qNoise: A generator of non-Gaussian colored noise

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Abstract

We introduce a software generator for a class of *colored* (self-correlated) and *non-Gaussian* noise, whose statistics and spectrum depend upon only two parameters, q and τ. Inspired by Tsallis' nonextensive formulation of statistical physics, this so-called q-distribution is a handy source of self-correlated noise for a large variety of applications. The q-noise—which tends smoothly for $q = 1$ to Ornstein–Uhlenbeck noise with autocorrelation τ—is generated via a stochastic differential equation, using the Heun method (a second order Runge–Kutta type integration scheme). The algorithm is implemented as a stand-alone library in `c++`, available as open source in the `Github` repository. The noise's statistics can be chosen at will, by varying only parameter q: it has compact support for $q < 1$ (sub-Gaussian regime) and finite variance up to $q = 5/3$ (supra-Gaussian regime). Once q has been fixed, the noise's autocorrelation can be tuned up independently by means of parameter τ. This software provides a tool for modeling a large variety of real-world noise types, and is suitable to study the effects of correlation and deviations from the normal distribution in systems of stochastic differential equations which may be relevant for a wide variety of technological applications, as well as for the understanding of situations of biological interest. Applications illustrating how the noise statistics affects the response of a variety of nonlinear systems are briefly discussed. In many of these examples, the system's response turns out to be optimal for some $q \neq 1$.

Keywords: non-Gaussian, random process generator, stochastic differential equations

PROGRAM SUMMARY

Manuscript Title: "qNoise: A generator of non-Gaussian colored noise"
Authors: J. Ignacio Deza and Roberto R. Deza
Program Title: qNoise[1]
Journal Reference:
Catalogue identifier:
Licensing provisions: GNU General Public Licence, version 3
Programming language: C++
Computer: PC, Apple
Operating system: UNIX (Linux, Mac OSX, etc.)
RAM: Less than 1MB
Number of processors used:
Supplementary material:
Keywords: non-Gaussian, random process generator, stochastic differential equations
Classification:
External routines/libraries: Standard C++11 libraries.
Subprograms used:
*Catalogue identifier of previous version:**
*Journal reference of previous version:**
*Does the new version supersede the previous version?:**
Nature of problem: Generation of non-Gaussian non-white random processes for a wide variety of uses.

Solution method: Numerically solving a differential equation inspired by Tsallis' nonextensive formulation of statistical physics[2, 3].

*Reasons for the new version:**

*Summary of revisions:**

Restrictions:

Unusual features:

*Additional comments:*When using this library as a source of noise for integrating stochastic ODE, match the time step to avoid problems with the time-scale.

Running time: On an average laptop, it typically generates about 1 million random numbers in a second.

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[1] http://www.github.com/ignaciodeza/qnoise
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* Items marked with an asterisk are only required for new versions of programs previously published in the CPC Program Library.

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1. Introduction

Most studies on noise-induced phenomena [1, 2] have assumed the noise source to have Gaussian distribution, either "white" (memoryless) or "colored" (red, pink, etc). Although customarily accepted without criticisms on the basis of the central limit theorem, the true rationale behind this assumption lies in the possibility of obtaining some analytical results, and avoiding many difficulties arising in generating and handling non-Gaussian noise. There is however experimental evidence that at least in some cases (particularly in sensory and biological systems) non-Gaussian noise sources may add desirable features to noise-induced phenomena (e.g. robustness, fault tolerance [3]). These findings add practical interest to the task of finding viable ways to deal with non-Gaussian noise.

Here we introduce a light-weighted generator for non-Gaussian, colored stochastic processes. The expected applications of this software are as diverse as the modeling of some types of vibration or fluctuation which are typically non-Gaussian, the generation of noise which is naturally confined to a domain, or the investigation of the response of many dynamical systems embedded in noise, as the latter departs from being Gaussian.

The main features of noise obeying Tsallis' statistics are summarized in Sec. 2. Section 3 is devoted to the description of the software architecture. Section 4, to a statistical analysis of the generated noise in the qualitatively different cases, ending with a discussion on the q-dependence of the variance and effective self-correlation time. The measured self-correlation times of the obtained series are compared with a fitting expression [4]. In Sec. 5, a very brief review is made of works which have investigated non-Gaussian noise-induced phenomena.

2. q-noise with Tsallis' statistics

The exponentially self-correlated Gaussian noise $\eta(t)$ known after Ornstein and Uhlenbeck (OU noise, or "colored" Gaussian noise) can be *dynamically* generated through the differential equation

$$\tau \dot{\eta} = -\eta(t) + \xi(t), \tag{1}$$

where $\xi(t)$ is centered Gaussian white noise with variance D, namely

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\,\xi(t') \rangle = 2D\,\delta(t - t').$$

This way, η's self-correlation time turns out to be τ.

A straightforward generalization of Eq. (1) was proposed some time ago [5] as a model for *correlated* diffusion:

$$\tau \dot{\eta} = -\frac{d}{d\eta} V_q(\eta) + \xi(t) \tag{2}$$

where the potential V_q is given by:

$$V_q(\eta) = \frac{D}{\tau (q - 1)} \ln \left[1 + \frac{\tau (q - 1)}{D} \frac{\eta^2}{2} \right], \tag{3}$$

As much as the OU noise allows to explore spectral effects within the class of exponentially correlated noise, this generalization provides moreover a device to explore statistics effects by varying just one parameter (namely q, at constant τ and D).

The stationary properties of η (including its autocorrelation function) being thoroughly described elsewhere [4, 6, 7, 8, 9], we here summarize the main results. Using the Fokker–Planck formalism, one obtains the stationary probability distribution

$$P_q^{\text{st}}(\eta) = \frac{1}{Z_q} \left[1 + \frac{\tau (q - 1)}{D} \frac{\eta^2}{2} \right]^{\frac{1}{1-q}}, \tag{4}$$

which can be normalized only for $q < 3$ (Z_q is a normalization factor). The first moment $\langle \eta \rangle$ always vanishes [4, 6, 7, 8, 9] and the second moment,

$$\langle \eta^2 \rangle = \frac{2D}{\tau (5 - 3q)}, \tag{5}$$
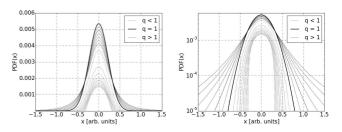
is finite only for $q < 5/3$.

Figure 1: Stationary q-noise pdf for $0 < q < 1$ (dotted line), $q = 1$ (bold line) and $1 < q < 1.6$ (single line). The right panel show the same plot in semilogarithmic scale. Notice the pdf is compact-supported for $q < 1$, Gaussian for $q = 1$ and fat-tailed for $q > 1$.

Some properties of the noise are summarized in Fig. 1. The bold line depicts the Gaussian limit ($q = 1$). The weaker full lines show that for $q > 1$, the second moment is larger than the Gaussian limit D/τ. For $q < 1$ (dotted lines) the distribution has a cut-off and is only defined for

$$|\eta| < \eta_c \equiv \sqrt{\frac{2D}{\tau (1 - q)}}. \tag{6}$$

In order to emphasize different aspects, the same distributions are shown in linear and semilogarithmic scales (Fig. 1, left and right panels respectively).

The autocorrelation time τ_q of the process $\eta(t)$ in its stationary regime, also diverges for $\to 5/3 \approx 1.66$. Far from its divergence point, it can be approximated as [4]

$$\tau_q \approx \frac{2\tau}{5 - 3q}. \tag{7}$$

When $q \to 1$, η becomes a Gaussian colored noise, namely the Ornstein–Uhlenbeck process $\xi_{OU}(t)$, with correlations

$$\langle \xi_{OU}(t)\,\xi_{OU}(t') \rangle = \frac{D}{\tau} \exp\left(-\frac{|t - t'|}{\tau} \right), \tag{8}$$

and probability distribution

$$P^{\text{st}}(\xi_{OU}) = Z^{-1} \exp\left(-\frac{\tau}{D} \frac{\xi_{OU}^2}{2} \right). \tag{9}$$

2.1. τ and $\langle \eta^2 \rangle$ dependence on q

Equations Eqs. (5) and (7) tell us that for $q \neq 1$, $\langle \eta^2 \rangle$ and τ_q do not attain their values (D and τ respectively) in a normal distribution. Rather, they both diverge at $q = 5/3$ (white squares in both panels of Fig. 2). It is however desirable to have a generator able to approximately keep constant the characteristics of these properties with respect of q, at least sufficiently far away from the divergence point. This can be very useful to study the effects of the statistics due to changes in q keeping τ and variance constant.

This way an *effective* τ_q and $\langle \eta^2 \rangle$ can be defined by dividing τ by Eq. (7) before integration and $\langle \eta^2 \rangle$ by Eq. (5) after integration. The filled circles in both panels of Figure 2 show this dependence for both τ and $\langle \eta^2 \rangle$, and how the system becomes independent of q for the range $0 < q < 1.5$ approximately. For $q > 1.5$, the proximity to the divergence point $q = 5/3$ (shown with a dotted line) makes this approximation fail.
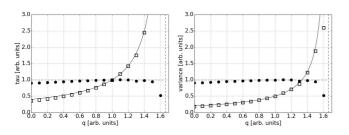

Figure 2: (Left) Dependence of τ on q. The white squares show the measured dependence, fitted by Eq. (7) (full line). The black dots show the behavior of an effective τ in order to make it independent of q. (Right) Equivalently for the variance of the noise, using Eq. (5).

3. Software Description

The noise generator [10] is implemented in `c++` as class, depending on standard libraries only. It generates random numbers using functions in the built-in `<random>` class. It includes Gaussian white noise, Gaussian colored noise (Orstein-Uhlenbeck), and two versions of non-Gaussian non-white noise. One where τ and $\langle \eta^2 \rangle$ *depend* on q (as in Eqs. 5 and 7) and a normalized version where this effect has been compensated to the first order, sufficiently far away from $q = 5/3$ (as shown as black dots in figure 2) This battery of functions would allow for modeling a wide variety of scenarios and be suitable for many applications, some of which are detailed in the last section of this communication. By default, the software uses the Mersenne-Twister generator [11] which provides a very long ($2^{19937} - 1$) pseudo-random number cycle. Hence it is advised to seed the generator *only once* to avoid spurious correlations.

3.1. Seeding

There are two ways for seeding the random number generator:

```
void seedTimer ()
void seedManual (unsigned UserSeed)
```

The timer seeding is automatic, and for simple applications no seeding is necessary. A manual seeding for each thread is advised when using multi-threading.

3.2. Functions

The class presents four public member functions:

```
1. double gaussWN ()
2. double orsUhl (double x, double tau, double H)
3. double qNoise (double eta, double tau, double q,
double H, double sqrt_H (optional))
4. double qNoiseNorm (double eta, double tau, double q,
double H, double sqrt_H (optional))
```

The first function is a wrapper for the normal distribution, implemented in the `<random>` standard library. It is presented as a function of this class for convenience.

The second function is an implementation of the Orstein-Uhlenbeck noise. It accepts three parameters. The previous value of the noise (since it is a Markov process), the autocorrelation time τ of the noise, and the integration time H (necessary for setting the adequate timescale of the noise).

The third function implements the q-noise distribution. It accepts the same variables as the `orsUhl` function plus q (the noise statistics), and sqrt_H as an optional variable. If H is constant, explicitly setting `sqrt_H` $= \sqrt{H}$ will avoid its calculation every time the function is invoked.

Finally, the fourth function is a wrapper for the third function. Here τ is given by Equation (7) and the resulting noise is divided using Eq. (5) in order to compensate for the dependence of both τ and the variance of the noise on q. See section 2.1 for an analysis of this effect.

4. Properties of the generated noise

Bounded domain ($q < 1$)

Bounded-domain noise is widespread in nature, and has multiple applications for modeling and control [9][1]. The infra-Gaussian noise considered here can be addressed as a small deviation from Gaussianity, allowing a perturbative approach. In Sec. 5.1, an example of a infra-Gaussian noise is shown, in a resonante trap. Another use is as a source of noise whose distribution is quasi-normal but identically zero outside a boundary.

The software insures that the domain is bounded, checking for out-of-bound values. This necessary test (especially for highly correlated noise) has been implemented and documented in the code.

Gaussian case ($q = 1$)

The Gaussian case behaves exactly as an Orstein-Uhlenbeck noise, being fully compatible with it for the whole range of τ. As shown in the introduction, the limit $q \rightarrow 1$ recovers the

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[1]In practice, physical noise has bounded domain because arbitrarily large fluctuations are strongly suppressed. Nonetheless, Gaussian noise has many desirable theoretic properties which allow for analytical results.
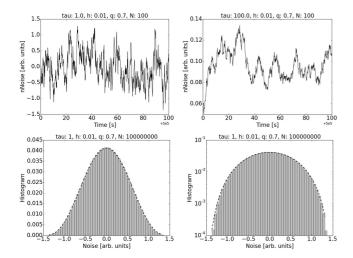
Figure 3: q-noise for $q = 0.7$) (bounded domain), and integration step $h = 0.01$. The top panels show a sample of the noise generated noise, for (left) $\tau = 1s$ and (right) $\tau = 100s$. Notice that in the right figure the noise is not centered around zero as its performing a very long excursion (bigger than the sample) given its very high autocorrelation. Both histograms in the bottom panels show the same data, coinciding with the sample on top-left. ($\tau = 1s$). Although in the linear histogram of the left it cannot be clearly seen, the semilogarithmic plot on the right clearly show the bounded domain. The dotted points show the theoretical distribution as in Fig. 1 for the same parameters agreeing excellently with the histogram of the data.

Figure 4: q-noise for $q = 1$) (Gaussian Behavior), and integration step $h = 0.01$. The top panels show a sample of the noise generated noise, for (left) $\tau = 1s$ and (right) $\tau = 100s$. Notice that in the right figure the noise is not centered around zero as its performing a very long excursion (bigger than the sample) given its very high autocorrelation. Both histograms in the bottom panels show the same data, coinciding with the sample on top-left. ($\tau = 1s$). Although in the linear histogram of the left shows a bell-shaped distribution, this is not enough to demonstrate Gaussianity. However the semilogarithmic plot of the right is parabolic demonstrating this fact. The dotted points show the theoretical distribution as in Fig. 1 for the same parameters agreeing excellently with the histogram of the data.

Gaussian noise, and all limits converge to it, See from Eqs. (5)-(9). This limit allows to explore regions arbitrarily near the normal distribution. It can be used to model small deviations from it due to some underlying physical phenomenon. As the value of q can be changed continuously and dynamically, this scheme also allows to model departures from the normal distribution due to of long time-scale fluctuations, by slowly varying $1 - \epsilon < q < 1 + \epsilon$ as a more realistic model for a small noisy system.

To compute the purely Gaussian case from the general case and set $q = 1$, is generally not advisable due to the longer computation times. An extensive battery of tests has been run in order to compare it to the Orstein-Uhlenbeck noise, and all of the results were successfully recovered. As presented above, `orsUhl`, a function for generating Orstein-Uhlenbeck noise for a variable τ is included in this package and its results are equivalent to using the non-normalized `qNoise` function for $q = 1$ at a fraction of the computation time.

Supra-Gaussian noise ($1 < q < 5/3$)

The supra-Gaussian (also called fat-tail) noise presented here is of the class of finite variance. This is a usually poorly studied class of noise. The Supra-Gaussian noise generally considered in the bibliography tends to be *Lévy-like*, where the variance is infinite [2].

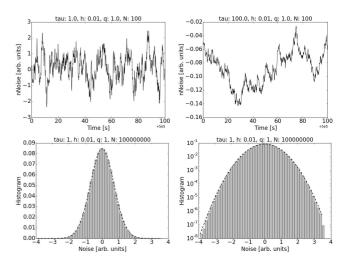The noise presented here is of finite variance. The long excursions are however much longer and much more frequent than in the Gaussian case. This case is the most frequently used in the applications of non-Gaussian noise presented below, as it allows to model many realistic systems outside of equilibrium.

5. Applications

Stochastic resonance

This is a phenomenon occurring in some nonlinear systems, whereby enhancing the response to a weak external signal may require *increasing* the noise intensity. An often resorted-to measure is the *signal-to-noise ratio* at the input frequency ω (denoted by R).

The main numerical and theoretical results are [12, 13]: (1) for fixed τ, the maximum R increases with decreasing q; (2) for given q, the optimal noise intensity (the one maximizing R) decreases with q and its value is approximately independent of τ; (3) for fixed noise intensity, the optimal value of q is independent of τ and in general turns out to be $q_{op} \neq 1$. A simple stochastic resonance experiment with a non-Gaussian *white* noise [3] confirmed most of these predictions.

Brownian motors

A class of nonequilibrium systems with both potential technological applications and biological interest are the so called "ratchets", in which the breakdown of spatial and/or temporal symmetry induces directional transport. Their transport properties can be studied by means of the Langevin equation

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - V'(x) - F + \xi(t) + \eta(t), \qquad (10)$$

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[2] The q-noise presents infinite variance for $q > 5/3$ but the description of this behavior is outside the scope of this article.
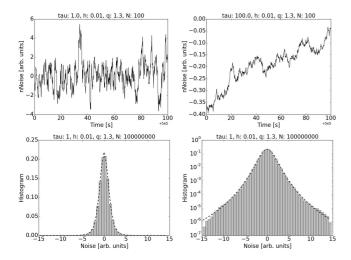
Figure 5: q-noise for $q = 1.3$) (supra-Gaussian Behavior), and integration step $h = 0.01$. The top panels show a sample of the noise generated noise, for (left) $\tau = 1s$ and (right) $\tau = 100s$. Notice that in the right figure the noise is not centered around zero as its performing a very long excursion (bigger than the sample) given its very high autocorrelation. Both histograms in the bottom panels show the same data, coinciding with the sample on top-left. ($\tau = 1s$). Although in the linear histogram of the left shows a bell-shaped distribution, this could make us suspect Gaussianity. The semilogarithmic plot of the right, however, show a supra-Gaussian behavior. The dotted points show the theoretical distribution as in Fig. 1 for the same parameters agreeing excellently with the histogram of the data.

with m the particle's mass, γ the friction constant, $V(x)$ the (sawtooth-like) ratchet potential, F a constant "load" force, and $\xi(t)$ the thermal noise, satisfying $\langle \xi(t)\xi(t') \rangle = 2\gamma T \delta(t - t')$.

The system is kept out of thermal equilibrium by the time-correlated forcing $\eta(t)$ (with zero mean), so allowing to rectify the motion. The q-dependence of the usual measures of performance has been studied: the mean current $J \equiv \langle dx/dt \rangle$ and the efficiency ε (the ratio of the work per unit time done against F, to the mean power injected by η).

In the overdamped regime ($m = 0, \gamma = 1$), J is found to grow monotonically with q whereas ε is maximized for some $1 < q < 5/3$. For $m \neq 0$, ratchets exhibit mass-separation capabilities which are enhanced by non-Gaussian noise [14, 15]. In [16], effects of biological and technological relevance have been found in a model for the transport properties of motor proteins when departing from Gaussian behavior: J is maximized not only by an optimal noise intensity but also by an optimal $q \neq 1$.

5.1. Resonant gated trapping

Stochastic resonance, which is essentially a threshold phenomenon, plays also a relevant role in ionic transport through cell membranes. In [17], a "toy model" considering the simultaneous action of a deterministic and a stochastic external field on the trapping rate of a gated imperfect trap, was studied by assuming Tsallis' noise with $q < 1$: the bounded character of the pdf contributed positively to the rate of overcoming the threshold, and such rate remained at about the same order within a larger range of values than if η had been a white noise.

Noise-induced transition

A genetic model exhibiting a reentrance from a disordered state to an ordered one, and again to a disordered state as τ varies from 0 to ∞ showed moreover a strong shift in the transition line, as q departed from $q = 1$. The transition was anticipated for $q > 1$, while it was retarded for $q < 1$ [18].

Noise-induced phase transition

It results that fat-tail noise distributions ($q > 1$) *counteract* the effect self-correlation (namely, they *advance* the ordering boundary as D is increased at constant coupling), and compact-support ones ($q < 1$) *enhance* it (they *retard* the ordering boundary). Particular interest rises the effect of ($q < 1$) multiplicative noise on the susceptibility: it shifts from being larger on the *ordering* boundary to being larger on the *disordering* boundary [19, 20].

Broad-spectrum energy harvesting

In piezoelectric energy harvesting from noise, a system obeying a square-well potential can strongly profit from the large correlated excursion occurring for $q > 1$ [21].

6. Conclusions

A software is presented that generates a class of non-Gaussian, colored noise. This noise can be easily generated during numerical experiments, or fed to experiments via an interface. The software has been uploaded to the online repository Github including examples, together with an open source license. As examples and motivation for its use, we have provided instances of noise-induced phenomena arising when the system is submitted to (colored and non-Gaussian) noise sources with Tsallis' q-statistics. The above discussed results show that non-Gaussian noise can significantly change the system's response in many noise-induced phenomena, as compared with the Gaussian case. Moreover, in all the cases presented here, the system's response was either enhanced or altered in a relevant way for values of q departing from Gaussian behavior. In other words, the *optimum response* occurred for $q \neq 1$. Clearly, the study of the change in the response of other related noise-induced phenomena when subject to such a kind of non-Gaussian noise will be of great interest.

Acknowledgments

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