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- can do better implicitly

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Diffusion equation: D(u,x)

- $u_j^{n+1} = \frac{\Delta t}{\Delta x^2}\left[D(u^n_{i+1/2}, x_{i+1/2})(u^{n+1}_{i+1} - u^{n+1}_i) - D(u^n_{i-1/2}, x_{i-1/2})(u^{n+1}_i - u^{n+1}_{i-1})\right]$
- $D(u^n_{i+1/2}, x_{i+1/2}) = \mathcal{M}\left[D(u^n_i, x_i), D(u^n_{i+1}, x_{i+1})\right]$
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