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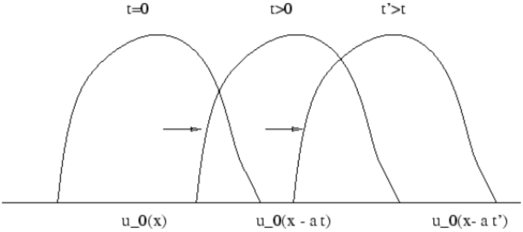
- $u(x,t) = u(x - vt, 0)$



Advection equation: FTCS, Lax

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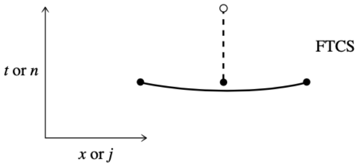
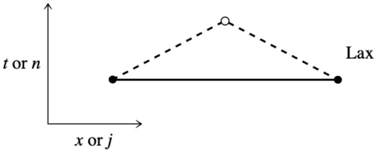
  $\xi = \frac{(e^{ik\Delta x} + e^{-ik\Delta x})}{2} - \frac{v\Delta t}{2\Delta x}(e^{ik\Delta x} - e^{-ik\Delta x}) = \cos k\Delta x - i\frac{v\Delta t}{\Delta x}\sin k\Delta x$
- $|\xi| < 1$ if $\Delta t \leq \Delta x / v$; Courant-Friedrichs-Lewy (CFL) condition
- Lax equivalent to $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = \frac{\Delta x^2}{2\Delta t}\nabla^2 u$





Advection equation: errors

- Amplitude, phase errors; amplification factor $\xi = Ae^{i\phi}$
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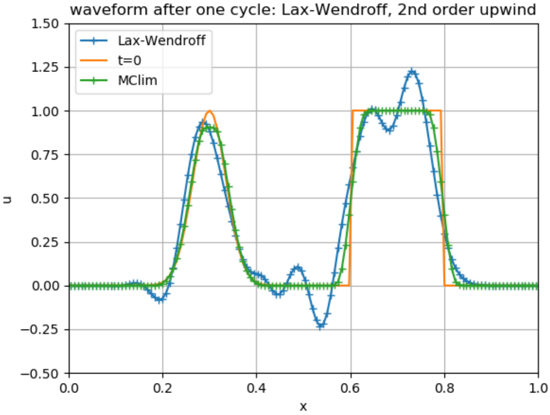
- shocks from nonlinearity! artificial/numerical viscosity
- identify amplitude, phase error in the following



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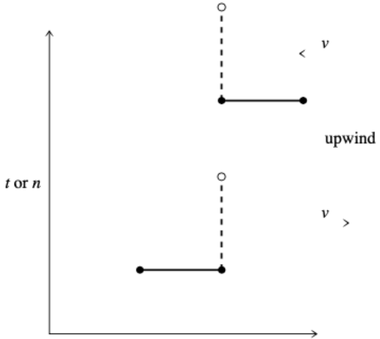
- upwind differencing (only 1st order accurate):

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\Delta t} &= -v_j^n \frac{u_j^n - u_{j-1}^n}{\Delta x}, \text{ for } v_j^n \geq 0 \\ &= -v_j^n \frac{u_{j+1}^n - u_j^n}{\Delta x}, \text{ for } v_j^n < 0 \end{aligned}$$



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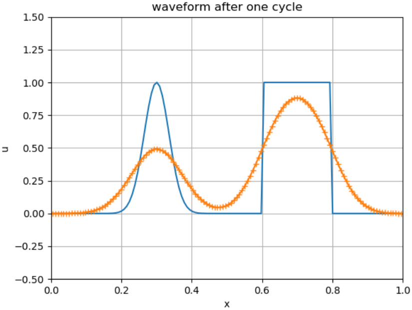
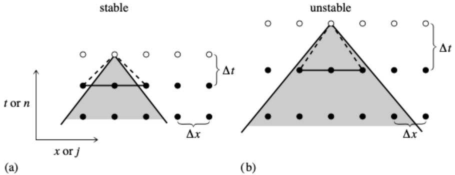
- for numerical stability, again, $\Delta t \leq \Delta x / v$ (CFL)





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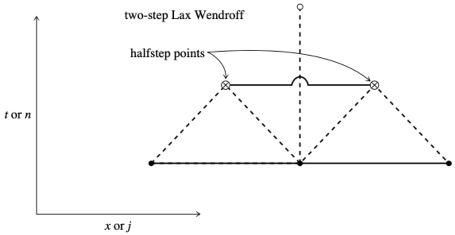

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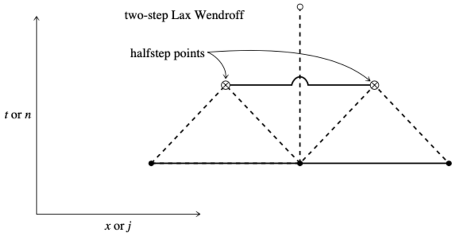

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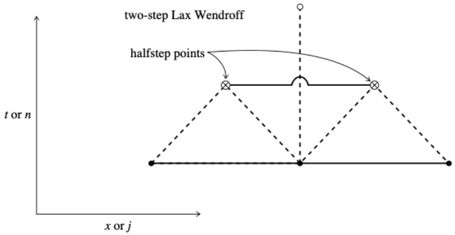

- again $\Delta t \leq \Delta x / v$ for stability (CFL)
- problem: oscillations at discontinuities!

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- numerical instability won't go away with resolution!

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- an unsplit scheme; one can split by directions

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- **Alternating-direction implicit (ADI)**: split into two steps of size $\Delta t/2$ & update x/y implicitly alternately
- $u_{j,l}^{n+1/2} = u_{j,l}^{n} + \frac{D\Delta t}{2\Delta^2}\left(\delta_x^2 u_{j,l}^{n+1/2} + \delta_y^2 u_{j,l}^{n}\right)$
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  $\delta_x^2 u_{j,l} \equiv u_{j+1,l} - 2u_{j,l} + u_{j-1,l}$
- Two tridiagonal eqs. to solve every Δt!

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&\cdots & &\cdots \\
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- first order accurate in time; can be made 2nd by Strang splitting $(\mathcal{A}/2)\mathcal{B}(\mathcal{A}/2)$
- $\frac{u^{n+1}}{u^n} = e^{(\mathcal{A}+\mathcal{B})\Delta t} \approx$ $1 - (\mathcal{A}+\mathcal{B})\Delta t + (\mathcal{A}^2 + \mathcal{B}^2 + \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A})\frac{\Delta t^2}{2} + ...$
- operators $\mathcal{A}$ & $\mathcal{B}$ do not commute in general!