Q1)

Ans $(a,b)\ R\ (c,d)$. iff $ad = bc$.

To Test whether R is equivalence relation or not, we need to show R is Reflexive, Symmetric and Transitive.

① Reflexivity :

$(a,b)\ R\ (a,b)$

LHS : ab

RHS : ba

LHS = RHS. Thus, $(a,b) R (a,b)$ is in the set.

Thus, R is Reflexive

② Symmetric.

$(a,b)\ R\ (c,d) \Rightarrow ad = bc$

$(c,d)\ R\ (a,b) \Rightarrow cb = da$

Thus, if $(a,b) R (c,d)$ then $(c,d) R (a,b)$.

Hence, R is Symmetric

③ Transitive

we have to prove

if $(a,b) R (c,d)$ and $(c,d) R (e,f)$ then $(a,b) R (e,f)$

Given : $(a,b) R (c,d) \Rightarrow ad = bc \rightarrow (1)$

$(c,d) R (e,f) \Rightarrow cf = de \rightarrow (2)$

Now,

$$ad = bc$$

$$a \cdot \frac{\cancel{c}f}{e} = b\cancel{c}$$

$$af = \cancel{b}eb \rightarrow (3)$$

From (3), we can say $(a,b) R (e,f)$.

Thus, R is Transitive

Thus, as R is Reflexive, Symmetric & Transitive,

R is an equivalence relation on set $Z \times Z$.

iii) If R is a relation in the set of Integers $\mathbb{Z}$ defined by xRy, where $xRy = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : (x-y)$ is divisible by $7\}$, then find all the distinct equivalence classes of the relation R.

Ans:- The set of all elements that are related to an element a of A, where R is defined as an equivalence relation on set A, is called the Equivalence class of a. The Equivalence Class of a with respect to R is denoted by $[a]_R$.

Let us first prove that R is an equivalence Relation.

① For Reflexive:

If xRx, then $(x-x)$ is divisible by 7

'0' is divisible by 7 $\therefore$ R is a Reflexive Relation.

② For Symmetric:

If $xRy \Rightarrow yRx$

i.e, $x-y = 7K$, $K \in \mathbb{Z}$

$\therefore y-x = 7(-K)$, $K \in \mathbb{Z}$ $\therefore$ R is also a Symmetric Relation

③ For Transitive

If xRy and $yRz \Rightarrow xRz$

i.e, $x-y = 7K_1$ — (1)

$y-z = 7K_2$ — (2)

(1) + (2)

$x-z = 7(K_1+K_2)$, i.e, $(x-z)$ is a multiple of 7

$\therefore$ R is a Transitive Relation.

$\therefore$ R is an Equivalence Relation.

Now, we will find the distinct equivalence classes of Relation R.

i.e, $[a] = \{x \in \mathbb{Z} \mid xRa\}$, for each integer a.

$= \{x \in \mathbb{Z} \mid x-a = 7K\}$ for some integer K $\}$

for a=0 ~~of~~ $[0] = \{x \in \mathbb{Z} \mid x = 7K+0$, for some integer K $\}$

$= \{\ldots -14, -7, 0, 7, 14 \ldots\}$

a=1 $[1] = \{x \in \mathbb{Z} \mid x = 7K+1$, for some integer K $\}$

$= \{\ldots -13, -6, 1, 8, 15 \ldots\}$

a=2 $[2] = \{x \in \mathbb{Z} \mid x = 7K+2$, for some integer K $\}$

$= \{\ldots -12, -5, 2, 9 \ldots\}$

$\underline{a=3}$ $\quad [3] = \{x \in \mathbb{Z} \mid x = 7k+3, \text{ for some integer } k\}$

$= \{\ldots -11, -4, 3, 10 \ldots\}$

$\underline{a=4}$ $\quad [4] = \{x \in \mathbb{Z} \mid x = 7k+4, \text{ for some integer } k\}$

$= \{\ldots -10, -3, 4, 11 \ldots\}$

$\underline{a=5}$ $\quad [5] = \{x \in \mathbb{Z} \mid x = 7k+5, \text{ for some integer } k\}$

$= \{\ldots -9, -2, 5, 12 \ldots\}$

$\underline{a=6}$ $\quad [6] = \{x \in \mathbb{Z} \mid x = 7k+6, \text{ for some integer } k\}$

$= \{\ldots -8, -1, 6, 13 \ldots\}$

Hence, $[0], [1], [2], [3], [4], [5], [6]$ represents the distinct Equivalence classes of Equivalence Relation R.

5 Find the smallest relation containing the Relation $R = \{(1,3), (1,4), (2,2), (4,1)\}$ on $\{1,2,3,4\}$ that is (a) reflexive on $\{1,2,3,4\}$ and symmetric (b) symmetric and transitive (c) reflexive on $\{1,2,3,4\}$ and transitive (d) an equivalence relation on $\{1,2,3,4\}$

Ans: a) Reflexive Closure: The reflexive closure of R is, $R \cup \Delta = R \cup \{(a,a) \mid a \in A\}$

(Δ is diagonal Relation on A) $\therefore R \cup \Delta = \{(1,3),(1,4),(2,2),(4,1)\} \cup \{(1,1),(2,2),(3,3),(4,4)\}$

$= \{(1,3),(1,4),(2,2),(4,1),(1,1),(3,3),(4,4)\}$

Symmetric Closure:

Symmetric Closure of $(R \cup \Delta)$ is $(R \cup \Delta) \cup (R \cup \Delta)^{-1}$

$\Rightarrow (R\cup\Delta)^{-1} = \{(b,a) \mid (a,b) \in (R \cup \Delta)\}$

$= \{(3,1),(4,1),(2,2),(1,4),(1,1),(3,3),(4,4)\}$

$\therefore (R\cup\Delta) \cup (R\cup\Delta)^{-1} = \{(1,1)(2,2)(3,3)(4,4)(1,3)(3,1)(1,4)(4,1)\}$ Ans.

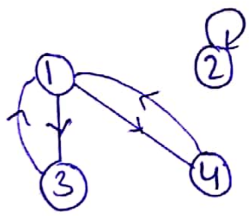
b) Symmetric Closure of R is $R \cup R^{-1}$

$\Rightarrow R^{-1} = \{(3,1),(4,1),(2,2),(1,4)\}$

$\therefore R \cup R^{-1} = \{(1,3)(3,1)(1,4)(4,1)(2,2)\cancel{(1,4)}\cancel{(4,1)}\}$ {as (1,4) and (4,1) were redudant}

→ Transitive Closure of R.

$R^{(n)}[i,j] = R^{(n-1)}[i,j]$ or $(R^{(n-1)}[i,n]$ and $R^{(n-1)}[n,j])$ [Using Warshall Theorem]



$R^0 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $R^1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

$R^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$, $R^3 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$, $R^4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

$\therefore$ Transitive Closure $= \{(1,1)(1,3)(1,4)(2,2)(3,1)(3,3)(3,4)(4,1)(4,3)(4,4)\}$

$\therefore$ Symmetric & Transitive Closure $= \{(1,1)(2,2)(3,3)(4,4)(1,3)(3,1)(1,4)(4,1)(3,4)(4,3)\}$ Ans

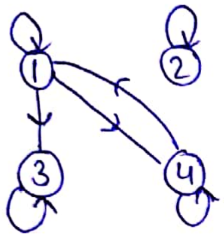
(c) Reflexive closure

$R \cup \Delta = \{(1,3)\ (1,4)\ (2,2)\ (4,1)\} \cup \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$

$= \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,3)\ (1,4)\ (4,1)\}$

Transitive Closure of $R \cup \Delta$

$R^n[i,j] = R^{n-1}[i,j]$ or $(R^{n-1}[i,n]$ and $R^{n-1}[n,j])$ [using Warshall Theorem]



$$R^0 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad R^1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad R^2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$R^3 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad R^4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$\therefore$ Transitive Closure $= \{(1,1)\ (1,3)\ (1,4)\ (2,2)\ (3,3)\ (4,1)\ (4,3)\ (4,4)\}$

$\therefore$ Reflexive & Transitive closure $= \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,3)\ (1,4)\ (4,1)\ (4,3)\ \cancel{(4,4)}\}$

(as (4,4) is redundant)

d) <u>Reflexive Closure</u>

$R \cup \Delta = \{(1,3)\ (1,4)\ (2,2)\ (4,1)\} \cup \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$

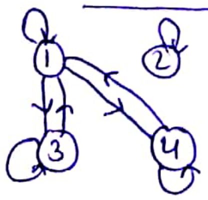
$= \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,3)\ (1,4)\ (4,1)\}$

<u>Symmetric Closure</u>

$R \cup R^{-1} = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,3)\ (1,4)\ (4,1)\} \cup \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (3,1)\ (4,1)\ (1,4)\}$

$= \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,3)\ (1,4)\ (3,1)\ (4,1)\}$

<u>Transitive Closure</u>



$$R^0 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad R^1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad R^2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$R^3 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad R^4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$\therefore$ Transitive Closure $= \{(1,1)\ (1,3)\ (1,4)\ (2,2)\ (3,1)\ (3,3)\ (3,4)\ (4,1)\ (4,3)\ (4,4)\}$

$\therefore$ for Equivalence Relation $=$ (Reflexive closure) $\cup$ (Symmetric Closure) $\cup$ (Transitive Closure)

Ans. $= \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,3)\ (1,4)\ (3,1)\ \cancel{(3,3)}\ (3,4)\ (4,1)\ (4,3)\ \cancel{(4,4)}\}$

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Prove that power set of natural numbers, $P(N)$, is not countable using diagonalization argument.

is: To prove the uncountability of power set of natural numbers, $P(N)$, we will use the Cantor's diagonalization theorem.

→ Cantor's Power Set Theorem states that if s is any set, then there is an injection from s to $P(s)$ but no bijection, so $|s| < P(s)$. In Particular it follows that $P(N)$ is uncountable.

So, let's try to prove it by Contradiction,

Let's assume that $P(N)$ is countable

$\forall x$, $S \subseteq P(N)$ by a decimal number of the form

$0.x_0 x_1 \ldots x_n \ldots$

where each $x_n \in \{0,1\}$ such that $x_n = 0$ if $n \in S$ and $x_n = 1$ if n does not belong to A.

According to our assumption that $P(N)$ is countable, so there exist a bijective function $f: N \to P(N)$ defined for all $n \in N$ by

$f(n) = S_n$

So, we will list the sets $\{S_1, S_2, S_3 \ldots S_n \ldots\}$ and thier decimal representations as well.

$S_1 \quad 0.x_{1,1}\, x_{1,2}\, x_{1,3} \ldots x_{1,n}$

$S_2 \quad 0.x_{2,1}\, x_{2,2}\, x_{2,3} \ldots x_{2,n}$

$\vdots$

$S_n \quad 0.x_{n,1}\, x_{n,2}\, x_{n,3} \ldots x_{n,n}$

So, now let us construct a decimal number such that $x = 0.x_1 x_2 x_3 \ldots$

$x_1 = 0$ if $x_{1,1} = 1$ & $x_2 = (x_{2,2})'$, $x_3 = (x_{3,3})'$ i.e, if $x_{2,2} = 1$, we make $x_2 = 0$ & viceversa,

So, in general we take $x_n = 0$ if $x_{n,n} = 1$ & $x_n = 1$ if $x_{n,n} = 0$

i.e, the new number x differ from all the above mentioned numbers in atleast one decimal place.

So, x does not maps & it fails the principle of Bijection, So it is a contradiction to our assumption.

Hence, $P(N)$ is uncountable.

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Q.5 Determine whether or not the following set is countable : the set $A = \{a^2 \mid a \in N\}$ where N is the set of natural number

Ans: We say, a given set S is countable if there is one-to-one correspon-dance with the natural number N.

∴ To Prove that the given set A is a countable set we need to establish a Bijection between N and A, where N represents a set of Natural Numbers and A represents a set of squares of Natural Numbers.

i.e, $f : N \to A$

$\therefore \quad f(a) = a^2$

To Show this if f is a one-one function.

$f(a) = f(b)$, where $a, b \in N$

$a^2 = b^2$

Now, Taking square root of both sides, we get

$a = b$

which indicates that f is one-to-one function.

Now, to prove that f is onto,

$\forall n \in A \quad \exists p \in N \Rightarrow f(a) = n$

$n = a^2$

Now, since f is both one-one & onto, it is Bijection.

i.e, A is a countable set as it has one-to-one correspondance with N i.e, set of Natural Numbers.

∴ $f(n)$ is Bijection

$\Rightarrow$ $A = \{a^2 \mid a \in N\}$ is countable

Q6 Lemma 1: set of all computer programs in any Particular language is countable.

proof: For any finite alphabets, there are finite no: of strings of length n

$\forall\ n \in N$. ~~the~~ — ①

The union of countable number of countable sets is countable — ②

From ① & ② There are countable no: of strings from any finite alphabet set.

set of all programs is a subset of set of all strings of a finite alphabet (language)

∴ set of all computer programs in any particular language is countable.

To prove that there exist an uncomputable function is same as proving that there are uncountably many different functions from a countably infinite set to itself.

Funcs ↕ Progs \ I/p's	1	2	3	4	- - - -
1	1	0	1	0	- - -
2	1	0	1	1	- - -
3	1	1	0	0	- -
4	1	1	1	0	- -
⋮	⋮	⋮	⋮	⋮	

proof by contradiction!

say no: of Func's are countable, Then they can be arranged in one-one correspondence with programs (Lemma 1)

Assume for countable no: of I/p's

If prog accept I/p then the matrix shows 1 else it shows 0

consider the diagnol of that ~~string~~ Matrix

consider a string which inverts every bit in diagnol of matrix

say S = 0 1 1 1 - - - . This state cannot be represented by any prog as it atleast differ with all the other states by atleast one-bit.

∴ Functions and programs are not in bijection.

i.e, no: of Functions possible are uncountable

$\Rightarrow$ $\exists$ function which is not computable.

Q 7) A: Let $f: A \to B$ and $g: B \to C$ be two functions. su Prove that if the composite function $gof: A \to C$ is injective the f is injective.

Soln: A function is said to be injective when a_1 and $a_2 \in A$ and $F(a_1) = F(a_2)$ then $a_1 = a_2$

Let x and $y \in A$.

Let say $f(x) = f(y)$ for some $x, y \in A$

~~The~~ $gof = g(f(x))$

$\therefore \quad g(f(x)) = g(f(y))$

$\therefore \quad gof(x) = gof(y)$

So this implies that if $\cancel{x=y}$ ~~then~~ gof is injective when $x = y$.

$\therefore$ Thus ~~the~~ ~~fs~~ proved that f is injective.

7B Find a function g such that $h = gof$ and $h(x) = 10x + 10$, $f(x) = 2x + 1$. all functions are defined over the set $\mathbb{R}$ of real numbers where gof is the composite function.

Soln: $f(x) = 2x+1 \qquad h(x) = 10x + 10$

$h(x) = gof = \cancel{g(2x+1)}\ g(f(x)) = g(2x+1)$

$g(x) = h(x)$

$g(2x+1) = 10x + 10$

$k(2x+1) + c = 10x + 10$

$2k = 10 \qquad k + c = 10$

$k = 5 \qquad c = 5$

So function $g(x) = 5x + 5 \quad \forall \mathbb{R}$

(3) To prove set of real no R is infinite.

sol:- we proof it by contradiction:-

so we assume that R is finite, which means A sets is said to be finite if there is a 1-1 correspondence between S and a proper subset of S.

=) subset of a countable set is also countable.

subset i.e [0, 1] should also be countable.

let us try to list between [0, 1]:

$r_1 = 0 \cdot d_{11} d_{12} d_{13} \cdots$

$r_2 = 0 \cdot d_{21} d_{22} d_{23} \cdots$

$r_3 = 0 \cdot d_{31} d_{32} d_{33} \cdots$

$r_4 = 0 \cdot d_{41} d_{42} d_{43} \cdots$

$r_5 = 0 \cdot d_{51} d_{52} d_{53} \cdots$

where $d_{ij} = \{0, 1, 2, 3, 4, \ldots 9\}$

$$d_{ij}^{\circ} = \begin{cases} 8 & \text{if } d_{ij} \neq 8 \\ 9 & \text{if } d_{ij} = 9 \end{cases}$$

so we are generated no that was not present in the previous list.

→ All numbers (real) between 0 and 1 can't be listed so they are uncountable. Any set with an uncountable subset is also uncountable.

so we can say that real no set is uncountable proved.

Q.9 You know that if $f(x)$ and $g(x)$ are functions from the $R \to R$ (the real numbers), then $f(x)$ is $\Theta(g(x))$ if and only if there are po[sitive] constants k, C_1 and C_2 such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$ whenev[er] $x > k$. Now show that $3x^2+x+1$ is $\Theta(3x^2)$ by directly finding th[e] constants k, C_1 and C_2. Express this Θ relationship using a picture showing the functions $3x^2+x+1$, $C_1 \cdot (3x^2)$ and $C_2 \cdot (3x^2)$ and the constant k on the x-axis, where C_1, C_2 and k are the constants found earlier to show that $3x^2+x+1$ is $\Theta(3x^2)$.

Ans: f is $\Theta(g)$, if f is both $O(g)$ and $\Omega(g)$

→ Upperbound (O)

$f(n) = O(g(n)) \iff \exists\, c_1, n_0$ such that

$0 \leq f(n) \leq c_1 g(n) \quad , \forall n \geq n_0$

→ Lower Bound (Ω)

$f(n) = \Omega(g(n)) \iff \exists\, c_2, n_0$ such that

$f(n) \geq c_2 g(n) \quad \forall n \geq n_0$

⇒ if $f(n) = \Theta(g(n)) \quad \exists\, c_1, c_2, n_0$, such that

$c_1(g(n)) \leq f(n) \leq c_2(g(n))$

→ We know that $f(x)$ & $g(x)$ are functions from $R \to R$, then

$f(x) = \Theta(g(x))$ iff $\exists$ +ve constants k, c_1 & c_2 such that

$c_1 g(x) \leq f(x) \leq c_2 g(x)$ whenever $x > k$.

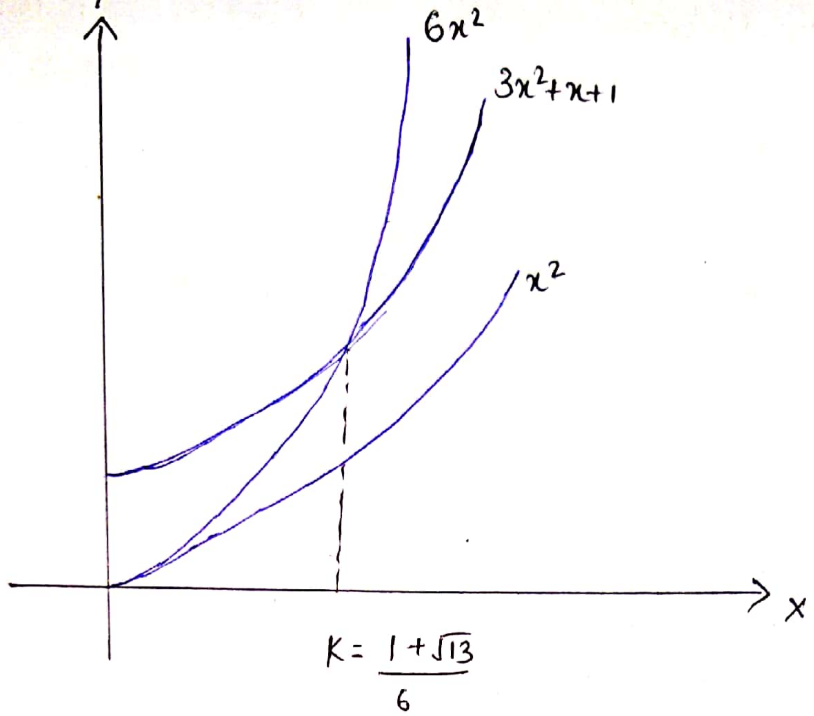
Now, we have to show that $3x^2+x+1$ is $\Theta(3x^2)$ by directly finding constants k, c_1 & c_2.

Now, let us suppose $c_2 = 2$ & $c_1 = \frac{1}{3}$

$\therefore \quad c_1 g(x) \leq f(x) \leq c_2 g(x)$

$\frac{1}{3} \times 3x^2 \leq (3x^2+x+1) \leq 2 \times 3x^2$

$x^2 \leq (3x^2+x+1) \leq 6x^2$



<u>Proof for K:</u>

$$6x^2 = 3x^2 + x + 1$$

$$-3x^2 + x + 1 = 0$$

$$\therefore \quad x = \frac{-1 \pm \sqrt{1+4\times 3}}{2\times(-3)}$$

$$x = \frac{-1 \pm \sqrt{13}}{-6} \qquad \therefore \quad x = \frac{1+\sqrt{13}}{6} \quad (\text{or}) \quad x = \frac{1-\sqrt{13}}{6}$$

Ans.

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10. A) Arrange the functions $(1.5)^n$, n^{100}, $(\log n)^3$, $\sqrt{n}\log n$, 10^n, $(n!)^2$, and $n^{99}+n^{98}$ in a list so that each function is big-O of next function. Give brief justification.

Sol.
- $\underline{(1.5)^n}$ is exponential. time complexity. (base 1.5)
- $\underline{(n^{100})}$ is polynomial time complexity. $(O(n^{100}))$
- $\underline{(\log n)^3}$ is logarithmic time complexity
- $\underline{\sqrt{n}\log n}$ is linearithmic where time complexity is greater that $O(n^{1/2})$ but clower than $O(n)$
- $\underline{10^n}$ is exponential with base 10.
- $\underline{(n!)^2}$ has factorial time complexity.
- $\underline{n^{99}+n^{98}}$ has polynomial time complexity $(O(n^{99}))$

Now arrangement of omplexities is as fallows:

$$\underline{(\log n)^3, \sqrt{n}\log n, n^{99}+n^{98}, n^{100}, 1.5^n, 10^n, (n!)^2}$$

Where each function is big O of next function.

Q 10 (B)

For n terms product is of form

$$P = 1 \times 3 \times 5 \times 7 - - - \times (2n-1)$$

$$P = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 - - - - \times (2n-2) \times (2n-1)}{2 \times 4 \times 6 \times - - - (2n-2)}$$

$$P = \frac{(2n-1)! \times 2n}{2^{n-1} (n-1)! \; 2n}$$

$$P = \frac{(2n)!}{2^n (n!)}$$

As $n!$ can be estimated as n^n i.e $n! = O(n^n)$

$$P \leq \frac{(2n)^{2n}}{2^n (n^n)}$$ [as rate of growth of numerator is greater than denominator $P \leq$ ratio]

$$P \leq 2^n \cdot n^n$$

$\therefore$ $\boxed{P = O(2^n \cdot n^n)}$