classmate

Date 17.9.2019

Page 1

CARDINALITY OF SETS

dfn inverse:

let $f: A \to B$ be a 1-1 correspondence (bijection)

the inverse fn. of f is $f^{-1}: B \to A$ such that

$f^{-1}(b) = a \iff f(a) = b$

dfn let $g: A \to B$ and $f: B \to C$ be functions

then $f \circ g : A \to C$ is called the composition of g with f

$\forall a \in A \quad f \circ g(a) = f(g(a))$

ex let $f: \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2 + 2$

$g: \mathbb{R} \to \mathbb{R}$ be $g(x) = 3x + 4$

then: $g \circ f : \mathbb{R} \to \mathbb{R}$

$g \circ f(x) = g(f(x)) = 3(x^2+2) + 4$

$= 3x^2 + 10$

$f \circ g : \mathbb{R} \to \mathbb{R}$

$f \circ g(x) = f(g(x)) = (3x+4)^2 + 2$

$= 9x^2 + 16 + 24x + 2$

$= 9x^2 + 24x + 18$

note $f \circ g(x) \neq g \circ f(x)$

thm ~~let~~ let $f: A \to B$ and $g: B \to C$

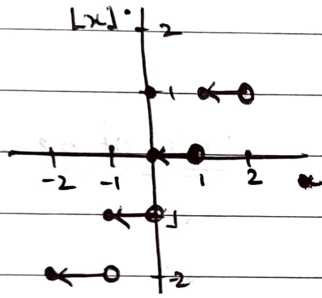
if f maps A onto B, and g maps B onto C,

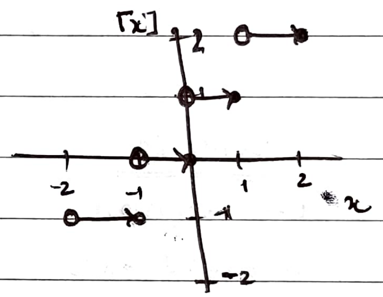
$\therefore g \circ f$ maps A onto C.

thm: if f and g are 1-1 functions (injections)
then $g \circ f : A \to C$ is also 1-1 function. <PROVE>

special fns: floor = $\lfloor x \rfloor$
→ assigns to x the largest integer $\leq x$

ceiling = $\lceil x \rceil$
→ assigns to x smallest integer $\geq x$.





exponential (e^x)
$y = a^x$, $a > 1$

logarithmic ($\ln x$)
$y = \log_b x$ ($\log_2 x$), $b > 1$

CARDINALITY

$\log_b a$ — for finite sets = no. of elements in the set

$\frac{\log a}{\log b}$ — dfn: the sets A and B have the same cardinality iff there is a 1-1 correspondence (bijection) from A to B.

when they are equal $|A| = |B|$

dfn if there is a 1-1 function (injection) from $A \to B$, the cardinality relationship

$|A| \leq |B|$

COUNTABLE SETS

infinite nos. of elements

countable — countably infinite

1-1 corresp. between $\mathbb{N}$ and the set in question

uncountable set — uncountably infinite

infinite set S is countable

$|S| = \aleph_0$ hebrew alphabet (aleph-nought)

ex show that the set of odd positive integers is a countable set.

$$\begin{array}{cccccc} 1 & 3 & 5 & 7 & 9 & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & 5 & \dots \end{array}$$

Need to establish a bijection between $\mathbb{N}$ and S

$f: \mathbb{N} \to S$ (bijection)

consider:

$f(n) = 2n - 1$ sets up 1-1 correspondance

↳ to prove 1-1 and onto..

show that f is a 1-1 fonction

$f(n) = f(m)$ where $n, m \in \mathbb{N}$
Show that $n = m$

$2n - 1 = 2m - 1$
$\Rightarrow n = m$
} f is a 1-1 function (injective).

show that f is onto. (ii) verify that f is onto.

$\forall m \in S; \exists n \in \mathbb{N}$ → i can find a pre image in domain
→ $f(n) = m$

$2n - 1 = m$

$\Rightarrow$ ~~n~~ $n = \frac{m+1}{2}$ Use k strategy.

since f is 1-1 and onto, it is bejection.

ex set of all integers $\mathbb{Z}$ is countable.

$f: \mathbb{N} \to \mathbb{Z}$

1	2	3	4	5	...
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
0	1	-1	2	-2	...

$$f(n) = \begin{cases} \frac{n}{2} & \text{when } n \text{ is even} \\ -\frac{n-1}{2} & \text{when } n \text{ is odd} \end{cases}$$

verification that f is 1-1 and onto.

Cardinality = $\aleph_0$ aleph-nought.

set of rational nos. $\mathbb{Q}$ is countable !

$\frac{p}{q}$ $\quad p, q \in \mathbb{Z}$

$\quad q \neq 0$.

ex show that the set of rational nos. ~~is~~ $\mathbb{Q}^+$ is countable.

$q=1$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$
$q=2$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$
$q=3$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$
$q=4$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$
$q=5$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$

Date 17.9.2019
Page 6

$$Q^+ = \underbrace{\frac{1}{1}}_{2}, \underbrace{\frac{1}{2}, \frac{2}{1}}_{3}, \underbrace{\frac{3}{1}, \frac{1}{3}}_{4}, \underbrace{\frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}}_{5}$$

$p+q \longrightarrow$ 2 3 4 5

since all the +ve rational nos. are listed once (can verify) $\Rightarrow Q^+$ is countable.

VERIFY

$\frac{21}{25}$ is in the bunch where $p+q = 46$.

ex to show that this is a bijection

all the real nos. not countable

not recurring not rational

(cantor's diagonalization argument)

UNCOUNTABLE SETS

$\mathbb{R}$, $(0,1) \longrightarrow$ uncountable.

thm set of $\mathbb{R}$ is countable.

Proof by ~~condtradiction~~ contradiction.

let us assume $\mathbb{R}$ is countable.

$\Rightarrow$ subset $(0,1)$ is also countable

$\Rightarrow$ elements of $(0,1)$ can be listed in natural order.

$r_1 = 0.\ d_{11}\ d_{12}\ d_{13}\ d_{14} \ldots$

$r_2 = 0.\ d_{21}\ d_{22}\ d_{23}\ d_{24} \ldots$

$r_3 = 0.\ d_{31}\ d_{32}\ d_{33}\ d_{34} \ldots$

where $d_{ij} \in \{0, 1, \ldots 9\}$.

now, construct a real no. $r = 0.d_1 d_2 \ldots$

where $d_i = \begin{cases} 3 & \text{if } d_{ii} \neq 3 \\ 4 & \text{if } d_{ii} = 3 \end{cases}$

$r_1 = 0.341259$ $\qquad$ $r = 0.434\ldots$

$r_2 = 0.451219$

$r_3 = 0.123456$

the new real no. r is not equal to any of $r_1, r_2, \ldots$ as r differs from decimal expansion of r_i in the d_{ii}^{th} position; $\forall i$

$\exists$ a real number $r \in (0,1)$ that is not on the list.

$\Rightarrow$ contradiction with the enumerability assumption of $(0,1)$

Set of real nos. in $(0,1)$ is uncountable.

Since $(0,1) \subset \mathbb{R}$ and uncountable, therefore $\mathbb{R}$ is also uncountable (Cantor's diagonalization argument).

thm: if A and B are countable sets, then $A \cup B$ is also countable (Check rosen book for proof).

thm: (Schröder – bernstein theorm).

if A and B are sets with the cardinality
$|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$ (bijection).

ex: Show that $|(0,1)| = |(0,1]|$

- find an injective fn: $f\ (0,1) \to (0,1]$

consider $f(x) = x$ $\qquad (0,1) \subset (0,1]$

$\Rightarrow |(0,1)| \leq |(0,1]|$

find an injective fn: $g\ (0,1] \to (0,1)$

consider $g(x) = \frac{x}{2}$ $\qquad (0,\frac{1}{2}] \subset (0,1)$

$\Rightarrow |(0,1]| \leq |(0,1)|$

$\Downarrow$

Schröder B.

$|(0,1)| = |(0,1]|$