CARDINALITY OF SETS

dfn inverse:

let $f: A \to B$ be a 1-1 correspondence (bijection)

the inverse function of f is

$f^{-1}: B \to A$ such that

$f^{-1}(b) = a \iff f(a) = b$

dfn let $g: A \to B$ and $f: B \to C$ be functions

the $f \circ g : A \to C$ is called the composition of g with f.

$\forall a \in A \quad f \circ g(a) = f(g(a))$

ex let $f: \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2 + 2$

$g: \mathbb{R} \to \mathbb{R}$ be $g(x) = 3x + 4$

then: $g \circ f : \mathbb{R} \to \mathbb{R}$

$g \circ f(x) = g(f(x)) = 3(x^2+2) + 4$

$= 3x^2 + 10$

$f \circ g : \mathbb{R} \to \mathbb{R}$

$f \circ g(x) = f(g(x)) = (3x+4)^2 + 2$

$= 9x^2 + 16 + 24x + 2$

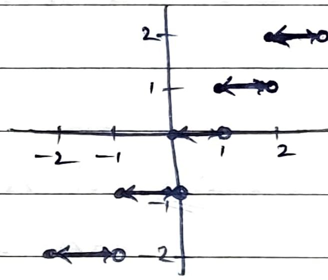
$= 9x^2 + 24x + 18$

note $f \circ g(x) \neq g \circ f(x)$

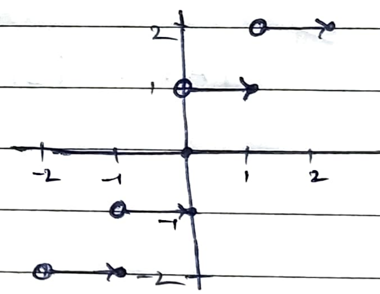
thm let $f: A \to B$ and $g: B \to C$
if f maps A onto B and g maps B onto C,
$g \circ f$ maps A onto C.

thm if f and g are 1-1 functions (injections) then
$g \circ f : A \to C$ is also 1-1 function <PROVE>.

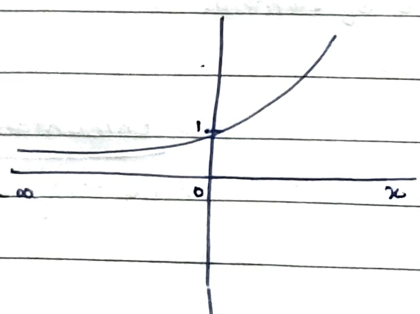
Special fns floor $= \lfloor x \rfloor$
→ assigns to x the largest integer $\leq x$



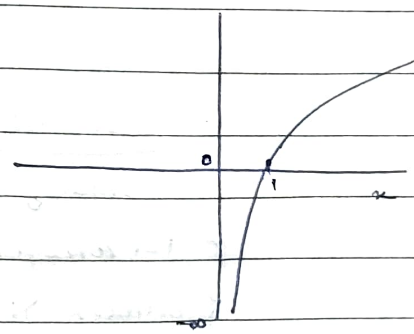
ceiling $= \lceil x \rceil$
→ assigns to x smallest integer $\geq x$



exponential $y = a^x$, $a > 1$



logarithm $y = \log_a x$, $a > 1$



CARDINALITY

for finite sets = no of elements in the set

dfn the sets A and B have the same cardinality iff there is a 1-1 correspondence (bijection) from A to B.

when they are equal, $|A| = |B|$

dfn if there is a 1-1 function (injection) from A to B, the cardinality relationship is

$|A| \leq |B|$

COUNTABLE SETS

infinite no. of elements

countable	uncountable
countably infinite	uncountably infinite

1-1 correspondence between $\mathbb{N}$ and the set in question.

infinite set S is countable

$|S| = \aleph_0$ (aleph-naught)

hebrew alphabet

ex Show that the set of odd positive integers is a countable set.

1	2	3	4
|	|	|	|
1	3	5	7

need to establish a bijection between $\mathbb{N}$ and S

$f: \mathbb{N} \to S$ (bijection)

Consider $f(n) = 2n-1$

sets up 1-1 correspondence

- show that f is a 1-1 function

$f(n) = f(m)$, where $n, m \in \mathbb{N}$

to show that $n = m$

$2n-1 = 2m-1$

$n = m$

$\therefore f$ is a 1-1 function (injection)

- show that f is onto

$\forall m \in S;\ \exists n \in \mathbb{N}$

(given any odd integer, i can find its preimage in domain)

$f(n) = m$

$2n-1 = m$

$\Rightarrow n = \dfrac{m+1}{2}$

$\therefore$ f is an onto (surjective) function.

Since f is 1-1 and onto, its a bijection.

$\therefore$ set of odd positive integers is a countable set.

ex set of all integers $\mathbb{Z}$ is countable.

$f: \mathbb{N} \to \mathbb{Z}$

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \\ | & | & | & | & | & \cdots \\ 0 & 1 & -1 & 2 & -2 & \end{array}$$

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{when } n \text{ is even} \\ -\dfrac{n-1}{2} & \text{when } n \text{ is odd} \end{cases}$$

to show that f is 1-1 and onto.

cardinality $= \aleph_0$ (aleph-nought)

ex set of rational numbers $\mathbb{Q}$ is countable!

$\frac{p}{q}$ $\quad p, q \in \mathbb{Z}$

$q \neq 0$

to show that the set of rational numbers $\mathbb{Q}^+$ is countable.

$$
\begin{array}{llllll}
 & p+q \; 2 & 3 & 4 & 5 & 6 \\
q=1 & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} \; \ldots \\
q=2 & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} \; \ldots \\
q=3 & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} \; \ldots \\
q=4 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} \; \ldots \\
q=5 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} \; \ldots
\end{array}
$$

since all the positive rational numbers are listed once (can verify)

$\Rightarrow$ $\mathbb{Q}^+$ is countable.

$\frac{21}{25}$ in the bunch where $p+q = 46$.

EXERCISE to show that this is a bijection.

all the real nos. not countable

- not recurring
- not rational

cantor's diagonalization argument.

UNCOUNTABLE SETS

$\mathbb{R}$, $(0,1) \longrightarrow$ uncountable

thm set of $\mathbb{R}$ is uncountable

proof by contradiction

let us assume $\mathbb{R}$ is countable

$\Rightarrow$ subset $(0,1) \subset \mathbb{R}$ is also countable.

$\Rightarrow$ elements of $(0,1)$ can be listed in natural order.

$r_1 = 0.\ d_{11}\ d_{12}\ d_{13}\ d_{14} \ldots$

$r_2 = 0.\ d_{21}\ d_{22}\ d_{23}\ d_{24} \ldots$

$r_3 = 0.\ d_{31}\ d_{32}\ d_{33}\ d_{34} \ldots$

where $d_{ij} \in \{0, 1, \ldots 9\}$

now, construct a new real number

$r = 0.d_1 d_2 d_3 d_4 \ldots$

where $d_i = \begin{cases} 3 & \text{if } d_{ii} \neq 3 \\ 4 & \text{if } d_{ii} = 3 \end{cases}$

the new real number r is not equal to any of this $r_1, r_2, r_3, \ldots$

as r differs from decimal expansion of r_i in the d_{ii}th position $\forall i$.

there exist

$\exists$ a real number $r \in (0,1)$ that is not on the list.

$\Rightarrow$ contradiction with the enumerability assumption of $(0,1)$

set of real numbers $(0,1)$ is uncountable.

since $(0,1) \subset \mathbb{R}$ and uncountable, therefore $\mathbb{R}$ is also uncountable.

CANTOR'S DIAGONALIZATION ARGUMENT

thm: if A and B are countable sets, then $A \cup B$ is also countable (check rosen book for proof).

thm: schröder - bernstein theorm

if A and B are sets with the cardinality $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$ (bijection)

both A & B are 1-1 functions

very complex proof

ex. show that $|(0,1)| = |(0,1]|$

find an injective function $f: (0,1) \to (0,1]$

consider $f(x) = x$

$(0,1) \subset (0,1]$ domain is smaller

$\Rightarrow |(0,1)| \leq |(0,1]|$

find an injective function $g: (0,1] \to (0,1)$

consider $g(x) = \frac{x}{2}$

$(0, \frac{1}{2}] \subset (0,1)$

$\Rightarrow |(0,1]| \leq |(0,1)|$

using schröder-bernstein's theorm

$|(0,1)| = |(0,1]|$