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A variational iteration method for solving Troesch's problem

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ABSTRACT

Troesch's problem is an inherently unstable two-point boundary value problem. A new and efficient algorithm based on the variational iteration method and variable transformation is proposed to solve Troesch's problem. The underlying idea of the method is to convert the hyperbolic-type nonlinearity in the problem into polynomial-type nonlinearities by variable transformation, and the variational iteration method is then directly used to solve this transformed problem. Only the second-order iterative solution is required to provide a highly accurate analytical solution as compared with those obtained by other analytical and numerical methods.

1. Introduction

In this paper, we consider a two-point boundary value problem, Troesch's problem, defined by

$$\begin{aligned} &y'' = \mu \sinh(\mu y) \\ &y(0) = 0, \quad y(1) = 1 \end{aligned} \tag{1}$$

where μ is a positive constant. This problem arises in an investigation of the confinement of a plasma column by radiation pressure [1] and also in the theory of gas porous electrodes [2,3]. The closed-form solution to Eq. (1) in term of the Jacobian elliptic function $sc(n|m)$ has been given [4] as

$$y(x) = \frac{2}{\mu} \sinh^{-1}\left[\frac{y'(0)}{2} sc(\mu x|m)\right], \tag{2}$$

where $m = 1 - \frac{1}{4}(y'(0))^2$ and satisfies the transcendental equation

$$sc(\mu|m)(1-m)^{\frac{1}{2}} = \sinh\left(\frac{\mu}{2}\right). \tag{3}$$

It is obvious that $y(x)$ has a singularity located at a pole of $sc(\mu x|m)$ or approximately at [4,5]

$$x_s = \frac{1}{\mu} \ln\left(\frac{8}{y'(0)}\right), \tag{4}$$

which implies that, if $y'(0) > 8e^{-\mu}$, then the singularity lies within the integration range. This results in the problem being very difficult to solve, and this difficulty increases as n increases.

Various numerical methods such as the Monte Carlo method [6], a combination of the multipoint shooting method with the continuation and perturbation technique [7], the quasilinearization method [8,9], the simple shooting method [10–13],

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the method of transformation groups [14], the invariant imbedding method [15] and the inverse shooting method [16] have been successfully applied to this problem and they have yielded results that vary in accuracy. The advantage of iterative approximate methods recently developed over these numerical methods is that they can provide highly accurate approximate or even exact solutions, which are valid on the whole domain, with less computational work. However, most of them such as the Adomian decomposition method [17,18], the variational iteration method [19], and the modified homotopy perturbation method [20] fail to solve Troesch's problem for $\mu > 1$. More recently, the differential transform method has been used in [21] to solve this problem for $\mu = 5$ and 10 with great success. Although highly accurate solutions are obtained for both cases, the number of terms required for convergence in the series solution and the size of computational work increase significantly.

In this paper, we present a new approach in which the hyperbolic-type nonlinearity in Troesch's problem is first converted into polynomial-type nonlinearities by variable transformation, and the variational iteration method is then used to solve this transformed problem in a straightforward manner. The calculated results show that only the second-order iterative solution is required to provide a highly accurate analytical solution as compared with those obtained by other analytical and numerical methods for a wide range of μ. Moreover, the simple first-order analytical approximation is accurate enough for $\mu > 5$ where the problem is more difficult to solve. It is expected that this approach can be extended to other inherently unstable two-point boundary value problems with hyperbolic nonlinearity.

2. Variational iteration method

The variational iteration method [22–24] has been proved to be one of the useful techniques to solve linear and nonlinear ordinary differential equations (ODEs) with a fast convergence rate and small calculation error. Another important advantage is that this method can be applied directly to nonlinear ODEs without requiring linearization, discretization or perturbation.

Consider the following general nonlinear differential equation:

$$Ly(x) + Ny(x) = g(x), \tag{5}$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is a known analytical function. Then, according to the variational iteration method, a correction functional can be constructed as

$$y_{n+1}(x) = y_n(x) + \int_{x_0}^{x} \lambda[Ly_n(s) + Ny_n(s) - g(s)]\mathrm{d}s, \tag{6}$$

where x_0 is a constant and λ is a general Lagrange multiplier. Making the above correction functional stationary, we obtain

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_{x_0}^{x} \lambda[Ly_n(s) + Ny_n(s) - g(s)]\mathrm{d}s. \tag{7}$$

In order to easily identify the Lagrange multiplier, the nonlinear term Ny_n is considered as a restricted variation, i.e., $\delta Ny_n = 0$. Hence, Eq. (7) becomes

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_{x_0}^{x} \lambda[Ly_n(s) - g(s)]\mathrm{d}s. \tag{8}$$

In general, the Lagrange multiplier, λ, can be readily identified by using the stationary conditions derived from the above equation.

3. Analysis of the method

As stated previously, direct application of the variational iteration method to Troesch's problem fails for $\mu > 1$. To overcome this difficulty, a new dependent variable originally suggested in [25] and recently modified in [13] is introduced as

$$u(x) = \tanh\left(\frac{\mu y(x)}{4}\right) \tag{9}$$

or equivalently

$$y(x) = \frac{4}{\mu}\tanh^{-1}(u(x)), \tag{10}$$

from which we find

$$y' = \frac{4}{\mu(1-u^2)}u' \tag{11}$$

$$y'' = \frac{4}{\mu(1-u^2)}u'' + \frac{8u}{\mu(1-u^2)^2}(u')^2. \tag{12}$$

As a result, Troesch's problem becomes

$$\begin{aligned}&(1-u^2)u''+2u(u')^2=\mu^2u(1+u^2)\\&u(0)=0,\quad u(1)=\tanh\left(\frac{\mu}{4}\right).\end{aligned}\tag{13}$$

Although this transformed Troesch problem seems more complex than the original problem, it is not as stiff as the original one since it contains only polynomial nonlinear terms without any hyperbolic nonlinear term [13].

According to the variational iteration method, we can construct a correction functional as

$$u_{n+1}(x)=u_n(x)+\int_0^x\lambda[(1-u^2)u''+2u(u')^2-\mu^2u(1+u^2)]\mathrm{d}s.\tag{14}$$

Making the above correction functional stationary, we obtain

$$\delta u_{n+1}(x)=\delta u_n(x)+\delta\int_0^x\lambda[(1-u^2)u''+2u(u')^2-\mu^2u(1+u^2)]\mathrm{d}s.\tag{15}$$

Since the nonlinear terms can be considered as a restricted variation, Eq. (15) becomes

$$\delta u_{n+1}(x)=\delta u_n(x)+\delta\int_0^x\lambda[u_n''(s)-\mu^2u_n(s)]\mathrm{d}s.\tag{16}$$

This yields the following stationary conditions:

$$\begin{aligned}&\delta u_n:1-\lambda'(x)=0\\&\delta u_n':\lambda(x)=0\\&\delta u_n:\lambda''(s)-\mu^2\lambda(s)=0.\end{aligned}\tag{17}$$

Therefore, the Lagrange multiplier can be easily identified as

$$\lambda(s)=\frac{\sinh[\mu(s-x)]}{\mu}.\tag{18}$$

Consequently, we obtain the following iteration formula:

$$u_{n+1}(x)=u_n(x)+\int_0^x\frac{\sinh[\mu(s-x)]}{\mu}[(1-u^2)u''+2u(u')^2-\mu^2u(1+u^2)]\mathrm{d}s.\tag{19}$$

We begin with the initial (zero-order) approximation $u_0(x)=ax$, where a is a unknown constant. We obtain

$$u_1(x)=\frac{a(4a^2+\mu^2)\sinh(\mu x)-a^3\mu x(4+\mu^2x^2)}{\mu^3}\tag{20}$$

$$u_2(x)=\frac{1}{630\mu^9}\left[35a\left(C_1-C_2\cosh(2\mu x)+C_3\sinh(\mu x)+C_4\sinh(2\mu x)\right)-C_5\cosh(\mu x)\right]\tag{21}$$

and so on. Note that

$$\begin{aligned}&C_1=18a^8\mu x(358880+59880\mu^2x^2+3000\mu^4x^4+72\mu^6x^6+\mu^8x^8)-9a^4\mu^3x(8a^2+\mu^2)(76+7\mu^2x^2)\\&C_2=3a^4\mu x(4a^2+\mu^2)^2(76+3\mu^2x^2)\\&C_3=520a^4\mu^4+18a^2\mu^6+18\mu^8+a^6\mu^2\left[1415-9\mu^2x^2(241+59\mu^2x^2+4\mu^4x^4)\right]\\&\qquad-4a^8\left[1618073+9\mu^2x^2(241+59\mu^2x^2+4\mu^4x^4)\right]\\&C_4=8a^4(4a^2+\mu^2)^2(38+9\mu^2x^2)\\&C_5=630a^3\mu^3(4a^2+\mu^2)(8a^2+\mu^2)x-3a^7\mu(4a^2+\mu^2)(18585+14630\mu^2x^2+1806\mu^4x^4+60\mu^6x^6)x.\end{aligned}\tag{22}$$

Imposing the boundary condition at $x=1$ on the above nth-order approximant u_n, we obtain a polynomial equation for a:

$$u_n(1)=\tanh\left(\frac{\mu}{4}\right).\tag{23}$$

Mathematica has a built-in command to solve this polynomial equation. Once the unknown constant a has been determined, the corresponding nth-order iterative solution of the original Troesch problem follows immediately as

$$y_1(x)=\frac{4}{\mu}\tanh^{-1}(u_1(x))\tag{24}$$

$$y_2(x)=\frac{4}{\mu}\tanh^{-1}(u_2(x))\tag{25}$$

and so on.

Table 1
Calculated value a of the transformed Troesch problem for $\mu = 1, 3, 5$ and 10.

n	$\mu = 1$	$\mu = 3$	$\mu = 5$	$\mu = 10$
0	2.449186624 (−01)	6.351489524 (−01)	8.482836400 (−01)	9.866142982 (−01)
1	2.107902101 (−01)	1.901230233 (−01)	5.714422626 (−02)	8.958443705 (−04)
2	2.113028553 (−01)	1.917845338 (−01)	5.719076883 (−02)	8.958444636 (−04)

Table 2
Initial slope $y'(0)$ of Troesch's problem for $\mu = 1, 3, 5$ and 10.

n	$\mu = 1$	$\mu = 3$	$\mu = 5$	$\mu = 10$
0	9.796746496 (−01)	8.468652698 (−01)	6.786269120 (−01)	3.946457193 (−01)
1	8.431608405 (−01)	2.534973644 (−01)	4.571538101 (−02)	3.583377482 (−04)
2	8.452114213 (−01)	2.557127118 (−01)	4.575261507 (−02)	3.583377854 (−04)
Exact [4]	8.452026845 (−01)	2.556042136 (−01)	4.575046116 (−02)	3.583377707 (−04)

Table 3
Solution of Troesch's problem with $\mu = 10$.

x	$y(x)$ [15]	$y_1(x) = \frac{4}{\mu}\tanh^{-1}(u_1(x))$	$y_2(x) = \frac{4}{\mu}\tanh^{-1}(u_2(x))$
0.000	0	0	0
0.100	4.211183679705 (−05)	4.211189501276 (−05)	4.211189936715 (−05)
0.200	1.299639238293 (−04)	1.299641033085 (−04)	1.299641161162 (−04)
0.300	3.589778855481 (−04)	3.589783710236 (−04)	3.589784021976 (−04)
0.400	9.779014227050 (−04)	9.779027043800 (−04)	9.779027740037 (−04)
0.500	2.659017178062 (−03)	2.659020349167 (−03)	2.659020496335 (−03)
0.600	7.228924695208 (−03)	7.228930931326 (−03)	7.228931229141 (−03)
0.700	1.966406025665 (−02)	1.966406256917 (−02)	1.966406314118 (−02)
0.800	5.373032958567 (−02)	5.373032846396 (−02)	5.373032947016 (−02)
0.900	1.521140787863 (−01)	1.521140752185 (−01)	1.521140767248 (−01)
0.925	2.020016854925 (−01)	2.020016825843 (−01)	2.020016841879 (−01)
0.950	2.762677349042 (−01)	2.762677326887 (−01)	2.762677343596 (−01)
0.970	3.722643330645 (−01)	3.722643317016 (−01)	3.722643333887 (−01)
0.980	4.482330386284 (−01)	4.482330376655 (−01)	4.482330393321 (−01)
0.990	5.740764982493 (−01)	5.740764989151 (−01)	5.740765004923 (−01)
0.995	6.901149417369 (−01)	6.901149440197 (−01)	6.901149454322 (−01)
0.997	7.657697261350 (−01)	7.657697277452 (−01)	7.657697289833 (−01)
0.998	8.180328282850 (−01)	8.180328296454 (−01)	8.180328307176 (−01)
0.999	8.889931171768 (−01)	8.889931177557 (−01)	8.889931185202 (−01)
1.000	1.0 (+00)	9.999999999997 (−01)	9.999999999996 (−01)

4. Results and discussion

Since $u(0) = 0$, it follows from Eq. (11) that

$$y'(0) = \frac{4}{\mu}u'(0) \tag{26}$$

which implies that

$$y'(0) = \frac{4}{\mu}a. \tag{27}$$

After solving Eq. (23) for μ =1, 3, 5 and 10 using the built-in utility in *Mathematica* and taking the real root, we obtain the approximate values for a and $y'(0)$ as shown in Table 1 and Table 2, respectively. Note that n is the order of the iterative solution. Obviously, the calculated value for $y'(0)$ will converge to the exact value with increasing order of iterative solution. This result demonstrates that our approach has overcome the difficulty arising in using the variational iteration method alone.

Furthermore, the values of $y(x)$ computed using first-order and second-order iterative solutions u_1 with $a = 0.0008958443705$ and u_2 with $a = 0.0008958444636$, respectively, for $\mu = 10$ are listed in Table 3 and compared with the numerical solution obtained by invariant imbedding method [15]. It can be seen that the results obtained using both approximate solutions agree very well with the numerical solution with a relative error of less than 1.5×10^{-4}%. This excellent agreement shows that the first-order iterative approximate solution is good enough to provide highly accurate results for large values of μ where the problem is more difficult to solve.

5. Conclusions

An efficient algorithm based on the variational iteration method and variable transformation has been successfully applied to Troesch's problem. Only the second-order iterative solution is required to provide a highly accurate analytical solution as compared with those obtained by other analytical and numerical methods. Furthermore, for large values of μ in which case the problem is more difficult to solve, the simple first-order iterative analytical solution is accurate enough. It is expected that other inherently unstable two-point boundary value problems with hyperbolic nonlinearity can be successfully solved by this approach.

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