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GENERATION OF COLORED NOISE

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In this work, we describe a simple Markovian algorithm to generate a typical sample path of colored noise described by an Ornstein–Uhlenbeck process. The algorithm works equally well to simulate a real or complex disorder potential with exponentially decaying covariance and higher correlation functions given by Wick's theorem. As an input, we only need independent Gaussian random numbers which can easily be generated by the well-known Box–Muller algorithm. Finally, we discuss an alternative method which can also be used to generate non-Gaussian colored noise.

Keywords: Generation of Colored Noise; Ornstein–Uhlenbeck Process.

1. Introduction

Fluctuations of the relevant degrees of freedom in nonequilibrium statistical physics are usually taken into account by adding a stochastic force $X(t)$ to the deterministic equations of motion. The prototype stochastic differential equation has the form:

$$\frac{dv}{dt} = -a(v,t) + b(v,t)X(t)\,. \tag{1}$$

Here, $v(t)$ is the relevant variable of interest, which usually is a function of time,[a] and $a(v,t)$ and $b(v,t)$ are certain known functions, which depend on the specific problem at hand. Equation (1) is known as the Langevin equation[1,2] and can easily be generalized to a matrix equation. It was first introduced by Langevin to describe Brownian motion.[3] In the case of Brownian motion, the relevant variable $v(t)$ is the velocity of a heavy particle of unit mass, $a(v,t) \equiv \alpha \cdot v$ is the dissipative force due to friction, and $b(v,t)X(t) \equiv X(t)$ is an *additive* random force.

The fluctuating random force is often called *noise* and can be of different origin. *Internal forces* such as thermal fluctuations are usually assumed to be Gaussian

[a]In condensed matter systems, however, the disorder is often considered to be stationary such that in one-dimensional systems, a space-coordinate can play the role of time.

with very small correlation times τ. Since a finite expectation value $\langle X(t)\rangle = \bar{x}$ can be incorporated into $a(v,t)$, it is no restriction to assume

$$\langle X(t)\rangle = 0\,, \tag{2}$$

where $\langle\cdots\rangle$ signifies averaging over the probability distribution of $X(t)$. A Gaussian stochastic process with standard deviation σ and correlation time τ is characterized by the covariance:

$$\langle X(t)X(t')\rangle = \sigma^2 e^{-|t-t'|/\tau}\,, \tag{3}$$

and all higher moments given by Wick's theorem. This process is called the Ornstein–Uhlenbeck process,[4] which by Doob's theorem (see for example van Kampen[2]) is essentially the only stationary Gaussian Markov process. The white-noise limit may be taken by letting τ approach 0 while keeping the quantity $D \equiv \sigma^2\tau$ constant. In this limit, the covariance becomes diagonal, such that disorder at different times is uncorrelated and

$$\langle X(t)X(t')\rangle = 2D\delta(t-t')\,. \tag{4}$$

While the white noise limit usually leads to a good approximation of *internal fluctuations*, in the case of *external fluctuations*, the relevant variables can vary substantially over the correlation time τ. In this case, it is essential to consider colored noise, i.e., finite τ. Unfortunately, in most cases, the finite correlation time leads to a serious complication when trying to solve the Langevin equation. Techniques which turn out to be successful in solving the white noise limit can only be applied after coupling the stochastic equations of motion to an extra equation, which takes care of the finite correlation time. Often, the only way out lies in a numerical simulation of the stochastic process. It is therefore important to find a method to generate typical disorder realizations. In the following, we describe a very simple algorithm to generate a concrete sample path of the Ornstein–Uhlenbeck process with finite correlation time τ, which can be useful in various applications. This algorithm can already be found in similar form in the mathematical literature on stochastic processes[5] and was also used in different physical situations such as the Kramers problem (see for example Refs. 6 and 7). For the generation of spatio-temporal colored noise, see Ref. 8.

2. Simple Algorithm to Generate Gaussian Colored Noise

Independent Gaussian random numbers Z_n with zero mean and unit variance can be generated by the Box–Muller algorithm.[9,b] The following recursive algorithm (which we will refer to as Algorithm I) maps these onto real correlated Gaussian

[b] Note that the Box–Muller algorithm needs a good number generator to generate independent uniformly distributed random numbers.

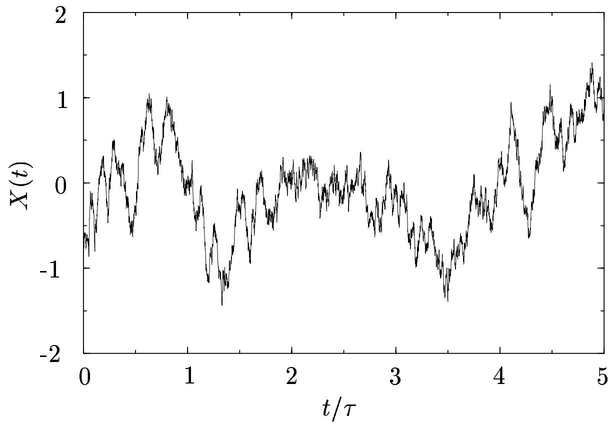

Fig. 1. Sample path of $X(t)$ with $\sigma = 1$ as a function of t/τ.

random numbers $X_n \equiv X(t_n)$ at the sample points t_n $(t_0 < t_1 < \cdots < t_{N-1})$ with $\langle X_n \rangle = 0$ and $\langle X_m X_n \rangle = \sigma^2 e^{-|t_m - t_n|/\tau}$:

$$X_0 = \sigma Z_0 \,, \tag{5}$$

$$X_n = \rho_n X_{n-1} + \sqrt{1 - \rho_n^2}\, \sigma Z_n \,, \tag{6}$$

where the correlation coefficients ρ_n are given by $\rho_n = e^{-|t_n - t_{n-1}|/\tau}$. Setting $\rho_0 = 0$, Eq. (5) is also included in Eq. (6). A sample path generated by this algorithm is presented in Fig. 1.

Using $\langle Z_n \rangle = 0$ and $\langle Z_m Z_n \rangle = \delta_{mn}$, it is easy to see recursively from Eqs. (5) and (6) that the first two moments of X_n are in fact given by $\langle X_n \rangle = 0$ and $\langle X_m X_n \rangle = \sigma^2 e^{-|t_m - t_n|/\tau}$. Because the X_n's are given by a linear combination of the Gaussian random variables X_n and a linear combination can only turn one Gaussian distribution into another Gaussian distribution,[5] the X_n's also have to be Gaussian random variables. Higher correlation functions are therefore given by Wick's theorem.[2] Obviously, the sample path in Fig. 1 is consistent with the above correlation functions: X_n is centered around zero with variance $\langle X_n^2 \rangle \approx \sigma^2 = 1$, and for $|t_n - t_m| \gtrsim \tau$, we have $\langle X_n X_m \rangle \approx 0$.

It is also easy to generalize the above Algorithm I to a complex disorder potential. In this case, one would like to have $\langle X(t) \rangle = 0$, $\langle X(t) X^*(t') \rangle = \sigma^2 e^{-|t-t'|/\tau}$ and $\langle X(t) X(t') \rangle = 0$. Generating $\operatorname{Re} X_n$ and $\operatorname{Im} X_n$ independently as before, one sees that to get the desired correlation functions, in Eqs. (5) and (6), one has to replace σ by $\sigma/\sqrt{2}$. Since $X_n = \operatorname{Re} X_n + i \operatorname{Im} X_n$ only depends linearly on $\operatorname{Re} X_n$ and $\operatorname{Im} X_n$, the complex X_n are also Gaussian random variables.

3. Non-Gaussian Colored Noise

The above Algorithm I is very simple and proves to be successful in generating *Gaussian* colored noise. However, external fluctuations do not have to be Gaussian, and there might be a need to generate a typical chain characterized by different statistics. Let us now describe an algorithm based on an expansion of a stochastic process in terms of harmonic functions,[10] which in the following we will refer to as Algorithm II. If $S(\omega)$ represents the power spectrum of the stochastic process, a typical sample path may be generated for large N by (see Refs. 11 and 13):

$$X(t) = \sqrt{2} \sum_{n=0}^{N-1} [2S(\omega_n)\Delta\omega]^{1/2} \cos(\omega_n t + \phi_n) \,. \tag{7}$$

Here, the ϕ_n are independent random phases, which are uniformly distributed over the interval $(0, 2\pi)$, $\Delta\omega = \omega_{\max}/N$, where $\omega_{\max}$ is an upper cutoff of the noise spectrum, and $\omega_n = n\Delta\omega$. Algorithm II has the advantage that it is applicable to an arbitrary given spectrum $S(\omega)$. In the case of Gaussian colored noise, the spectrum can be found by taking the Fourier transform of Eq. (3), resulting in:

$$S(\omega) = \frac{1}{\pi} \frac{\sigma^2 \tau}{1 + \omega^2 \tau^2} \,. \tag{8}$$

In comparison to Algorithm I, which unfortunately only works to generate a sample path of an Ornstein–Uhlenbeck process, Algorithm II has the disadvantage that for it to become accurate, both $\omega_{\max}$ and then N have to be chosen sufficiently large. Even when using a fast Fourier transform, which results in $\mathcal{O}(N \log_2 N)$ operations,[9] this can lead to large computation times. In addition, the sample paths are always periodic with period $2\pi N/\omega_{\max}$, which can lead to further complications. For a more quantitative comparison between the two Algorithms, see the Comment by Manella and Palleschi[12] on Ref. 13.

4. Summary

In summary, we have described a very simple algorithm to simulate a real or complex Ornstein–Uhlenbeck process and an alternative algorithm, which is not restricted to generate *Gaussian* colored noise. When generating Gaussian colored noise, the advantage of the former in comparison to the latter is that it takes advantage of the Markov property of the Ornstein–Uhlenbeck process: to generate X_n, we only need to know X_{n-1}. When numerically solving an initial value problem of a stochastic differential equation, the disorder may be simultaneously generated with the propagation of the desired solution. In addition, arbitrary long chains can be easily generated. We have used the described algorithm to generate the fluctuating order parameter field $\Delta(x) \equiv X(t)$ of the so-called fluctuating gap model (see Refs. 14 and 15). The above method enabled us to calculate the density of states for arbitrary correlation lengths $\xi \equiv \tau$ with unprecedented numerical accuracy. The algorithm, however, should be useful in all contexts, where there is a need to generate colored noise described by an Ornstein–Uhlenbeck process.

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