EXTENDED MASTER'S THEORM

$$T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$$

$a = b = 2$; $f(n) = \left(\frac{n}{\lg n}\right)$

extended master's theorm (emm)

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$ $\quad\quad T(1) = d$

has solutions:

(1) if $f(n) = O\left(n^{\log_b a} (\log_b n)^{\alpha}\right)$

with $\alpha < -1$ then $T(n) = \Theta\left(n^{\log_a b}\right)$

(2) if $f(n) = \Theta\left(n^{\log_b a} (\log_b n)^{-1}\right)$ [here $\alpha = -1$]

then $T(n) = \Theta\left(n^{\log_b a} \cdot \log_b (\log_b n)\right)$

(3) if $f(n) = \Theta\left(n^{\log_a b} (\log_b n)^{\alpha}\right)$; with $\alpha > -1$

then $T(n) = \Theta\left(n^{\log_b a} (\log_b n)^{\alpha+1}\right)$

(4) if $f(n) = \Omega\left(n^{\log_b a + \epsilon}\right)$ for some $\epsilon > 0$

then $T(n) = \Theta(f(n))$

provided $\exists$ a const. $k < 1$ $\ni$ $af\left(\frac{n}{b}\right) < k f(n)$ $\forall$ suff. larger (regularity cond.)

$$T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\lg n}$$

$a = b = 2 \quad f(n) = \frac{n}{\lg n}$

EMM GMM

4' $\longleftrightarrow$ 3

1' $\longleftarrow$ includes case 1

[think about how case 1 is included in 1']

3' with $\alpha = 0$ is case 2

Case 2'

$$T(n) = \theta(n \cdot \lg \lg n)$$

ex $T(n) = 2T\left(\frac{n}{2}\right) + n(\lg n)^2$

ex $T(n) = 3T\left(\frac{n}{2}\right) + n(\lg n)^2$

CHANGE OF VARIABLES

ex $T(n) = 2T(n-1) + 1$ ① $T(0) = 0$

let $s(n) = T(n) + 1$

$s(n) - 1 = 2(s(n-1) - 1) + 1$

$s(n) - 1 = 2s(n-1) - 2 + 1$

$s(n) = 2s(n-1)$

$\Rightarrow s(n) = 2^n$

$s(n) = T(n) + 1$

$2^n = T(n) + 1$

$T(n) = 2^n - 1$

ex $T(n) = 2T(\lfloor \sqrt{n} \rfloor) + \log n$

ignoring rounding off issues

let $m = \log_2 n \Rightarrow n = 2^m$

$T(2^m) = 2T(2^{m/2}) + m$

$S(m) = 2S\left(\frac{m}{2}\right) + m$

$\Rightarrow$ master theorm case 2 $\quad m^{\log_2 2}$ vs m

$S(m) = \Theta(m \log m)$

$\Rightarrow T(n) = T(2^m) = S(m) = O(m \lg m)$

$T(n) = O(\lg n . \lg \lg n)$

ex $T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + 1.5n$

guess the complexity and prove it by induction.

comes from: finding the median of n elements.
supposed to be a linear time algorithm.

our guess $T(n) = O(n)$

assume $T(n) \leq kn$

$\overset{?}{k = 60}$; $T(n) \leq 60n$

$T\left(\frac{n}{5}\right) \leq 12n$

$T\left(\frac{7n}{10}\right) \leq 42n$

$T(n) \leq 69n$

$T\left(\frac{n}{5}\right) \leq k\frac{n}{5}$

$T\left(\frac{7n}{10}\right) \leq \frac{7kn}{10}$

$T(n) \leq k\frac{n}{5} + \frac{7kn}{10} + 15n$

$\leq n\left(\frac{k}{5} + \frac{7k}{10} + 15\right)$

$\leq n\left(\frac{9k}{10} + 15\right)$

$\frac{9k}{10} + 15 = k$

$\frac{k}{10} = 15$

$k = 150$

RECURRENCE RELATION AND FIBONACCI SERIES

$F(n) = F(n-1) + F(n-2)$

$F(0) = 1$

$F(1) = 1$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} F(1) \\ F(0) \end{bmatrix} = \begin{bmatrix} F(2) \\ F(1) \end{bmatrix}$$

$T = CDC^{-1}$

diagonal matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F(1) \\ F(0) \end{bmatrix} = \begin{bmatrix} F(n+1) \\ F(n) \end{bmatrix}$$

$T = CDC^{-1}$

$T^n = CD^nC^{-1}$

$V_n = A^n V_0 \longrightarrow \begin{bmatrix} f(1) \\ f(0) \end{bmatrix}$

$A = CDC^{-1}$

$AC = CD$

$D = C^{-1}AC$

$A_{n \times n}$ is diag. if there is an invertible $C_{n \times n}$ such that $C^{-1}AC$ is a diag. matrix.

$A_{n \times n}$ $\quad$ $C_{n \times n}$

$v_1, v_2, \ldots v_n$ $\quad$ $D_{n \times n}$

$AC = CD$

iff λ_1 are e-values of A

v_j is corresponding e-vector.

$$AC = A \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$$

$$= \begin{bmatrix} Av_1 & Av_2 & \ldots & Av_n \end{bmatrix}$$

$$CD = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_n \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | \\ \lambda v_1 & \lambda v_2 & \lambda v_n \\ | & | & | \end{bmatrix}$$

$$\boxed{Av_1 = \lambda_1 v_1}$$

$Ax = \lambda x$

eigen value

eigen vector

th $A_{n \times n}$ is diag. iff
it has n linearly independent eigen vectors.

$A = CDC^{-1}$ —— full rank matrix of order n
all columns are linearly independent.

$A^n = \lambda_i^n$

$f(A) = f(\lambda_i)$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow CDC^{-1}$$

~~f(0)~~

$A v = \lambda v$

$(A - \lambda I) v = 0$

$\det (A - \lambda I) = 0$

... bla bla bla.

$$\therefore f(n) = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

$$f(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$f(0) = \frac{1}{\sqrt{5}}\left(1-1\right) = 0 \checkmark$$

$$f(1) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1 \checkmark$$

<u>Solved</u>

RECURRENCE RELATION FOR FIBONACCI SERIES

$$\begin{cases} f(n) = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right] \\ f(0) = 0 \qquad f(1) = 1 \end{cases}$$