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A Modified Continuation Method for the Numerical Solution of Nonlinear Two-Point Boundary Value Problems by Shooting Techniques

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Summary. A modification of the well-known continuation (or homotopy) method for actual computation is worked out. Compared with the classical method, the modification seems to be a more reliable device for supplying useful initial data for shooting techniques. It is shown that computing time may be significantly reduced in the numerical solution of sensitive realistic two-point boundary value problems.

Introduction

In both pure and applied mathematics the principle of continuation has been employed with great success (for a detailed historical review cf. Ficken [13]; more recent investigations have been done e.g. by Rheinboldt [23], Avila [1], or Feilmeier [12], Wacker [28]). The present paper deals with a special version of the *continuation* or *homotopy method* designed to overcome serious difficulties in the numerical solution of nonlinear two-point boundary value problems (cf. Keller [15], p. 146, Bulirsch [3] and Stoer/Bulirsch [26], p. 177, for multiple shooting, or e.g. Roberts/Shipman [24], Chapter 7, for quasilinearization).

The general idea of using the classical continuation method in actual computation will be recalled (this application seems to date back to work of Lahaye [17] in 1934, cf. also Ortega/Rheinboldt [18], Chapter 7.5). Let

$$F(x)=0 \tag{0.1}$$

be an operator equation having a unique (unknown) solution x^*. Let $\bar{x}^*$ denote the (unique) given solution of a somehow "familiar" operator equation

$$\bar{F}(x)=0.$$

Then the numerical solution of problem (0.1) may be attacked by constructing a *homotopy* (or *one-parameter operator imbedding*)

$$H(x;\tau)=0 \quad \text{for } \tau\in[0,1] \tag{0.2}$$

with

$$x(\tau)\big|_{\tau=0}\equiv\bar{x}^*, \qquad x(\tau)\big|_{\tau=1}\equiv x^*.$$

Starting with $\tau_0:=0$ and $x(\tau_0)\equiv\bar{x}^*$, a *homotopy chain* of subproblems

$$H(x;\tau_i)=0 \quad i=1,2,\dots$$

is solved: taking $x(\tau_{i-1})$ as initial data for the i^{th} subproblem, the parameter τ_i is selected so that a given iterative method (such as Newton's method) will converge to the solution $x(\tau_i)$. In actual computation, some trial and error experiments may be necessary to establish the sequence $\{\tau_i\}$. If $\tau_N = 1$ is reached (within a tolerable number N of homotopy steps), the solution $x^* \equiv x(\tau_N)$ of the original problem (0.1) is obtained.

In real life applications, two essential difficulties may occur. First, the continuation method may fail to succeed, if the *homotopy path* $x(\tau)$ terminates at some $\tau^* < 1$. This case is usually avoided, if a "natural" parameter of an underlying scientific or engineering model is selected as homotopy parameter. (Otherwise, a different choice of the operator imbedding should be made.) Second, even if the existence of a unique solution $x(\tau)$—depending continuously on τ—is guaranteed for all $\tau \in [0, 1]$, the *homotopy stepsizes*

$$\Delta \tau_i := \tau_{i+1} - \tau_i \tag{0.3}$$

may decrease below a reasonable bound. As a consequence, computing time may increase beyond any level adequate to the original problem (0.1).

It is the purpose of this paper to suggest a modification of the classical continuation method that may often help to overcome the latter difficulty. In § 1, the algorithm is derived on the basis of some elementary theoretical considerations. In § 2, the performance of the classical and the modified continuation method is compared by solving realistic test problems using a standard multiple shooting algorithm [5]. As it turns out, the proposed modification speeds up the continuation method significantly.

1. Derivation of the Algorithm

It is well-known that the singularities of the solutions of nonlinear ordinary differential equations depend on the initial values (*movable singularities*). For this reason, a careful choice of the initial guess of the solution data is required in shooting techniques. This may be supplied by means of continuation methods.

In general, the homotopy parameter τ may appear in the system of n ordinary differential equations

$$y' = f(x, y; \tau) \qquad x \in [a, b], \ \tau \in [0, 1] \tag{1.1.a}$$

and the two-point boundary conditions

$$r(y(a), y(b); \tau) = 0. \tag{1.1.b}$$

Recall that in multiple shooting techniques the interval $[a, b]$ is suitably subdivided, say

$$a = x_1 < x_2 < \cdots < x_{m-1} < x_m = b \qquad (m \text{ nodes}).$$

Upon estimating $(m-1)$ initial values

$$s_j^0(\tau) \approx y(x_j; \tau) \quad j = 1, \ldots, m-1,$$

$(m-1)$ initial value problems are solved. Thus $(m-1)$ solutions

$$y(x; x_j, s_j; \tau) \quad \text{for} \quad x \in [x_j, x_{j+1}], \qquad j = 1, \ldots, m-1$$

are obtained. The n-vectors s_j have to be determined so that the following $n(m-1)$ conditions hold:

(1.2.a) continuity conditions (for $m>2$)

$$F_j(s_j, s_{j+1}; \tau) := y(x_{j+1}; x_j, s_j; \tau) - s_{j+1} = 0 \qquad j=1, \ldots, m-2,$$

(1.2.b) boundary conditions:

$$F_{m-1}(s_1, s_{m-1}; \tau) := r(s_1, y(x_m; x_{m-1}, s_{m-1}; \tau); \tau) = 0.$$

The conditions (1.2) define a system of $n(m-1)$ nonlinear algebraic equations, say

$$(1.3) \qquad F(s; \tau) := \begin{pmatrix} F_1(s_1, s_2; \tau) \\ \vdots \\ F_{m-2}(s_{m-2}, s_{m-1}; \tau) \\ F_{m-1}(s_1, s_{m-1}; \tau) \end{pmatrix} = 0 \quad \text{with} \quad s := \begin{pmatrix} s_1 \\ \vdots \\ s_{m-1} \end{pmatrix}.$$

In the standard multiple shooting algorithm [5] as used here, this system is solved numerically by the *modified Newton method*. With $J(s; \tau)$ denoting the Jacobian matrix of the mapping $F(s; \tau)$ (assumed to be nonsingular here) iterates s^k (for $k=1, 2, \ldots$) are obtained from:

$$(1.4) \qquad \begin{aligned} s^{k+1} &:= s^k + \lambda_k \Delta s^k, \qquad 0 < \lambda_k \leqq 1 \\ \Delta s^k &:= -J(s^k; \tau^i)^{-1} F(s^k; \tau^i) \end{aligned}$$

where τ^i is fixed, until $s(\tau^i)$ is computed to sufficient accuracy. Then τ^i is replaced by some value τ^{i+1} selected ad hoc.

Instead of varying s and τ separately, one might want to construct a *simultaneous iteration in both s and* τ. This can be done defining an *extended* two-point boundary value problem:

$$(1.5.a) \qquad \begin{aligned} y' &= f(x, y; \tau) \qquad x \in [a, b] \\ \tau' &= 0, \end{aligned}$$

$$(1.5.b) \qquad \begin{aligned} r(y(a), y(b); \tau(b)) &= 0 \\ h(\tau(b)) &= 0 \end{aligned}$$

where $h(\tau)$ is a C^2-function satisfying

(1.6) (a) $h(1)=0$,

(b) $h'(\tau) \neq 0$ for all $\tau \in [0, 1]$,

(c) $h(0) > 0$ and $h(\tau)$ convex for $\tau \in [0, 1]$

or

$h(0) < 0$ and $h(\tau)$ concave for $\tau \in [0, 1]$.

The following elementary result is important for actual computation:

Replacing the boundary value problem (1.1) *by the extended boundary value problem* (1.5) *means just replacing the modified Newton iteration* (1.4) *by the iteration*

$$(1.7.a) \qquad \begin{aligned} s^{k+1} &:= s^k + \hat{\lambda}_k(\Delta s^k + \Delta \tau^k \widehat{\Delta s}^k), \qquad 0 < \hat{\lambda}_k \leqq 1 \\ \tau^{k+1} &:= \tau^k + \hat{\lambda}_k \Delta \tau^k \end{aligned}$$

where

$$\text{(1.7.b)} \qquad \begin{aligned} \widehat{\Delta s}^k &:= -J(s^k;\tau^k)^{-1} F_\tau(s^k;\tau^k) \\ \Delta\tau^k &:= -\frac{h(\tau^k)}{h'(\tau^k)}. \end{aligned}$$

Moreover,

$$\text{(1.7.c)} \qquad \tau^{k+1} \leqq 1 \quad \forall k.$$

Proof. Upon replacing (1.1) by (1.5), the nonlinear system (1.3) is extended by the following additional conditions:

(1.8.a) continuity condition

$$\tau_{j+1} - \tau_j = 0 \qquad j = 1, \ldots, m-1$$

(1.8.b) boundary condition

$$h(\tau_m) = 0$$

where τ_j denotes the value of the homotopy parameter at the node x_j. Obviously, (1.8.a) and the index m in (1.8.b) can be dropped. So the following nonlinear system remains to be solved:

$$\text{(1.9)} \qquad \begin{aligned} F(s;\tau) &= 0 \\ h(\tau) &= 0. \end{aligned}$$

From (1.9), the result (1.7.a, b) is readily obtained. For the special case of linear $h(\tau)$ compare e.g. Leder [16]. With the assumptions (1.6) on $h(\tau)$, one verifies

$$\Delta\tau^k \leqq 1 - \tau^k \Rightarrow \tau^{k+1} = \tau^k + \hat\lambda_k \Delta\tau^k \leqq 1. \quad \diamond$$

Remark. Same result, if $\tau(b)$ in (1.5.b) is replaced by $\tau(a)$.

Further insight into the multiple shooting algorithm for (1.1) and (1.5) may be gained from the following

Transition table (notation close to [26])

$$(1.1) \rightarrow (1.5)$$

$$A = \frac{\partial r}{\partial y_a} \rightarrow \bar A := \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$B = \frac{\partial r}{\partial y_b} \rightarrow \bar B := \left(\begin{array}{c|c} B & r_\tau \\ \hline 0 & h'(\tau) \end{array}\right)$$

$$G_j := \frac{\partial y(x_{j+1}; x_j, s_j; \tau)}{\partial s_j} \rightarrow \bar G_j := \left(\begin{array}{c|c} G_j & F_{j,\tau} \\ \hline 0 & 1 \end{array}\right)$$

functional matrix

$$E := A + BG_{m-1} \cdots G_1 \rightarrow \bar E := \bar A + \bar B \bar G_{m-1} \cdots \bar G_1 = \left(\begin{array}{c|c} E & w \\ \hline 0 & h'(\tau) \end{array}\right)$$

with

$$w := r_\tau + BG_{m-1} F_{m-2,\tau} + \cdots + BG_{m-1} \cdots G_2 F_{1,\tau}.$$

With the definitions above one readily obtains

$$\text{(1.10)} \qquad \det(\bar E) = h'(\tau) \det(E).$$

Since $h'(\tau) \neq 0$ from (1.6.b), the functional matrix $\bar{E}$ for the modified continuation method is nonsingular, if and only if the functional matrix E for the classical continuation method is nonsingular.

Remark 1. It is well-known that the determinant is not an appropriate quantity for numerical analysis. Rather one uses (cf. [5]) an estimate of the (scaled) *condition number* together with a special norm representing the *sensitivity* (scaled):

$$\|E\| := \max_k \sqrt{\sum_i \left(\frac{\partial r_i}{\partial s_{1k}}\right)^2}.$$

Note that in the solution point both of these numbers do not depend on the shooting method employed, but are characteristic of the *problem*. (They depend, however, on the *shooting direction* as illustrated in example III of § 2.)

Remark 2. It was shown in [26] that quasi-linearization can be obtained from multiple shooting (in connection with Newton's method) by a limiting process

$$m \to \infty.$$

Hence, the results (1.7) and (1.10) apply—mutatis mutandis—to quasilinearization as well.

It is clear from (1.7) that—depending on the choice of $h(\tau)$—*any* homotopy stepsize $\Delta\tau^k$ may be induced. So one cannot expect to have a theoretically satisfying homotopy stepsize control. In applications, however, it may be worth trying the extension (1.5), if a parameter $\bar{\tau}$ (to be selected ad hoc) is introduced. Then $h(\tau)$ is chosen to be linear without loss of generality:

$$h(\tau) := \tau - \bar{\tau} \qquad \bar{\tau} \in]0, 1]. \tag{1.11}$$

As a consequence, if $\hat{\lambda}_q = 1$ for some iterate s^q, then iteration (1.7) reduces to (1.4), i.e.

$$\Delta\tau^{q+1} = \Delta\tau^{q+2} = \cdots = 0, \qquad \tau^{q+1} = \tau^{q+2} = \cdots = \bar{\tau}.$$

Hence, in order to generate at least one parameter iterate

$$\tau^k \neq \bar{\tau} \qquad (k > 0),$$

the modified Newton method should be started with some relaxation factor

$$\hat{\lambda}_0 < 1. \tag{1.12}$$

This is assured in the standard multiple shooting algorithm [5] used here; the strategy for determining further values of $\hat{\lambda}_k$ is found in [8] (compare also Table 4 in § 2).

A simple *geometric interpretation of the first iterate* in (1.7) is given in Figure 3. It is assumed that the mapping $F(s;\tau)$ in (1.3) defines a (unique) homotopy path $s(\tau)$ depending continuously on τ for all $\tau \in [0, 1]$. Then $s(\tau)$ satisfies (cf. Davidenko [6])

$$\dot{s}(\tau) = -J(s;\tau)^{-1} F_\tau(s;\tau). \tag{1.13}$$

Let s^0 denote the initial data used to start the iteration (1.7) and let

$$s^0 \equiv s(\tau^0).$$

Then

$$\Delta s^0 = 0, \quad \widehat{\Delta s}^0 = \dot{s}(\tau^0).$$

So the first Newton correction in (1.7) can be interpreted as the *tangent direction of the homotopy path* $s(\tau)$ in τ^0.

For the solution of sensitive problems, the following property is important: the modified continuation method starts at $(s(\tau^0), \tau^0)$ whereas the classical continuation method starts at $(s(\tau^0), \bar{\tau})$ which may be "far away" from the homotopy path (compare example IV in § 2, Figure 4 and Figure 5).

The technique of extending (1.1) to (1.5) applies just as well for an imbedding with respect to *more than one* parameter. Let τ, σ denote two imbedding parameters (as realized in example II in § 2), then (1.7) is replaced by

$$\begin{aligned} s^{k+1} &:= s^k + \hat{\lambda}_k (\Delta s^k + \Delta \tau^k \widehat{\Delta s}_\tau^k + \Delta \sigma^k \widehat{\Delta s}_\sigma^k) \\ \tau^{k+1} &:= \tau^k + \hat{\lambda}_k \Delta \tau^k \\ \sigma^{k+1} &:= \sigma^k + \hat{\lambda}_k \Delta \sigma^k \end{aligned} \tag{1.14}$$

where (dropping the index k)

$$\begin{aligned} \Delta s &:= -J(s; \tau, \sigma)^{-1} F(s; \tau, \sigma) \\ \widehat{\Delta s}_\tau &:= -J(s; \tau, \sigma)^{-1} F_\tau(s; \tau, \sigma) \\ \widehat{\Delta s}_\sigma &:= -J(s; \tau, \sigma)^{-1} F_\sigma(s; \tau, \sigma). \end{aligned}$$

2. Numerical Results

The following experiments were run on the TR 440 of the Leibniz-Rechenzentrum der Bayerischen Akademie der Wissenschaften. The computations were performed in FORTRAN single precision with a 38 bit mantissa (examples I-III) and FORTRAN double precision with an 84 bit mantissa (example IV). The two-point boundary value problems were solved by the standard version of a multiple shooting algorithm [5]. The classical continuation method (*old*) and the modified continuation method (*new*) were compared using equivalent initial data and equal prescribed relative precision of the solution data. In the examples tested, the number of initial value problems (*trajectories*) solved in order to find the solution of the boundary value problem is a measure of the total amount of computation (counting each approximation of the Jacobian matrix by numerical differentiation for n trajectories).

Example I. (due to Troesch [27])

The two-point boundary value problem is given by

$$\begin{aligned} y'' &= \tau \sinh(\tau y), \qquad \tau > 0 \\ y(0) &= 0, \qquad y(1) = 1. \end{aligned} \tag{2.1}$$

The underlying physical model describes the confinement of a plasma column by radiative pressure. The mathematical problem had been discussed by several authors (cf. e.g. Scott [25], p. 74/75, where further references are listed, too). The exact solution of the problem in terms of Jacobian elliptic functions can be found in [26], p. 169. In case one should want to solve this problem directly, it is recommended to employ the computationally economic, rapidly convergent method of the arithmetic-geometric mean (cf. Bulirsch [2]). The widespread interest in this example as a test problem comes from the following fact: the solution of the associated initial value problem has a *singularity* at some point $x_\tau > 1$ with x_τ approaching 1^+ as τ increases. So one might expect serious numerical difficulties for multiple shooting methods.

The problem was solved numerically for successively increasing values of τ (homotopy parameter). In Table 1 the results obtained are compared, using the *old* and *new* continuation method (additional trivial differential equation and linear boundary condition for τ).

Table 1. Comparison of *old* and *new* continuation method (fixed set of 13 nodes)

$\tau^0 \to \tau$	number of trajectories		sensitivity at $\bar{\tau}$ (as defined in § 1)
	old	*new*	
$2 \to 3$	7	9	$0.4\,E\,1$
$6 \to 7$	10	10	$0.5\,E\,2$
$7 \to 8$	fail[a]	9	$0.8\,E\,2$
$9 \to 10$	—	10	$0.2\,E\,3$
$15 \to 16$	—	19	$0.2\,E\,9$
$16 \to 17$	—	fail[a]	—

[a] fail means: the standard multiple shooting algorithm [5] suggests a new node to be inserted. (No attempts to optimize the choice of nodes or to maximize $\bar{\tau}$ have beeen made. One might proceed beyond that point as far as permitted by the relative machine precision.)

The situation as described in Table 1 was found to be typical for a wide class of examples: in less sensitive problems the *old* method is more economic, whereas in sensitive problems the *new* method is significantly superior to the *old* one.

Example II. Thin Shallow Spherical Shell.

In this example the elastic stability of thin shallow spherical shells subject to uniform pressure is studied. It has been investigated by Reissner [20], Weinitschke [29]. The mathematical formulation leads to a so-called *singular* two-point boundary value problem (cf. Stoer/Bulirsch [26], p. 177, and [22]).

$$\text{(2.2.a)} \qquad \begin{aligned} f'' &= -\mu^2 g + f g - 3\frac{f'}{x} - 2\gamma \\ g'' &= \mu^2 f - \frac{1}{2} f^2 - 3\frac{g'}{x}, \end{aligned}$$

$$
(2.2.b)\quad \begin{array}{ll} f'(0)=g'(0)=0 & \text{(regularity conditions)} \\ f(1)=0,\ g'(1)+(1-\nu)\,g(1)=0 & \text{(clamped edge conditions).} \end{array}
$$

Notation (for further notation see [29]):

x	normalized polar angle,
$f(x)$	normalized angular deflection,
$g(x)$	normalized stress,
ν	Poisson's ratio (here $\nu=\frac{1}{3}$).

The *load parameter* γ and the *parameter* μ^2 (characterizing the *geometry of the shell*) are chosen as natural homotopy parameters for actual computation. A peculiarity of the problem is that the homotopy path with respect to γ (for fixed μ^2) terminates at some limiting value γ_B (*buckling load*, depending on μ^2) which is unknown a priori. In Table 2 the two algorithms *old* (classical continuation method) and *new* (modified continuation method with additional trivial differential equation and linear boundary condition for γ) are compared.

Table 2. One-parameter imbedding with respect to γ (for fixed μ^2)

$\gamma_0 \to \bar{\gamma}$	number of trajectories		μ^2	γ_B
	old	*new*		(due to [21])
1000 → 11000	24	20	121	≧11880
1000 → 16000[a]	31[a]	see result		
16000 → 19900	23	below	144	≧19960
1000 → 19900	fail	27		

[a] homotopy stepsize approximately maximized

Table 3 shows a comparison of the *old* and *new* algorithm, if a two-parameter imbedding with respect to γ *and* μ^2 is constructed (two additional trivial differential equations and linear boundary conditions).

Table 3. Two-parameter imbedding with respect to γ and μ^2

$(\gamma_0, \mu_0^2) \to (\bar{\gamma}, \bar{\mu}^2)$	number of trajectories		$\bar{\gamma}_B(\bar{\mu}^2)$
	old	*new*	(due to [21])
(1000,144) → (13000,169)	24	29	
→ (14000,169)	fail	31	≧27630
→ (27000,169)	fail	37	
(1000,144) → (14000,196)	35	29	
→ (15000,196)	fail	29	≧37650
→ (37000,196)	fail	58	
(1000,144) → (40000,225)	fail	fail	≧50040

As an example of the performance of the new algorithm, the iterates γ_k and μ_k^2 are listed in Table 4 together with the relaxation factors $\hat{\lambda}_k$ as suggested by the strategy due to [8].

Table 4. Iterates of imbedding parameters (γ, μ^2) together with relaxation factors $\hat{\lambda}_k$ for $(1000, 144) \to (27000, 169)$

k	0	1	2	3	4
$\hat{\lambda}_k$	0.01	0.484	0.234	1.00	...
γ_k	1000	1260,...	13710,...	16820,...	27000
μ_k^2	144	144,25..	156,2...	159,2...	169

In this example, augmenting the number of differential equations and boundary conditions (from 4 to 6) did not change the order of magnitude of the *condition number* of the iteration matrix (varying here between $1.E\,04$ and $1.E\,07$). The norm of the first Newton correction, however, was considerably reduced (scaled norm: *old* $\approx 1.E\,04$, *new* $\approx 1.E\,02$, compare Fig. 3). A significant expansion of the domain of convergence is observed.

Example III (due to Holt [14])

Consider the system of 5 differential equations ($n := -0.1$, $s := 0.2$)

$$
\begin{aligned}
y_1' &= y_2\\
y_2' &= y_3\\
y_3' &= -0.5\,(3-n)\,y_1 y_3 - n\,y_2^2 + 1 - y_4^2 + s\,y_2 \qquad (2.3.\mathrm{a})\\
y_4' &= y_5\\
y_5' &= -0.5\,(3-n)\,y_1 y_5 - (n-1)\,y_2 y_4 + s(y_4 - 1)
\end{aligned}
$$

with 5 boundary conditions

$$
(2.3.\mathrm{b}) \qquad y_1(0) = y_2(0) = y_4(0) = 0, \quad y_2(\tau) = 0, \quad y_4(\tau) = 1 \quad (\tau > 0).
$$

The original problem reported in [14] is a so-called *asymptotic* boundary value problem, i.e. with final time $\tau = \infty$ in (2.3.b). Instead, the problem is solved for finite τ—with τ being as large as possible for the algorithm employed. In [14], Holt claimed that conventional shooting techniques (available in 1964) failed to solve his problem; that is why he recommended the alternative use of difference methods. In [24], Roberts/Shipman gave an extensive discussion and a solution of the problem by means of shooting techniques in connection with a classical continuation method with respect to τ. Starting with initial data for $\tau = 11.3$ (the largest value they had obtained by the Goodman/Lance method of adjoints), they proceeded to solve the problem in several steps until the utmost value $\tau = 13.3$ (obtained by means of quasi-linearization).

In Figure 1, the results of the *old* and *new* continuation method are compared (using the standard multiple shooting algorithm [5], shooting in *forward* direction). In order to give an impression of the domains of convergence of the two different

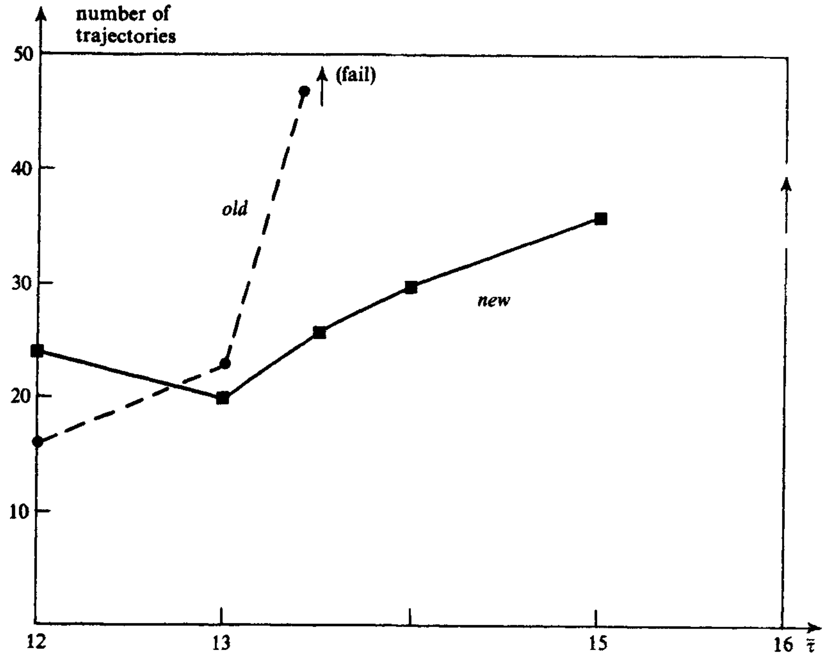

Fig. 1. Amount of computation for the *old* and *new* continuation method (one *single* step $11.3 \to \bar{\tau}$, no attempts to maximize $\bar{\tau}$ by several homotopy steps). Standard algorithm [5], shooting in *forward* direction.

methods, the amount of computation for one *single* homotopy step

$$\tau^0 = 11.3 \to \bar{\tau}$$

is drafted versus $\bar{\tau}$. Like in the examples studied before, the new method exhibited an expanded domain of convergence. The problem, however, turned out to be rather ill-conditioned: at $\bar{\tau}=15$ one obtains $cond_2(E) \approx 10^{12}$ (from *new*). As a consequence, the single precision computations in forward direction were terminated at this point.

A totally different quantitative behavior of both of the continuation methods was found when shooting in *backward* direction (see Fig. 2); in this direction the problem appeared to be comparatively well-conditioned ($cond_2(E) \approx 10^5$ at $\bar{\tau}=40$). Note the square in Figure 2 representing the scale of Figure 1 for comparison. It is shown that in this case even the classical continuation method proceeded successfully far beyond any value of $\bar{\tau}$ reached before. Nevertheless, the modified continuation method turns out be considerably superior.

In this example, the condition number of the matrix E (in forward direction) was significantly increased by adding the trivial differential equation; for the backward direction the associated effect was negligible. The utmost value of the homotopy parameter obtained in *several* steps by the method proposed here was $\bar{\tau}=132$. (As indicated by the condition number and the sensitivity, *single* precision computation had to be terminated at that value.)

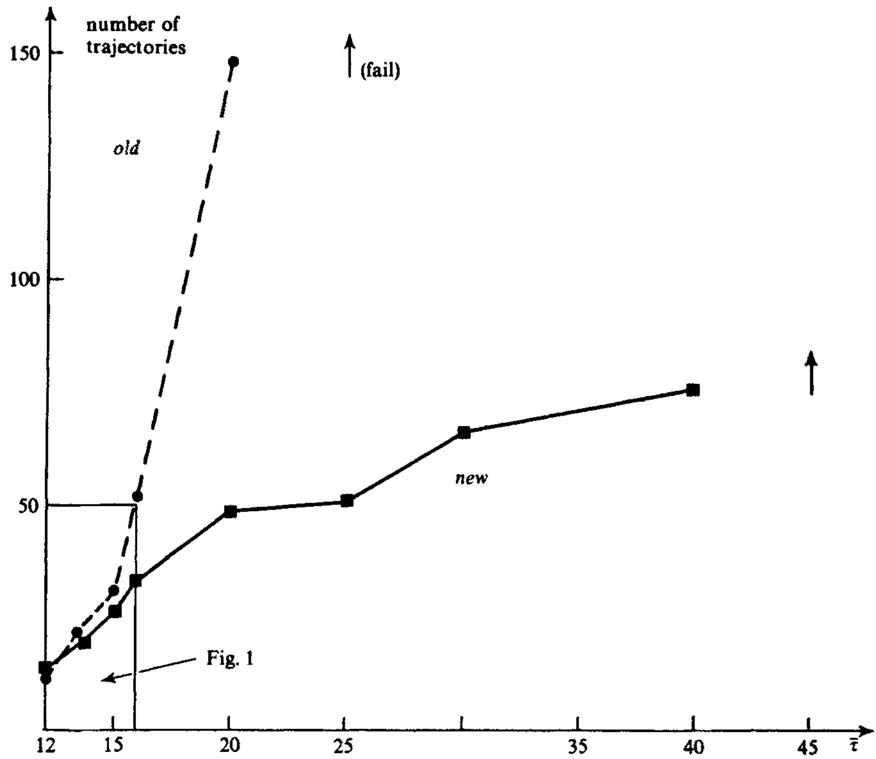

Fig. 2. Amount of computation for the *old* and *new* continuation method (one *single* step $11.3 \to \bar{\tau}$). Standard algorithm [5], shooting in *backward* direction. (utmost value obtained by single precision computations in *several* steps: $\bar{\tau} = 132.0$)

Example IV. Optimal descent of the second stage of a Space Shuttle subject to reradiative heating constraints.

The underlying physical model of this optimal control problem is due to Dickmanns [10]. The quantity to be maximized is the so-called *lateral range* of the second stage of a Space Shuttle returning to Earth. As described in [11], this is important from a technical point of view, since increasing lateral range will allow increased return frequencies of the vehicle from a given orbit to given landing sites. The numerical solution of the mathematical problem had been attacked in [19] by means of the *classical* continuation method using multiple shooting techniques especially trimmed to handle *ill-conditioned* two-point boundary value problems (cf. [7]). Upon employing the *modified* continuation method, it was for the first time possible to obtain the complete solution of this problem.

An extensive presentation of the whole model would be beyond the scope of this paper; it can be found in [9]. Here the following details are given. Let $\Lambda(T)$ denote the *cross-range angle* to be maximized, where the *total flight time* T is free ($\dot{T} = 0$). For the *physical quantities* (*velocity* v, *heading angle* χ, *flight path angle* γ, *cross-range angle* Λ, *height* h, *down-range angle* Θ) the following differential equa-

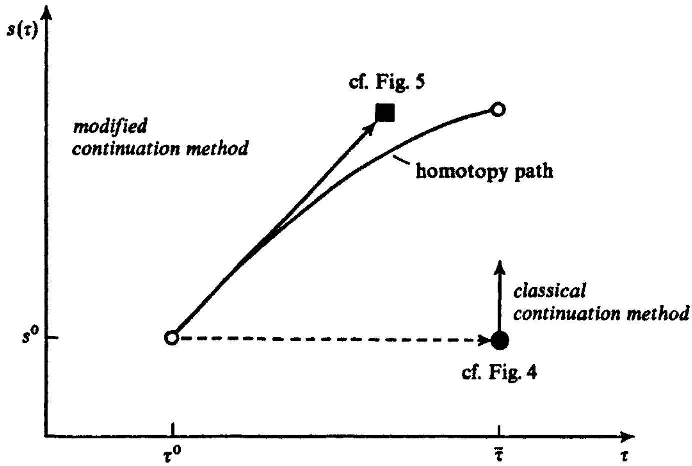

Fig. 3. Geometric interpretation of the first iterates of the classical (●) and the modified continuation method (■)

tions hold:

(2.4.a)
$$\begin{aligned}
\dot{v} &= -\left(c_{W_0}+k\cdot c_A^n\right)\frac{F\varrho_0}{2m}e^{-\beta h}v^2 - g_0\left(\frac{R}{R+h}\right)^2\sin\gamma\\
\dot{\chi} &= c_A\frac{F\varrho_0}{2m}e^{-\beta h}v\frac{\sin\mu}{\cos\gamma} - \frac{v}{R+h}\cos\gamma\cos\chi\tan\Lambda\\
\dot{\gamma} &= c_A\frac{F\varrho_0}{2m}e^{-\beta h}v\cos\mu - \left(\frac{g_0}{v}\left(\frac{R}{R+h}\right)^2 - \frac{v}{R+h}\right)\cos\gamma\\
\dot{\Lambda} &= \frac{v}{R+h}\cos\gamma\sin\chi\\
\dot{h} &= v\cdot\sin\gamma\\
\dot{\Theta} &= \frac{v}{R+h}\cdot\frac{\cos\gamma\cos\chi}{\cos\Lambda}
\end{aligned}$$

with control variables μ (*aerodynamic bank angle*) and c_A (*lift coefficient*). The following 9 boundary conditions are prescribed:

(2.4.b)
$$\begin{aligned}
&v(0)=7.85, \quad v(T)=1.116, \quad \chi(0)=0., \quad \gamma(0)=-1.25\frac{\pi}{180},\\
&\gamma(T)=-2.7\frac{\pi}{180}, \quad \Lambda(0)=0., \quad h(0)=95., \quad h(T)=30, \quad \Theta(0)=0.
\end{aligned}$$

For the lift coefficient c_A three inequality constraints are required, among which the following ***reradiative heating constraint*** is active at the solution point:

(2.4.c)
$$c_A \leqq c_{AH} := \sum_{i=1}^{5} B_i(h)\,H_i(h,v) + \Delta c_{AH}$$

where B_i, H_i are given functions characterising the vehicle. The parameter Δc_{AH} describing different levels of the ***maximum permitted skin temperature*** of the

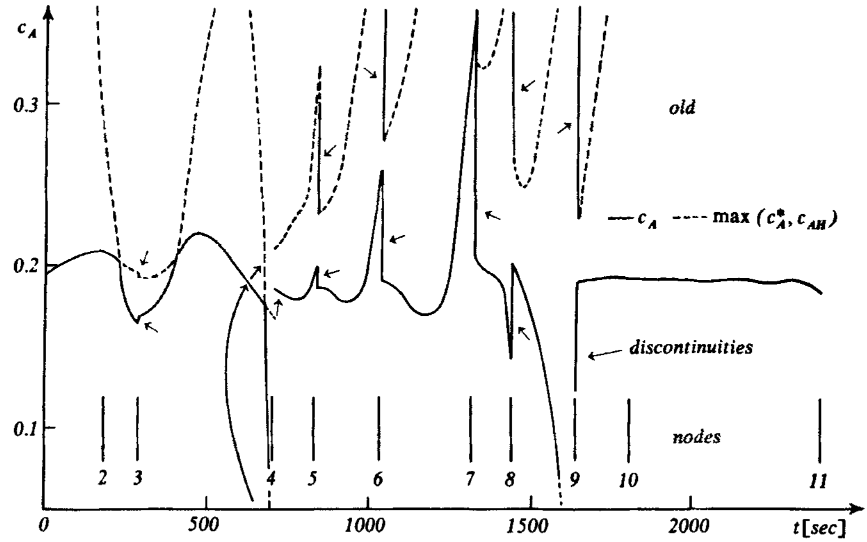

Fig. 4. Δc_{AH}: 0.0085 → 0.0, starting trajectory (similar to first iterate) of lift coefficient $c_A(t)$ (control variable) for *classical continuation method* (point ● in Fig. 3)

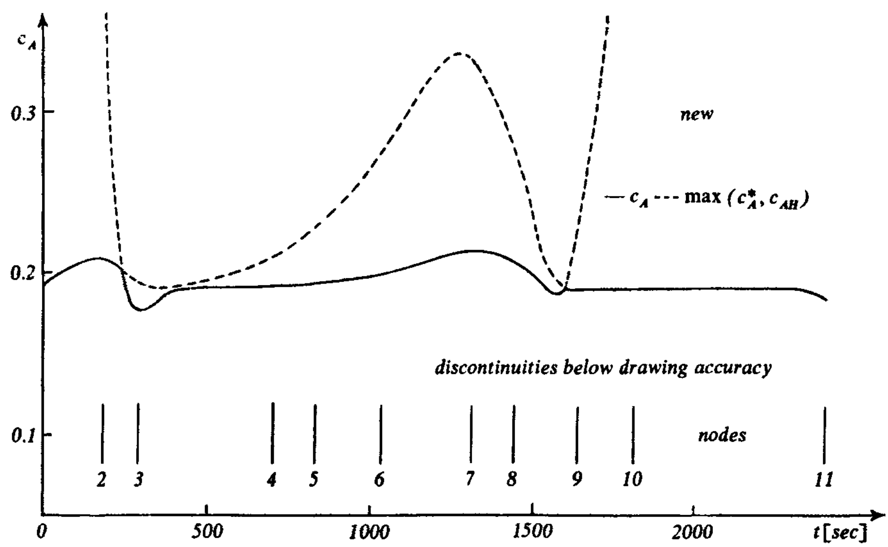

Fig. 5. Δc_{AH}: 0.0085 → 0.0, first iterate of $c_A(t)$ for *modified continuation method* (point ■ in Fig. 3)

Space Shuttle was selected as *homotopy parameter*. From the first and second variation one obtains 6 rather lengthy nonlinear differential equations for the *adjoint variables* λ_v, λ_λ, λ_γ, λ_Λ, λ_h, λ_Θ together with 4 *transversality conditions*. (cf. [7]) Moreover, one obtains expressions for the control variables μ, c_A in

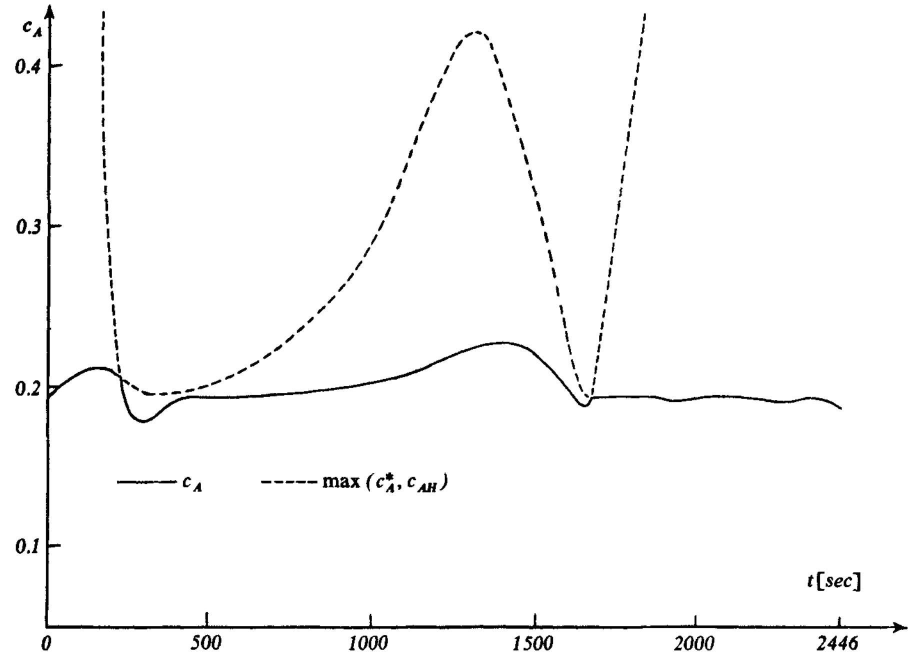

Fig. 6. Lift coefficient $c_A(t)$: solution for $\Delta c_{AH}=0$

terms of the adjoint variables, e.g.

$$c_A = c_A^* := \left(-\frac{w}{v\cdot\lambda_v\cdot n\cdot k}\right)^{\frac{1}{n-1}} \quad \text{if } c_A^* < c_{AH}$$

with

$$w := \left[\left(\frac{\lambda_x}{\cos\gamma}\right)^2 + \lambda_\gamma^2\right]^{1/2}.$$

This yields a two-point boundary value problem with 13 differential equations and 13 boundary conditions.

In order to give an impression of the numerical problems encountered in actual computation the estimates of the condition numbers of the iteration matrix E (cf. § 1) are listed in Table 5 for selected values of the homotopy parameter. In [19] the homotopy chain had been terminated at $\Delta c_{AH}=0.0080$, because the amount of the computation had increased beyond any tolerable level (for a detailed discussion of the technical results see [11]). In Table 6 the *old* and *new* continuation method are compared.

Table 5. Condition number of iteration matrix E for different values of Δc_{AH} (due to [19])

Δc_{AH}	0.072	0.054	0.025	0.008
$\text{cond}_2(E)$	0.5 E 10	0.9 E 12	3.0 E 16	1.0 E 19

(unconstrained problem for comparison: $\text{cond}_2(E) \approx 0.2\,E\,10$)

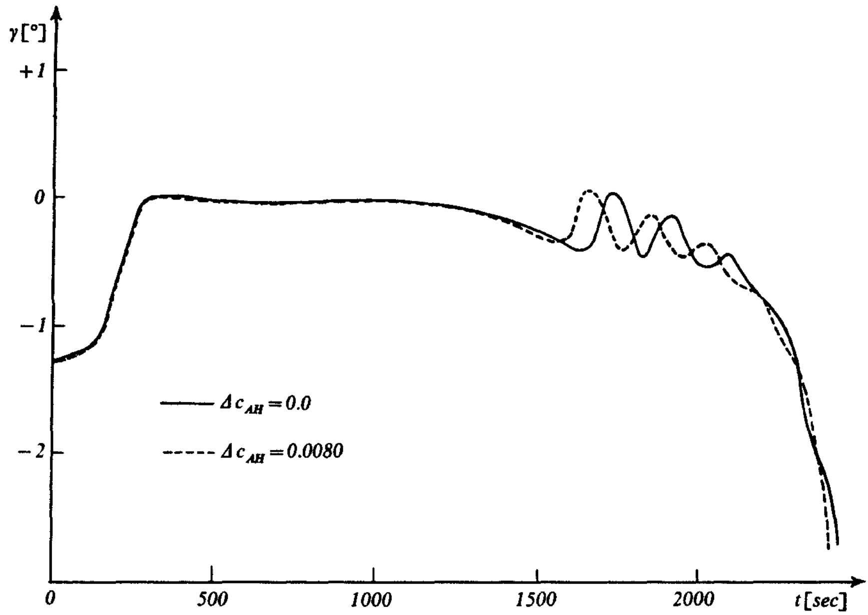

Fig. 7. Flight path angle $\gamma(t)$: solutions for $\Delta c_{AH}=0.0080$ and for $\Delta c_{AH}=0.0$

Table 6. One-parameter imbedding with respect to Δc_{AH}

$\Delta c_{AH}^{(0)} \to \overline{\Delta c}_{AH}$	number of trajectories[a]	
	old	*new*
0.0085 → 0.0080	114	48
0.0085 → 0.00	fail	89

[a] Remark: On an IBM 370/165 the computing time for one trajectory is about 2.8 sec

An interesting insight into the qualitative behavior of the two continuation methods may be gained from Figure 4 and Figure 5 where the graphs of the first iterates of the control variable $c_A(t)$ are drawn (compare also Fig. 3). In the starting trajectory of the classical method (Fig. 4) the structure of $c_A(t)$ is heavily disturbed. In the first iterate of the modified method (Fig. 5) the structure of $c_A(t)$ is preserved (compare the graph of $c_A(t)$ for $\Delta c_{AH}=0.$ in Fig. 6).

The effect of pushing down Δc_{AH} from 0.0080 to 0.0 can be studied in Figures 6 and 7: here the graphs of the most important quantities, i.e. the flight path angle $\gamma(t)$ and the lift coefficient c_A (control variable) are presented.

In this example, the new method brings a considerable expansion of the domain of convergence. Assuming a constant successful homotopy stepsize 0.0005 for the classical method (extrapolated from 0.0085 to 0.0), an acceleration factor of about $0.0085 \cdot 114/(0.0005 \cdot 89) \approx 22$ can be estimated from Table 6.

The new continuation method permits further advances in the computational investigation of certain connected problems of outstanding technical interest. For example, the problem could be solved even for *negative* values of Δc_{AH} (results of technical interest will be published elsewhere) and for a refined physical model including additional heating terms.

Conclusion

The modified continuation method as proposed here may bring significant progress in the numerical solution of nonlinear two-point boundary value problems by multiple shooting techniques. This was demonstrated by solving realistic examples. Main features observed are: a) the implementation of the method is enticingly simple, b) augmenting the order of the system does not cause additional numerical difficulties, c) the domain of convergence of the modified Newton method is expanded, d) computing time is significantly reduced. Although a general guarantee of accelerated convergence cannot be given, the method seems worth trying in cases where the classical method turns out to be inefficient.

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