Ordinary Differential Equations

- Varying the step size.
- Verlet method.
- Leapfrog method.

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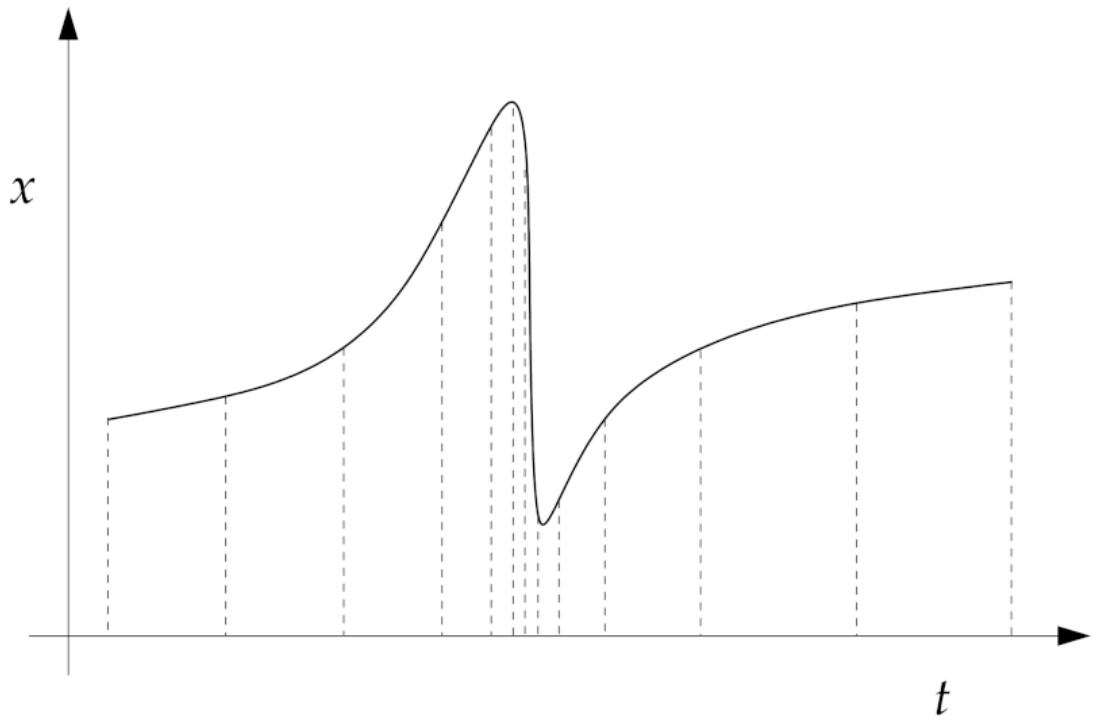

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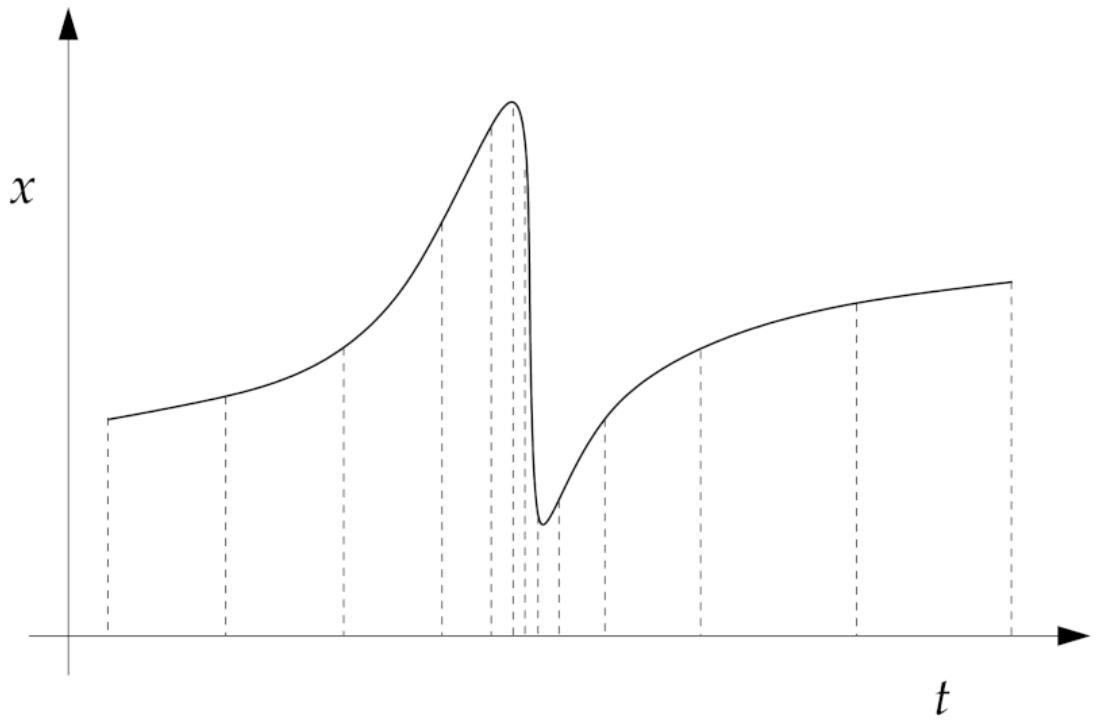
- In the regions the solution is varying rapidly, one need points that are more closely spaced!



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- Illustrate this approach using 4th order Runge Kutta.

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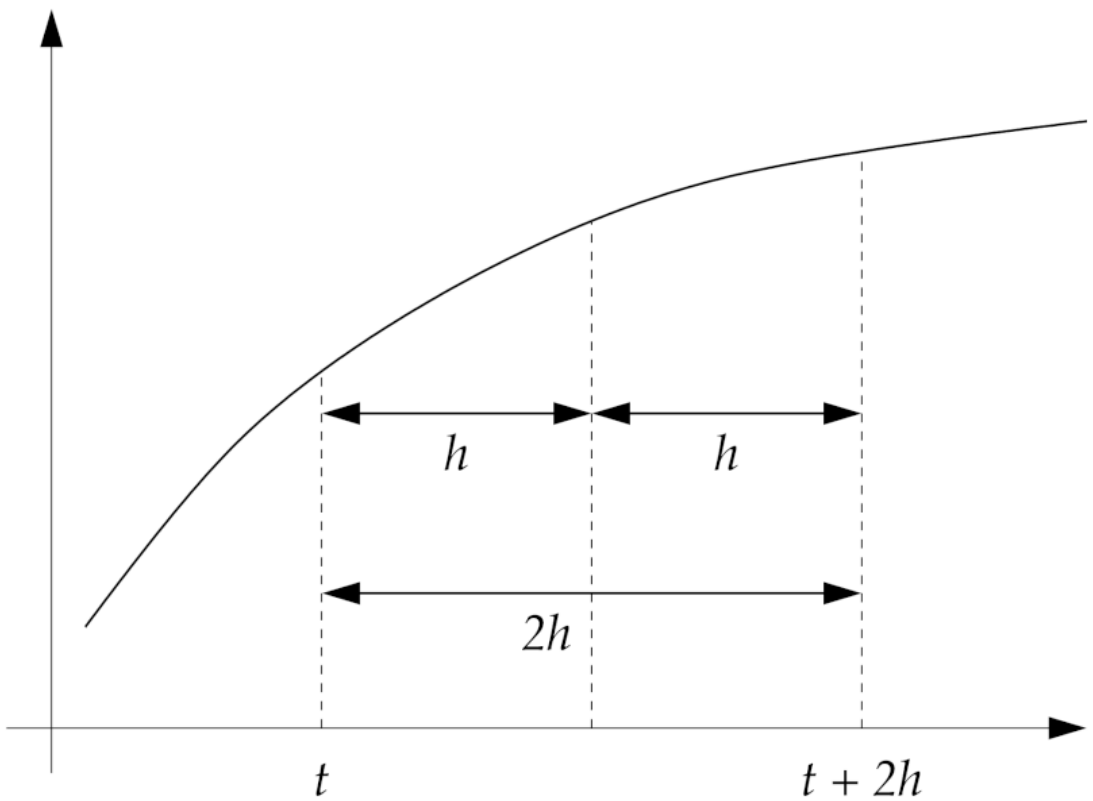
- Start with some initial value of h – and do first two steps of the solution. So – starting at $x(t)$ we have an estimate for $x(t + 2h)$.



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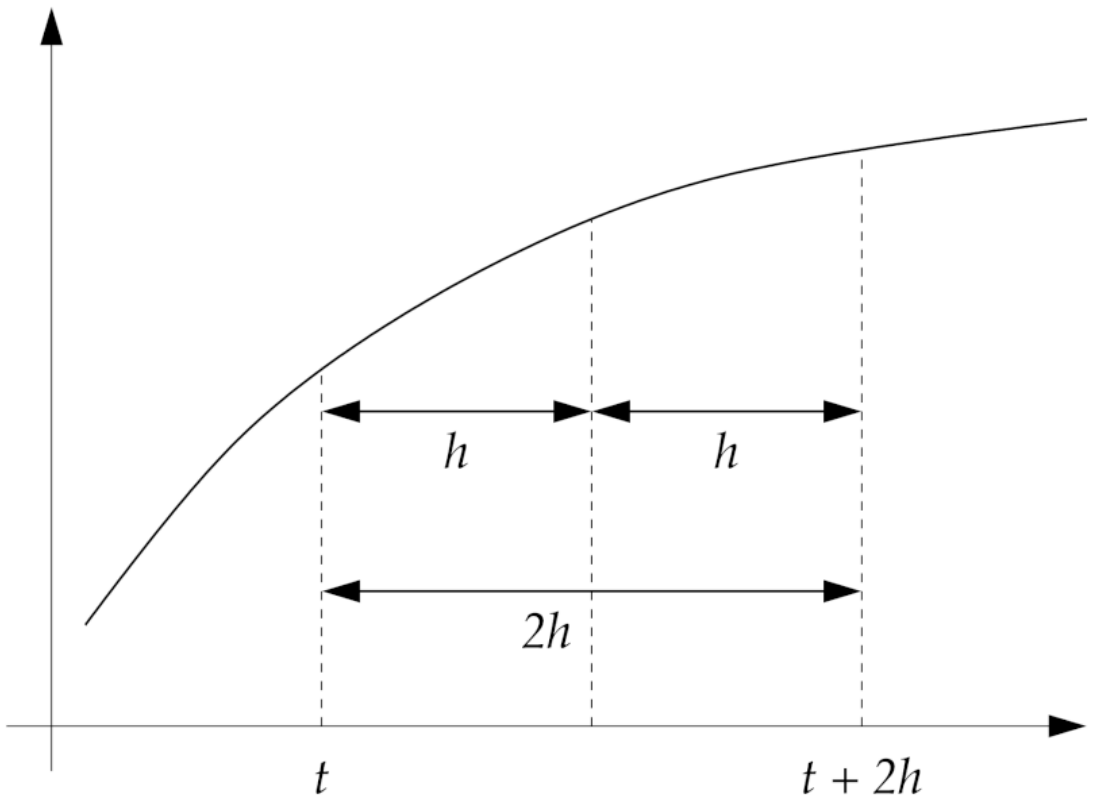
- Now going back to t again – we reach $x(t + 2h)$ in just one Runge-Kutta step.



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- Equating the two, the error ϵ on steps of size h is:

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- If our target accuracy **per unit time** is δ, the error for each step is $h'\delta$.

$$h' = h\left(\frac{30h\delta}{|x_1 - x_2|}\right)^{1/4}.$$

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- This method is very widely used! All black-box solvers use something like this.

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- This can be easily incorporated in the code and can give a modest improvement in the accuracy of the results.

The Leapfrog method

Consider:

$$\frac{dx}{dt} = f(x,t)$$

- In the second order Runge-Kutta method, one estimates the value of $x(t+h)$ by using the slope at the midpoint $f(x(t+\frac{1}{2}h), t+\frac{1}{2}h)$. But as one does not know the value at the midpoint, it is calculated using Euler's method:

$$\begin{aligned} x(t+\tfrac{1}{2}h) &= x(t) + \tfrac{1}{2}hf(x,t) \\ x(t+h) &= x(t) + hf(x(t+\tfrac{1}{2}h), t+\tfrac{1}{2}h) \end{aligned}$$

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- The leapfrog method is a variant of this idea – this method starts out the same way as Runge-Kutta with a half-step to the midpoint, followed by a full step to calculate $x(t+h)$ – but then rather than calculating the next midpoint value from $x(t+h)$, it is calculated from the previous midpoint value $x(t+\frac{1}{2}h)$.

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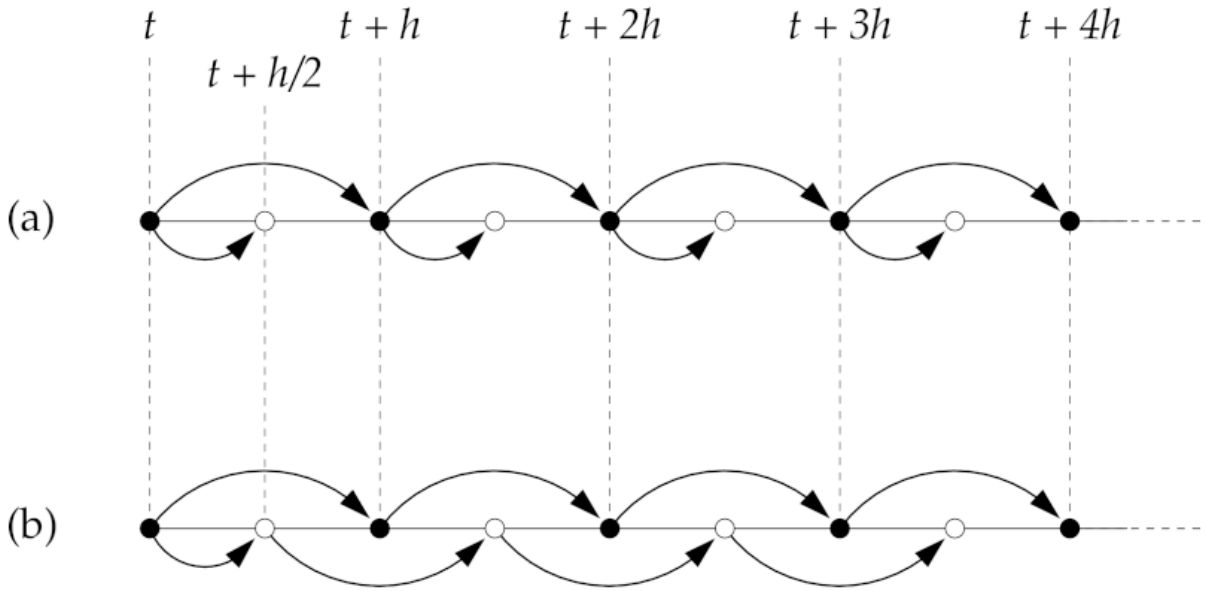

$$x(t+h) = x(t) + hf(x(t+\tfrac{1}{2}h), t+\tfrac{1}{2}h)$$
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- It is time reversal symmetric, which makes it useful for physics problems where energy conservation is important.
- The error is even in the step size h, which makes it an ideal starting point for Richardson extrapolation!

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- Energy is conserved does not mean it is constant over some cycle – it may oscillate – but will not drift.

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- The values of x are only available at integer multiples of h, while the velocity is at half integer multiples.

The Verlet Method

Given the initial values of x and v at some time t :

$$\begin{aligned} v(t+\tfrac{1}{2}h) &= v(t) + \tfrac{1}{2}hf(x(t),t) \\ x(t+h) &= c(t) + hv(t+\tfrac{1}{2}h) \\ k &= hf(x(t+h),t+h) \\ v(t+h) &= v(t+\tfrac{1}{2}h) + \tfrac{1}{2}k \\ v(t+\tfrac{3}{2}h) &= v(t+\tfrac{1}{2}h) + k \end{aligned}$$

Notice in the above equations, $v(t+h)$ is calculated even though it is not really needed. This is done to ensure that one can calculate total energy at any time, where both x and v are available.