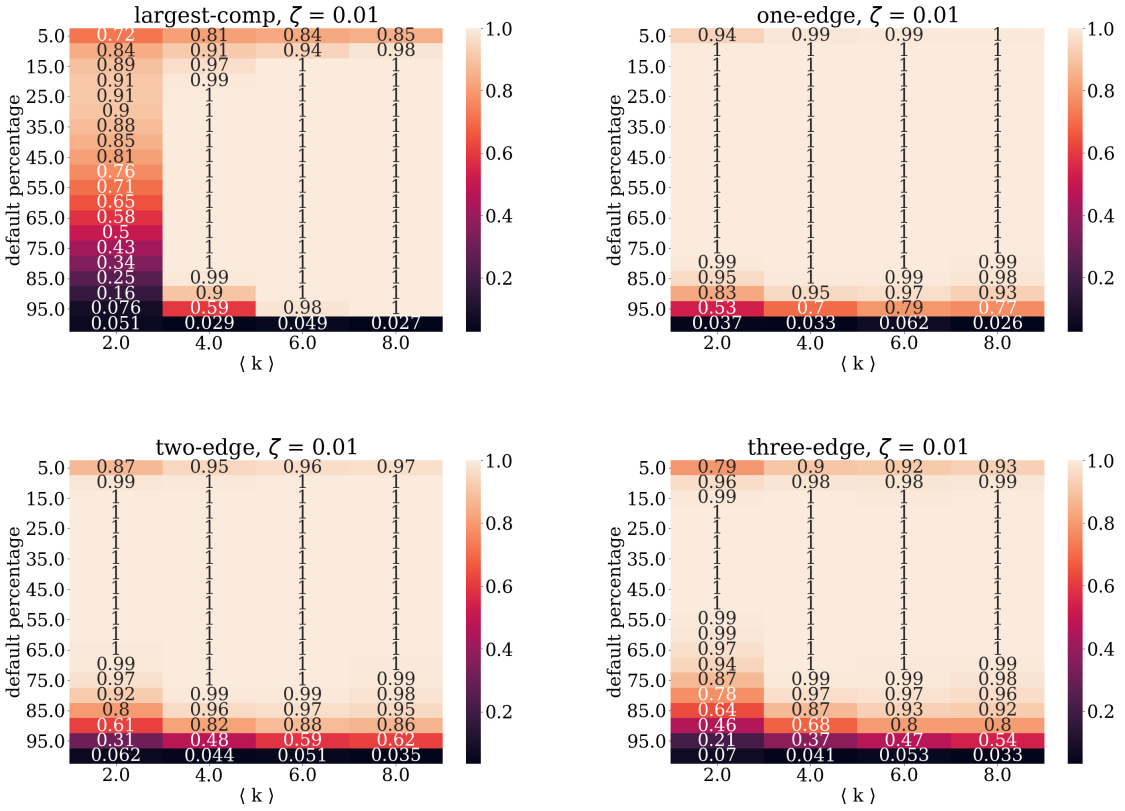


FIG. 16: KS-test of the results for a SI process with $\zeta = 0.01$ on a network of $N = 1000$ vertices.

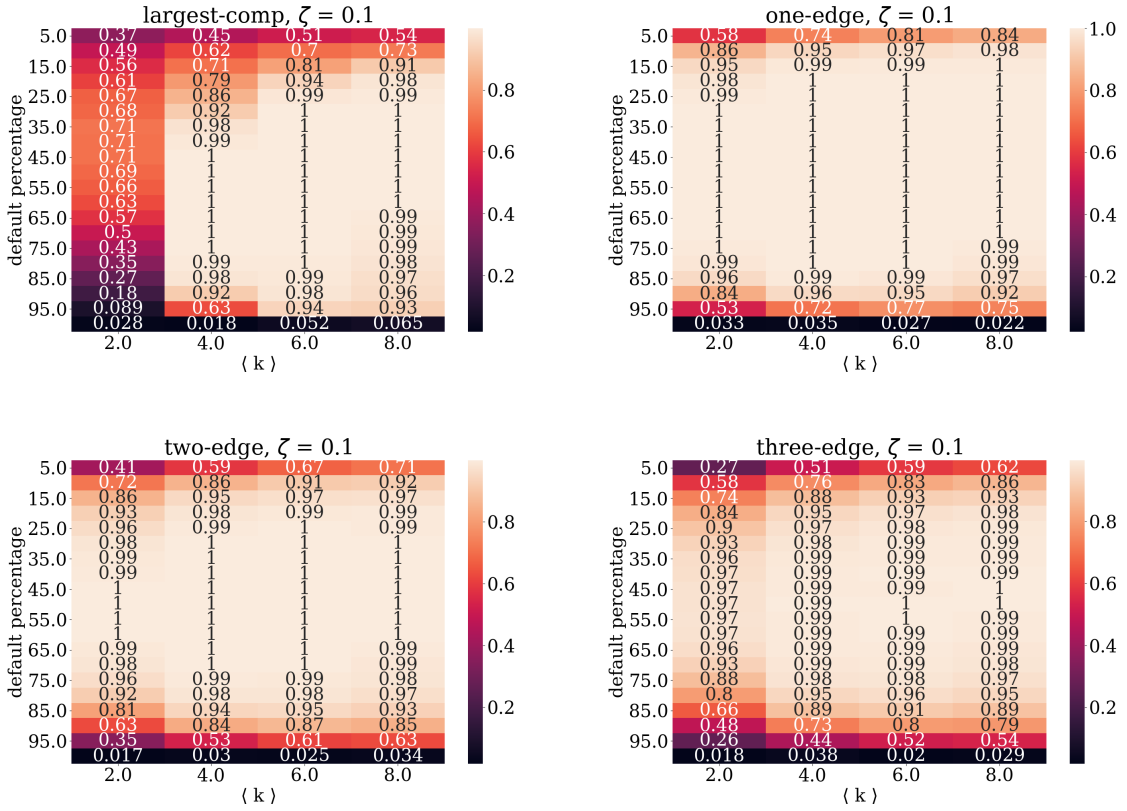


FIG. 17: KS-test of the results for a SI process with $\zeta = 0.1$ on a network of $N = 1000$ vertices.

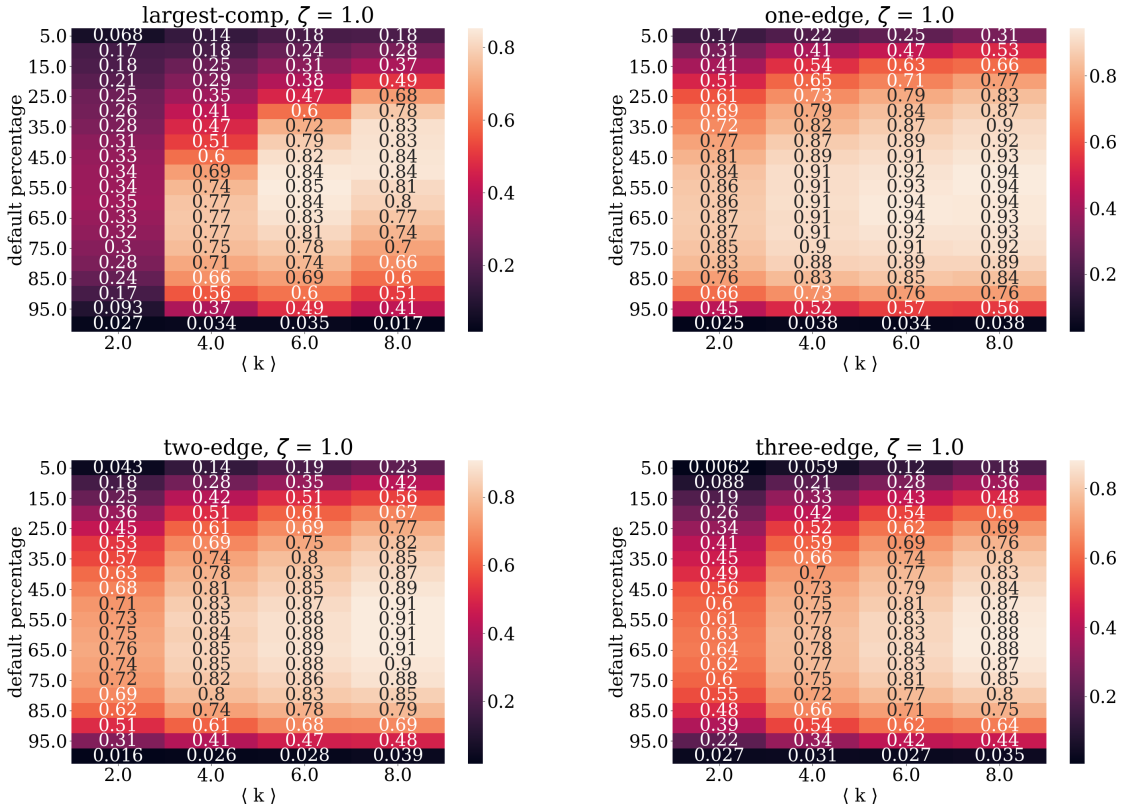


FIG. 18: KS-test of the results for a SI process with $\zeta = 1$ on a network of $N = 1000$ vertices.

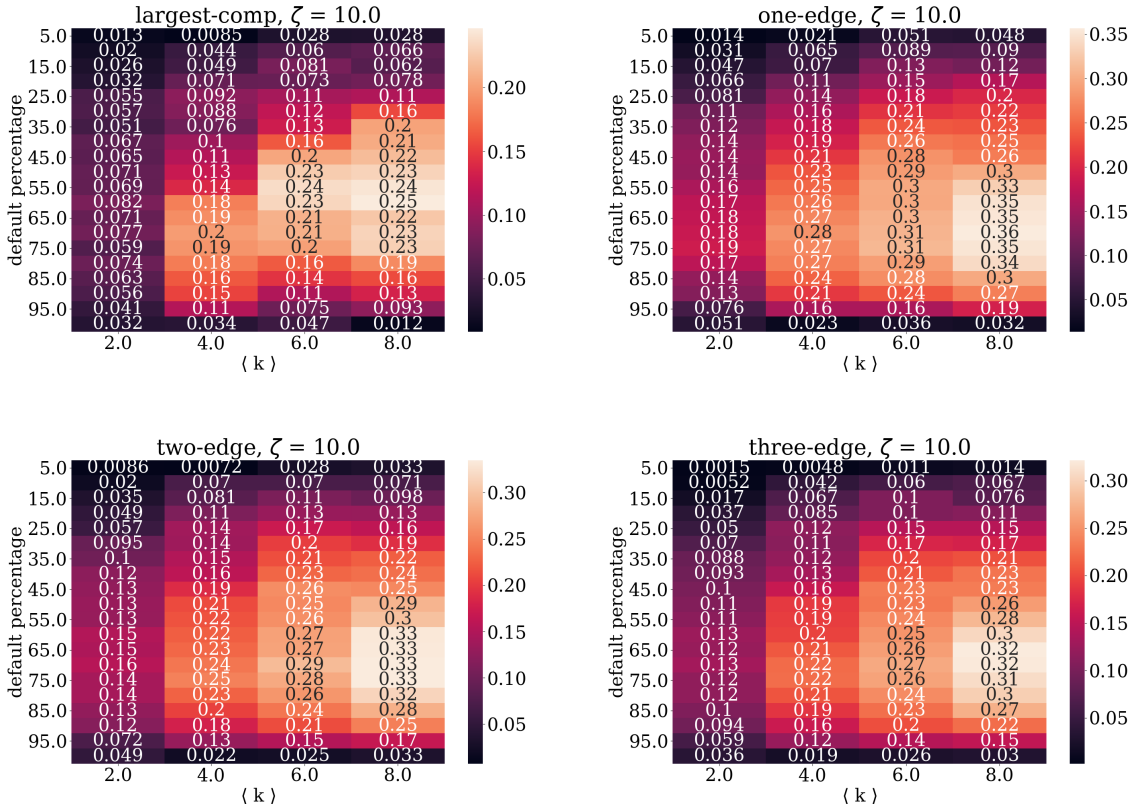


FIG. 19: KS-test of the results for a SI process with $\zeta = 10$ on a network of $N = 1000$ vertices.

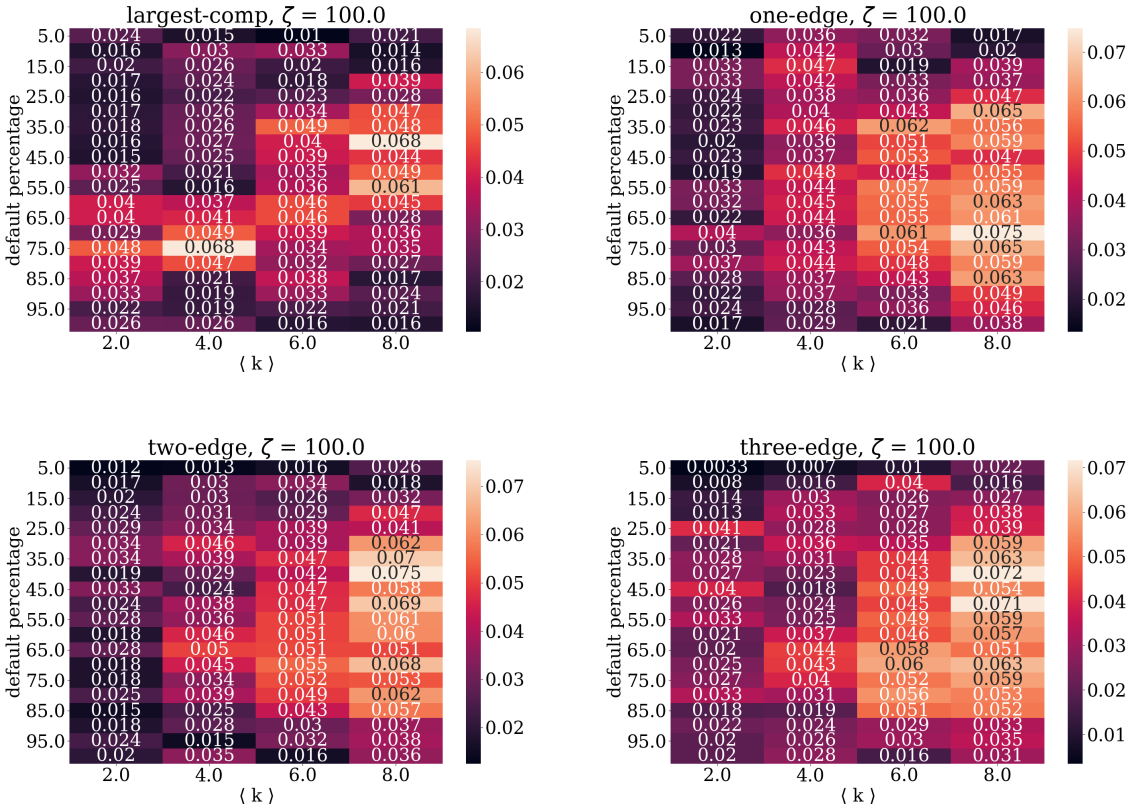


FIG. 20: KS-test of the results for a SI process with $\zeta = 100$ on a network of $N = 1000$ vertices.

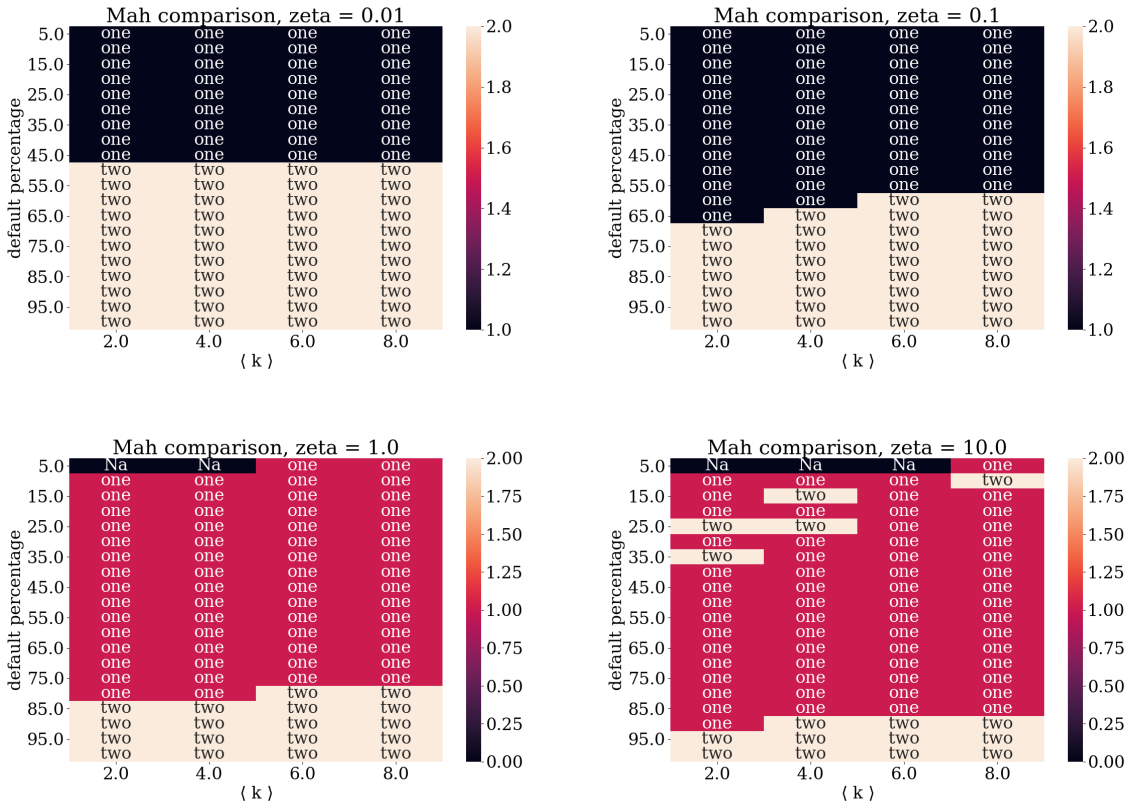


FIG. 21: Comparison of Mahalanobis distance of the results for an SI process on a network of $N = 1000$ vertices. The best statistically significant statistic for a given percentage, ζ and $\langle k \rangle$ is stated as "lc" for largest component, "one" for one-edge, "two" for two-edge, etc. A z-score greater than 1 is considered to be statistically significant, otherwise an "Na" value is put in the table.

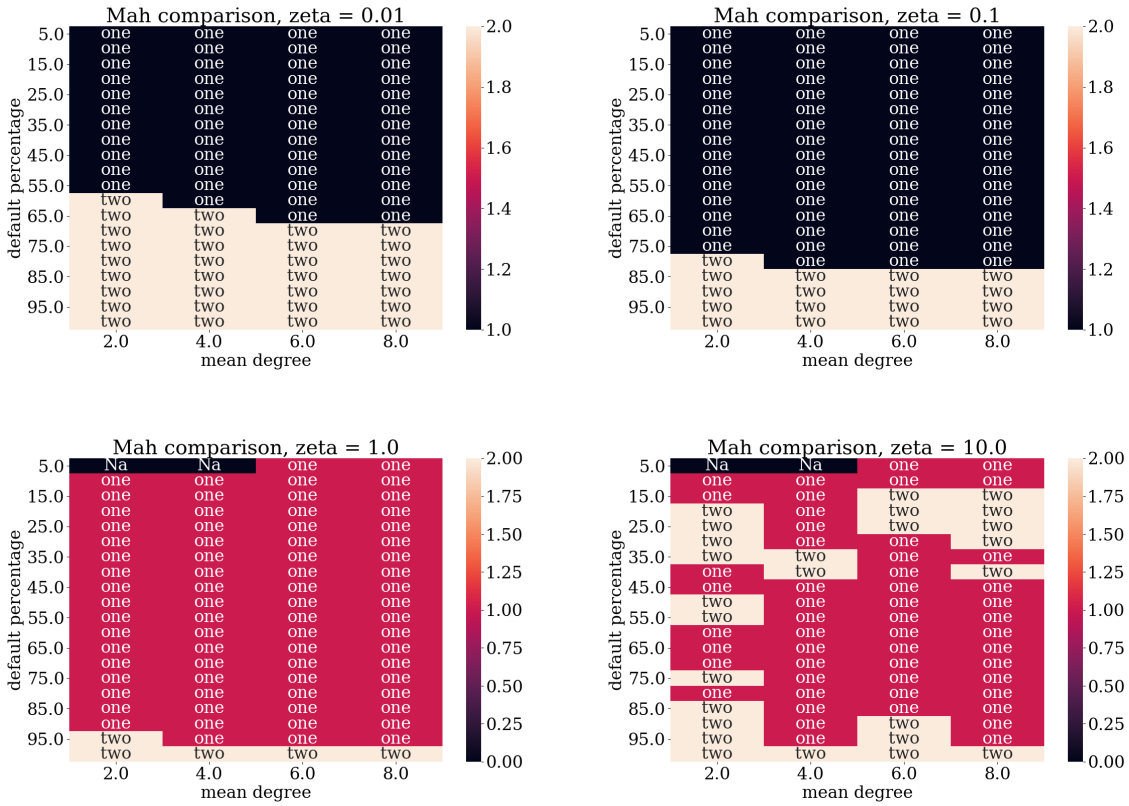


FIG. 22: Comparison of Mahalanobis distance of the results for a voter model process on a network of $N = 1000$ vertices. The best statistically significant statistic for a given percentage, ζ and $\langle k \rangle$ is stated as "lc" for largest component, "one" for one-edge, "two" for two-edge, etc. A z-score greater than 1 is considered to be statistically significant, otherwise an "Na" value is put in the table.