RECORRENCE RELATIONS

Chinese 6 ring puzzle
geometric series
tower of hanoi usage

GEOMETRIC SERIES

$$X = 1 + 2 + 2^2 + \ldots 2^{n-1}$$

$$2X = 2 + 2^2 + \ldots 2^{n-1} + 2^n$$

$$\underline{X = 2^n - 1}$$

Zeno' paradox

$$X = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

$$\frac{1}{2}X = \frac{1}{4} + \frac{1}{8} + \ldots$$

$$\frac{X}{2} = \frac{1}{2}$$

$$\Rightarrow \underline{X = 1}$$

$$X = \frac{1}{10} + \frac{1}{10^2} + \ldots$$

$$\frac{X}{10} = \frac{1}{10^2} + \ldots$$

$$\frac{9X}{10} = \frac{1}{10}$$

$$\Rightarrow X = \frac{1}{9}$$

$$X = \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \cdots$$

$$\frac{X}{10} = \frac{1}{10^2} + \frac{2}{10^3} + \cdots$$

$$\frac{9X}{10} = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots$$

$$\frac{9X}{10} = \frac{1}{9}$$

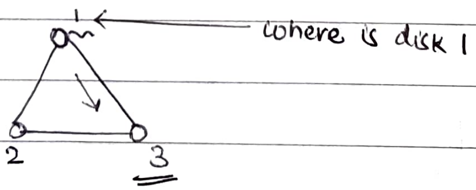
$$\Rightarrow X = \frac{10}{81}$$

ex $X = \frac{1}{10} + \frac{2^2}{10^2} + \frac{3^2}{10^3} + \frac{4^2}{10^4} + \cdots$

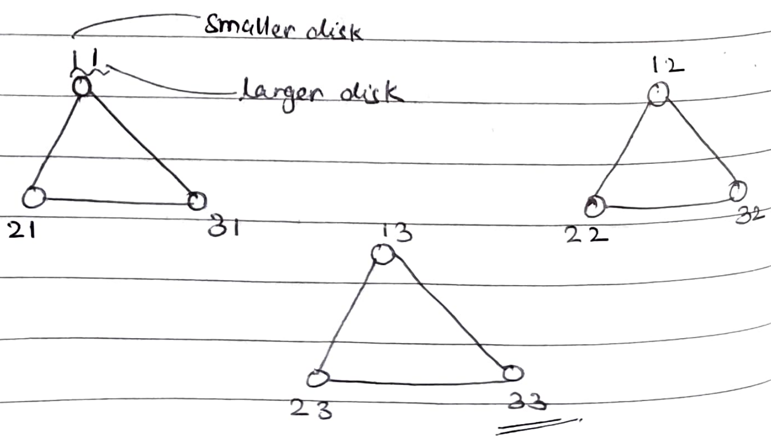
$\hookrightarrow$ try

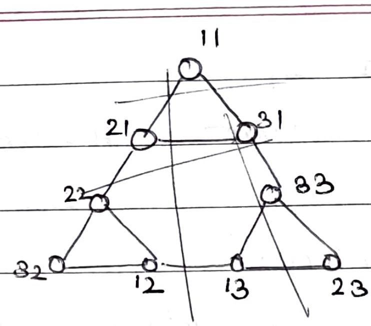
GRAPHICAL REPRESENTATION OF TOWER OF HANOI

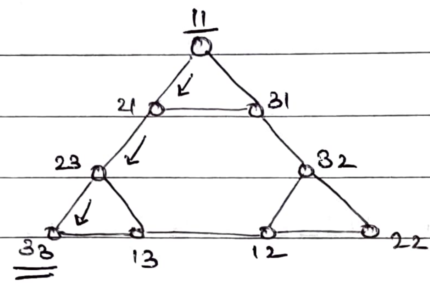
1 disk



2 disk







enumerates all possible moves

properties of graph.

11 → 21 → 23 → 33 (3)

degree of node:

no. of edges coming out of node

corners = 2 others = 3

recursive call + display move

111
↓
311
↓
321
↓
221
↓
223
↓
123
↓
133 → 333

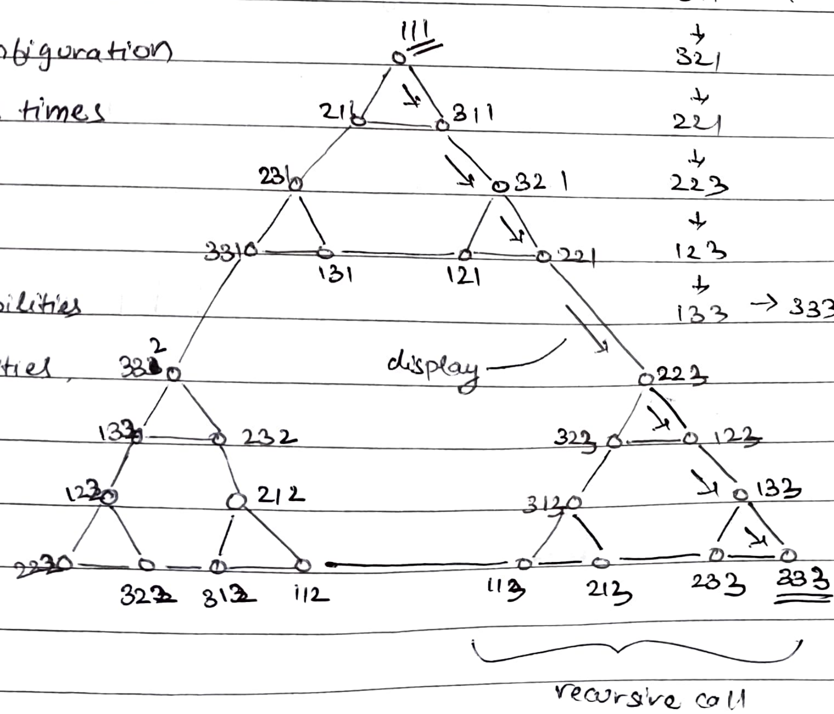
(7)
optima

for 3 disk configuration

copy above 3 times

at edges 2 possibilities

others 3 possibilities,



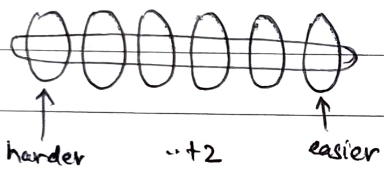
ex 4 disk problem

published ian stewart - puzzler martin gardener

Self similar patterns

typically discussed in fractals - design principle

CHINESE RING PUZZLE 6 RINGS



111 111

└ all rings on rod.

RULE

- start at right end
- go until you hit the first "1" bit
- ring left of this can come off

111 111 → 111101

111 111 → 111110

101101 → 101111

101101 → 101100

$$100100 \to 101100$$

$$100100 \to 100101$$

RECURSIVE PROCEDURE FOR CHINESE RINGS

111111
↓
011111
↓
001111
↓
. . .

110000
↓
010000
↓ reverse CR.
011111

CR (n)

* get the harder ring off
- CR(n-2) 4 rings taken off
- take out the hardest ring
- reverse CR(n-2)
- CR (n-1)

RECURRENCE RELATION

$$T(n) = 2T(n-2) + T(n-1) + 1$$

$$T(n) = 2T(n-2) + T(n-1) + 1$$

$$T(n-1) = 2T(n-3) + T(n-2) + 1$$

$$T(n) = 2T(n-2) + 2T(n-3) + T(n-2) + 1 + 1$$

$$= 3T(n-2) + 2T(n-3) + 2$$

× repeated substitution not a good idea.

base case

$T(1) = 1$ 1 1

$T(2) = 2$ 0 1^2

0 0^2

1 1 0 0 0 0

↓

0 1 0 0 0 0

↓

... reverse CR

↓

0 1 1 1 1 1

$T(1) = 1$

$T(2) = 2$

$T(3) = 2T(1) + T(2) + 1$

$= 2 + 2 + 1 = 5$

$T(4) = 2T(2) + T(3) + 1$

$= 4 + 5 + 1 = 10$

$T(5) = 2T(3) + T(4) + 1$

$= 10 + 10 + 1 = 21$

n	T(n)
1	1
2	2
3	5
4	10
5	21
6	42
7	85

our guesswork

our conjecture

n is odd : $T(n) = 2T(n-1) + 1$

n is even : $T(n) = 2T(n-1)$

this behaviour can be proved by induction.

PROOF BY INDUCTION

inductive assumption, it is true for n

$$T(n-1) = \begin{cases} 2T(n-2) & n-1 \text{ even} \\ 2T(n-2)+1 & n-1 \text{ odd} \end{cases}$$

strong induction vs weak induction

case n is even $\Rightarrow$ $(n-1)$ is odd
so by inductive assumption $T(n-1) = 2T(n-2)+1$

$$T(n) = 2\left(2T(n-2)+1\right)$$

$$= 4T(n-2) + 2 = 2T(n-1)$$

$\{ T(n) = 2T(n-2) + T(n-1) + 1 \}$

$$T(n-1) = 2T(n-2)+1$$

$$= 2\left(2T(n-3)\right)+1$$

$$= 4T(n-3)+1$$

$T(n-1) = T, \ldots$

n even $\quad T(n) = \frac{2}{3}\left(2^n - 1\right)$

n odd $\quad T(n) = \frac{1}{3}\left(2^{n+1} - 1\right)$

42 steps
for 8 ring CR