COMPUTABLE AND UNCOMPUTABLE FUNCTIONS

dfn a fn. is computable, if there is a computer program that evaluates (in some programming language) the values of the function.

dfn a fn which is not computable is an uncomputable fn.

thm set of all computer programs in any particular language is countable.

Proof for any finite alphabet, there are finite no. of strings of length n, for every $n \in \mathbb{N}$, from the theorm (can verify) which states the union of a countable number of countable sets is countable. (lemma 1)

enumerate in same order.

$\bigcup_{n=1}^{\infty} S_n$ is countable.

$\Rightarrow$ there are only a countable no. of strings from any finite alphabet

note, that the set of all computer programs (in a particular programming language) is a subset of the set of all strings of a finite alphabet, which is countable.

(lemma) $\rightarrow$ a subset of a countable set is also countable, we can conclude the set of all computer programs is countable.

thm show that there is no 1-1 correspondence (bijection) from the set of positive integers $(\mathbb{N})$ to the power set (set of all subsets) $P(\mathbb{N})$ $\mathbb{N} \not\leftrightarrow P(\mathbb{N})$

proof uses cantor's diagonalization argument
proof by contradiction.

on the contrary, we assume that
there is a bijection from $\mathbb{Z}^+$ to $P(\mathbb{Z}^+)$

$$P(\mathbb{Z}^+) = \{\phi, \{1\}, \{2\}, \ldots \{1,2\}, \{2,3\}, \ldots\}$$

we will come up with a scheme of enumerating them

$\mathbb{Z}^+$ $P(\mathbb{Z}^+)$

presence bit strings.

trial $1 \longleftrightarrow \{2\}$ 0 1 0 0 0 . . .

~~$2 \longleftrightarrow$~~

$2 \longleftrightarrow \{1,2\}$ 0 1 1 0 0 0 . . .

$3 \longleftrightarrow \{4,5,6\}$ 0 0 0 1 1 1 0 . . .

$4 \longleftrightarrow \{1,2,3,4\}$ 1 1 1 1 0 0 0 . . .

⋮

$a_1\ a_2\ a_3 \ldots$

$s = s_1\ s_2\ s_3\ s_4 \ldots$

1 0 1 0

▽

every element A of $\mathcal{P}(\mathbb{Z}^+)$ can be represented uniquely by the bit string $a_1 a_2 a_3 \ldots$ where $a_i = 1$ if $i \in A$
$a_i = 0$ if $i \notin A$

now consider a string $s = s_1 s_2 s_3 \ldots$ by setting s_i to be 1 - the ith bit of $f(i)$, such that s is not in the range of f.

therefore f cannot be 1-1 correspondence.
by contradiction, there is no 1-1 correspondence between $\mathbb{Z}^+$ onto $\mathcal{P}(\mathbb{Z}^+)$

- in general, for any set A, $|A| < |\mathcal{P}(A)|$

<u>continuum hypothesis</u> (cantor)

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \ldots$$

countably infinite | uncountable | uncountable.

$\aleph_0$ (aleph nought)

$2^{|\mathbb{N}_0|}$
$= 2^{\aleph_0}$
$= \mathfrak{c}$

$|\mathbb{R}|$

there is no other cardinality (countability) (cardinality nos.).

$$\aleph_0 < \aleph_1 < \aleph_2 < \ldots$$

a barber who shaves all men who does not shave themselves.

self-referential statements : bertrand russel.

gödel's incompleteness theorm.
↳ never consistent as well as complete (meta system)

douglas hofstadter (GEB) . cs
gödel, escher, bach
(mobius strip) (musician)

in general, we can verify that there are uncountably many different functions from a particular countably infinite set (say ℕ) to itself.

(uses cantor's diagonalization argument)
⇒ there are uncomputable functions in general,
there are more computations than there are programs

programs \ inputs	1	2	3	
1	1	0	1	. . .
2	1	0	0	. . .
3	1	1	0	. . .
⋮		. . .		

there are more problems, than there are programs.

links set theory as appn. to theory of computation.

SUMS

thm $1^3 + 2^3 + 3^3 + \dots n^3 = (1+2+3+\dots n)^2$

proof by induction

base case $n=1$: $1^3 = (1)^2$

let us assume it is true for $n=k$

$1^3 + 2^3 + \dots k^3 = (1+2+\dots k)^2$

we need to show that this is true for $k+1$ $(n=k+1)$

$1^3 + 2^3 + \dots k^3 + (k+1)^3 = (1+2+\dots k + (k+1))^2$

$LHS = (1+2+\dots k)^2 + (k+1)^3$

$= \left\{ k \left(\frac{k+1}{2} \right) \right\}^2 + (k+1)^3$

$= \frac{1}{4} \left\{ k^2 + 4(k+1) \right\} (k+1)^2$

$= \frac{1}{4} \left\{ k^2 + 4k + 4 \right\} (k+1)^2$

$$= \frac{(k+2)^2 (k+1)^2}{4} = \left\{ \frac{(k+1)(k+2)}{2} \right\}^2$$

$$= \left(1+2+\dots(k+1)\right)^2 = RHS \checkmark$$

by induction, the result is true for $n \in \mathbb{N}$

thm prove that $(1^2+2^2+\dots n^2) = \frac{n(n+1)(2n+1)}{6}$

look at ith term – simpler method.

$$1^2 = 1$$

$$1 + 3 = 4$$

$$1+3+5 = 9$$

$$\dots$$

$$1+3+\dots(2n-1) = n^2$$

$$n(1) + (n-1)(3) + \dots (1)(2n-1) = \sum_{i=1}^{n} i^2$$

$$\sum_{i=1}^{n} (n-i+1)(2i-1) = \sum i^2$$

$$\sum \left(2ni - n - 2i^2 + i + 2i - 1\right) = \sum i^2$$

$$-2\sum i^2 + (2n+3)\sum i - (n+1)\sum 1 = \sum i^2$$

$$3\sum i^2 = \frac{(2n+3)(n)(n+1)}{2} - (n+1)(n)$$

$$6\sum i^2 = n(n+1)(2n+3) - 2n(n+1)$$

$$6\sum i^2 = n(n+1)\{2n+3-2\}$$

$$\sum i^2 = \frac{n(n+1)(2n+1)}{6}$$

sum of squares using a different arg.

we will continue in recurrence relations . . .