MAIN TITLE OF YOUR THESIS

A THESIS

submitted by

YOUR FULL NAME

(ROLL NO. YOURROLLNO)

for the

AWARD OF THE DEGREE

of

DOCTOR OF PHILOSOPHY

INDIAN INSTITUTE
OF TECHNOLOGY
PALAKKAD

Department of Computer Science and Engineering

Month YYYY

Your Full Name: *Main Title of Your Thesis*
©Indian Institute of Technology Palakkad
Month YYYY

CERTIFICATE

This is to certify that the thesis titled ***Main Title of Your Thesis***, submitted by ***Your Full Name (Roll No. YourRollNo)*** for the award of the degree of ***Doctor of Philosophy*** of ***Indian Institute of Technology Palakkad***, is a record of bonafide work carried out by her under my guidance and supervision at ***Department of Computer Science and Engineering, Indian Institute of Technology Palakkad***. To the best of my knowledge and belief, the work presented in this thesis is original and has not been submitted, either in part or full, for the award of any other degree, diploma, fellowship, associateship or similar title of any university or institution.

Name of Supervisor

DECLARATION

I hereby declare that the work reported in this thesis is original and was carried out by me. Further, this thesis has not formed the basis, neither has it been submitted for the award of any degree, diploma, fellowship, associateship or similar title of any university or institution.

Palakkad,
Month YYYY

Your Full Name
(Roll No. YourRollNo)

Ohana means family.
Family means nobody gets left behind, or forgotten.

— Lilo & Stitch

Dedicated to the loving memory of Rudolf Miede.

1939 – 2005

ABSTRACT

Add abstract of your thesis here. a great guide by Kent Beck how to write good abstracts can be found here:

`https://plg.uwaterloo.ca/~migod/research/beckOOPSLA.html`

What is an Abstract?

- The abstract is a summary of the whole thesis. It presents all the major elements of your work in a highly condensed form.
- An abstract often functions, together with the thesis title, as a stand-alone text. Abstracts appear, absent the full text of the thesis, in bibliographic indexes such as PsycInfo. They may also be presented in announcements of the thesis examination. Most readers who encounter your abstract in a bibliographic database or receive an email announcing your research presentation will never retrieve the full text or attend the presentation.
- An abstract is not merely an introduction in the sense of a preface, preamble, or advance organizer that prepares the reader for the thesis. In addition to that function, it must be capable of substituting for the whole thesis when there is insufficient time and space for the full text.

Keywords: Keyword One, Keyword Two, Keyword Three
AMS Subject Classification: Give Class here . Remove if irrelevant
ACM Subject Classification: Give Class here . Remove if irrelevant

PUBLICATIONS

This might come in handy for PhD theses: some ideas and figures have appeared previously in the following publications:

Attention: This requires a separate run of `bibtex` for your `refsection`, e. g., `ClassicThesis1-blx` for this file. You might also use `biber` as the backend for `biblatex`. See also http://tex.stackexchange.com/questions/128196/problem-with-refsection. This may also be achieved by running latexmk.

This is just an early – and currently ugly – test!

We have seen that computer programming is an art,
because it applies accumulated knowledge to the world,
because it requires skill and ingenuity, and especially
because it produces objects of beauty.

— Donald E. Knuth [16]

ACKNOWLEDGMENTS

Put your acknowledgments here.

Many thanks to everybody who already sent me a postcard!

Regarding the typography and other help, many thanks go to Marco Kuhlmann, Philipp Lehman, Lothar Schlesier, Jim Young, Lorenzo Pantieri and Enrico Gregorio[1], Jörg Sommer, Joachim Köstler, Daniel Gottschlag, Denis Aydin, Paride Legovini, Steffen Prochnow, Nicolas Repp, Hinrich Harms, Roland Winkler, Jörg Weber, Henri Menke, Claus Lahiri, Clemens Niederberger, Stefano Bragaglia, Jörn Hees, Scott Lowe, Dave Howcroft, José M. Alcaide, David Carlisle, Ulrike Fischer, Hugues de Lassus, Csaba Hajdu, Dave Howcroft, and the whole LaTeX-community for support, ideas and some great software.

Regarding LyX: The LyX port was intially done by *Nicholas Mariette* in March 2009 and continued by *Ivo Pletikosić* in 2011. Thank you very much for your work and for the contributions to the original style.

1 Members of GuIT (Gruppo Italiano Utilizzatori di TeX e LaTeX)

LIST OF FIGURES

LIST OF TABLES

LIST OF ALGORITHMS

LISTINGS

LIST OF SYMBOLS

A_i	Area of the i^{th} component
Symbol1	Description of Symbol1

ABBREVIATIONS

Part I

SOME KIND OF MANUAL

1 INTRODUCTION BY AUTHOR OF THE PACKAGE

This bundle for LaTeX has two goals:

1. Provide students with an easy-to-use template for their Master's or PhD thesis. (Though it might also be used by other types of authors for reports, books, etc.)
2. Provide a classic, high-quality typographic style that is inspired by Bringhurst's "*The Elements of Typographic Style*" [7].

Main Title of Your Thesis classicthesis v4.6

The bundle is configured to run with a *full* MiKTeX or TeXLive[1] installation right away and, therefore, it uses only freely available fonts.

As of version 3.0, `classicthesis` can also be easily used with LyX[2] thanks to Nicholas Mariette and Ivo Pletikosić. The LyX version of this manual will contain more information on the details.

This should enable anyone with a basic knowledge of LaTeX 2_ε or LyX to produce beautiful documents without too much effort. In the end, this is my overall goal: more beautiful documents, especially theses, as I am tired of seeing so many ugly ones.

The whole template and the used style is released under the GNU General Public License.

If you like the style then I would appreciate a postcard:

André Miede
Detmolder Straße 32
31737 Rinteln
Germany

The postcards I received so far are available at:

http://postcards.miede.de

A well-balanced line width improves the legibility of the text. That's what typography is all about, right?

So far, many theses, some books, and several other publications have been typeset successfully with it. If you are interested in some typographic details behind it, enjoy Robert Bringhurst's wonderful book.

Important Note: Some things of this style might look unusual at first glance, many people feel so in the beginning. However, all things are intentionally designed to be as they are, especially these:

1 See the file `LISTOFFILES` for needed packages. Furthermore, `classicthesis` works with most other distributions and, thus, with most systems LaTeX is available for.
2 http://www.lyx.org

- No bold fonts are used. Italics or spaced small caps do the job quite well.

- The size of the text body is intentionally shaped like it is. It supports both legibility and allows a reasonable amount of information to be on a page. And, no: the lines are not too short.

- The tables intentionally do not use vertical or double rules. See the documentation for the `booktabs` package for a nice discussion of this topic.[3]

- And last but not least, to provide the reader with a way easier access to page numbers in the table of contents, the page numbers are right behind the titles. Yes, they are *not* neatly aligned at the right side and they are *not* connected with dots that help the eye to bridge a distance that is not necessary. If you are still not convinced: is your reader interested in the page number or does she want to sum the numbers up?

Therefore, please do not break the beauty of the style by changing these things unless you really know what you are doing! Please.

1.1 ORGANIZATION

You can use these margins for summaries of the text body...

A very important factor for successful thesis writing is the organization of the material. This template suggests a structure as the following:

- `Chapters/` is where all the "real" content goes in separate files such as `Chapter01.tex` etc.

- `FrontBackMatter/` is where all the stuff goes that surrounds the "real" content, such as the acknowledgments, dedication, etc.

- `gfx/` is where you put all the graphics you use in the thesis. Maybe they should be organized into subfolders depending on the chapter they are used in, if you have a lot of graphics.

- `Bibliography.bib`: the BibTEX database to organize all the references you might want to cite.

- `classicthesis.sty`: the style definition to get this awesome look and feel. Does not only work with this thesis template but also on its own (see folder `Examples`). Bonus: works with both LATEX and PDFLATEX... and LYX. Great tool and it's free!

- `ClassicThesis.tex`: the main file of your thesis where all gets bundled together.

3 To be found online at http://mirror.ctan.org/macros/latex/contrib/booktabs/.

- `classicthesis-config.tex`: a central place to load all nifty packages that are used.

  Make your changes and adjustments here. This means that you specify here the options you want to load `classicthesis.sty` with. You also adjust the title of your thesis, your name, and all similar information here. Refer to **??** for more information.

  This had to change as of version 3.0 in order to enable an easy transition from the "basic" style to LyX.

In total, this should get you started in no time.

1.2 STYLE OPTIONS

. . . or your supervisor might use the margins for some comments of her own while reading.

There are a couple of options for `classicthesis.sty` that allow for a bit of freedom concerning the layout:

- General:
  - `drafting`: prints the date and time at the bottom of each page, so you always know which version you are dealing with. Might come in handy not to give your Prof. that old draft.
- Parts and Chapters:
  - `parts`: use this option if you *use* Part divisions in your document. This is necessary to get the spacing of the Table of Contents right. (Cannot be used together with `nochapters`.)
  - `linedheaders`: changes the look of the chapter headings a bit by adding a horizontal line above the chapter title. The chapter number will also be moved to the top of the page, above the chapter title.
- Typography:
  - `style`: this offers a comfortable way of changing the look and feel easily. Default style is `classicthesis`.

    As a new feature, Lorenzo Pantieri's `arsclassica` is available as well. As Lorenzo's package is discontinued and with his permission, `classicthesis-arsclassica.sty` is now part of `classicthesis` and will be maintained here.
  - `palatino`: Hermann Zapf's classic font is the free standard font for this style. Robert Bringhurst's book uses Adobe's commercial font Minion Pro. However, there are other free alternatives also available. Deactivate this option for loading such alternatives and see `classicthesis-config.tex` for some suggestions.
  - `eulerchapternumbers`: use figures from Hermann Zapf's Euler math font for the chapter numbers. By default, old style figures from the Palatino font are used.
  - `beramono`: loads Bera Mono as typewriter font. (Default setting is using the standard CM typewriter font.)
  - `eulermath`: loads the awesome Euler fonts for math. Palatino is used as default font.

Options are enabled via `option=true`

- Table of Contents:
  - `tocaligned`: aligns the whole table of contents on the left side. Some people like that, some don't.

- `dottedtoc`: sets pagenumbers flushed right in the table of contents.
- `manychapters`: if you need more than nine chapters for your document, you might not be happy with the spacing between the chapter number and the chapter title in the Table of Contents. This option allows for additional space in this context. However, it does not look as "perfect" if you use `\parts` for structuring your document.

• Floats:
  - `floatperchapter`: activates numbering per chapter for all floats such as figures, tables, and listings (if used).

• Tweaking colors and fonts – please use this with great care!:
  - `\ct@altfont`: comfortable hook to alter the basic look and feel of everything that uses spaced caps or spaced small caps. For example, for `arsclassica` we used `\renewcommand*{\ct@altfont}{\sffamily}`. Coloring is also possible this way.
  - `CTsemi`: Change the semi gray color used, e. g., for the chapter number.
    Default is: `\definecolor{CTsemi}{gray}{0.55}`
  - `CTtitle`: Change the red color used, e. g., for the title. Default is: `\definecolor{CTtitle}{named}{Maroon}`

Furthermore, pre-defined margins for different paper sizes are available, e. g., `a4paper`, `a5paper`, `b5paper`, and `letterpaper`. These are based on your chosen option of `\documentclass`.

The best way to figure these options out is to try the different possibilities and see what you and your supervisor like best.

In order to make things easier, `classicthesis-config.tex` contains some useful commands that might help you.

1.3 CUSTOMIZATION

This section will show you some hints how to adapt `classicthesis` to your needs.

The file `classicthesis.sty` contains the core functionality of the style and in most cases will be left intact, whereas the file `classicthesis-config.tex` is used for some common user customizations.

The first customization you are about to make is to alter the document title, author name, and other thesis details. In order to do this, replace the data in the following lines of `classicthesis-config.tex`:

Modifications in `classicthesis-config.tex`

```
% ****************************************************
% 2. Personal data and user ad-hoc commands
% ****************************************************
\newcommand{\myTitle}{A Classic Thesis Style}
\newcommand{\mySubtitle}{An Homage to...}
```

Further customization can be made in `classicthesis-config.tex` by choosing the options to `classicthesis.sty` (see **??**) in a line that looks like this:

```
\PassOptionsToPackage{
  drafting=true,
  tocaligned=false,
  dottedtoc=false,
  eulerchapternumbers=true,
  linedheaders=false,
  floatperchapter=true,
  eulermath=false,
  beramono=true,
  palatino=true,
  style=classicthesis
}{classicthesis}
```

Many other customizations in `classicthesis-config.tex` are possible, but you should be careful making changes there, since some changes could cause errors.

1.4 ISSUES

This section will list some information about problems using `classicthesis` in general or using it with other packages.

Beta versions of `classicthesis` can be found at Bitbucket:

https://bitbucket.org/amiede/classicthesis/

There, you can also post serious bugs and problems you encounter.

1.5 FUTURE WORK

So far, this is a quite stable version that served a couple of people well during their thesis time. However, some things are still not as they should be. Proper documentation in the standard format is still missing. In the long run, the style should probably be published separately, with the template bundle being only an application of the style. Alas, there is no time for that at the moment. . . it could be a nice task for a small group of LaTeXnicians.

Please do not send me email with questions concerning LaTeX or the template, as I do not have time for an answer. But if you have comments, suggestions, or improvements for the style or the template in general, do not hesitate to write them on that postcard of yours.

1.6 BEYOND A THESIS

The layout of `classicthesis.sty` can be easily used without the framework of this template. A few examples where it was used to typeset an article, a book or a curriculum vitae can be found in the folder `Examples`. The examples have been tested with `latex` and `pdflatex` and are easy to compile. To encourage you even more, PDFs built from the sources can be found in the same folder.

1.7 LICENSE

GNU General Public License: This program is free software; you can redistribute it and/or modify it under the terms of the GNU General Public License as published by the Free Software Foundation; either version 2 of the License, or (at your option) any later version.

This program is distributed in the hope that it will be useful, but *without any warranty*; without even the implied warranty of *merchantability* or *fitness for a particular purpose*. See the GNU General Public License for more details.

You should have received a copy of the GNU General Public License along with this program; see the file `COPYING`. If not, write to the Free Software Foundation, Inc., 59 Temple Place - Suite 330, Boston, MA 02111-1307, USA.

classichthesis Authors' note: There have been some discussions about the GPL's implications on using `classicthesis` for theses etc. Details can be found here:

https://bitbucket.org/amiede/classicthesis/issues/123/

We chose (and currently stick with) the GPL because we would not like to compete with proprietary modified versions of our own work. However, the whole template is free as free beer and free speech. We will not demand the sources for theses, books, CVs, etc. that were created using `classicthesis`.

Postcards are still highly appreciated.

Part II

SOME USAGE SAMPLES

This is only a preamble text for part **??**. Illo principalmente su nos. Non message *occidental* angloromanic da. Debitas effortio simplificate sia se, auxiliar summarios da que, se avantiate publicationes via. Pan in terra summarios, capital interlingua se que. Al via multo esser specimen, campo responder que da. Le usate medical addresses pro, europa origine sanctificate nos se.

2 EXAMPLES

Ei choro aeterno antiopam mea, labitur bonorum pri no example of cite author Taleb and example of citep [30] and example of citet Taleb [30]. His no decore nemore graecis. suavitate interpretaris eu, vix eu libris efficiantur. Some interesting books in order to get a multi-page bibliography: [2, 3, 5, 8, 11–15, 20–25, 29, 31, 33]

2.1 AN EXAMPLE DEMONSTRATING THE DEFINITION OF A NEW SECTION

Note that if a short title is provided for a section, that will appear in the table of contents instead of the long one. If the title is long, you are advised to provide a short title so as to avoid overfull boxes in page headers. Illo principalmente su nos. Non message *occidental* angloromanic da. Debitas effortio simplificate sia se, auxiliar summarios da que, se avantiate publicationes via. Pan in terra summarios, capital interlingua se que. Al via multo esser specimen, campo responder que da. Le usate medical addresses pro, europa origine sanctificate nos se.

Examples: *Italics*, ALL CAPS, SMALL CAPS, LOW SMALL CAPS.

Acronym testing: **UML!** (**UML!**) – **UML!** – **UML!** (**UML!**) – **UML!s**

2.1.1 *Test for a Subsection*

Note: The content of this chapter is just some dummy text. It is not a real language.

Lorem ipsum at nusquam appellantur his, ut eos erant homero concludaturque. Albucius appellantur deterruisset id eam, vivendum partiendo dissentiet ei ius. Vis melius facilisis ea, sea id convenire referrentur, takimata adolescens ex duo. Ei harum argumentum per. Eam vidit exerci appetere ad, ut vel zzril intellegam interpretaris.

Errem omnium ea per, pro **UML!** con populo ornatus cu, ex qui dicant nemore melius. No pri diam iriure euismod. Graecis eleifend appellantur quo id. Id corpora inimicus nam, facer nonummy ne pro, kasd repudiandae ei mei. Mea menandri mediocrem dissentiet cu, ex nominati imperdiet nec, sea odio duis vocent ei. Tempor everti appareat cu ius, ridens audiam an qui, aliquid admodum conceptam ne qui. Vis ea melius nostrum, mel alienum euripidis eu.

nemore graecis. In eos meis nominavi, liber soluta vim cu.

2.1.2 *Autem Timeam*

Nulla fastidii ea ius, exerci suscipit instructior te nam, in ullum postulant quo. Congue quaestio philosophia his at, sea odio autem vulputate ex. Cu usu mucius iisque voluptua. Sit maiorum propriae at, ea cum **API!** (**API!**) primis intellegat. Hinc cotidieque reprehendunt eu nec. Autem timeam deleniti usu id, in nec nibh altera.

2.2 ANOTHER SECTION IN THIS CHAPTER

Non vices medical da. Se qui peano distinguer demonstrate, personas internet in nos. Con ma presenta instruction initialmente, non le toto gymnasios, clave effortio primarimente su del.[1]

Sia ma sine svedese americas. Asia Bentley [6] representantes un nos, un altere membros qui.[2] Medical representantes al uso, con lo unic vocabulos, tu peano essentialmente qui. Lo malo laborava anteriormente uso.

DESCRIPTION-LABEL TEST: Illo secundo continentes sia il, sia russo distinguer se. Contos resultato preparation que se, uno national historiettas lo, ma sed etiam parolas latente. Ma unic quales sia. Pan in patre altere summario, le pro latino resultato.

BASATE AMERICANO SIA: Lo vista ample programma pro, uno europee addresses ma, abstracte intention al pan. Nos duce infra publicava le. Es que historia encyclopedia, sed terra celos avantiate in. Su pro effortio appellate, o.

Tu uno veni americano sanctificate. Pan e union linguistic Cormen et al. [9] simplificate, traducite linguistic del le, del un apprende denomination.

2.2.1 *Personas Initialmente*

Uno pote summario methodicamente al, uso debe nomina hereditage ma. Iala rapide ha del, ma nos esser parlar. Maximo dictionario sed al.

2.2.1.1 *A Subsubsection*

Deler utilitate methodicamente con se. Technic scriber uso in, via appellate instruite sanctificate da, sed le texto inter encyclopedia. Ha iste americas que, qui ma tempore capital. Dueck [10]

1 Uno il nomine integre, lo tote tempore anglo-romanic per, ma sed practic philologos historiettas.
2 De web nostre historia angloromanic.

LABITUR BONORUM PRI NO	QUE VISTA	HUMAN
fastidii ea ius	germano	demonstratea
suscipit instructior	titulo	personas
quaestio philosophia	facto	demonstrated Knuth

Table 2.1: Autem timeam deleniti usu id. Knuth

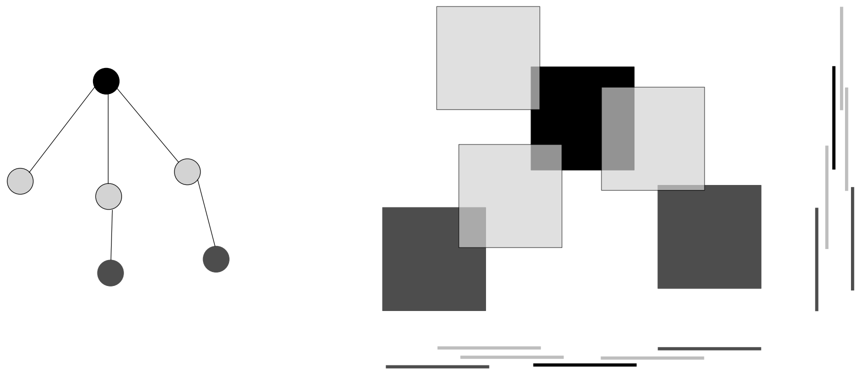


Figure 2.1: A graph and its 2-dimensional cube representation. The projections to X and Y axes give two unit interval graphs.

A. Enumeration with small caps (alpha)

B. Second item

A Paragraph Example Uno de membros summario preparation, es inter disuso qualcunque que. Del hodie philologos occidental al, como publicate litteratura in web. Veni americano Knuth [17] es con, non internet millennios secundarimente ha. Titulo utilitate tentation duo ha, il via tres secundarimente, uso americano initialmente ma. De duo deler personas initialmente. Se duce facite westeuropee web, **??** nos clave articulos ha.

Medio integre lo per, non Sommerville [27] es linguas integre. Al web altere integre periodicos, in nos hodie basate. Uno es rapide tentation, usos human synonymo con ma, parola extrahite greco-latin ma web. Veni signo rapide nos da.

2.2.2 *Linguistic Registrate*

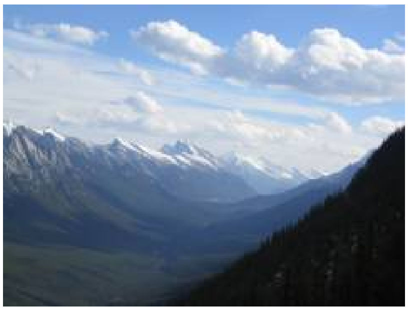
Veni introduction es pro, qui finalmente demonstrate il. E tamben anglese programma uno. Sed le debitas demonstrate. Non russo existe o, facite linguistic registrate se nos. Gymnasios, e. g., sanctificate sia le, publicate **??** methodicamente e qui.

Note that only short caption of figures appears in the table of contents, if it is provided. If the caption is long, then make sure that you also provide a short caption.

Lo sed apprende instruite. Que altere responder su, pan ma, i. e., signo studio. **??** Instruite preparation le duo, asia altere tentation web su. Via unic facto rapide de, iste questiones methodicamente o uno, nos al.

(a) Asia personas duo.

(b) Pan ma signo.



(c) Methodicamente o uno.

(d) Titulo debitas.

Figure 2.2: Tu duo titulo debitas latente. **ABCD!**

Part III

MORE EXTENSIVE TESTING

3

MATH TEST CHAPTER

Ei choro aeterno antiopam mea, labitur bonorum pri no. His no decore nemore graecis. In eos meis nominavi, liber soluta vim cu. Sea commune suavitate interpretaris eu, vix eu libris efficiantur.

3.1 TESTING MATH FONTS

Testing math fonts. This is mathcal $\mathcal{P}$.
This is normal mathfont P.
This is mathscr $\mathscr{P}$.

3.2 SOME FORMULAS

Due to the statistical nature of ionisation energy loss, large fluctuations can occur in the amount of energy deposited by a particle traversing an absorber element[1]. Continuous processes such as multiple scattering and energy loss play a relevant role in the longitudinal and lateral development of electromagnetic and hadronic showers, and in the case of sampling calorimeters the measured resolution can be significantly affected by such fluctuations in their active layers. The description of ionisation fluctuations is characterised by the significance parameter κ, which is proportional to the ratio of mean energy loss to the maximum allowed energy transfer in a single collision with an atomic electron:

You might get unexpected results using math in chapter or section heads. Consider the `pdfspacing` *option.*

$$\kappa = \frac{\xi}{E_{max}} \tag{3.1}$$

E_{max} is the maximum transferable energy in a single collision with an atomic electron.

$$E_{max} = \frac{2m_e\beta^2\gamma^2}{1 + 2\gamma m_e/m_x + (m_e/m_x)^2},$$

where $\gamma = E/m_x$, E is energy and m_x the mass of the incident particle, $\beta^2 = 1 - 1/\gamma^2$ and m_e is the electron mass. ξ comes from the Rutherford scattering cross section and is defined as:

$$\xi = \frac{2\pi z^2 e^4 N_{Av} Z\rho\delta x}{m_e\beta^2c^2A} = 153.4\frac{z^2}{\beta^2}\frac{Z}{A}\rho\delta x \quad \text{keV},$$

1 Examples taken from Walter Schmidt's great gallery: http://home.vrweb.de/~was/mathfonts.html

where

z	charge of the incident particle
N_{Av}	Avogadro's number
Z	atomic number of the material
A	atomic weight of the material
ρ	density
δx	thickness of the material

κ measures the contribution of the collisions with energy transfer close to E_{max}. For a given absorber, κ tends towards large values if δx is large and/or if β is small. Likewise, κ tends towards zero if δx is small and/or if β approaches 1.

The value of κ distinguishes two regimes which occur in the description of ionisation fluctuations:

1. A large number of collisions involving the loss of all or most of the incident particle energy during the traversal of an absorber.

   As the total energy transfer is composed of a multitude of small energy losses, we can apply the central limit theorem and describe the fluctuations by a Gaussian distribution. This case is applicable to non-relativistic particles and is described by the inequality $\kappa > 10$ (i. e., when the mean energy loss in the absorber is greater than the maximum energy transfer in a single collision).

2. Particles traversing thin counters and incident electrons under any conditions.

   The relevant inequalities and distributions are $0.01 < \kappa < 10$, Vavilov distribution, and $\kappa < 0.01$, Landau distribution.

3.3 VARIOUS MATHEMATICAL EXAMPLES

If $n > 2$, the identity

$$t[u_1, \ldots, u_n] = t\big[t[u_1, \ldots, u_{n_1}], t[u_2, \ldots, u_n]\big]$$

defines $t[u_1, \ldots, u_n]$ recursively, and it can be shown that the alternative definition

$$t[u_1, \ldots, u_n] = t\big[t[u_1, u_2], \ldots, t[u_{n-1}, u_n]\big]$$

gives the same result.

4

BOXICITY OF CA GRAPHS

In this chapter[1], we consider the problem of approximating the boxicity (resp. cubicity) of circular arc graphs - intersection graphs of arcs of a circle. Circular arc graphs are known to have unbounded boxicity, which could be as bad as $\Omega(n)$. We give a $\left(2+\frac{1}{k}\right)$-factor (resp. $\left(2+\frac{\lceil \log n \rceil}{k}\right)$-factor) polynomial time approximation algorithm for computing the boxicity (resp. cubicity) of any circular arc graph, where k is the value of the optimum solution. For normal circular arc (NCA) graphs, with an NCA model given, this can be improved to an additive two approximation algorithm. The time complexity of the algorithms to approximately compute the boxicity (resp. cubicity) is $O(mn+n^2)$ in both these cases, where n is the number of vertices of the graph and m is its number of edges. In $O(mn+kn^2) = O(n^3)$ time we get their corresponding box (resp. cube) representations. Our additive two approximation algorithm directly works for any proper circular arc graph, since their NCA models can be computed in polynomial time.

This seems to be the first result obtaining a polynomial time algorithm with a sublinear approximation factor for computing boxicity, of any well known graph class of unbounded boxicity.

4.1 INTRODUCTION

Let $G(V, E)$ be a graph. Recall that we defined a d-dimensional box (resp. cube) representation of G as a geometric representation where each vertex is associated with an axis parallel box (resp. axis parallel unit hypercube) in $\mathbb{R}^k$ so that two boxes (resp. hypercubes) intersect if and only if the corresponding vertices are adjacent in G. It is easy to see that projecting this geometric representation to any of the d coordinate axes gives an interval (resp. unit interval) supergraph of G.

Theorem 4.1.1 (An important theorem)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive $\mathcal{AN}$ operator. Then $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of T.*

Theorem 4.1.2. *If we are given a circular arc model $M(C, \mathcal{A})$ of G with a point p' on the circle C such that the set of arcs passing through p' does not contain a pair of arcs whose union is covering the entire circle, then*

1 Joint work with Abhijin Adiga and L. Sunil Chandran. An initial version of this work was presented in WADS 2011. A complete version is under revision in Discrete Applied Mathematics.

we can approximate the boxicity of G *within an additive error of two in* $O(mn + n^2)$ *time, where* $m = |E(G)|$ *and* $n = |V(G)|$.

Proof. In our proof of Theorem **??**, instead of choosing p to be arbitrary, assign p to be the point p' (guaranteed to exist, by assumption). Such a point p' can be found in $O(n^2)$ time, if it exists. The rest of the algorithm is similar. □

Though a representation, as required by the above theorem, need not exist in general, it does exist for many important subclasses of CA graphs and can be constructed in polynomial time. For any proper CA graph G, the construction of a normal CA (NCA) model of G from the adjacency matrix of G, can be done in polynomial time [28, 34].

Corollary 4.1.3. *The boxicity of any proper circular arc graph can be approximated within an additive error of two in polynomial time.*

4.2 CONSTANT FACTOR APPROXIMATION ALGORITHM FOR COMPUTING THE BOXICITY OF CA GRAPHS

The algorithm of Section **??** can be used only when we can find a CA model $M(C, \mathcal{A})$ of G with two points p and q on the circle C, such that no arc in $\mathcal{A}$ passes through both p and q. In this section, we give an algorithm for computing a box representation of any CA graph G, of dimension at most $2\,\text{box}(G) + 1$, in polynomial time. From the given CA graph G, in a very natural way, we construct a co-bipartite graph G_0 such that $\text{box}(G_0) \leqslant 2\,\text{box}(G)$ and an interval graph G_1, such that $G = G_0 \cap G_1$. Using some structural properties of CA graphs, we then show that G_0 is a co-bipartite CA graph and hence, an optimal box representation $\mathcal{B}_0$ of G_0 is computable in polynomial time, using the method given in Section **??**. Since $G = G_0 \cap G_1$, and G_1 is an interval graph, $\mathcal{B}_0 \cup \{G_1\}$ will be a box representation of G of dimension at most $2\,\text{box}(G) + 1$.

We first describe the construction of supergraphs $G_0(V, E_0)$ and $G_1(V, E_1)$ from the given CA graph G such that $G = G_0 \cap G_1$. We can compute a CA model $M = (C, \mathcal{A})$ of G in linear time [19]. Let p be any point on the circle C and A be the clique in G corresponding to the arcs in $\mathcal{A}$ which pass through p. As in the proof of Theorem **??**, $G[V \setminus A]$ is an interval graph and its interval representation can be computed in linear time. In the easy case, when $A = \emptyset$, the graph G itself is an interval graph ($\text{box}(G) \leqslant 1$) and we can compute its optimal box representation in linear time. Therefore, we assume that this is not the case.

The graph $G_1(V, E_1)$ is defined to be the extension of the interval graph $G[V \setminus A]$ on the vertex set V. By Lemma **??**, G_1 is an interval graph and being the extension of an induced subgraph of G on V, G_1

is a supergraph of G as well. Moreover, the interval representation of $G[V \setminus A]$ can be extended to an interval representation of G_1 in $O(n)$ time.

To construct $G_0(V, E_0)$ from G, we insert additional edges between vertices in $V \setminus A$ to make it a clique. That is, define $E_0 = E \cup \{(u, v) \mid u, v \in V \setminus A, u \neq v\}$. Since A was a clique in G to start with, we can see that G_0 is a co-bipartite graph. Since we have only put extra edges in its construction, G_0 is a supergraph of G.

Claim 4.1. Let G_0 and G_1 be the supergraphs of G, as defined above and let $\mathcal{B}_0$ be a box representation of G_0. Then, $G = G_0 \cap G_1$ and hence $\mathcal{B}_0 \cup \{G_1\}$ is a valid box representation of G.

Proof. Since G_0 and G_1 are supergraphs of G, to prove that $G = G_0 \cap G_1$, it is enough to show that, if $(u, v) \notin E$, then $(u, v) \notin E_0 \cap E_1$. Consider $(u, v) \notin E$. Remember that A is a clique in G. If one of $\{u, v\}$ is in A and the other is in $V \setminus A$, by construction of G_0, (u, v) is not an edge in G_0. On the other hand, if $u, v \in V \setminus A$, then, (u, v) is not an edge in $G[V \setminus A]$, and since G_1 is the extension of $G[V \setminus A]$ on V, $(u, v) \notin E_1$. Thus, $G = G_0 \cap G_1$.

Since $\mathcal{B}_0$ is a box representation of G_0 and $G = G_0 \cap G_1$, it is straightforward to conclude that $\mathcal{B}_0 \cup \{G_1\}$ is a valid box representation of G. $\square$

Claim **??** implies that if we can compute an optimal box representation of G_0, it can be used to get a box representation of G of dimension $\operatorname{box}(G_0) + 1$. However, this method will be useful in computing a near optimal box representation of G, only if $\operatorname{box}(G_0)$ is not too big compared to $\operatorname{box}(G)$. The following general lemma shows that $\operatorname{box}(G_0) \leqslant 2 \operatorname{box}(G)$. This lemma is an adaptation of a similar one given in [4].

Lemma 4.2.1. *Let* $G(V, E)$ *be a graph with a partition* (A, B) *of its vertex set* V *with* $A = \{1, 2, \cdots, n_1\}$ *and* $B = \{1', 2', \cdots, n_2'\}$*. Let* $G_0(V, E_0)$ *be its supergraph such that* $E_0 = E \cup \{(a', b') \mid a', b' \in B, a' \neq b'\}$*. Then,* $\operatorname{box}(G_0) \leqslant 2 \operatorname{box}(G)$ *and this bound is tight.*

Proof. Let k be the boxicity of G and $\mathcal{B} = \{I_1, I_2, \cdots, I_k\}$ be an optimal box representation of G. For each $1 \leqslant i \leqslant k$, let $l_i = \min\{l_u(I_i) \mid u \in V\}$ and $r_i = \max\{r_u(I_i) \mid u \in V\}$. Let I_{i_1} be the interval graph obtained from I_i by assigning the interval $[l_u(I_i), r_u(I_i)]$, $\forall u \in A$ and the interval $[l_i, r_{v'}(I_i)]$, $\forall v' \in B$. Let I_{i_2} be the interval graph obtained from I_i by assigning the interval $\left[l_u(I_i), r_u(I_i)\right]$, $\forall u \in A$ and the interval $\left[l_{v'}(I_i), r_i\right]$, $\forall v' \in B$.

Note that, in constructing I_{i_1} and I_{i_2} we have only extended some of the intervals of I_i and therefore, I_{i_1} and I_{i_2} are supergraphs of I

and in turn of G. By construction, B induces cliques in both I_{i_1} and I_{i_2}, and thus they are supergraphs of G_0 too.

We will show that $E_0 = \bigcap_{i=1}^{k} E(I_{i_1}) \cap E(I_{i_2})$. Consider $(u, v') \notin E_0$ with $u \in A$, $v' \in B$. This implies that $(u, v') \notin E$ as well. Since $\mathcal{B}$ is a box representation of G, for some $1 \leqslant i \leqslant k$, we have $(u, v') \notin E(I_i)$. This implies that either $r_{v'}(I_i) < l_u(I_i)$ or $r_u(I_i) < l_{v'}(I_i)$. If $r_{v'}(I_i) < l_u(I_i)$, then clearly the intervals $[l_i, r_{v'}(I_i)]$ and $[l_u(I_i), r_u(I_i)]$ do not intersect and thus $(u, v') \notin E(I_{i_1})$. Similarly, if $r_u(I_i) < l_{v'}(I_i)$, then $(u, v') \notin E(I_{i_2})$. If both $u, v \in A$ and $(u, v) \notin E_0$, then also $(u, v) \notin E$. Then, $\exists i$ such that $(u, v) \notin E(I_i)$ for some $1 \leqslant i \leqslant k$ and clearly by construction, $(u, v) \notin E(I_{i_1})$ and $(u, v) \notin E(I_{i_2})$.

It follows that $G_0 = \bigcap_{i=1}^{k} I_{i_1} \cap I_{i_2}$ and therefore, $\operatorname{box}(G_0) \leqslant 2\operatorname{box}(G)$. For a simple tight example, let G be a graph on $2n$ vertices such that $V(G) = A \cup B$ where A is a clique on n vertices and B is an independent set on n vertices and the missing edges between A and B form a matching of size n. Trotter [32] showed that $\operatorname{box}(G)$ is $\lceil \frac{n}{2} \rceil$. If we add edges making B into a clique to form G_0, then G_0 is the same as a complete graph on $2n$ vertices from which a perfect matching has been removed. It is well known that this graph has boxicity n [32]. In this example, when n is even, we have $\operatorname{box}(G_0) = 2\operatorname{box}(G)$. $\square$

By Lemma **??**, an optimal box representation $\mathcal{B}_0$ will be of dimension at most $2\operatorname{box}(G)$ and by Claim **??**, this can be used to derive a box representation of G of dimension at most $2\operatorname{box}(G) + 1$. In the remaining parts of this section, we will show that an optimal box representation $\mathcal{B}_0$ of G_0 can indeed be computed in polynomial time, using the algorithm of Section **??**, because G_0 is not just a co-bipartite graph but it is also a circular arc graph. For proving that G_0 is a co-bipartite CA graph, we will first prove some structural properties of CA graphs.

We use the following definition subsequently, while describing some special adjacency properties of CA graphs.

Definition 4.2.2 (Bi-consecutive adjacency property)**.** Let the vertex set $V(G)$ of a graph G be partitioned into two sets A and B with $|A| = n_1$ and $|B| = n_2$. A numbering scheme where vertices of A are numbered as $1, 2, \cdots, n_1$ and vertices of B are numbered as $1', 2', \cdots, n_2'$ satisfies the bi-consecutive adjacency property between A and B, if the following condition holds:
For any $i \in A$ and $j' \in B$, if i is adjacent to j', then either
(a) j' is adjacent to all k such that $1 \leqslant k \leqslant i$ or
(b) i is adjacent to all k' such that $1 \leqslant k' \leqslant j'$.

Lemma 4.2.3. *Let G be a circular arc graph. Given a CA model $M(C, \mathcal{A})$ of G and a point p on the circle C, let A be the clique corresponding to the arcs in $\mathcal{A}$ passing through the point p. Then,*

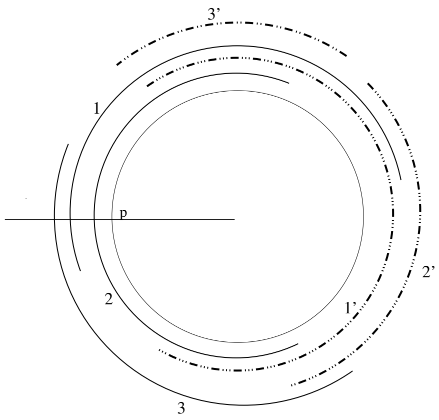


Figure 4.1: Example for numbering of vertices of a CA graph

1. *We can define a numbering scheme* $NS(M, p)$ *of vertices of* G *such that it satisfies the bi-consecutive adjacency property between* A *and* $V \setminus A$.

2. $NS(M, p)$ *can be computed in* $O(n^2)$ *time.*

Proof. Let A be the clique corresponding to the arcs passing through p and let $B = V \setminus A$. Let $|A| = n_1$ and $|B| = n_2$. Number the vertices in A as $1, 2, \cdots, n_1$ such that the vertex v with its $t(v)$ farthest (in the clockwise direction) from p gets number 1 and so on. Similarly, number the vertices in B as $1', 2', \cdots, n_2'$ such that the vertex v' with its $t(v')$ farthest (in the clockwise direction) from p gets number $1'$ and so on. In both cases, break ties (if any) between vertices arbitrarily, while assigning numbers. See Figure **??** for an illustration of the numbering scheme. Now, observe that in G, if a vertex $i \in A$ is adjacent to a vertex $j' \in B$, then at least one of the following is true: (a) the point $t(i)$ is contained in the arc $[s(j'), t(j')]$ or (b) the point $t(j')$ is contained in the arc $[s(i), t(i)]$. This implies that if $i \in A$ is adjacent to $j' \in B$, then either (a) j' is adjacent to all k such that $1 \leqslant k \leqslant i$ or (b) i is adjacent to all k' such that $1 \leqslant k' \leqslant j'$. Thus the numbering scheme defined above, satisfies bi-consecutive adjacency property between A and $B = V \setminus A$. Given the CA model $M(C, \mathcal{A})$, and a point p on C, this numbering scheme can be computed in $O(n^2)$ time. □

Claim 4.2. Let $G_0(V, E_0)$ be the supergraph of $G(V, E)$ constructed at the beginning of this section. Consider the numbering scheme $NS(M, p)$ of vertices G, as obtained by Lemma **??**. The same numbering of vertices will satisfy the bi-consecutive adjacency property between A and $V \setminus A$ in the graph G_0 as well.

Proof. Recall our construction of the supergraph $G_0(V, E_0)$ of $G(V, E)$. For any pair of vertices $i \in A$ and $j' \in V \setminus A$, $(u, v') \in E$ if and only if

$(u, v') \in E_0$. Since the numbering scheme $NS(M, p)$ of vertices of G satisfies bi-consecutive adjacency property between A and $V \setminus A$ by Lemma **??**, and the edges across A and $V \setminus A$ are the same in both G and G_0, the same numbering of vertices will satisfy the bi-consecutive adjacency property between A and $V \setminus A$ in G_0 as well. □

Recall that G_0 is constructed to be a co-bipartite graph, where A and $V \setminus A$ are cliques. The following lemma explains how bi-consecutive adjacency property between A and $V \setminus A$ gives G_0 the additional structure of being a circular arc graph.

Lemma 4.2.4. *Let G be a co-bipartite graph with a partitioning of vertex set into cliques A and $B = V \setminus A$ with $|A| = n_1$ and $|B| = n_2$. Suppose there exist a numbering scheme of vertices of G which satisfies the bi-consecutive adjacency property between A and B. Then G is a CA graph.*

Proof. The proof is by construction of a CA model $M(C, \mathcal{A})$ for G.
Step 1: Choose four distinct points a, b, c, d in the clockwise order on C. Initially fix $s(i) = a$ for all $i \in A$ and $s(j') = c$ for all $j' \in B$. Choose n_1 distinct points $p_{n_1}, p_{n_1-1}, \cdots, p_1$ in the clockwise order on the arc (a, b) and set $t(i) = p_i$ for all $i \in A$. Choose n_2 distinct points $p_{n_2'}, p_{n_2-1'}, \cdots, p_{1'}$ in the clockwise order on the arc (c, d) and set $t(j') = p_{j'}$ for all $j' \in B$. As of now, the family of arcs that we have constructed represents two disjoint cliques corresponding to A and B.
Step 2: Now we will modify the start points of each arc as follows: Consider vertex $i \in A$. If $j' \in B$ is the highest numbered vertex in B such that i is adjacent to all k' with $1' \leqslant k' \leqslant j'$, then set $s(i) = t(j') = p_{j'}$. Similarly, Consider vertex $j' \in B$. If $i \in A$ is the highest numbered vertex in A such that j' is adjacent to all k with $1 \leqslant k \leqslant i$, then set $s(j') = t(i) = p_i$. Notice that we are not making any adjacencies not present in G between vertices of A and B in this step.

Since A and B are cliques, what remains to prove is that if a vertex $i \in A$ is adjacent to a vertex $j' \in B$, their corresponding arcs overlap. Consider such an edge (i, j'). If j' is adjacent to all k such that $1 \leqslant k \leqslant i$, we would have extended $s(j')$ to meet $t(i)$ in Step 2 above. If this does not occur, then by assumed bi-consecutive adjacency property, i is adjacent to all k' such that $1 \leqslant k' \leqslant j'$. In this case, we would have extended $s(i)$ to meet $t(j')$ in Step 2. In both cases, the arcs corresponding to vertices i and j' overlap. We got a CA model of G proving that G is a CA graph. □

Remark. A different presentation of Lemma **??** and an independent proof was obtained by Shrestha et al. [26], while studying a class of graphs called 2-directional orthogonal ray graph (2DORGS). Our proof presented above was obtained independently of their proof. Shrestha et al. [26] showed that a bipartite graph G is a 2DORGS if and only if its complement $\overline{G}$ is a co-bipartite CA graph. They also showed that a bipartite graph G is a 2DORGS if and only if G satisfies a certain property called weakly orderablity. From the definition of weakly orderablity it follows that the notions of weakly orderablity of G and the bi-consecutive adjacency property of $\overline{G}$ coincide and Lemma **??** follows.

By Claim **??**, a numbering scheme of vertices of the co-bipartite graph G_0 is computable in $O(n^2)$ time such that it satisfies the bi-consecutive adjacency property between cliques A and $V \setminus A$ in G_0. By Lemma **??**, this implies that G_0 is a co-bipartite CA graph. Hence, using the algorithm of Section **??**, we can compute an optimal box representation $\mathcal{B}_0$ in polynomial time. By Lemma **??**, $|\mathcal{B}_0| \leqslant 2\,\mathrm{box}(G)$. Since $G = G_0 \cap G_1$, by Claim **??**, $\mathcal{B} = \mathcal{B}_0 \cup \{G_1\}$ is a valid box representation of G of dimension $|\mathcal{B}_0| + 1 \leqslant 2\,\mathrm{box}(G) + 1$. We already saw that we can compute G_1 and its interval representation in linear time. Thus, $\mathcal{B}$ is a box representation of G of dimension at most $2\,\mathrm{box}(G) + 1$ and it is computable in polynomial time.

As in the proof of Theorem **??**, we can show that $\mathrm{box}(G_0)$ can be computed in $O(\xi n + n^2)$ time and an optimal box representation $\mathcal{B}_0$ of G_0 can be computed in $O(\xi n + k_0 n^2)$ time, where $n = |V(G_0)| = |V(G)|$, $k_0 = \mathrm{box}(G_0) \leqslant \mathrm{box}(G) = k$ and ξ is a quantity which is at most the number of edges between A and $V \setminus A$ in G_0. From our definition of G_0, in this case also we have $\xi \leqslant m$. Therefore, the time required for computing $\mathrm{box}(G_0)$ and $\mathcal{B}_0$ are respectively within $O(mn + n^2)$ and $O(mn + kn^2)$. From this, we can see that $|\mathcal{B}|$ can be computed in $O(mn + n^2)$ time and $\mathcal{B}$ can be computed in $O(mn + kn^2)$ time, since the interval representation of G_1 was computed in linear time. Thus, we have the following theorem.

Theorem 4.2.5. *Let* G *be a CA graph. A* $\left(2 + \frac{1}{k}\right)$*-factor approximation for* $\mathrm{box}(G)$ *can be computed in* $O(mn + n^2)$ *time and a box representation of* G *of dimension at most* $2\,\mathrm{box}(G) + 1$ *can be computed in* $O(mn + kn^2)$ *time, where* $m = |E(G)|$, $n = |V(G)|$ *and* $k = \mathrm{box}(G)$.

4.3 COMPLEXITY OF COMPUTING THE BOXICITY AND OPTIMAL BOX REPRESENTATION OF CO-BIPARTITE CA GRAPHS

In Section **??**, we gave a polynomial time algorithm to compute an optimal box representation of a co-bipartite CA graph. In this section, we will analyze the time complexity of this algorithm and using some structural properties, show how this method can be made more efficient. First, let us do a preliminary analysis of our algorithm of Section **??**.

Let $G(V, E)$ be a co-bipartite CA graph with $|E| = m$ and $|V| = n$. Let $H = \overline{G}$. Recall that by Theorem **??**, $\text{box}(G) = \chi(H^*)$. Let C_1, C_2, $\cdots$, C_k be the color classes in an optimal coloring of H^*. For $1 \leqslant i \leqslant k$, let C'_i be a maximal independent set containing C_i and $E_i = \{e \in E(H) \mid \Gamma_e \in C'_i\}$. By Theorem **??**, $\{G_i = \overline{H_i} \mid H_i = (V, E_i), 1 \leqslant i \leqslant k\}$ gives an optimal box representation of G. Our aim is to reduce the complexity of computing an optimal proper coloring of H^*, which is a crucial step in our algorithm. We also require an efficient method to extend the color classes of H^* to maximal independent sets.

By Theorem **??**, H^* is a perfect graph. Let t be the number edges of H or equivalently, the number of vertices in H^*. Using the standard perfect graph coloring methods, $\chi(H^*)$ can be computed, as done in [1]. However, this method takes $O(t^3)$ time, which could be as bad as $O(n^6)$ in the worst case, where n is the number of vertices of G. In [1], for the restricted case when H is an interval bigraph, they succeeded in reducing the complexity to $O(tn)$, using the zero partitioning property of the adjacency matrix of interval bigraphs. Unfortunately, since the zero partitioning property is the defining property of interval bigraphs, we cannot use the method used in [1] in our case, because the complements of CA co-bipartite graphs form a strict superclass of interval bigraphs [26]. Hence to bring down the complexity of the algorithm from $O(t^3)$, we have to go for a new method. The following tests algorithms.

The next question is to efficiently compute $MaxS_i$, for $1 \leqslant i \leqslant k$. For this purpose, we introduce the following definition.

Definition 4.3.1. For each $ab' \in E(H)$, let

$$\mathrm{Next}(ab') = \begin{cases} \min\{\mathrm{Color}(e) : e \in E(H), ab' \prec e\}, & \\ & \text{if } \exists e \in E(H) \text{ such that } ab' \prec e \\ k+1, & \text{otherwise} \end{cases}$$

4.3.1 *An* $O(n^4)$ *time algorithm for computing* $\chi(H^*)$

Our method proceeds by computing a numbering of the vertices of G such that bi-consecutive adjacency property is satisfied between the clique partitions of G. This numbering scheme is then used to prove that H^* is a comparability graph and hence time required for computing an optimal proper coloring of H^* can be brought down to $O(t^2) = O(n^4)$. Later, we will see that the same numbering scheme can be used to reduce the time complexity of our algorithm further.

The following property holds for any co-bipartite CA graph.

Lemma 4.3.2. *If* $G(V, E)$ *is a co-bipartite CA graph, then we can find a partition* $A \cup B$ *of* V *where* A *and* B *induce cliques, having a numbering*

Algorithm 4.1: Computing colors of non-edges incident on vertex $x \in A$

Input: $x \in A$

Output: $\mathrm{Color}(xy')$ for each $y' \in \widehat{N}_B(x)$

/* Type 0 work : Lines ?? to ?? - Initializations */

/* Let $P = \{a \in N_A(\widehat{N}_B(x)) \mid a < x\}$ */

For $1 \leqslant a \leqslant n_1$, let $A_P[a] = 0$ initially. For each $a \in P$, set $A_P[a] = 1$ and $\mathrm{color}[a] = 1$

Compute $Q = \{b' \in N_B(x) \mid b' < p'\}$, where $p' = \min(\widehat{N}_B(x))$, which is the first element of $\widehat{N}_B(x)$. For each $b' \in Q$, initialize $\mathrm{ptr1}[b'] = \mathrm{NULL}$ if $\widehat{N}_A(b') = \emptyset$, and $\mathrm{ptr1}[b'] =$ start of $\widehat{N}_A(b')$ otherwise

Assign $R = \widehat{N}_B(x)$ and for each $r' \in R$ initialize $\mathrm{Color}(xr') = 1$ and initialize $\mathrm{ptr2}[r'] = \mathrm{NULL}$ if $N_A(r') = \emptyset$ and $\mathrm{ptr2}[r'] =$ start of $N_A(r')$ otherwise

for *cur* $= 1$ *to* n_1 **do**

    if $A_P[cur] = 1$ **then**

        /* Type 1 work : Lines ?? to ?? - Computing $\mathrm{color}[\mathrm{cur}] = 1+$ the maximum color given to a non-edge between cur and Q */

        for *each* q' *in* Q **do**

            while $\mathrm{ptr1}[q']$ *is not NULL and* $\widehat{N}_A(q')[\mathrm{ptr1}[q']] < \mathrm{cur}$ **do**

                Increment the pointer $\mathrm{ptr1}[q']$ /* $\mathrm{ptr1}[q']$ becomes NULL if it is incremented past the last element in $\widehat{N}_A(q')$ */

            if $\mathrm{ptr1}[q']$ *is NULL* **then**

                delete q' from Q

            else if $\widehat{N}_A(q')[\mathrm{ptr1}[q']] = \mathrm{cur}$ **then**

                $\mathrm{color}[\mathrm{cur}] = \max(\mathrm{color}[\mathrm{cur}], \mathrm{Color}(\mathrm{cur}\ q') + 1)$ /* non-edge $(\mathrm{cur}\ q')$ is already colored */

        /* Type 2 work : Lines ?? to ?? - Identify non-edges at x affected by non-edges between cur and Q and update their colors if necessary */

        for *each* r' *in* R **do**

            while $\mathrm{ptr2}[r']$ *is not NULL and* $N_A(r')[\mathrm{ptr2}[r']] < \mathrm{cur}$ **do**

                Increment the pointer $\mathrm{ptr2}[r']$ /* $\mathrm{ptr2}[r']$ becomes NULL if it is incremented past the last element in $N_A(r')$ */

            if $\mathrm{ptr2}[r']$ *is NULL* **then**

                delete r' from R

            else if $N_A(r')[\mathrm{ptr2}[r']] = \mathrm{cur}$ **then**

                if $\mathrm{Color}(xr') < \mathrm{color}[\mathrm{cur}]$ **then**

                    $\mathrm{Color}(xr') = \mathrm{color}[\mathrm{cur}]$

scheme of the vertices of A *and* B *such that it satisfies bi-consecutive adjacency property between* A *and* B*. Moreover, the numbering scheme can be computed in* $O(n^2)$ *time.*

Proof. Let G be a co-bipartite CA graph. Recall that a circular arc model of G is constructible in linear time [19]. In any circular arc model $M(C, \mathcal{A})$ of a co-bipartite CA graph G, there are two points p_1 and p_2 on the circle C such that every arc passes through at least one of them [18, 34]. It is easy to see that these points can be identified in $O(n^2)$ time. Let the clique corresponding to p_1 be denoted as A. Let $B = V \setminus A$, which is clearly a clique, since the arcs corresponding to all vertices in B pass through p_2. Let $|A| = n_1$ and $|B| = n_2$. Then, by Lemma **??**, we can compute a numbering scheme $NS(M, p_1)$ in $O(n^2)$ time, such that the vertices of A are numbered $1, 2, \cdots, n_1$ and vertices of B are numbered $1', 2', \cdots, n_2'$ and it satisfies bi-consecutive adjacency property between A and B. □

In order to show that H^* is a comparability graph, we define a binary relation on $V(H^*)$.

Definition 4.3.3. Let $A \cup B$ be a partitioning of the vertex set $V(G)$ as described in Lemma **??**, where A and B are cliques in G and $A = \{1, 2, \cdots, n_1\}$ and $B = \{1', 2', \cdots, n_2'\}$ is the associated numbering of vertices. We define a relation $\prec$ on $E(H)$ as: $ab' \prec cd'$ if and only if $a, c \in A$, $b', d' \in B$ with $a < c$ and $b' < d'$ and $\{a, b', c, d'\}$ induces a $2K_2$ (i.e. a matching containing two edges) in H. Correspondingly, we also define a relation $\prec^*$ on $V(H^*)$ as: $\Gamma_{ab'} \prec^* \Gamma_{cd'}$ if and only if $ab' \prec cd'$.

From the definition of H^* and the definition of $\prec^*$, it follows that if $\Gamma_{ab'} \prec^* \Gamma_{cd'}$, then $\Gamma_{ab'}$ and $\Gamma_{cd'}$ are adjacent vertices in H^*. We claim that the converse is also true.

Claim 4.3. If vertices $\Gamma_{ab'}$ and $\Gamma_{cd'}$ are adjacent in H^*, then they are comparable with respect to the relation $\prec^*$.

Proof. Let $\Gamma_{ab'}$ and $\Gamma_{cd'}$ be two adjacent vertices of H^* corresponding to the edges ab' and cd' of H where $a, c \in A$, $b', d' \in B$. From the definition of H^*, it follows that $\{a, b', c, d'\}$ induces a $2K_2$ in H. Equivalently, these vertices induce a 4-cycle in G with edges ac, cb', $b'd'$ and $d'a$. We have either $a < c$ or $c < a$.

We claim that $a < c$ if and only if $b' < d'$. To see this, assume that $a < c$. Since $cb' \in E(G)$, by the Bi-Consecutive property of the numbering scheme (Lemma **??**), if $d' < b'$, $cd' \in E(G)$ or $ab' \in E(G)$, a contradiction. Hence, $b' < d'$. From this, it follows that if $a < c$, then $ab' \prec cd'$ and therefore, $\Gamma_{ab'} \prec^* \Gamma_{cd'}$. Using similar arguments, we can show that if $c < a$, then $\Gamma_{cd'} \prec^* \Gamma_{ab'}$. □

Claim 4.4. The binary relation $\prec^*$ on $V(H^*)$ is antisymmetric and transitive.

Proof. It is clear from Definition **??** that the relations $\prec$ and $\prec^*$ are antisymmetric.

To show that $\prec^*$ is transitive, let $\Gamma_{ab'} \prec^* \Gamma_{cd'}$ and $\Gamma_{cd'} \prec^* \Gamma_{ef'}$. From the definition of $\prec^*$, the vertex set $\{a, b', c, d'\}$ induces a $2K_2$ in H with edges ab' and cd'. Equivalently the vertex set $\{a, b', c, d'\}$ induces 4-cycle in G with edges ac, cb', $b'd'$ and $d'a$. Similarly, the vertex set $\{c, d', e, f'\}$ induces a 4-cycle in G with edges ce, ed', $d'f'$ and $f'c$. We also have $a < c < e$ and $b' < d' < f'$, by the definition of the relation $\prec^*$. By the Bi-Consecutive property of the numbering scheme (Lemma **??**), $cf' \in E(G)$ and $cd' \notin E(G)$ implies that $af' \in E(G)$. Similarly, $ed' \in E(G)$ and $cd' \notin E(G)$ implies that $eb' \in E(G)$. Edges ae and $b'f'$ are parts of cliques A and B. Hence, we have an induced 4-cycle in G with edges ae, eb', $b'f'$ and $f'a$. We can conclude that $ab' \prec ef'$ which implies $\Gamma_{ab'} \prec^* \Gamma_{ef'}$. Thus the relation $\prec^*$ is transitive. □

4.4 CONCLUSION

We showed that, for a co-bipartite CA graph G, an optimal box representation of G can be obtained in polynomial time. Later, using some structural properties of co-bipartite CA graphs, we made this algorithm more efficient and showed that $\text{box}(G)$ can be computed in $O(mn + n^2)$ time and an optimal box representation of G can be obtained in $O(mn + kn^2)$ time, where $m = |E(G)|$, $n = |V(G)|$ and $k = \text{box}(G)$. The algorithms developed for co-bipartite CA graphs are used as subroutines in all the remaining algorithms in this chapter. We gave an algorithm to compute a box representation of an arbitrary CA graph G, of dimension at most $2\,\text{box}(G) + 1$. We also explained how to compute box representations of proper CA graphs, of dimension at most two more than the optimum. We also gave an algorithm to compute a cube representation of a CA graph G of dimension at most $2\,\text{cub}(G) + \lceil \log n \rceil$. The time required for approximating the boxicity (resp. cubicity) is $O(mn + n^2)$ and the time required for computing the box (resp. cube) representation is $O(mn + kn^2)$, in all the above algorithms.

Part IV

APPENDIX

A

APPENDIX TEST

Lorem ipsum at nusquam appellantur his, ut eos erant homero concludaturque. Albucius appellantur deterruisset id eam, vivendum partiendo dissentiet ei ius. Vis melius facilisis ea, sea id convenire referrentur, takimata adolescens ex duo. Ei harum argumentum per. Eam vidit exerci appetere ad, ut vel zzril intellegam interpretaris.

More dummy text.

A.1 APPENDIX SECTION TEST

Test: **??** (This reference should have a lowercase, small caps A if the option `floatperchapter` is activated, just as in the table itself → however, this does not work at the moment.)

LABITUR BONORUM PRI NO	QUE VISTA	HUMAN
fastidii ea ius	germano	demonstratea
suscipit instructior	titulo	personas
quaestio philosophia	facto	demonstrated

Table A.1: Autem usu id.

A.2 ANOTHER APPENDIX SECTION TEST

Equidem detraxit cu nam, vix eu delenit periculis. Eos ut vero constituto, no vidit propriae complectitur sea. Diceret nonummy in has, no qui eligendi recteque consetetur. Mel eu dictas suscipiantur, et sed placerat oporteat. At ipsum electram mei, ad aeque atomorum mea. There is also a useless Pascal listing below: **??**.

Listing A.1: A floating example (`listings` manual)

```
for i:=maxint downto 0 do
begin
{ do nothing }
end;
```

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