RATES OF GROWTH OF FUNCTIONS

UPPER BOUND O

there exists

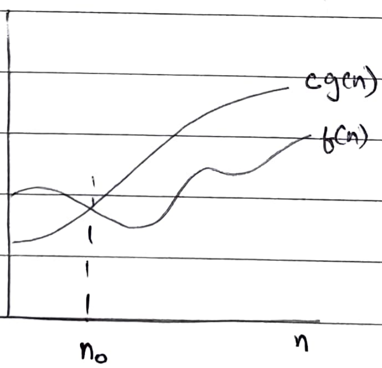
$f(n) = O(g(n)) \iff \exists\ c, n_0$ such that

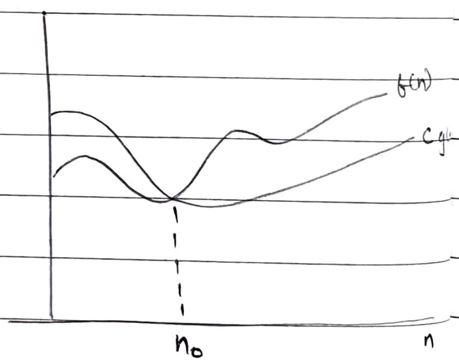
$0 \le f(n) \le cg(n) \ \forall n \ge n_0$

LOWER BOUND Ω

$f(n) = \Omega(g(n)) \iff \exists\ c, n_0$ such that

$f(n) \ge cg(n) \ge 0 \quad \forall n \ge n_0$





SAME ORDER Θ

$f(n) = \Theta(g(n))$

$\Updownarrow$

$\exists\ c_1, c_2, n_0$ such that

$c_1 g(n) \le f(n) \le c_2 g(n) \quad \forall n \ge n_0$

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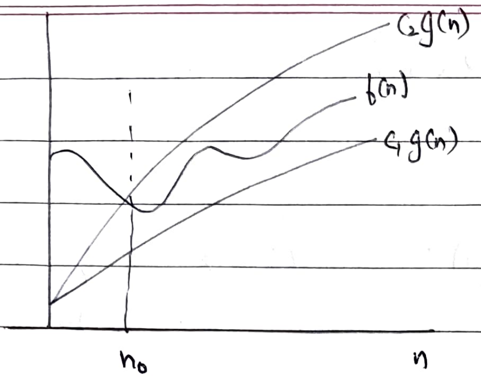

ex $f(n) = \frac{n^2}{2} - \frac{n}{2}$

show that $f(n) = O(n^2)$

$\frac{n^2}{2} - \frac{n}{2} \leq \boxed{}\, n^2 \qquad \forall\, n \geq \boxed{}$

$c = 1 \qquad\qquad n_0 = 1$

Show that $f(n) = \Omega(n^2)$

$\frac{n^2}{2} - \frac{n}{2} \geq \boxed{}\, n^2$

$\frac{n^2}{2} - \frac{n}{2} \geq \frac{n^2}{3} \qquad c = \frac{1}{3}$

$\frac{n^2}{6} - \frac{n}{2} \geq 0$

$n(n-3) \geq 0$

$n \geq 3 \qquad n_0 = 3$

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ex $\frac{n^2+3n}{2}$ is $\Theta(n^2)$

$\frac{n^2}{2} + \frac{3n}{2} \leq 4n^2 \quad \forall n \geq 1$ } upper bound

$\leq n^2 \quad \leq 3n^2$

$\frac{n^2}{2} + \frac{3n}{2} \geq \frac{n^2}{2} \quad \forall n \geq 1$

$c_1 = 4 \quad c_2 = \frac{1}{2} \quad n_0 = 1$

$\frac{n^2+3n}{2}$ is $\Theta(n^2)$

ex $f(n) = 2n^4 + 5n^3 + \frac{n^2}{3} - n + 7$ is $O(?)$

$\leq 2n^4 \quad \leq 5n^4 \leq n^4 \quad \leq n^4$

$\leq 9n^4 \quad \forall n \geq 1 \quad O(n^4)$

$\downarrow$
c

ex is $2^n = \Theta(2^{n+1})$?

$2^n \leq c_1 2^{n+1}$

$2^n \geq c_2 2^{n+1}$

$1 \leq C_1 \cdot 2$

$\rightarrow C_1 \geq \frac{1}{2} \longrightarrow C_1 = 1$

$2^n \geq C_2 \, 2^{n+1}$

$C_2 \leq \frac{1}{2} \longrightarrow C_2 = \frac{1}{2}$

ex is $2^{2n} = O(2^n)$?

$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \frac{2^{2n}}{2^n} = 2^n \rightarrow \infty$ (no!)

ex is $2^n = O(n^2)$

$\lim_{n \to \infty} \frac{2^n}{n^2}$ vs $\lim_{n \to \infty} \frac{n^2}{2^n}$

ex show that $\log(n!) = \Theta(n \log n)$ → comparision sorting no other info known about numbers

$\log n! \leq C_1 n \log n$

$\log n! \geq C_2 n \log n$

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$\log n + \log(n-1) + \dots \log 1 \leq n \log n$

$c_1 = 1$ for this.

$(\log 1 + \log n) + (\log 2 + \log(n-1)) +$

$\dots \left(\log\left\lfloor \frac{n}{2} \right\rfloor + \log\left\lceil \frac{n}{2} \right\rceil\right)$

$\Rightarrow \log n + \log 2(n-1) + \dots \log \frac{n}{2} \cdot \frac{n}{2} \geq \frac{n}{2} \log \frac{n}{2}$

$RHS = \frac{1}{2}\left(n \log n - n \log 2\right)$

$\geq \frac{1}{4} n \log n$ $\qquad\qquad$ $\geq \frac{1}{2} n \log n$

$c_2 = \frac{1}{4}$ $\quad n_0 = 4$ $\qquad\qquad$ $c_2 = \frac{1}{2}$ $\quad n_0 = 2$

$\frac{1}{4} n \log n \leq \frac{1}{2}\left(n \log n - n \log 2\right)$

$\Rightarrow n \log n \leq 2n \log n - 2n \log 2$

$\Rightarrow 2n \log 2 \leq n \log n$

$\Rightarrow \log n \geq 2$

$\Rightarrow n \geq 4$?

$n \geq 2$

, $n_0 = 2$

alternative:

$\log n + \log(n-1) + \dots \log 1$

$n! = n(n-1)(n-2)\dots 1$

$n! > n(n-1)(n-2)\dots \frac{n}{2}$

$n! > \frac{n}{2} \cdot \frac{n}{2} \dots \frac{n}{2}$

$n! > \left(\frac{n}{2}\right)^{n/2}$

$\log n! \geq \frac{n}{2} \log \frac{n}{2}$

ex $n \log n^2$ vs $\log n^n$

$\Theta(n \log n)$

ex $n^{\log n}$ vs $(\log n)^n$

let $n = 2^y$

$\Rightarrow \lg n = y$

$n^y = (2^y)^y \qquad (y)^{2^y}$

$2^{y^2} \qquad (2^{\lg y})^{2^y}$

$y = 2^{\lg y}$

2^{y^2} vs $2^{2^y \lg y}$

↳ much bigger.

$y^2 < 2^y$

RECURSION; INDUCTION; RECORRENCE RELATION

compound interest, tower of hanoi

$m_0 = 100$

interest = 10%.

$m_1 = 110$

$m_2 = 121$

$\vdots$

$\underbrace{m_i = 1.1\, M_{i-1}}_{\text{recurrence relation.}} \qquad (M_{i-1} + 0.1\, M_{i-1})$

① brute force:
repeated substitution:

$m_i = 1.1\, m_{i-1}$

$m_{i-1} = 1.1\, m_{i-2}$

$\vdots$

base case

we are looking for closed form solution for this recurrence relation.

$m_i = (1.1)^2 m_{i-2}$

... do this r times
go backwards r steps

$m_i = (1.1)^r m_{i-r}$

base case is reached when $i-r=0$

$i-r=0$
$\Rightarrow r=i$

$\therefore m_i = (1.1)^i m_0$

$\Rightarrow m_n = (1.1)^n m_0$ closed form solution
dont have to compute telescopically.

ex binary search : number guess game
20Q object guess game y/n

$$T_n = 1 + T_{\frac{n}{2}}$$

closed form solution

$$T_{\frac{n}{2}} = 1 + T_{\frac{n}{4}}$$

$$T_n = 2 + T_{\frac{n}{4}}$$

.... r times

$$T_n = r + T_{\frac{n}{2^r}}$$

base case $T_1 = 0$

$$\frac{n}{2^r} = 1$$

$$\Rightarrow n = 2^r$$

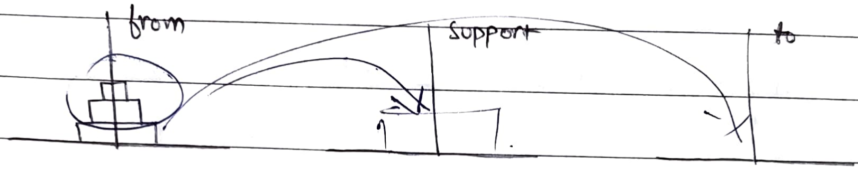
$$\Rightarrow r = \lg n$$

$$\therefore T_n = \lg n + \cancel{T_1}^{0}$$

$$T_n = \log_2 n \qquad \langle RR \rangle$$

TOWER OF HANOI

buddhist elites 64 blocks ⟶ enlightenment



3 disk problem → 2 disk problem

```
TOH (n, from, to, supp)
  if n>0
  {
      TOH (n-1, from, supp, to)
      display ('move disk from {from} to {to}')
      TOH (n-1, supp, to, from)
  }
```

$T_0 = 0$

$T_1 = 1$

$T_n = 2T_{n-1} + 1$ ⟨RR⟩

→ recursive calls

$T_{n-1} = 2T_{n-2} + 1$

$T_n = 2(2T_{n-2} + 1) + 1$

$= 2^2 T_{n-2} + 2 + 1$

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$$T_n = 2^3 T_{n-3} + \underbrace{2^2 + 2^1 + 2^0}_{2^3 - 1}$$

$$T_n = 2^3 T_{n-3} + 2^3 - 1$$

$$T_n = 2^r T_{n-r} + 2^r - 1$$

to hit base case $n - r = 0 \Rightarrow r = n$

$$T_n = 2^n \cancel{T_0}^{\,0} + 2^n - 1$$

$$T_n = 2^n - 1$$

GS: $\dfrac{2^{n-1}\left((1/2)^n - 1\right)}{1/2 - 1}$

$= 2^{n-1}\left(1 - 1/2^n\right) 2$

$= 2^n\left(1 - \dfrac{1}{2^n}\right)$

$= 2^n - 1$

$2^{n-1} + 2^{n-2} + \dots 1 = 2^n - 1$