- Fourier Transforms.
- Discrete Fourier Transform.

Fourier Transforms

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- Allows one to break down functions/signals into their component parts and analyze, smooth or filter them.
- Also allows one to perform certain kinds of calculations and solve certain differential equations.

Fourier Series

Any periodic function $f(x)$ defined on a finite interval $0 \leq x \leq L$ can be written as a Fourier series (assuming that the function is bounded and has at most a finite number of discontinuities).

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- If the function is even (i.e. symmetric) about the midpoint $(x = \frac{L}{2})$ then one can write the cosine series:

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- If the function is odd (i.e. antisymmetric) about the midpoint $(x = \frac{L}{2})$ then one can write the sine series:

$$f(x) = \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{2\pi k x}{L}\right)$$

Fourier Series – periodic vs non periodic

- For a general function (with no special symmetry), one can write the series:

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- If the function in not periodic, and we are only interested in a portion of this non periodic function over a finite interval, 0 to L, we can just take that portion and repeat it to create a periodic function!
- Then the Fourier coefficients will only give the correct information about the function in the interval 0 to L. Outside this interval, the function will be just repeated (and may not have anything to do with the original function.

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$$\int_0^L f(x)\exp\left(-i\frac{2\pi kx}{L}\right)dx = \sum_{k'=-\infty}^{\infty}\gamma_{k'}\int_0^L \exp\left(i\frac{2\pi(k'-k)x}{L}\right)dx$$

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If $k' \neq k$:

$$\begin{aligned}\int_0^L \exp\left(i\frac{2\pi(k'-k)x}{L}\right)dx &= \frac{L}{i2\pi(k'-k)}\left[\exp\left(i\frac{2\pi(k'-k)x}{L}\right)\right]_0^L \\ &= \frac{L}{i2\pi(k'-k)}[e^{i2\pi(k'-k)}-1] \\ &= 0\end{aligned}$$

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Thus, given a function $f(x)$, we can find the Fourier coefficients γ_k, or given the coefficients, we can find the function $f(x)$ – we can go back and forth freely between the function and the Fourier coefficients.

- For some functions $f(x)$, the Fourier coefficients can be determined analytically.

Discrete Fourier Transform (DFT)

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- There are, however, many cases where this is not doable – the integral is not doable because the function is too complicated or the function $f(x)$ may not even be known in analytic form (for eg. if it is a signal measured in the laboratory experiment).

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- In such cases, the integral can be evaluated numerically.

Discrete Fourier Transform (DFT)

Applying the trapezoidal rule for integration (N slices of width $h = L/N$) to calculate γ_k:

$$\gamma_k = \frac{1}{L}\frac{L}{N}\left[\frac{f(0)}{2} + \frac{f(L)}{2} + \sum_{n=1}^{N-1} f(x_n)\exp\left(-i\frac{2\pi k x_n}{L}\right)\right]$$

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This formula can be used to evaluate the coefficients on a computer. A simpler way to write this is as:

$$\gamma_k = \frac{1}{N}\left[\sum_{n=0}^{N-1} y_n\exp\left(-i\frac{2\pi k n}{N}\right)\right]$$

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The quantities γ_k and c_k only differ by the constant $1/N$ factor. For our purpose they are both equal, and we define the latter as the definition of DFT.

Discrete Fourier Transform (DFT)

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\begin{aligned}
\sum_{k=0}^{N-1} c_k \exp\left(i\frac{2\pi kn}{N}\right) &= \sum_{k=0}^{N-1}\sum_{n'=0}^{N-1} y_{n'} \exp\left(-i\frac{2\pi kn'}{N}\right)\exp\left(i\frac{2\pi kn}{N}\right) \\
&= \sum_{n'=0}^{N-1} y_{n'} \sum_{k=0}^{N-1} \exp\left(i\frac{2\pi k(n'-n)}{N}\right) \\
&= \sum_{n'=0}^{N-1} y_{n'} \frac{\exp\left(i2\pi(n'-n)\right)-1}{\exp\left(i2\pi(n'-n)/N\right)-1} = \sum_{n'=0}^{N-1} y_{n'} N\delta_{n,n'} \\
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Equivalently,

$$y_n = \frac{1}{N}\sum_{k=0}^{N-1} c_k \exp\left(i\frac{2\pi kn}{N}\right)$$

This is called the "Inverse Discrete Fourier Transform."

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- It is important to appreciate that unlike the original Fourier series, the discrete version only gives us the sample values at $y_n = f(x_n)$. It tells us nothing about the value of the function $f(x)$ in between the points.

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- It is important to appreciate that unlike the original Fourier series, the discrete version only gives us the sample values at $y_n = f(x_n)$. It tells us nothing about the value of the function $f(x)$ in between the points.
- So, two different functions with same values at the sample points will have the same DFT – no matter what they do in between the points!

DFT for real functions

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$$\begin{aligned}
c_{N-r} &= \sum_{n=0}^{N-1} y_n \exp\left(-i\frac{2\pi(N-r)n}{N}\right) \\
&= \sum_{n=0}^{N-1} y_n \exp(-i2\pi n)\exp\left(i\frac{2\pi rn}{N}\right) \\
&= \sum_{n=0}^{N-1} y_n \exp\left(i\frac{2\pi rn}{N}\right) = c_r^*
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Thus: $c_{N-1} = c_1^*$ $c_{N-2} = c_2^*$ and so forth. That means that Fourier coefficients c_k of a real function only has to be calculated for $0 \le k \le \frac{N}{2}$.

```python
from numpy import zeros
from cmath import exp, pi

def dft(y):
    N = len(y)
    c = zeros(N//2 +1, complex)
    for k in range(N//2+1):
        for n in range(N):
            c[k] += y[n]*exp(-2j*pi*k*n/N)
return c
```

One thing to notice about the DFT is that we can shift the sample points along the x-axis, and not much changes. Suppose:

$$x'_n = x_n + \Delta = \frac{n}{N}L + \Delta$$

Discrete Fourier Transform – positions of sample points

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$$x'_n = x_n + \Delta = \frac{n}{N}L + \Delta$$

Then the DFT is:

$$\begin{aligned} c_k &= \sum_{n=0}^{N-1} f(x_n + \Delta)\exp\left(-i\frac{2\pi k(x_n + \Delta)}{L}\right) \\ &= \exp\left(-i\frac{2\pi k\Delta}{L}\right)\sum_{n=0}^{N-1} f(x'_n)\exp\left(-i\frac{2\pi k x_n}{L}\right) \\ &= \exp\left(-i\frac{2\pi k\Delta}{L}\right)\sum_{n=0}^{N-1} y'_n\exp\left(-i\frac{2\pi k n}{N}\right) \end{aligned}$$

Discrete Fourier Transform – positions of sample points

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But this is the same as the original DFT except for a (k-dependent) phase factor. Thus the DFT is really independent of where we choose to place the samples – only the coefficients change by a phase factor.

Two-dimensional Fourier Transforms

Functions of two variables $f(x, y)$ can also be Fourier transformed, using a two dimensional Fourier transform. This simply means that one first transform with respect to one variable and then the other!

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Suppose that we have an $M \times N$ grid points of samples y_{mn}.
We first transform on each of the M rows:

$$c'_{ml} = \sum_{n=0}^{N-1} y_{mn} \exp\left(-i\frac{2\pi ln}{N} \right)$$

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Then we take the l^{th} coefficient of each row and Fourier transform them :

$$c_{kl} = \sum_{m=0}^{M-1} c'_{ml} \exp\left(-i\frac{2\pi km}{M}\right)$$

Two-dimensional Fourier Transforms

Alternatively, we can write a single expression for the complete Fourier transform in two dimensions:

$$c_{kl} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} y_{mn} \exp\left[-i2\pi \left(\frac{ln}{N} + \frac{km}{M} \right) \right]$$

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The corresponding inverse transform is:

$$y_{mn} = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} c_{kl} \exp\left[i2\pi \left(\frac{ln}{N} + \frac{km}{M} \right) \right]$$