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FIBONACCI SERIES GF.

$$F(x) = \frac{1}{1-x-x^2} \qquad \frac{1}{-x^2-x+1}$$

$$\text{roots} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{1-4\cdot-1\cdot1}}{2\cdot-1} = \frac{1 \pm \sqrt{1+4}}{-2}$$

$$= -\frac{1 \pm \sqrt{5}}{2} =$$

$$F(x) = \frac{1}{\left(x - \frac{1 \pm \sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right)}$$

$$F(x) = \frac{1}{(x+a)(x+b)} = \frac{C}{(x+a)} + \frac{D}{x+b}$$

$$C(x+b) + D(x+a) = 1$$

$$(C+D)x \quad (C+D)x + Cb + Da = 1$$

$$\Rightarrow C+D = 0 \quad \text{and} \quad Cb + Da = 1$$

$$F(x) = \frac{1}{(1-\alpha x)(1-\beta x)} = \frac{C}{1-\alpha x} + \frac{D}{1-\beta x}$$

$$\Rightarrow C(1-\beta x) + D(1-\alpha x) = 1$$

$$\Rightarrow C+D - x(\alpha C + \beta$$

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$C(1-\beta x) + D(1-\alpha x) = 1$

$\Rightarrow (C+D) - x(C\beta + D\alpha) = 1$ $\qquad \alpha = \frac{1}{1+\sqrt{5}}$

$\Rightarrow C+D = 1 \qquad C\beta + D\alpha = 0$

$\cancel{\Rightarrow C\beta = - D\alpha.}$

C

$\left(\frac{1+\sqrt{5}}{2} - x\right)\left(\frac{1-\sqrt{5}}{2} - x\right)$

$\left(1 - \frac{2x}{1+\sqrt{5}}\right)\left(1 - \frac{2x}{1-\sqrt{5}}\right)$

$\alpha = \frac{2}{1+\sqrt{5}} \qquad \beta = \frac{2}{1-\sqrt{5}}$

$\therefore \frac{2C}{1+\sqrt{5}} = - \frac{2D}{1-\sqrt{5}} \qquad \Rightarrow \sqrt{5}(C-D) = 1$

$C-D = \frac{1}{\sqrt{5}}$

$C(1-\sqrt{5}) = -D(1+\sqrt{5})$

$C+D = 1$

$C - \sqrt{5}C = -D - \sqrt{5}D$

$\Rightarrow \cancel{C+D} = \sqrt{5}C - \sqrt{5}D \qquad 2C = \frac{1}{\sqrt{5}} + 1$

1

$C = \frac{1+\sqrt{5}}{2\sqrt{5}}$

$$C = \frac{1+\sqrt{5}}{2\sqrt{5}} \qquad D = 1 - C = 1 - \frac{\left(\frac{1}{\sqrt{5}} + 1\right)}{2}$$

$$\therefore D = \frac{\sqrt{5}-1}{2\sqrt{5}} \qquad = \cancel{1} - \frac{1}{2\sqrt{5}} - \cancel{\frac{1}{2}}$$

$$= \frac{1}{2}\left\{1 - \frac{1}{\sqrt{5}}\right\} = \frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{5}}$$

$$D = -\frac{1-\sqrt{5}}{2\sqrt{5}}$$

$$F(x) = \frac{1}{1-ax} \qquad A_n = a^n x^n$$

$$\alpha = \frac{2}{1+\sqrt{5}} \qquad \beta = \frac{2}{1-\sqrt{5}}$$

$$C = \frac{1+\sqrt{5}}{2\sqrt{5}} \qquad D = -\frac{1-\sqrt{5}}{2\sqrt{5}}$$

$$A_n = C\alpha^n x^n + D\beta^n x^n$$

$$= \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{2}{1+\sqrt{5}}\right)^n x^n - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{2}{1-\sqrt{5}}\right)^n x^n$$

CATALAN NOS.

(i) how many binary trees can be forme
n nodes?

(ii) ways of balancing pairs of parentheses

(iii) ways of associating n+1 mat. mul

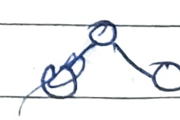
$$M_n = \sum_{i=1}^{n-1} M_i \cdot M_{n-i} \quad \text{recurrence reln.}$$

$$M_4 = \sum_{i=1}^{3} M_i \cdot M_{n-i}$$

$$= \underset{1}{M_1} \cdot \overset{2}{\underset{2}{M_3}} + \underset{1}{M_2} \cdot \overset{1}{\underset{1}{M_2}} + \underset{2}{M_3} \cdot \overset{2}{\underset{1}{M_2}} \quad \Big\} \; 5$$

$$M_3 = \underset{1}{M_1} \cdot \underset{1}{M_2} + \underset{1}{M_2} \cdot \underset{1}{M_1} \quad \Big\} \; 2$$

$B_0 = 1 \qquad B_1 = 1 \qquad B_2 = 2$



$B_2 = B_0 \cdot B_2 + B_1 \cdot B_1$

$B_2 = B_0 \cdot B_2 + B_1 \cdot \overset{B_0}{B_1} + B_2 \cdot B_0$

$B_2 = B_0 \cdot B_1 + B_1 \cdot B_0$

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$B_3 = \underset{1}{B_0} \cdot \underset{2}{B_2} + \underset{1}{B_1} \cdot \underset{1}{B_1} + \underset{2}{B_2} \cdot \underset{1}{B_0} \Big\} 5$

$B_4 = \underset{1}{B_0} \cdot \underset{5}{B_3} + \underset{1}{B_1} \cdot \underset{2}{B_2} + \underset{2}{B_2} \cdot \underset{1}{B_1} + \underset{5}{B_3} \cdot \underset{1}{B_0} \Big\} 14$

$=$

$C(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \dots$

$B_5 = B_0$

$C^2(x) = 1 + 2x + 5x^2 + 14x^3 + \dots$

CATALAN NOS.

$xC^2(x) = x + 2x^2 + 5x^3 + 14x^4 + \dots$

n	C_n
0	1
1	1
2	2
3	5
4	14
5	42
6	132

$xC^2(x) = C(x) - 1$

$\Rightarrow xC^2(x) - C(x) + 1 = 0$

$\Rightarrow C(x) = \dfrac{1 \pm \sqrt{1 - 4 \cdot x \cdot 1}}{2x}$

$\Rightarrow C(x) = \dfrac{1 + \sqrt{1-4x}}{2x}$

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choose $c(x) = \frac{1-\sqrt{1-4x}}{2x}$

let $g(x) = \sqrt{1-4x}$ taylor series expansion.

$g(0) + g'(0)x + g''(0)\frac{x^2}{2!} + g'''(0)\frac{x^3}{3!} + \dots$

$g(0) = 1$

$g'(0) = \frac{1}{2}(1-\cancel{4x})^{-1/2}(-4)\{0\} = \frac{1}{2} \cdot -4 = -2.$

$g''(0) = -\cancel{\frac{1}{4}}(1-4x)^{-3/2}(-4)\cancel{(-4)} \{0\} = -4.$

$g'(0) = \frac{1}{2}(1-4x)^{-1/2}(-4)\{0\}$

$g''(0) = \frac{1}{2}\left(-\frac{1}{2}\right)(1-4x)^{-3/2}(-4)^2\{0\}$

$g'''(0) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1-4x)^{-5/2}(-4)^3\{0\}$

$g''''(0) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(1-4x)^{-7/2}(-4)^4\{0\}$

$\therefore g^n(0) = -\frac{4^n}{2^n}\{1\cdot3\cdot5\cdot7\dots(2n-3)\}.$

$= -2^n\{1\cdot3\cdot5\cdot7\dots(2n-3)\}.$

$$\therefore \frac{g^{(n)}(0)}{n!} = \frac{-2^n\,(1\cdot 3\cdot 5\cdot 7\cdots(2n-3))}{n!} \qquad n \geq 2$$

$$g(x) = g(0) + g'(0)x + g''(0)\frac{x^2}{2!} + \cdots$$

$$= 1 - 2x - \frac{2^2x^2}{2!} - \frac{2^3(1\cdot 3)x^3}{3!} - \frac{2^4(1\cdot 3\cdot 5)x^4}{4!} - \cdots$$

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

$$= \frac{1}{2x}\left\{2x + \frac{2^2x^2}{2!} + \frac{2^3(1\cdot 3)x^3}{3!} + \frac{2^4(1\cdot 3\cdot 5)x^4}{4!} + \cdots\right)$$

$$= 1 + \frac{2x^2}{2!} + \frac{2^2(1\cdot 3)x^2}{3!} + \frac{2^3(1\cdot 3\cdot 5)x^3}{4!} + \cdots$$

$=$

$$\therefore \text{nth term} = \frac{2^n(1\cdot 3\cdot(2n-1))x^n}{(n+1)!}$$

$$= \frac{1}{n+1}\,\frac{2^n(1\cdot 3\cdot(2n-1))\,x^n}{n!}$$

$$= \frac{1}{n+1}\,\frac{2n!}{n!\,n!}\cdot x^n$$

remember socks picking.

$$= \frac{1}{n+1}\binom{2n}{n}x^n$$

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EU

gcd (17 7) = gcd 7 3 = gcd 3 1 = gcd 1 0

gcd 17 7 → 1

gcd 7 3

gcd 3 1

gcd 1 0

$1 = 7 - 2(3)$

$= 7 - 2\{17 - 2(7)\}$

$= 7 - 2(17) + 4(7)$

$= 5(7) - 2(17)$ ✓

$1 = 5(17) - 12(7)$

ex 100, 46

gcd 100 46

gcd 46 8

gcd 8 6

gcd 6 2

gcd 2 0

$2 = 1(8) - 1(6)$

$= \cancel{(46 - 5(8))}$

$= 8 - (46 - 5(8))$

$= (100 - 2(46))$

$- \{46 - 5(100 - 2(46))\}$

$= 100 - 2(46) - (46 - 5(100) + 10(46))$

$= 100 - 2(46) - (46) + 5(100) - 10(46)$

$= 6(100) - 13(46)$

$2 = 6(100) - 13(46)$

$2 = 87(46) - 40(100)$

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$ax - by = \gcd(a, b)$

general:
$a(x \pm nb) - b(y \pm na) = \gcd(a, b)$

special case ($n = 1$)

$a(x \pm b) - b(y \pm a) = 1$

$ax \cancel{-ab} - by \cancel{+ab} = 1$

bulls eye 7 4 = 1.
$(7-1).(4-1) = 18$.

if given a, b with $\gcd(a, b) = 1$
then for every $n \geq (a-1)(b-1)$
18 . $(4+7)$ -
there exist x & $y \geq 0$ st.
$ax + by = n$.

Pierre fermat : (1600's) little theorm.

LAST theorm $x^n + y^n = z^n$ $n > 2$.

proved in 1999 by wiles.

$a^{p-1} \equiv 1 \pmod{p}$.

ex $a = 6$ $p = 5$

(FLT)

$6^{5-1} = 6^4 = 216$

$\times 6$

1296

$p = 1$

0 → 0.

1.6	1	
2.6	2	36.0.1.
3.6	3	
4.6	4	
5.6	0	
6.6	1	
7.6	2	
8.6	3	
9.6	4	
10.6	0	
11.6	1	
12.0		

$a^{p-1} - 1$ = multiple of p.

a 2a 3a 4a $(p-1)a$

$a^{p-1}(1 \cdot 2 \cdot 3 \ldots (p-1))$

p = 17 e : 5 gcd 5 16 = 1

x = 6 $x^e \bmod p = 6^5$

gcd e p = 1

x = 6 $x^e \bmod p$ = $6^5 \bmod 17$

decoding = 7776 mod 17

= 7

= y

d p

13 17

$d \cdot e \equiv 1 \bmod p-1$

5d = 1 mod p-1

5 · 13 · 1 md 16

13
5
65 16

48
16
64

a^{p-1} =

17.5
34 44 51 55 (4) 17(3).

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$p = 17 \quad e = 5 \quad d = 13.$

$(e \cdot d) = 1 \bmod p-1$

$(x^e)^d = x \bmod p$

$x^{e \cdot d} = x \bmod p$

$x^{4 \cdot 16 + 1} = x^{4 \cdot 16} \cdot x$

$a^{p-1} = 1 \bmod p$

$a^{16} = 1 \bmod p.$

$5d = 1 \bmod 16$

$5d - 1 = 16k$

$5d - 16k = 1$

$1 = 1(16) - 3(5)$

$1 = \underset{d}{\underline{13}}(5) - \underset{k}{\underline{4}}(16)$

$d = 16.$