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Efficient Generation of Colored Noise

STEVEN M. KAY

***Abstract*—A new technique is presented for efficiently generating colored noise. Instead of discarding initial samples to account for the transient, the approach proposed here is to set the initial conditions of the filter so that the output process will be stationary. It is shown that the Levinson-Durbin algorithm provides an efficient means for determining these initial conditions.**

I. Introduction

Colored noise is usually generated by passing white noise through a filter with a rational transfer function, i.e., one having poles and zeros. However, since the input to the filter must start at some time $k = 0$, the output process will consist of an undesired transient and the desired stationary random process. One way to alleviate this problem is to discard the output process samples until the transient dies out. However, the transient can be shown to be proportional to the impulse response length. Thus, if the filter has poles near the unit circle, i.e., the impulse response is long, this procedure will lead to increased computation.

Another solution to this problem has been proposed by Levin [1]. He specifies auxiliary random variables to be added to the output process, which act to compensate for the lack of an input sequence before $k = 0$, and hence yield the same output as if the input process had existed from $k = -\infty$. His technique requires determination of the filter impulse response, which necessitates one to find the poles of the filter or at least perform synthetic division. These computations can be quite lengthy.

A new technique is described for generating the stationary random process. Instead of using auxiliary random variables, one need only specify the initial conditions for the filter, i.e., its state at $k = 0$. By using the Levinson-Durbin algorithm, it is shown how these initial conditions can be simply generated.

II. Theoretical Background

Assume we wish to generate colored noise $Z(k)$ by filtering white noise $u(k)$ through a pole-zero filter, i.e.,

$$Z(k) = -\sum_{l=1}^{p} a_l Z(k-l) + \sum_{l=0}^{q} b_l u(k-l), \qquad k \geq 0 \tag{1}$$

where $b_0 = 1$ and $q \leq p$.

Alternately, $Z(k)$ can be decomposed as follows:

$$S(k) = -\sum_{l=1}^{p} a_l S(k-l) + u(k)$$

$$Z(k) = \sum_{l=0}^{q} b_l S(k-l), \qquad k \geq 0. \tag{2}$$

note that $S(k)$ is the output of an all pole filter, i.e., it is an autoregressive (AR) process. To generate $Z(k)$ for $k \geq 0$ using (2), we need to know $\{S(-1), S(-2), \cdots, S(-p)\}$. It can be shown [2] that if $X(-1) = [S(-1)\, S(-2) \cdots S(-p)]^T$ is a zero mean vector and if

$$E(X(-1)\, X(-1)^T) = R_S$$

where $[R_S]_{ij} = R_S(i-j)$ is the autocorrelation function of the *desired* time series $S(k)$, then $S(k)$ as computed by (2) will be a wide sense stationary process.

III. Generation of Initial Conditions

$X(-1) = [S(-1) \cdots S(-p)]^T$ can be generated by linearly transforming a vector $V = [V(1)\; V(2) \cdots V(p)]^T$ of independent zero mean, unit variance random variables by an appropriate matrix transformation A, i.e.,

$$X(-1) = AV \tag{3}$$

We choose A such that $E[X(-1)\, X(-1)^T] = R_S$. Thus

$$R_S = E[AV(AV)^T] = AIA^T = AA^T. \tag{4}$$

Thus, we need only perform a square rooting of the covariance matrix R_S. Since R_S is Toeplitz, the Levinson-Durbin algorithm will easily yield A. Thus, it can be shown that [3]:

$$B^T R_S B = P \tag{5}$$

where

$$B = \begin{bmatrix} 1 & a_1^{(1)} & a_2^{(2)} & \cdots & a_{p-1}^{(p-1)} \\ 0 & 1 & a_1^{(2)} & \cdots & a_{p-2}^{(p-1)} \\ & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$P = \operatorname{diag}(P_0, P_1, \cdots, P_{p-1})$$

and

$a_i^{(j)}$ is the ith coefficient of the jth-order prediction error filter, and P_j is the prediction error power of the jth-order prediction error filter.

Note that $a_i^{(p)} = a_i$. Then,

$$R_S = B^{T^{-1}} P B^{-1}$$

Defining $\sqrt{P} = \operatorname{diag}(\sqrt{P_0}, \sqrt{P_1}, \cdots, \sqrt{P_{p-1}})$, we have

$$A = B^{T^{-1}} \sqrt{P}$$

B^T is always invertible since the determinate of B^T equals 1. Now,

$$X(-1) = B^{T^{-1}} \sqrt{P}\, V$$

or

$$(\sqrt{P})^{-1} B^T X(-1) = V$$

where it has been assumed that $P_i \neq 0, i = 0, 1, \cdots, p-1$ which will be the case if the all pole filter has all its poles inside the unit circle.

Thus, we need to solve a set of linear equations to obtain $X(-1)$. However, since $(\sqrt{P})^{-1} B^T$ is lower triangular, $X(-1)$ can be found efficiently. Evaluating the matrix, we have:

$$(\sqrt{P})^{-1} B^T = \begin{bmatrix} \frac{1}{\sqrt{P_0}} & 0 & \cdots & 0 \\ \frac{a_1^{(1)}}{\sqrt{P_1}} & \frac{1}{\sqrt{P_1}} & \cdots & 0 \\ \vdots & & & \\ \frac{a_{p-1}^{(p-1)}}{\sqrt{P_{p-1}}} & \frac{a_{p-2}^{(p-1)}}{\sqrt{P_{p-1}}} & \cdots & \frac{1}{\sqrt{P_{p-1}}} \end{bmatrix}$$

Solving for $X(-1)$ we have:

$$S(-1) = \sqrt{P_0}\, V(1)$$

$$S(-k) = \sqrt{P_{k-1}}\, V(k) - \sum_{l=1}^{k-1} a_{k-l}^{(k-1)} S(-l), \qquad k = 2, 3, \cdots, p. \tag{6}$$

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Thus $\{S(-1), \cdots, S(-p)\}$ are found recursively.

Finally, one must have knowledge of B and P. In many instances one is given an autocorrelation function or power spectral density and wishes to obtain an AR process with an autocorrelation function or power spectral density approximating the given one. A well known procedure for doing this is to solve the Yule-Walker equations for the AR parameters for a certain order model. Typically, then the equations would be solved using the Levinson-Durbin algorithm and so B and P would already be available. If, however, an ARMA process is desired to yield an approximation to the given autocorrelation function or power spectral density or if the $\{a_k\}$ have not been found by the Levinson-Durbin algorithm, B and P can still be found easily. In this case one can use the "step-down" procedure [3]. With $a_i^{(p)} = a_i$, $i = 1, 2, \cdots, p$ we have for $j = p, p-1, \cdots, 2$

$$a_i^{(j-1)} = \frac{a_i^{(j)} - k_j a_{j-i}^{(j)}}{1 - k_j^2}, \quad i = 1, 2, \cdots, j-1 \tag{7a}$$

with

$$k_j = a_j^{(j)}.$$

This procedure yields B. To find P we have

$$P_0 = \frac{\sigma_u^2}{\prod_{i=1}^{p} (1 - k_i^2)} \tag{7b}$$

and for $j = 1, 2, \cdots, p-1$

$$P_j = P_{j-1}(1 - k_j^2).$$

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Signal Design for Detection of Targets in Clutter

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***Abstract*—It is shown that the joint optimization of the transmitter and receiver for detection of targets in the clutter (reverberation) limited environment reduces to optimization of the transmitted signal. The optimum receiver is determined by the transmitted signal and external environment.**

I. INTRODUCTION

The design of signals and filters for optimum detection of targets in clutter (reverberation) has been an active research topic for some time. The published work on this topic has either dealt with design of optimum signals, optimum filters, or joint optimization of the signal and filter [1]-[9].

The purpose of this note is to demonstrate that the joint optimization of the signal (transmitter) and filter (receiver) in the clutter limited interference reduces to the optimization of the transmitted signal only. This simplifying observation has not been explicitly pointed out in the literature. The joint signal-filter optimization reduces to the signal optimization because the optimum filter is determined by the transmitted signal, the clutter (reverberation) scattering function, and white noise intensity. Only the design of the transmitted signal is under the system designer's control.

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In this note, we assume that the total interference consists of two uncorrelated Gaussian processes; the colored clutter process and a white noise process. Optimum receiver is defined to be the maximum-likelihood receiver, and the signal optimization maximizes the detection index for the maximum-likelihood receiver.

II. OPTIMUM RECEIVER AND SIGNAL

The structure of the maximum-likelihood receiver for incoherent detection of a slowly fluctuating point target in colored Gaussian interference is shown in Fig. 1 [10]. The receiver computes the test statistic

$$|l|^2 = \left| \int_{T_1}^{T_2} r(t)\, g^*(t)\, dt \right|^2 \tag{1}$$

and compares it to a threshold. In (1), $r(t)$ is the complex envelope of the received signal and $g(t)$ is the impulse response of the optimum filter, i.e., mismatched filter, which satisfies the integral equation

$$\int_{T_1}^{T_2} R_n(t,u)\, g(u)\, du = \sqrt{E_t}\, f_d(t) \tag{2}$$

where E_t is the energy of the transmitted signal, $f_d(t)$ is a delayed and Doppler shifted replica of the transmitted signal $f(t)$, and $R_n(t,u)$ is the covariance function of the total interference. By this assumption, the total interference consists of clutter and white noise. It is assumed that the clutter and white noise are uncorrelated. Hence, the interference covariance function is

$$\begin{aligned} R_n(t,u) &= E\{n(t)\, n^*(u)\} \\ &= R_c(t,u) + N_0 \delta(t-u) \end{aligned} \tag{3}$$

where $R_c(t,u)$ is the covariance of the clutter, N_0 is the white noise intensity, and $\delta(t-u)$ is the delta function. Substituting (3) into (2), the integral equation for $g(t)$ becomes

$$g(t) = \frac{\sqrt{E_t}}{N_0} f_d(t) - \frac{1}{N_0} \int_{T_1}^{T_2} R_c(t,u)\, g(u)\, du. \tag{4}$$

The clutter covariance can be expressed by

$$R_c(t,u) = E_t \int_{-\infty}^{\infty} f(t-\tau)\, R_R(t-u,\tau)\, f^*(u-\tau)\, d\tau \tag{5}$$

or

$$R_c(t,u) = E_t \iint_{-\infty}^{\infty} f(t-\tau)\, S_R(\phi,\tau)\, f^*(u-\tau) \exp[j2\pi\phi(t-u)]\, d\phi\, d\tau \tag{6}$$

where $R_R(t,\tau)$ is the correlation function of the clutter scattering channel, and $S_R(\phi,\tau)$ is the clutter scattering function. A useful performance measure of the optimum receiver is

$$\begin{aligned} \Delta_{OPT} &= \frac{E\{|l|^2/H_1\} - E\{|l|^2/H_0\}}{E\{|l|^2/H_0\}} \\ &= \frac{\overline{E}_r}{\sqrt{E_t}} \int f_d^*(t)\, g(t)\, dt \end{aligned} \tag{7}$$

where $E\{\cdot\}$ denotes expectation and $\overline{E}_r$ is expected received energy. Obviously, $\overline{E}_r$ is proportional to the transmitted energy. Note that in the case of white noise Δ_{OPT} becomes $\overline{E}_r/N_0$. Hence, Δ_{OPT} can be considered a generalized signal to interference ratio. For a given false alarm probability, the probability of detection increases monotonically with Δ_{OPT}. The objective of system optimization is to maximize Δ_{OPT} under appropriate constraints.

The integral in (4) is a Fredholm integral equation of the second kind. If the interval (T_1, T_2) is finite and if $f_d(t)$ and $R_c(t,u)$ are continuous