SOLVING RECURRENCE RELATIONS

CHINESE RING PUZZLE

$T(n) = 2T(n-2) + T(n-1) + 1$ —— (1)

$T(1) = 1 \qquad T(2) = 2$

$$T(n) = \begin{cases} 2T(n-1) + 1 & \text{if } n \text{ is odd} \\ 2T(n-1) & \text{if } n \text{ is even} \end{cases}$$ —— (2).

closed form solution

case 1: n is even

$(n-1)$ is odd, so

$T(n-1) = 2T(n-2) + 1$ —— from (2)

sub. $T(n-1)$ in (1)

$T(n) = 2T(n-2) + 2T(n-2) + 1 + 1$

$T(n) = 4T(n-2) + 2$

SUBSTITUTION METHOD

$T(n) = 4T(n-2) + 2$

$T(n-2) = 4T(n-4) + 2$

$$\therefore\ T(n) = 4T(n-4) + 8 + 2$$

$$= 4^2\, T(n-4) + 10$$

$$T(n) = 4^2\, T(n-4) + 2(4) + 2$$

$$T(n) = 4^3\, T(n-6) + 2(4)^2 + 2(4) + 2$$

r times:

$$T(n) = 4^r\, T(n-2r) + 2(1 + 4 + \ldots 4^{r-1})$$

$$= 4^r\, T(n-2r) + 2\left(\frac{4^r - 1}{3}\right)$$

to hit the base case $n - 2r = 2$

$$\Rightarrow r = \frac{n-2}{2}$$

$$T(n) = 2^{n-2}\, T(n-(n-2)) + \frac{2}{3}\left(2^{n-2} - 1\right)$$

$$= 2^{n-2}\, T(2) + \frac{2}{3}\left(2^{n-2} - 1\right)$$

$$= 2^{n-2} \cdot 2 + \frac{2}{3}\left(2^{n-2} - 1\right)$$

$$= 2^{n-1} + \frac{2^{n-1} - 2}{3}$$

$$= 2^{n-1} + \frac{2^{n-1} - 1}{3} = \frac{4}{3}2^{n-1} - \frac{2}{3}$$

$\therefore T(n) = \frac{2}{3}(2^n - 1)$

CR-6. $\frac{2}{3}(2^6 - 1) = \frac{2}{3} \times (64-1) = \frac{2 \times 63}{3} = \underline{\underline{42}}$ steps.

hamming distance ... gray code ... uninteresting graph.

case 2: n is odd

$(n-1)$ is even, so from ②.

$T(n-1) = 2T(n-2)$

sub. $T(n-1)$ in ①.

$T(n) = 2T(n-2) + 2T(n-2)$

$T(n) = 4T(n-2) + 1$

base case $T(1) = 1$

Soln. $T(n) = \frac{1}{3}(2^{n+1} - 1)$

METHODS FOR SOLVING RRs

1. repeated substitution
2. master theorm
3. change of variables
4. guess the answer
   prove by induction
5. linear homogenous eqns. & heterogenous eqns.
6. linear algebra methods (diagonalization)

ex $T(n) = 2T\left(\frac{n}{2}\right) + n$

$T(1) = 1$

repeated substitution. ... r times.

$T(n) = 2^r T\left(\frac{n}{2^r}\right) + n \times r$

$\frac{n}{2^r} = 1 \Rightarrow n = 2^r \Rightarrow r = \lg n$

$T(n) = n + n \lg n$

$T(n) = \Theta(n \lg n)$

ex $T(n) = 3T\left(\frac{n}{2}\right) + n$

$T(1) = 1$

repeated substitution...

$$T(n) = 3^r T\left(\frac{n}{2^r}\right) + n * \left(1 + 3 + \left(\frac{3}{2}\right)^2 + \dots \left(\frac{3}{2}\right)^{r-1}\right)$$

$$= 3^r T\left(\frac{n}{2^r}\right) + n\left(\frac{(3/2)^r - 1}{3/2 - 1}\right)$$

$$= 3^r T\left(\frac{n}{2^r}\right) + 2n\left(\left(\frac{3}{2}\right)^r - 1\right)$$

$n = 2^r \Rightarrow r = \lg n$

$$T(n) = 3^{\lg n} + 2n\left(\left(\frac{3}{2}\right)^{\lg n} - 1\right)$$

$$T(n) = n^{\log_2 3} + 2n\left(n^{\log_2 3/2} - 1\right)$$

ex

$$= n^{\lg 3} + 2n\left(n^{\lg 3 - 1} - 1\right)$$

$$= n^{\lg 3} + 2\left(n^{\lg 3} - n\right)$$

$$= 3n^{\lg 3} - 2n$$

$$T(n) = \Theta\left(n^{\lg 3}\right)$$

ex $T(n) = 3T\left(\frac{n}{2}\right) + n^2$

repeated substitution . . .

$T(n) = n^{\lg 3} + n^2\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \dots \left(\frac{3}{4}\right)^{r-1}\right)$

$T(n) = \Theta(n^2)$.

MASTER THEORM

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Soln. of $T(n)$
(or $T(n)$ asymptomatically bounded as follows)

critical exponent / ratio.

① if $f(n) = O\left(n^{\log_b a - \varepsilon}\right)$

for some $\varepsilon > 0$, then $T(n) = \Theta\left(n^{\log_b a}\right)$

② if $f(n) = \Theta\left(n^{\log_b a}\right)$ then $T(n) = \Theta\left(n^{\log_b a} . \lg n\right)$

③ if $f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)$ for some $\varepsilon > 0$, then $T(n) = \Theta(f(n))$

and if $af\left(\frac{n}{b}\right) \leq kf(n)$, for some const. $k < 1$
and for sufficiently large n,

regularity cond.

ex $T(n) = 2T\left(\frac{n}{2}\right) + n$

Case 2

ex $T(n) = 3T\left(\frac{n}{2}\right) + n$

$n = O\left(n^{\log 3/2}\right)$ case ①.

$\Theta\left(n^{\log_2 3}\right)$

$n^{\lg 3}$ vs n^2 (Case 3)

$T(n) = \Theta(n^2)$

reg. cond.. $3f\left(\frac{n}{2}\right) \le k\, f(n)$

$\frac{3n^2}{4} \le k\, n^2$ $k = 0.9$.

satisfied.

by case 3.

T(n) of the order of f(n)

ex $T(n) = 9T\left(\frac{n}{3}\right) + n$

case 1 $T(n) = \Theta(n^3)$

ex $T(n) = T\left(\frac{3n}{5}\right) + 1$

case 2 $T(n) = \Theta(\lg n)$

ex $T(n) = 3T\left(\frac{n}{4}\right) + n\lg n$

$n^{\log_4 3} \to 0.8$ vs $n\lg n$.

reg. cond. $af\left(\frac{n}{b}\right) \leq k f(n)$

$3f\left(\frac{n}{4}\right) \leq k f(n)$

$\frac{3n}{4} \lg \frac{n}{4} \leq k n \lg n$

$\frac{3n}{4}(\lg n - 2) \leq k n \lg n$.

$\frac{3}{4} n\lg n - \frac{3}{2} \leq k n \lg n$

$k = \frac{3}{4}$

case 3. $T(n) = \Theta(n \lg n)$

ex $T(n) = 2T\left(\frac{n}{2}\right) + n\log n$

$n^{\lg 2}$ vs $n \log n$

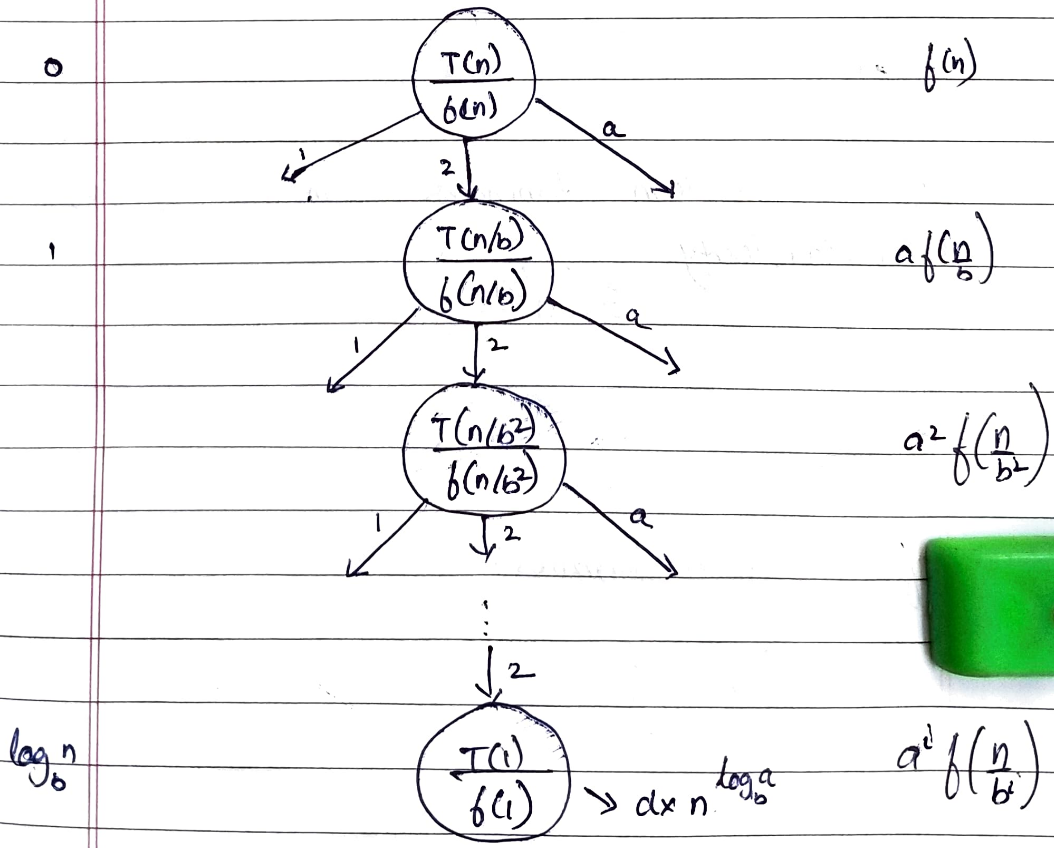
$\log n < n^{\varepsilon}$ for any $\varepsilon > 0$

$f(n)$ is not polynomially larger than n
we cant apply master's theorm.

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$T(1) = d.$

recursion tree



root node

total levels $= \log_b n + 1$

no. of leaf nodes $= a^{\log_b n} = n^{\log_b a}$

critical exponent.

$T(n) =$ sum of all the elements.

$$= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) + d\, n^{\log_b a}$$

$a \geq 1$

$b > 1$

$$T(n) = \underbrace{\sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)}_{①} + \underbrace{\Theta\left(n^{\log_b a}\right)}_{②}.$$

case 1

If the second term dominates (sum of leaf nodes)
then complexity $\Theta\left(n^{\log_b a}\right)$

case 2

If neither term dominates:

$\Theta\left(n^{\log_b a} \cdot \log n\right)$

Case 3

if the first term dominates (sum of internal nodes)

$\Theta(f(n))$

work on a small numerical example where you generate a recursion tree. draw recursion tree for computing.

$T(n) = 2T\left(\frac{n}{3}\right) + 5n$ when $n = 27$.

$T(1) = 7$.