Higher order & implicit methods for ODEs

- Higher order method using Leap-frog: modified-midpoint & Bulirsch-Stoer methods
- Stiff equations
- Higher order semi-implicit extrapolation method

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- This gives only even terms in the cumulative error with h^2 leading order – Gragg's modified-midpoint method.

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- Combine them using Richardson extrapolation

$$
\begin{array}{lccccccc}
n=1: & R_{1,1} & & & & & & \\
 & & \searrow & & & & & \\
n=2: & R_{2,1} & \rightarrow & R_{2,2} & & & & \\
 & & \searrow & & \searrow & & & \\
n=3: & R_{3,1} & \rightarrow & R_{3,2} & \rightarrow & R_{3,3} & & \\
 & & \searrow & & \searrow & & \searrow & \\
n=4: & R_{4,1} & \rightarrow & R_{4,2} & \rightarrow & R_{4,3} & \rightarrow & R_{4,4} \\
 & \underbrace{\phantom{R_{4,1}}}_{\substack{\text{Modified}\\ \text{midpoint}}} & & \multicolumn{5}{c}{\underbrace{\phantom{R_{4,2} \rightarrow R_{4,3} \rightarrow R_{4,4}}}_{\text{Richardson extrapolation}}}
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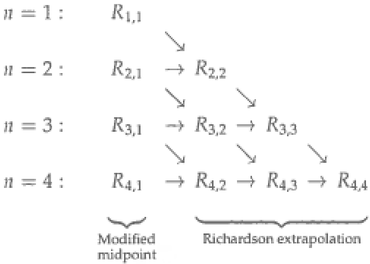
- typically n shouldn't be more than 10 or so



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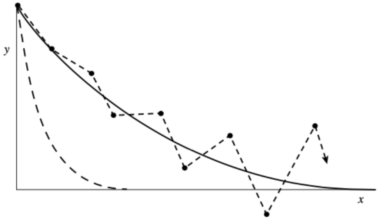
- Analytic solution: $u = 2e^{-x} - e^{-1000x}$; $v = -e^{-x} + e^{-1000x}$; one term decays 1000 times faster!



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