COMPUTABLE AND UNCOMPUTABLE FUNCTIONS

dfn a function is computable, if there is a computer program that evaluates (in some programming language) the values of the function.

dfn a function which is not computable is an uncomputable function.

thm the set of all computer programs in any particular language is countable.

for any finite alphabet, there are finite no. of strings of length n, for every $n \in \mathbb{N}$, from the theorm (can verify) which states the union of a countable no. of countable sets is countable (lemma1).

enumerate in some order.

$\bigcup_{n=1}^{\infty} S_n$ is countable

⇒ there are only a countable no. of strings from any finite alphabet.

note, that the set of all computer programs (in a particular programming language) is a subset of the set of all the strings of a finite alphabet which is countable?

(lemma2) → a subset of a countable set is also countable
we can conclude the set of all computer programs is countable

thm Show that there is no 1-1 correspondence (bijection) from the set of +ve integers ($\mathbb{N}$) to the power set (set of all subsets) $P(\mathbb{N})$ $\mathbb{N} \not\leftrightarrow P(\mathbb{N})$

proof uses cantor's diagonalization argument
proof by contradiction.

on the contrary, we assume that
there is a bijection from $\mathbb{Z}^+$ to $P(\mathbb{Z}^+)$

$$P(\mathbb{Z}^+) = \{ \phi, \{1\}, \{2\}, \dots \{1,2\}, \{2,3\}, \dots \}$$

we will come up with a scheme of enumerating these.

$\mathbb{Z}^+$ $P(\mathbb{Z}^+)$

presence bit strings

trial

$1 \longleftrightarrow \{2\}$ 0 1 0 0 0 ...

$2 \longleftrightarrow \{1,2\}$ 1 1 0 0 0 ...

$3 \longleftrightarrow \{4,5,6\}$ 0 0 0 1 1 1 0 ...

$4 \longleftrightarrow \{1,2,3,4\}$ 1 1 1 1 0 0 0 0 ...

$\vdots$

$a_1 \; a_2 \; a_3$

$s = s_1 \; s_2 \; s_3 \; s_4$

1 0 1 0

every element A of $P(\mathbb{Z}^+)$ can be represented uniquely by the bit string $a_1 a_2 a_3 \ldots$ ~~where $a_i = 1$~~

where $a_i = 1$, if $i \in A$

$a_i = 0$, if $i \notin A$

now consider a string $s = s_1 s_2 s_3 \ldots$ by setting s_i to be $1 -$ the i^{th} bit of $f(i)$; such that s is not in the range of f.

therefore f cannot be 1-1 correspondence by contradiction, there is no 1-1 correspondence between $\mathbb{Z}^+$ onto $P(\mathbb{Z}^+)$.

- in general for any set A, $|A| < |P(A)|$

Continuum hypothesis (Cantor)

countably infinite		uncountable		uncountable	
$\lvert\mathbb{N}\rvert$	$<$	$\lvert P(\mathbb{N})\rvert$	$<$	$\lvert P(P(\mathbb{N}))\rvert$	$< \ldots$

$$|P(\mathbb{N})| \updownarrow |\mathbb{R}|$$

$$\aleph_0 < \aleph_1 < \aleph_2 < \ldots$$

there is no other countability (cardinality nos.)

$\aleph_0$ (aleph-nought)

$$2^{|\mathbb{N}_0|} = 2^{\aleph_0} = C$$

a barber who shaves all men who do not shave themselves.

↳ self-referential statements
bertrand russel

never consistent as well as complete · meta system

↳ gödel's incompleteness theorm

douglas hofstadter (GEB) - GS
godel, escher, bach
(mobius strip) (musician)

in general, we can verify that there are uncountably many different functions from a particular countably many infinite set (say $\mathbb{N}$) to itself.

(uses cantor's diagonalization argument)
⇒ there are uncomputable functions in general.

there are more computations than there are programs.

programs \ inputs	1	2	3	4	5 ...
1	1	0	1	...	
2	1	0	0	...	
3	1	1	0	...	
4	...	...	...		
⋮					

there are more problems than there are programs.

links set theory as application to theory of computation

SUMS

thm $1^3 + 2^3 + 3^3 + \dots n^3 = (1+2+3+\dots n)^2$

proof by induction

base case $n=1$:

$1^3 = 1^2$ ✓

let us assume its true for $n=K$

$1^3 + 2^3 + \dots K^3 = (1+2+\dots K)^2$

we need to show that this is true for $(k+1)$.

$$1^3 + 2^3 + \dots k^3 + (k+1)^3 = (1 + 2 + \dots k + (k+1))^2$$

$$LHS = (1+2+\dots k)^2 + (k+1)^3$$

$$= \left\{k\left(\frac{k+1}{2}\right)\right\}^2 + (k+1)^3$$

$$= (k+1)^2 \left\{\left(\frac{k}{2}\right)^2 + (k+1)\right\}$$

$$= \left(\frac{k+1}{2}\right)^2 \left\{k^2 + 4k + 4\right\}$$

$$= \left(\frac{k+1}{2}\right)^2 \left(k+2\right)^2$$

$$= \left\{\frac{(k+1)(k+2)}{2}\right\}^2$$

$$= \{1 + 2 + \dots k + (k+1)\}^2 = RHS \checkmark$$

by induction, the result is true for $n \in \mathbb{N}$

thm prove that $(1^2 + 2^2 + \dots n^2) = \dfrac{n(n+1)(2n+1)}{6}$

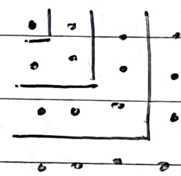
induction is a bad strategy for this.

look @ i^{th} term — simpler method.

$1^2 = 1$  1

$2^2 = 4$  $1 + 3$

$3^2 = 9$  $1 + 3 + 5$

$4^2 = 16$  $1 + 3 + 5 + 7$

(geometric – old books)
(detour)

$$1 + 3 + \dots (2n-1) = n^2$$

$$n(1) + (n-1)(3) + \dots (1)(2n-1)$$

$$\sum_{i=1}^{n} (n-i+1)(2i-1) = \sum_{i=1}^{n} i^2$$

$$\Rightarrow \sum (2in - n - 2i^2 + i + 2i - 1) = \sum i^2$$

$$\Rightarrow \sum \left(-2i^2 + i(2n+3) - (n+1)\right) = \sum i^2$$

$$\Rightarrow 3\sum i^2 = (2n+3)\sum i - (n+1)\sum 1$$

$$\Rightarrow 3\sum i^2 = (2n+3)\left(\frac{n(n+1)}{2}\right) - (n+1)n$$

$$\Rightarrow 6\sum i^2 = (2n+3)\, n(n+1) - 2n(n+1)$$

$$\Rightarrow 6\sum i^2 = n(n+1)\left\{2n+3 - 2\right\}$$

$$\Rightarrow \sum i^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum of squares using a different arg
we will continue in recurrence relations.