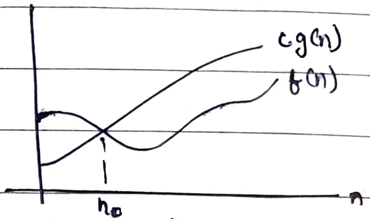
RATES OF GROWTH OF FUNCTIONS

UPPER BOUND O

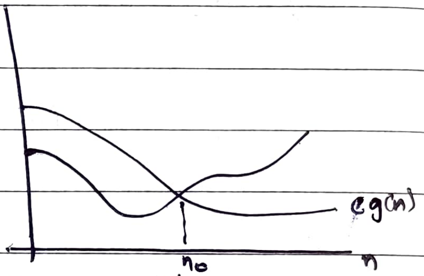


$f(n) = O(g(n))$

$\Updownarrow$

$\exists\ c, n_0$ such that $0 \leq f(n) \leq c\,g(n),\ \forall\ n \geq n_0$

LOWER BOUND Ω

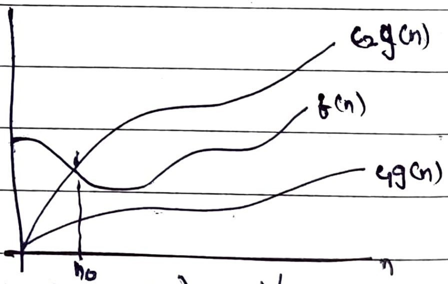


$f(n) = \Omega(g(n))$

$\Updownarrow$

$\exists\ c, n_0$ such that $f(n) \geq c\,g(n) \geq 0\ \ \forall\ n \geq n_0$

SAME ORDER Θ



$f(n) = \Theta(g(n))$

$\Updownarrow$

$\exists\ c_1, c_2, n_0$ st. $c_1 g(n) \leq f(n) \leq c_2 g(n),\ \forall\ n \geq n_0$

ex $f(n) = \dfrac{n^2}{2} - \dfrac{n}{2}$

Show that $f(n) = O(n^2)$

$\dfrac{n^2}{2} - \dfrac{n}{2} \leq \underline{c}\,n^2 \qquad \forall\ n \geq \underline{n_0}$

$c = 1 \qquad\qquad n_0 = 1$

show that $f(n) = \Omega(n^2)$

$\frac{n^2}{2} - \frac{n}{2} \geq cn^2 \quad \forall\ n \geq n_0$

$\frac{n^2}{2} - \frac{n}{2} \geq \frac{n^2}{3} \qquad c = \frac{1}{3} \qquad n_0 = 3$

$\frac{n^2}{6} - \frac{n}{2} \geq 0$

$n(n-3) \geq 0$

$n \geq 3$

ex $\frac{n^2+3n}{2}$ is $\Theta(n^2)$

$\frac{n^2}{2} + \frac{3n}{2} \leq cn^2 \quad \forall\ n \geq n_0$

$\leq n^2 \quad \leq 3n^2 \qquad 4n^2 \qquad n_0 = 1$

$c = 4$

$\frac{n^2}{2} + \frac{3n}{2} \geq \frac{n^2}{2}, \quad \forall\ n \geq 1$

$c_1 = 4 \qquad n_0 = 1$

$c_2 = 1/2$

$\frac{n^2+3n}{2}$ is $\Theta(n^2)$

ex $f(n) = 2n^4 + 5n^3 + \frac{n^3}{3} - n + 7$ is $O(?)$

$\leq 2n^4 \quad \leq 5n^4 \quad \leq n^4 \quad \leq n^4$

$\leq 9n^4 \quad \forall\ n \geq 1 \qquad O(n^4)$

ex is $2^n = \Theta(2^{n+1})$?

$2^n \leq c_1 2^{n+1} \qquad 1 \leq c_1 \times 2 \qquad c_1 \geq 1/2$ (1)

$2^n \geq c_2 2^{n+1} \qquad 2^n \geq c_2 2^{n+1} \qquad c_2 \leq 1/2$ (1/2)

ex is $2^{2n} = \Theta(2^n)$?

$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \frac{2^{2n}}{2^n} = 2^n \to \infty$ no!

ex $2^n \quad n^2$

$\lim_{n \to \infty} \frac{2^n}{n^2}$ vs $\lim_{n \to \infty} \frac{n^2}{2^n}$

ex show that $\log(n!) = \Theta(n \log n)$

comparision, sorting
no info known about
nos.

$\log n! \leq c_1 n \log n$

$\log n! \geq c_2 n \log n$

$\log n + \log(n-1) + \ldots \log 1 \leq n \log n \quad (c_1 = 1)$

$(\log n + \log 1) + (\log (n-1) + \log 2)$

$+ \ldots (\log \frac{n}{2} \cdot \log \frac{n}{2}) \geq \frac{n}{2} \log \frac{n}{2}$

$\geq \frac{1}{2}(n \log n - n \log 2)$

$\geq \frac{1}{4} n \log n / \frac{1}{2} n \log n$

$c_2 = 1/4 \quad n_0 = 4 \quad c_2 = 1/2 \quad n_0 = 2$

$\frac{1}{4} n \log n \leq \frac{1}{2} n \log n - \cancel{2} n \log 2$

$\log n \leq 2 \log n - \log 2$

$1 \leq \log n$

$n \geq 2 \quad n_0 = 2$

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$\log n + \log(n-1) + \log(n-2) + \dots \log 1$

$n! = n(n-1)(n-2)\dots 1$

$n(n-1)(n-2)\dots 1 \geq n(n-1)(n-2)\dots \frac{n}{2}$

$> \frac{n}{2} \cdot \frac{n}{2} \dots \quad \Big\} \left(\frac{n}{2}\right)^{n/2}$

$n! \geq \left(\frac{n}{2}\right)^{n/2}$

$\log n! \geq \frac{n}{2} \log \frac{n}{2}$

ex $n \log n^2$ vs $\log n^n$

$\Theta(n \log n)$

ex $n^{\log n}$ vs $(\log n)^n$

let $n = 2^y$ $\quad \log n = y$

$n^y = (2^y)^y \quad (y)^{2^y}$

$2^{y^2} \quad (2^{\log_2 y})^{2^y}$

2^{y^2} vs $\underbrace{2^{2^y \log_2 y}}_{\text{much bigger}}$

$y^2 < 2^y$