notes the first zero of the derivative of $J_\nu(x)$, $N_\nu(x)$ is negative and increasing. Hence for $j'_\nu/\nu \geqq x \geqq 1$, a simple bound for $N_\nu(\nu x)$ is

$$|N_\nu(\nu x)| \leqq |N_\nu(\nu)| \leqq 1.73|J_\nu(\nu)|. \tag{20}$$

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NEWTON'S METHOD IN BANACH SPACES[1]

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In this note we show that if f is a mapping between Banach spaces which is in class C' in the sense of Hildebrandt and Graves [4],[2] then the equation $f(x)=0$ may be solved by an iterative process:

$$x_{n+1} = x_n - [f'(z_n)]^{-1}f(x_n), \qquad n = 0, 1, 2, \cdots,$$

provided that the initial guess x_0 and the arbitrarily selected points z_n are sufficiently close to the solution desired. Here the derivative is taken in the sense of Fréchet. If we let $z_n = x_n$, $n = 0, 1, 2, \cdots$, we obtain the usual Newton process; if $z_n = x_0$, $n = 0, 1, 2, \cdots$, we obtain what is sometimes called the modified Newton process. Naturally, in any application, the computer would determine the z_n so as to minimize effort.

This result is closely related to recent theorems of Kantorovič [5; 6; 7] and Mysovskih [8; 9], although these authors assumed the existence and boundedness of the second Fréchet derivative of f. In turn for this assumption, they were able to establish more rapid convergence. Under the assumption of analyticity, Stein [10] eliminated explicit mention of the second derivative, which is desirable

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[2] Numbers in brackets refer to the list of references at the end.

since it plays no role in the iteration. Fenyö [1] treated the Newton process in the case when the first derivative satisfies a Lipschitz condition at x_0, and the modified process without this restriction.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be (real or complex) Banach spaces and let f map an open set of $\mathfrak{X}$ into $\mathfrak{Y}$. We recall that the *Fréchet derivative* $f'(x_0)$ of f at a point x_0 in the domain of f is the bounded linear operator of $\mathfrak{X}$ into $\mathfrak{Y}$ such that[3]

$$\lim_{|h|\to 0} |h|^{-1} |f(x_0 + h) - f(x_0) - f'(x_0)(h)| = 0,$$

provided such an operator exists. If $\mathfrak{G}$ is an open set in $\mathfrak{X}$, we say, following Hildebrandt and Graves [4], that f is in *class* $C'(\mathfrak{G})$ if $f'(x)$ exists for $x \in \mathfrak{G}$, and if the mapping $x \to f'(x)$ is a continuous map of $\mathfrak{G}$ into the space $\mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$ of bounded linear operators from $\mathfrak{X}$ to $\mathfrak{Y}$ equipped with the uniform operator topology.

If f is in class $C'(\mathfrak{G})$, let $\delta(x_0, \epsilon)$ denote the *modulus of continuity* of f' at x_0. This means that if[3] $|x - x_0| \leqq \delta(x_0, \epsilon)$ then $|f'(x) - f'(x_0)| \leqq \epsilon$. Further, let $\mathfrak{S}(x_0, \alpha)$ denote the sphere $\{x \in X: |x - x_0| < \alpha\}$.

LEMMA 1. *If f is in class $C'(\mathfrak{S}(x_0, \alpha))$ and if x_1 and x_2 are such that $|x_i - x_0| \leqq \delta(x_0, \epsilon)$, then*

$$(*) \qquad |f(x_1) - f(x_2) - f'(x_0)(x_1 - x_2)| \leqq \epsilon |x_1 - x_2|.$$

PROOF. It follows from the general form of Taylor's theorem, due to Graves [2, p. 173], that

$$f(x_1) - f(x_2) = \int_0^1 f'[tx_1 + (1 - t)x_2](x_1 - x_2)dt,$$

from which the statement follows.

LEMMA 2. *Let f be in class $C'(\mathfrak{S}(x_0, \alpha))$ and suppose that $f'(x_0)$ has a bounded inverse. For any $\lambda > |[f'(x_0)]^{-1}|$, there is a number $\beta \leqq \min\{1, \alpha\}$ such that*

(1) *if $|x - x_0| \leqq \beta$, then the operator $f'(x)$ has an inverse and $|[f'(x)]^{-1}| < \lambda$;*

(2) *if $|x_i - x_0| \leqq \beta$, $i = 1, 2, 3$, then*

$$(**) \qquad |f(x_1) - f(x_2) - f'(x_3)(x_1 - x_2)| \leqq (1/2\lambda) |x_1 - x_2|.$$

PROOF. Choose δ_1 such that if $|x - x_0| \leqq \delta_1$, then $|f'(x) - f'(x_0)| \leqq (4\lambda)^{-1}$. This implies that if $|x - x_0| \leqq \delta_1$, then $f'(x)$ has an inverse

[3] Since no confusion can result, we use a single vertical bar to denote the norm of an element of the spaces $\mathfrak{X}$ and $\mathfrak{Y}$ and the norm of an operator between these two spaces.

operator. Since the function $|T^{-1}|$ is an upper semi-continuous function for T in the open set of invertible operators in $\mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$ (see [3, p. 112]), there is an $\alpha_1 \leqq \min \{\alpha, \delta_1\}$ such that if $|x-x_0| \leqq \alpha_1$, then $|[f'(x)]^{-1}| < \lambda$. This proves (1). If $\beta = \min \{1, \alpha_1, \delta(x_0, 1/4\lambda)\}$, it follows from Lemma 1 and the above that if $|x_i - x_0| \leqq \beta$, then (2) holds.

THEOREM. *Let f: $\mathfrak{S}(x_0, \alpha) \to \mathfrak{Y}$ be in class $C'(\mathfrak{S}(x_0, \alpha))$, and suppose that $f'(x_0)$ has an inverse with $|[f'(x_0)]^{-1}| < \lambda < \infty$. Let $|f(x_0)| < \beta/2\lambda$, where β is as in Lemma 2, and let z_n, $n = 0, 1, 2, \cdots$, be completely arbitrary points with $|z_n - x_0| \leqq \beta$. Then the sequence $\{x_n\}$ obtained by the iterative process*

$$x_{n+1} = x_n - [f'(z_n)]^{-1} f(x_n), \qquad n = 0, 1, 2, \cdots,$$

converges to a solution $\dot{x}$ of the equation $f(x) = 0$. Further, $|\dot{x} - x_0| \leqq \beta$ and is the only solution of the equation in this neighborhood of x_0. The rapidity of the convergence is given by $|x_n - \dot{x}| < 2^{-n}\beta$, $n = 0, 1, 2, \cdots$.

PROOF. By definition, $|x_1 - x_0| \leqq \lambda |f(x_0)| < \beta/2$. Further

$$f(x_1) = f(x_1) - f(x_0) - f'(z_0)(x_1 - x_0),$$

and by (**) of Lemma 2, we have that

$$|f(x_1)| \leqq (1/2\lambda) |x_1 - x_0|.$$

By induction, suppose that $x_1, \cdots, x_n$ have been chosen such that for $i = 1, \cdots, n$ we have

$$(\mathrm{a}_i) \qquad |x_i - x_0| < \beta,$$

$$(\mathrm{b}_i) \qquad |x_i - x_{i-1}| \leqq \lambda |f(x_{i-1})|,$$

$$(\mathrm{c}_i) \qquad |f(x_i)| \leqq (1/2\lambda) |x_i - x_{i-1}|.$$

Then, since $x_{n+1} = x_n - [f'(z_n)]^{-1} f(x_n)$, it follows that

$$|x_{n+1} - x_n| \leqq \lambda |f(x_n)|,$$

which is (b_{n+1}). Thus $|x_{n+1} - x_n| < (1/2)|x_n - x_{n-1}|$. Iterating (b_i) and (c_i), we conclude readily that

$$(\dagger) \qquad \begin{aligned} |x_{n+1} - x_0| &\leqq \left\{ \sum_{i=0}^{n} 2^{-i} \right\} |x_1 - x_0| \\ &< \{1 - 2^{-(n+1)}\}\beta. \end{aligned}$$

This proves (a_{n+1}). Since

$$f(x_{n+1}) = f(x_{n+1}) - f(x_n) - f'(z_n)(x_{n+1} - x_n),$$

the validity of (a_{n+1}) permits the use of (**) to conclude that

$$|f(x_{n+1})| \leqq (1/2\lambda)|x_{n+1} - x_n|,$$

which is (c_{n+1}). Thus the inductive steps may be continued. Further, for any integers n and p,

$$(\dagger)\qquad \begin{aligned} |x_{n+p} - x_n| &\leqq \sum_{i=1}^{p} |x_{n+i} - x_{n+i-1}| \\ &\leqq \lambda |f(x_0)| 2^{-n} \left\{ \sum_{i=0}^{p-1} 2^{-i} \right\} < 2^{-n}\beta, \end{aligned}$$

and hence $\{x_n\}$ is a Cauchy sequence converging to an element $\dot{x} \in \mathfrak{X}$. In view of (a_i), we have $|\dot{x} - x_0| \leqq \beta$. From ($c_i$) it follows that $f(\dot{x}) = 0$. To see that $\dot{x}$ is the unique solution of $f(x) = 0$ in this neighborhood, let $\ddot{x}$ be another solution with $|\ddot{x} - x_0| \leqq \beta$. We have

$$|\dot{x} - \ddot{x}| = |[f'(x_0)]^{-1} f'(x_0)(\dot{x} - \ddot{x})| \leqq \lambda |f'(x_0)(\dot{x} - \ddot{x})|.$$

By (*) we have $|f'(x_0)(\dot{x} - \ddot{x})| \leqq (1/2\lambda)|\dot{x} - \ddot{x}|$, and hence $|\dot{x} - \ddot{x}| \leqq (1/2)|\dot{x} - \ddot{x}|$ which is a contradiction unless $\dot{x} = \ddot{x}$. Finally the inequality concerning the speed of convergence follows from (†) upon letting p approach infinity.

We wish to point out to the reader the similarity between what we have done and Theorems 1 and 2 of Graves [**3**].

Since the modified Newton process (i.e., when $z_n = x_0$) avoids the necessity of computing a new inverse at each step, it appears to be particularly convenient. By analysing the proof, we may see that Lemma 2 is never used unless the point z_n differs from x_0, so in the modified process we require only the first lemma. This permits better estimates:

COROLLARY. *In the modified Newton process where $z_n = x_0$, $n = 0, 1, 2, \cdots$, the number β may be chosen to be* $\min\{1, \alpha, \delta(x_0, 1/2\lambda)\}$.

REMARK. Since the calculation of the inverse operators $[f'(z_n)]^{-1}$ is generally difficult, it is worth observing that the above proof applies directly to assure the convergence of an iteration

$$x_{n+1} = x_n - \overset{-1}{T_n} f(x_n), \qquad n = 0, 1, 2, \cdots,$$

for any sequence $\{T_n\}$ of bounded operators such that

$$|T_n - f'(x_0)| < 1/4\lambda, \qquad |\overset{-1}{T_n}| < \lambda.$$

For application of the Newton process, we refer the reader to Kantorovič [**6**; **7**].

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