Q1: Prove, without using the Venn diagram that $A-B$, $B-A$ and $A\cap B$ are pairwise disjoint, where A and B are two sets.

Ans 1 Proof :- Let A and B be the two sets

(i) $(A-B)\cap(B-A)=\phi \Rightarrow (A-B)$ & $(B-A)$ are disjoint.

$(A\cap\bar{B})\cap(B\cap\bar{A})$ [As $A-B = A\cap\bar{B}$]

$= A\cap(\bar{B}\cap B)\cap\bar{A}$ [By Associative law]

$= A\cap\phi\cap\bar{A}$ [By complement law $A\cap\bar{A}=\phi$]

$= \phi\cap\bar{A} = \phi$ [By domination law $A\cap\phi=\phi$]

(ii) $(A-B)\cap(A\cap B) = \phi \Rightarrow (A-B)$ & $A\cap B$ are disjoint

$= (A\cap\bar{B})\cap(A\cap B)$

$= (A\cap A)\cap(B\cap\bar{B})$ [Using Associative & commutative law]

$= A\cap\phi$ [By domination ~~complement~~ law]

$= \phi$

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(iii) $(B-A) \cap (A \cap B) = \phi \Rightarrow B-A$ & $A \cap B$ are disjoint sets

$= (B \cap \bar{A}) \cap (A \cap B)$

$= B \cap (\bar{A} \cap A) \cap B$ [Associative law]

$= B \cap \phi \cap B$ [Complement law $A \cap \bar{A} = \phi$]

$= \phi$ [Domination law]

from (i), (ii) & (iii),

$(A-B)$, $(B-A)$ & $(A \cap B)$ are pairwise disjoint sets.

② $(A \cap \phi) \cap A \Rightarrow \phi \cap A \Rightarrow \phi$. Hence proved.

Q2 Let $(x, y) \in A \times (B \cup C)$. Then, by definition of cartesion product, $x \in A \wedge y \in (B \cup C)$

$\Rightarrow x \in A \wedge (y \in B \vee y \in C)$

$\Rightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$ [using distributive property of $\wedge$ over $\vee$]

$\Rightarrow (x,y) \in (A \times B) \vee (x,y) \in (A \times C)$ [By defn of cartesion product]

$\Rightarrow (x,y) \in (A \times B) \cup (A \times C)$

So. $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

→ Let $(x,y) \in (A \times B) \cup (A \times C)$

[By defn of union]

$((x,y) \in (A \times B)) \vee ((x,y) \in (A \times C))$

[By defn of cartesion product]

$(x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$

[By distributive property]

$x \in A \wedge (y \in B \vee y \in C)$

$x \in A \wedge (y \in (B \cup C))$

$(x,y) \in A \times (B \cup C)$

So, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

Hence, $A \times (B \cup C) = (A \times B) \cup (A \times C)$

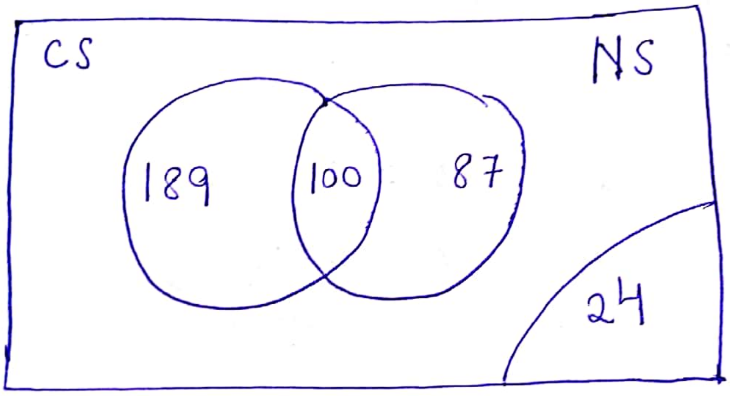
Q3 $\sqrt{50} = 7$. Number of primes $\leq 7 \Rightarrow 2, 3, 5, 7 \Rightarrow 4$

$$\pi(50) = 50 - 1 + \pi(\sqrt{50}) - \left(\left\lfloor \frac{50}{2} \right\rfloor + \left\lfloor \frac{50}{3} \right\rfloor + \left\lfloor \frac{50}{5} \right\rfloor + \left(\left\lfloor \frac{50}{7} \right\rfloor\right)\right) + \left(\left\lfloor \frac{50}{2\cdot 3} \right\rfloor + \left\lfloor \frac{50}{2\cdot 5} \right\rfloor + \left\lfloor \frac{50}{2\cdot 7} \right\rfloor + \left\lfloor \frac{50}{3\cdot 5} \right\rfloor + \left\lfloor \frac{50}{3\cdot 7} \right\rfloor + \left\lfloor \frac{50}{5\cdot 7} \right\rfloor\right) - \left(\left\lfloor \frac{50}{2\cdot 3\cdot 5} \right\rfloor + \left\lfloor \frac{50}{2\cdot 3\cdot 7} \right\rfloor + \left\lfloor \frac{50}{3\cdot 5\cdot 7} \right\rfloor\right) + \left(\left\lfloor \frac{50}{2\cdot 3\cdot 5\cdot 7} \right\rfloor\right)$$

$$= 50 - 1 + 4 - (58) + (22) - (2) + 0$$

$$= 15$$

Q4



$|CS| \rightarrow$ Students taking Computer Science Course
$|NS| \rightarrow$ Students taking Natural Science Course

$$|CS \cup NS| = |CS| + |NS| - |CS \cap NS|$$
$$= 289 + 187 - 100$$
$$= 376.$$

Students taking Neither of Courses $= |U| - |CS \cup NS|$
$= 400 - 376$
$= 24.$

Q5: Let $P(n)$ be the statement "any postage of ₹35 or more can be paid using ₹5 or ₹9 stamps."

There are 5 base cases here, as follows:

$P(35) = 5(7) + 9(0)$

$P(36) = 5(0) + 9(4)$

$P(37) = 5(2) + 9(3)$

$P(38) = 5(4) + 9(2)$

$P(39) = 5(6) + 9(1)$

Note: after 5 base cases, you can see that we reached 35+4 and the next case is handled by considering 35+5 (40) and the cycle can now repeat.

Since the proof depends on multiple previous instances, this is strong induction.

Strong Inductive Hypothesis or assumption here is:

Let $P(k)$ hold for a, b

in $35 \le k \le j$, where $j \geq 39$.

i.e., k can be expressed as $k = 5a + 9b$, for integers $a, b \geq 0$.

Need to prove inductive step that $P(k+1)$ is true.

~~If $j \geq 39$, then if~~ We see that $P(k-4)$ is true since $35 \le k-4 \le k$

$\therefore$, by inductive assumption, $k - 4 = 5a + 9b$

adding 5 both sides; $k - 4 + 5 = 5(a+1) + 9b$

$k + 1 = 5(a+1) + 9b$.

$\therefore P(k+1)$ is true.

$\therefore$ By induction $P(n)$ is true for any $n \geq 35$. ∎

Prove that n is odd integer if and only if n^2 is an odd integer.

Ans Let p be a statement that n is an odd integer and

Q. 6 q be the statement that n^2 is an odd integer.

P.T $\Rightarrow p \Leftrightarrow q$

Proof: $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

① Assume that n is an odd integer, then by definition $n = 2k+1$ for some integer k. We will now use this to show that n^2 is also an odd integer.

$$n^2 = (2k+1)^2$$
$$= (2k+1)(2k+1)$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2+2k)+1 \quad \text{since 2 is common factor}$$
$$= \text{even} + 1$$
$$= \text{odd}$$

Hence we have shown that n^2 has the form of an odd integer since $2k^2+2k$ is an integer. $\therefore p \rightarrow q$.

② Now let n^2 be an odd integer. so $n^2 = 2q+1$.

$$n^2 = 2q+1 \quad (q \text{ is even})$$
$$n^2 = 2(2k^2+2k)+1 \quad q = 2k^2+2k$$
$$= 4k^2+4k+1$$
$$= (2k+1)^2$$
$$n = 2k+1.$$

Hence we have shown that n has the form of an odd integer.

$\Rightarrow q \rightarrow p$

$\Rightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

$\equiv p \leftrightarrow q$

Hence Proved.

Q7 Let a, b, and c be integers such that $a^2+b^2=c^2$. Prove that at least one of a and b is even.

Ans: Proof by contradiction –

Suppose that a & b both are odd.

Then,

$a = 2k+1$

$b = 2l+1$

where k, l are integers, and therefore

$a^2+b^2 = (2k+1)^2+(2l+1)^2$

$= 4k^2+4k+1+4l^2+4l+1$

$= 4(k^2+k+l^2+l)+2$

Case I: let c be even.

then it is divisible by 2 and so c^2 is divisible by 4.

However a^2+b^2 is equal to a multiple of 4 & 2 and so it is not divisible by 4.

Case II: let c be odd,

then so is c^2.

However in above a^2+b^2 is even.

and Thus Both are contradiction.

Therefore given statement is true.

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Q8: Because every Boolean function can be represented using the boolean operators: product (.), sum (+), and negation (¬) we say that $\{\neg, \cdot, +\}$ is functionally complete. We can find a smaller set of functionally complete operators. This can be done if one of the three operators of this set can be expressed in terms of the other two.
Now, show that the set $\{\cdot, \neg\}$ is functionally complete.

Ans 8: A set is functionally complete if the operators in the set can fully implement all operators in $\{\neg, \cdot, +\}$ or any other functionally complete set.

Let Our set $S = \{\cdot, \neg\}$ and A, B be two boolean variables

then $A \cdot B$ is implemented {Trivially}

$\neg A$ is implemented {Trivially}

$A+B = \neg(\neg(A+B))$ {Double negation law}

$= \neg(\neg A \cdot \neg B)$ {De morgan's law}

Since, A+B is implemented via operators in set S.
We conclude that Set $S = \{\cdot, \neg\}$ is functionally complete.

Q 9 A: $\oplus$

CNF: $(X \vee Y) \wedge (\neg X \vee \neg Y)$

DNF: $(\neg X \wedge Y) \vee (X \wedge \neg Y)$

Q 9B: $\oplus$ operator is not functionally complete.

- $\{\oplus, \neg\}$ is not complete. Intuitive explanation is as follows. If they were complete, we would be able to generate any four-bit sequence → a column of a truth table; [remember for two binary variables, for example, there are 16 possible functions we can define — corresponding to the 16 columns of truthtable] with these operators starting from, say, 0011, 0101 [see that both these starting sequences have even number of 0's or 1's]. $\oplus$ and $\neg$ would keep the number 1's even and so would never be able to generate 1000, for example (or other odd no. of 1s).

- $\{\oplus, \vee\}$ is not complete. Similar argument starting with 0011 or 0101, these operators $\oplus$ and $\vee$ cannot change the 0 in the first position of 0011 or 0101 to a 1, so they would not be able to generate any sequence starting with a 1, for example:

- $\{\oplus, \wedge\}$ is also not complete for a similar reason as above.

The sets $\{\oplus, \wedge, \neg\}$ and $\{\oplus, \vee, \neg\}$ are complete.

The formal arguments are too theoretical. One can look at Emil Post's results on functional completeness.

https://en.wikipedia.org/wiki/Functional_completeness.

Q10

To prove $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Need to prove (i) $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and (ii) $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

Q10 A) (i) Let $x \in \overline{A \cap B}$. Then x cannot be in both and B — at least one of the statements "$x \in \bar{A}$", "$x \in \bar{B}$" is true. Therefore $x \in \bar{A} \cup \bar{B}$ and hence $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$

(ii) Now let $x \in \bar{A} \cup \bar{B}$. Again, x cannot be in both A and B, so $x \notin A \cap B$; so $x \in \overline{A \cap B}$. Hence $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$.

From (i) and (ii), we can conclude that $\overline{A \cap B} = \bar{A} \cup \bar{B}$ ∎

Q10B: To prove $(B-A) \cup (C-A) = (B \cup C) - A$.

we need to show (i) $(B-A) \cup (C-A) \subseteq (B \cup C) - A$

and (ii) $(B \cup C) - A \subseteq (B-A) \cup (C-A)$.

(i) Let $x \in (B-A) \cup (C-A)$. Then $x \in (B-A) \vee x \in (C-A)$. Assume without loss of generality that $x \in B-A$. This implies that $x \in B \wedge x \notin A$. From this, We can conclude $x \in (B \cup C) - A$.

Hence $(B-A) \cup (C-A) \subseteq (B \cup C) - A$.

(ii) Let $x \in (B \cup C) - A$, then $x \notin A \wedge (x \in B \vee x \in C)$. Therefore from distributive property of logic; $(x \notin A \wedge x \in B) \vee (x \notin A \wedge x \in C)$.

~~$x \in (B-A)$~~

From commutativity of $\wedge$: $(x \in B \wedge x \notin A) \vee (x \in C \wedge x \notin A)$.

Therefore, $x \in (B-A) \vee x \in (C-A)$; $x \in (B-A) \cup (C-A)$.

Hence $(B \cup C) - A \subseteq (B-A) \cup (C-A)$.

From (i) and (ii), we can conclude: $(B-A) \cup (C-A) = (B \cup C) - A$.

∎