SOLVING RECURRENCE RELATIONS

CHINESE RING PUZZLE

$$T(n) = 2T(n-2) + T(n-1) + 1 \quad \text{—} \; ①$$

$T(1) = 1 \qquad T(2) = 2$

$$T(n) = \begin{cases} 2T(n-1) + 1 & \text{if } n \text{ is odd} \\ 2T(n-1) & \text{if } n \text{ is even} \end{cases} \quad \text{—} \; ②$$

closed form solution

case 1: n is even

$(n-1)$ is odd, so

$$T(n-1) = 2T(n-2) + 1 \quad \text{— from ②}$$

sub $T(n-1)$ in ①

$$\cancel{T(n) = 2(2T(n-2)+n)}$$

$$T(n) = 2T(n-2) + 2T(n-2) + 1 + 1$$

$$\Rightarrow T(n) = 4T(n-2) + 2$$

SUBSTITUTION METHOD

$$T(n) = 4T(n-2) + 2$$

$$T(n-2) = 4T(n-4) + 2$$

$$\therefore T(n) = 4(4T(n-4)+2)+2$$

$$= 4^2 T(n-4) + 2(4) + 2$$

$$T(n) = 4^3 T(n-6) + 2(4)^2 + 2(4) + 2$$

$$T(n) = 4^r T(n-2r) + 2\{4^{r-1} + \dots 1\}$$

$$= 4^r T(n-2r) + 2\left(\frac{4^r-1}{3}\right)$$

to hit base case $n-2r=2$

$$\Rightarrow r = \frac{n-2}{2}$$

$$T(n) = 2^{n-2} T(2) + 2\left(\frac{2^{n-2}-1}{3}\right)$$

$$= 2^{n-1} + \frac{2^{n-1}-2}{3}$$

$$= \frac{3.2^{n-1} + 2^{n-1} - 2}{3}$$

$$= \frac{2}{3}\left(2^n - 1\right)$$

$$\frac{2}{3}\left(2^6-1\right) = \frac{2}{3}(64-1) = 42 \text{ steps.}$$

hamming distance ... gray code ... uninteresting graph.

case 2: n is odd

(n-1) is even, so from (2)

$T(n-1) = 2T(n-2)$

sub. $T(n-1)$ in (1)

$T(n) = 2T(n-2) + 2T(n-2) + 1$

$= 4T(n-2) + 1$ ——— (4)

base case $T(1) = 1$

$T(n-2) = 4T(n-4) + 1$

$T(n) = 4(4T(n-4) + 1) + 1$

$= 4^2 T(n-4) + 4(1) + 1$

$T(n) = 4^r T(n-2r) + 4^{r-1} + \dots 1$

$= 4^r T(n-2r) + \left(\frac{4^r - 1}{3}\right)$

to hit base case $n - 2r = 1$

$\Rightarrow r = \frac{n-1}{2}$

$T(n) = 2^{n-1} \cdot 1 + \left(\frac{2^{n-1} - 1}{3}\right)$

$$\Rightarrow T(n) = \frac{3 \cdot 2^{n-1} + 2^{n-1} - 1}{3}$$

$$= \frac{2^{n+1} - 1}{3}$$

$$\therefore T(n) = \frac{1}{3}\left(2^{n+1} - 1\right)$$

METHODS OF SOLVING RRS

1. repeated substitution — divide & conquer
2. master theorm — shortcut for repeated substitution.
3. change of variables
4. guess the answer → prove by induction
5. linear homogeneous & heterogeneous eqns.
6. linear algebra methods (diagonalization) — X cantor's
   - eigen values
   - eigen vectors

ex1 $T(n) = 2T\left(\frac{n}{2}\right) + n$

$T(1) = 1$

repeated substitution ... r times.

$$T(n) = 2^r T\left(\frac{n}{2^r}\right) + nr$$

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$\frac{n}{2^r} = 1 \qquad n = 2^r \qquad \Rightarrow \underline{r = \lg n}$ (base 2)

$T(n) = n.1 + n.\lg n$

$T(n) = \Theta(n \lg n)$

ex 2 $T(n) = 3T\left(\frac{n}{2}\right) + n$

$T(1) = 1$

repeated substitution

$T(n) = 3^r T\left(\frac{n}{2^r}\right) + n\left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \ldots \left(\frac{3}{2}\right)^{r-1}\right)$

$= 3^r T\left(\frac{n}{2^r}\right) + n\left(\frac{(3/2)^r - 1}{3/2 - 1}\right)$

$= 3^r T\left(\frac{n}{2^r}\right) + 2n\left(\left(\frac{3}{2}\right)^r - 1\right)$

$\frac{n}{2^r} = 1 \Rightarrow n = 2^r \Rightarrow \underline{r = \lg n}$

$T(n) = 3^{\lg n} T(1) + 2n\left(\left(\frac{3}{2}\right)^{\lg n} - 1\right)$

$\Rightarrow T(n) = n^{\lg 3} + 2n\left(n^{\lg 3/2} - 1\right)$

exercise

$$\therefore T(n) = n^{\lg 3} + 2n\left(n^{\lg 3 - 1} - 1\right)$$

$$= n^{\lg 3} + 2\left(n^{\lg 3} - n\right)$$

$$= 3n^{\lg 3} - 2n$$

$$T(n) = \Theta\left(n^{\lg 3}\right)$$

ex3 $T(n) = 3T\left(\frac{n}{2}\right) + n^2$

repeated substitution

$$T(n) = n^{\lg 3} + n^2\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \dots \left(\frac{3}{4}\right)^{r-1}\right)$$

$$T(n) = \Theta(n^2)$$

MASTER THEOREM

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

soln. of $T(n)$
(or $T(n)$ asymptotically bounded as follows)

(1) if $f(n) = O\left(n^{\log_b a - \epsilon}\right)$

for some $\epsilon > 0$, then $T(n) = \Theta\left(n^{\log_b a}\right)$

(2) if $f(n) = \Theta\left(n^{\log_b a}\right)$ then $T(n) = \Theta\left(n^{\log_b a} \cdot \lg n\right)$

(3) if $f(n) = \Omega\left(n^{\log_b a + \epsilon}\right)$ for some $\epsilon > 0$, then

$T(n) = \Theta(f(n))$

if $a f\left(\frac{n}{b}\right) \leq k f(n)$ for some constant $k < 1$
and for sufficiently large n.

regularity condition.

ex1 $T(n) = 2T\left(\frac{n}{2}\right) + n$

case 2

ex2 $T(n) = 3T\left(\frac{n}{2}\right) + n$

$n = O\left(n^{\log_2 3}\right)$ ∴ case 1

$= \Theta\left(n^{\log_2 3}\right)$

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$n^{\lg 3}$ vs n^2 (Case 3)

$T(n) = \Theta(n^2)$

reg. cond. $3f\left(\frac{n}{2}\right) \leq k f(n)$

$\frac{3n^2}{4} \leq k n^2$

$k = 0.9$ (satisfied).

$\therefore$ by case 3
$T(n)$ is of the order of $f(n)$

ex $T(n) = 9T\left(\frac{n}{3}\right) + n$

case 1 $T(n) = \Theta(n^3)$

ex $T(n) = T\left(\frac{3n}{5}\right) + 1$ $n^{\log_{5/3} 1}$

case 2 $T(n) = \Theta(\lg n)$

ex $T(n) = 3T\left(\frac{n}{4}\right) + n\lg n$

$n^{\log_4 3} \to 0.8$ vs $n\lg n$

reg. cond.

$af\left(\frac{n}{b}\right) \leq kf(n)$

$3\frac{n}{4}\lg\frac{n}{4} \leq kn\lg n$

$\frac{3}{4}n(\lg n - 2) \leq kn\lg n$

$\frac{3}{4}n\lg n - \frac{3n}{2} \leq kn\lg n$

$k = \frac{3}{4}$

case 3 $T(n) = \Theta(n\lg n)$

ex $T(n) = 2T\left(\frac{n}{2}\right) + n\lg n$

$n^{\lg 2}$ vs $n\log n$

$\log n < n^{\epsilon}$

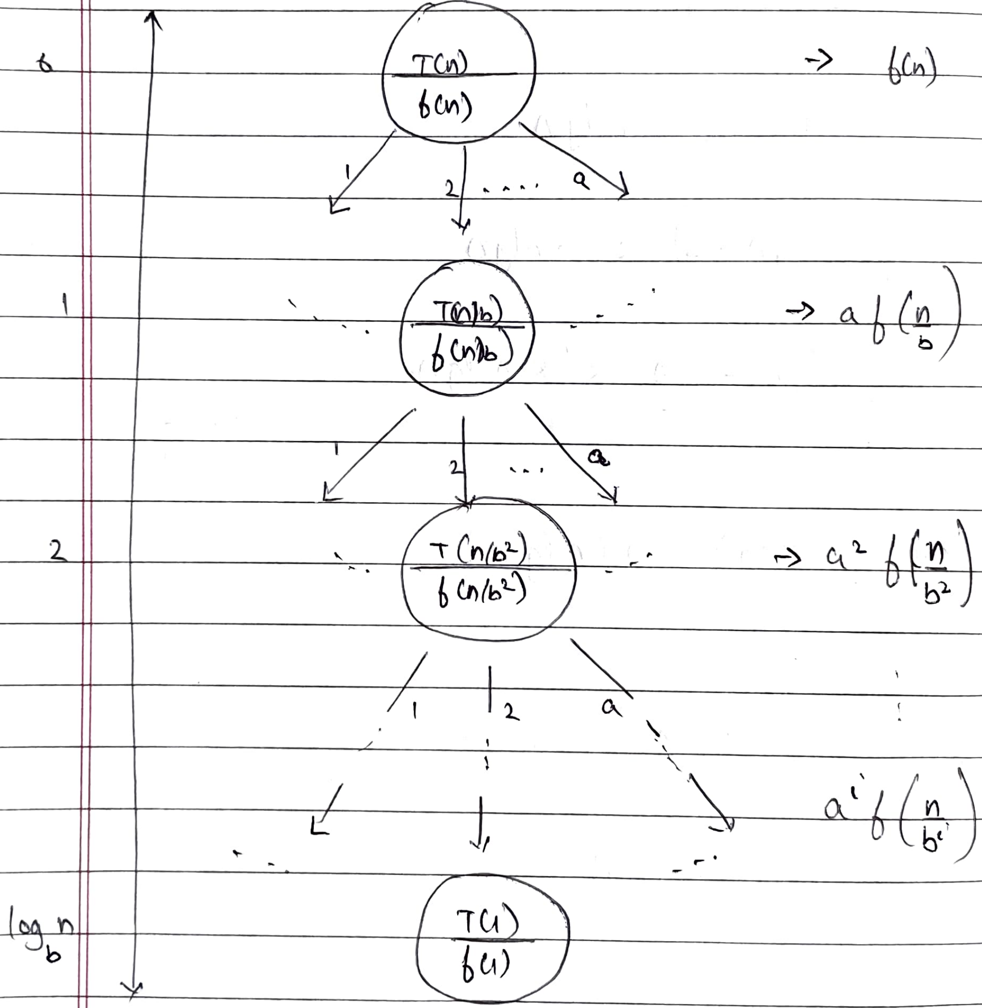
for any $\epsilon > 0$

$f(n)$ is not polynomially larger than n.
We can't apply master's theorm.

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

$$T(1) = d.$$

RECURSION TREE



root node

total levels $= \log_b n + 1$

critical exponent,

no. of leaf nodes $= a^{\log_b n} = n^{\log_b a}$

$T(n) =$ sum of all the elements

$$= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) + d \, n^{\log_b a}$$

$a \geq 1$

$b > 1$

$$T(n) = \underbrace{\sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)}_{1} + \underbrace{\Theta\left(n^{\log_b a}\right)}_{2}$$

Case 1.

if the second term dominates (sum of leaf nodes) then the complexity $\rightarrow \Theta\left(n^{\log_b a}\right)$

Case 2:

if neither dominates:

$\rightarrow \Theta\left(n^{\log_b a} . \log n\right)$

Case 3:

if the first term dominates: (sum of internal nodes)

$\rightarrow \Theta(f(n))$

Work on a small numerical example where you can generate a recursion tree. draw recursion tree for computing.

$T(n) = 2T\left(\frac{n}{3}\right) + 5n$ where $n = 27$

$T(0) = 7$

Σ		n								fib
1	1									1
2	1	1								1
4	1	2	1							2
8	1	3	3	1						3
16	1	4	6	4	1					5
32	1	5	10	10	5	1				8
64	1	6	15	20	15	6	1			13
128	1	7	21	35	35	21	7	1		fib.
256	1	8	28	56	70	56	28	8	1	

1, 1, 2, 3, 5, 8, 13, 21 sum fib.

A

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \dots \binom{n}{n} x^0 y^n$$

how to prove:

$$\binom{n}{0} + \binom{n}{1} + \dots \binom{n}{n} = 2^n$$

1 - by formula (factorial)
2 - by binomial theorm
3 - by induction
4 - by counting argument. (combinatorial proof)

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \dots \binom{n}{n} \checkmark$$

- by binomial theorm.

2^n = size of power set

~~$\binom{n}{0}$ = no. of subsets with size 0~~

$\binom{n}{0}$ = no. of ways to choose 0 1's
$\binom{n}{1}$ = no. of ways to choose 1 1's
$\binom{n}{n}$ = no. of ways to choose n 1's.

} sum of size of all subsets

= size of power set

= size of all subsets

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thm pascal's identity

let n, k be positive integers, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

row

column

combinatorial argument

Choose k people from n

= choose k people from $n-1$ without X

+ choose $k-1$ people from $n-1$ with X

to prove:

$$\binom{n}{0} + \binom{n}{1} + \dots \binom{n}{n} = 2^n$$

~~thus~~ proof by induction

base case $n=0$ $\quad 2^0 = 1$

$$\binom{0}{0} = 1$$

inductive assumption

let (I) be true for k

i.e., $\binom{k}{0} + \binom{k}{1} + \dots \binom{k}{k} = 2^k$

to prove (I) is true for $k+1$

i.e. $\binom{k+1}{0} + \binom{k+1}{1} + \dots \binom{k+1}{k} + \binom{k+1}{k+1} = 2^{k+1}$

$$LHS = \binom{K+1}{0} + \binom{K+1}{1} + \dots \binom{K+1}{K} + \binom{K+1}{K+1}$$

$$= \binom{K+1}{0} + \binom{K}{0} + \binom{K}{1} + \binom{K}{1} + \binom{K}{2} + \dots$$

$$+ \binom{K}{K-2} + \binom{K}{K-1} + \binom{K}{K-1} + \binom{K}{K} + \binom{K+1}{K+1}$$

$$= \binom{K}{0} + \binom{K}{0} + \binom{K}{1} + \binom{K}{1} + \binom{K}{2} + \dots$$

$$\dots \binom{K}{K-2} + \binom{K}{K-1} + \binom{K}{K-1} + \binom{K}{K} + \binom{K}{K}$$

$$= 2\left\{ \binom{K}{0} + \binom{K}{1} + \dots \binom{K}{K-1} + \binom{K}{K} \right\}$$

$$= 2 \cdot 2^K = 2^{K+1} = RHS \checkmark$$

thm $\binom{2n}{2} = 2\binom{n}{2} + n^2$

prove by counting (combinatorial argument)

~~choose~~ let there be 2 groups of people A, B each of size n.

LHS = no. of ways to choose people from groups A and B combined

= no. of ways to choose 2 people from group A
+ no. of ways to choose 2 people from group B
+ no. of ways to choose 1 person from group A and another person from group B.

$= \binom{n}{2} + \binom{n}{2} + n \times n$

$= 2\binom{n}{2} + n^2 = \text{RHS} \checkmark$

GENERALIZED PERM. (WITH ~~RECURSION~~ REPETITION)

ex how many 5 digit binary nos. are there.

$2 \times 2 \times 2 \times 2 \times 2 = 2^5$

ex how many base 3 nos. with atleast one zero of length n.

use to principle of exclusion

no. of nos. of base 3 − no. of base 3 nos. without zero

$= 3^n - 2^n$

ex how many base 10 nos. of length n in atleast 3 zeros.

$= 10^n - 9^n - n.9^{n-1} - \binom{n}{2} 9^{n-2}$

$= 10^n - \left\{ \binom{n}{0} 9^n + \binom{n}{1} 9^{n-1} + \binom{n}{2} 9^{n-2} \right\}$

ex BABALOO permutations (with repetitions)

$$\frac{7!}{2!\,2!\,2!\,1!}$$

$$P(n, n_1, n_2, n_3, \ldots n_k) = \frac{n!}{n_1!\, n_2!\, n_3! \ldots n_k!}$$

B A O L

$$\binom{7}{2} \times \binom{5}{2} \times \binom{3}{2} \times \binom{1}{1}$$

$$= \frac{7!}{5!\,2!} \cdot \frac{5!}{3!\,2!} \cdot \frac{3!}{1!\,2!} \cdot \frac{1!}{0!\,1!}$$

$$= \frac{7!}{2!\,2!\,2!\,1!}$$

COMBINATIONS WITH REPETITIONS

ex ₹5 for 5 biscuits from 2 jars.

$$\underbrace{X + Y}_{\text{jars}} = 5$$

jar1 jar2
B B | B B B
separator

$$\binom{6}{1} = 6$$

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distribution of m non-distinct objects into n distinct groups.