Continuous and discontinuous numerical solutions to the Troesch problem

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ABSTRACT

Accurate continuous and discontinuous solutions to the Troesch problem are calculated for a wide range of values of the parameter n by means of a simple numerical method, which is based on an energy conservation law and which employs an inverse shooting procedure. A physical interpretation is given to the discontinuous solutions and the existence of an infinite number of discontinuous solutions, for any value of n, is demonstrated.

1. INTRODUCTION

We consider the sensitive two-point boundary value problem

$$\ddot{y} = n \sinh ny \tag{1.1}$$

$$y(0) = 0, \quad y(1) = 1 \tag{1.2}$$

This nonlinear problem, which is connected with the investigation of the confinement of a plasma column by radiation pressure, was originally formulated by Troesch [1] in 1960. Since then the problem has received a fair amount of attention in the literature with no less than seven papers [2-8] appearing on the subject during the period 1972-76. The inherent difficulty arises from the fact that the associated initial value problem has a pole at

$$t_\infty = (1/n)\ \ell n\, [8/\dot{y}(0)] \tag{1.3}$$

This results in the problem being very difficult to solve by general purpose shooting methods, the difficulty increasing with increasing n.

The first published solution to the problem is that by Tsuda, Ichida and Kiyono [9] which appeared in 1967. They applied Monte Carlo path integral calculations and obtained solutions for $n \leqslant 5$. Thereafter in 1972 Roberts and Shipman used a combination of methods to solve the problem for n = 5, 6 and 10. Then followed a number of papers [3-8] ranging from the sophisticated to the very straightforward in approach and yielding results varying in accuracy. In the last paper, again by Roberts and Shipman [8], a closed form solution to the problem is presented in terms of Jacobian elliptic functions. In particular they pointed out that the solution exhibits a form of branching or bifurcation and they calculated the associated discontinuous solutions to the problem.

In this paper we develop a very simple numerical method which is ideally suitable to the problem at hand and which gives results of the same accuracy as the best which have so far appeared in the literature. In addition the method yields with equal ease the so-called discontinuous solutions of Roberts and Shipman. A physical interpretation is given to the discontinuous solutions and in particular it is shown that for each value of n an infinite number of discontinuous solutions may exist as opposed to the finite number allowed by the method of solution of Roberts and Shipman. Before discussing our method it is necessary to briefly summarise the latest work of Roberts and Shipman so that a comparison may be made.

2. CLOSED FORM SOLUTION [8]

By integrating (1.1) Roberts and Shipman obtained the first integral

$$\dot{y}^2 = 2 \cosh ny + C \tag{2.1}$$

The initial condition (1.2) gives the value of the constant C

$$C = \dot{y}^2(0) - 2 \tag{2.2}$$

The implicit solution of (1.1), (1.2) is then given by

$$t = \int_0^y dv/(C + 2 \cosh nv)^{1/2} \tag{2.3}$$

By introducing the change of variable z = inv/2 the solution may be written in elliptic integral form as

$$u = \int_0^\varphi d\theta/(1 - m \sin^2 \theta)^{1/2} \tag{2.4}$$

where

$u = [in\,(2 + C)^{1/2}/2]t,$

$\varphi = iny/2,$

$m = 4/(2 + C).$

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By carrying through the necessary further arguments they obtained the final solution in terms of Jacobian elliptic functions. This is given by

$$y(t) = (2/n)\sinh^{-1}\{[\dot{y}(0)/2]\mathrm{sc}[nt\,|\,1-\dot{y}^2(0)/4]\} \quad (2.5)$$

This expression satisfies the differential equation (1.1) and also the initial condition $y(0) = 0$. The boundary condition $y(1) = 1$ will be satisfied if m satisfies the transcendental equation

$$\sinh(n/2)/(1-m)^{1/2} = \mathrm{sc}(n|m) \quad (2.6)$$

where

$$m = 1 - \dot{y}^2(0)/4 \quad (2.7)$$

The usefulness of the closed form solution (2.5) is thus dependent on finding a numerical solution to equation (2.6). If the lefthand and righthand sides of (2.6) are plotted as functions of m, the intersections of the two curves are the roots of (2.6). The roots in turn give the missing initial condition $\dot{y}(0)$ through the relationship (2.7). Roberts and Shipman found that as n increases, more than one root was possible as m goes through the allowable range [0, 1]. Their analysis showed that (2.6) has k + 1 roots for

$$(2k+1)\pi/2 \leqslant n \leqslant (2k+3)\pi/2.$$

For example if n = 10 three roots are obtained. Only the solution corresponding to the largest value of m remains finite for $0 \leqslant t \leqslant 1$, the remaining solutions, termed "discontinuous solutions", having one or more poles in the interval [0, 1].

The work of Roberts and Shipman therefore revealed the surprising existence of more than one solution for values of n larger than $3\pi/2$. They have the property that they meet the boundary conditions (1.2) and satisfy the differential equation (1.1) everywhere in the interval [0, 1] with the exception of a finite number of points. A second feature revealed by their work and which complicates the calculation of the solution from the closed form, is the fact that as n grows the roots of (2.6) have a point of accumulation at m = 1. It thus becomes increasingly more difficult to extract the root corresponding to the continuous solution from the others.

3. NUMERICAL METHOD

Our approach to the problem is a very simple one by comparison to the above, yet surprisingly efficient and accurate. The idea was obtained from the work by La Budde and Greenspan [10] in which, in order to obtain high stability when integrating the equations of motion of classical mechanics, they developed numerical methods which conserve certain constants of motion such as the total energy and total angular momentum. This method is known as "discrete mechanics".

Seen from this point of view the Troesch problem is a simple dynamical one. Equation (1.1) may be considered as the equation of motion of a particle of mass one describing a path along a straight line in a conservative force field. Integrating one obtains the energy conservation law

$$\frac{1}{2}v_{k+1}^2 - \cosh ny_{k+1} = \frac{1}{2}v_k^2 - \cosh ny_k \quad (3.1)$$

where v_k and y_k respectively denote the velocity and the displacement at time t_k. We now develop a numerical method for this problem which incorporates the conservation law (3.1).

The model is discretised by dividing the time into intervals (t_k, t_{k+1}) of which the lengths $\Delta t_k = t_{k+1} - t_k$ need not be uniform. We then postulate that over each interval the acceleration a_k^* be constant and be chosen such that the energy conservation law (3.1) be satisfied. Now since a_k^* is uniform over (t_k, t_{k+1}) we have the kinematic relationship

$$2a_k^*(y_{k+1} - y_k) = v_{k+1}^2 - v_k^2 \quad (3.2)$$

from which follows

$$a_k^* = \frac{1}{2}(v_{k+1}^2 - v_k^2)/(y_{k+1} - y_k) \quad (3.3)$$

Further it is clear from (1.1) that for any initial velocity $\dot{y}(0) > 0$, the particle will move away from the origin with increasing speed. We may therefore divide the y-axis into steps Δy_k and for each step, with v_{k+1} being known from the energy relationship (3.1), calculate the corresponding acceleration a_k^*. The length of the time step then follows from the kinematic expression

$$v_{k+1} = v_k + a_k^*\,\Delta t_k \quad (3.4)$$

$$\Delta t_k = (v_{k+1} - v_k)/a_k^* \quad (3.5)$$

Substituting (3.3) into (3.5) we finally obtain

$$\Delta t_k = 2(y_{k+1} - y_k)/(v_{k+1} + v_k) \quad (3.6)$$

For a given value of $\dot{y}(0)$ we may therefore calculate the total time T taken to cover the distance from 0 to 1. If we now consider $\dot{y}(0)$ as a parameter an inverse shooting procedure may readily be carried out to yield a value of $\dot{y}(0)$ which corresponds to T = 1.

A few technical points remain to be cleared up. Since we expect the particle to move away from the origin with increasing speed we cannot choose Δy to be uniform since this will result in initial time steps which are too large compared to later time steps. We avoid this by requiring the time steps not to be greater than a prescribed value, say δ, throughout, that is

$$\Delta t_k = 2(y_{k+1} - y_k)/(v_{k+1} + v_k) \leqslant \delta \quad (3.7)$$

which implies

$$\Delta y_k \leqslant (v_{k+1} + v_k)\,\delta/2 \quad (3.8)$$

Choosing

$$\Delta y_k = v_k \delta \quad (3.9)$$

satisfies our requirement since we have that $v_{k+1} > v_k$. On the other hand Δy_k calculated by criterion (3.9) may also become too large. To prevent this we test whether $v_k \delta$ is greater than δ and if this is so we choose $\Delta y_k = \delta$. In this way we assure

$$\max_{0 \leqslant t \leqslant T} [\Delta t, \Delta y] \leqslant \delta \tag{3.10}$$

The final step, of course, is chosen such that $y_{final} = 1$ with the corresponding $\Delta y \leqslant \delta$.
The algorithm as developed up to this point works extremely well. We may, however, obtain greater accuracy for a particular prescribed value of δ if we use (3.6) as a predictor in the form

$$\Delta t_k^p = 2\Delta y_k / (v_{k+1} + v_k) \tag{3.11}$$

and apply a corrector which is obtained by making use of the following Taylor expansions

$$y_{k+1} = y_k + y_k' \Delta t + y_k'' \frac{\Delta t^2}{2!} + y_k''' \frac{\Delta t^3}{3!} + y_k^{IV} \frac{\Delta t^4}{4!} + 0(\Delta t^5) \tag{3.12}$$

and

$$v_{k+1} = y_k' + y_k'' \Delta t + y_k''' \frac{\Delta t^2}{2!} + y_k^{IV} \frac{\Delta t^3}{3!} + y_k^{V} \frac{\Delta t^4}{4!} + 0(\Delta t^5) \tag{3.13}$$

Multiplying (3.13) by $\Delta t/4$ and subtracting from (3.12) we obtain after rearrangement

$$y_{k+1} - y_k = (3v_k + v_{k+1})\Delta t/4 + y_k'' \Delta t^2/4 + y_k''' \Delta t^3/24 + 0(\Delta t^5) \tag{3.14}$$

Using the fact thay $y'' = n \sinh ny$ and $y''' = n^2 (\cosh ny) y'$ we have, after solving for Δt in the first term on the right hand side and using (3.11) in the terms involving higher powers of Δt, the following corrector equation

$$\Delta t_k^c = [4(y_{k+1} - y_k) - n(\sinh ny_k)(\Delta t_k^p)^2 - n^2 (\cosh ny_k) v_k (\Delta t_k^p)^3/6]/(3v_k + v_{k+1}) \tag{3.15}$$

4. NUMERICAL RESULTS

The results obtained in the case of the continuous solution for different values of n are listed in Tables I(a) and (b). The calculations were performed on an IBM S/370 model 145 computer using double precision. The final value of δ used was 0.001. The best available results of other authors are also listed.
The present method yields reliable results over a wide range of values of n, the results comparing very well with those quoted for other authors. The only exception is for the rather extreme value of $n = 20$ for which only one other [5] independent result is available. The calculations proceeded economically with approximate initial solutions being obtained with larger values of δ. The value of $\delta = 0.001$ was only used during the final iterations when convergence occurred rapidly.
To illustrate the influence of δ on the solution we list in Table II the values of $\dot{y}(0)$ obtained for three different values of δ in the case $n = 10$.

Table I(a)

n	$\dot{y}(0)$ This method	Other authors	Reference
2	0.518 621 195	0.518 621 220 0	[8]
4	0.111 880 165 7	0.111 880 166 2	[8]
6	0.179 509 503(10^{-1})	0.179 509 500(10^{-1})	[8]
8	0.258 716 98(10^{-2})	0.258 716 95(10^{-2})	[8]
10	0.358 337 9(10^{-3})	0.358 337 8(10^{-3})	[8]
12	0.489 106 6(10^{-4})	0.489 1(10^{-4})	[7]
15	0.244 455 7(10^{-5})	0.244 450(10^{-5})	[7]
20	0.2004(10^{-7})	0.165(10^{-7})	[5]

Table I(b)

n	$\dot{y}(1)$ This method	Other authors	Reference
2	2.406 939 826	2.406 939 711	[8]
4	7.254 583 575	7.254 582 910	[8]
6	20.035 757 90	20.035 753 67	[8]
8	54.579 834 46	54.579 823 77	[8]
10	148.406 421 2	148.406 421 2	[7]
12	403.426 314 7	403.426 314 7	[7]
15	1808.041 861	1808.041 861	[7]
20	22026.465 7	—	—

Table II

$n = 10$ δ		$\dot{y}(0)$
0.1	(no corrector)	0.2281(10^{-3})
0.01	(no corrector)	0.3556(10^{-3})
0.001		0.358 3379(10^{-3})

5. DISCONTINUOUS SOLUTIONS

We return to the discontinuous solutions of Roberts and Shipman. What is the meaning of these solutions and can we calculate them numerically ? A physical interpretation may perhaps again be obtained by considering (1.1) as the equation of motion of a particle.

Consider the particle to be released from the origin in the positive y-direction with a given initial velocity (energy). Under the influence of the force law it will reach an infinite displacement and speed within a finite time, say t_∞. We say it has reached a pole. We now define the path to be continued at t_∞^+ with a displacement of minus infinity and with the same velocity (and energy) as at t_∞. A centrosymmetric image, about the point $(t_\infty, 0)$, of the previous path is now executed and after a period $2t_\infty$ the particle returns to the origin and starts repeating its path. Discontinuities therefore occur at $t_\infty, 3t_\infty, 5t_\infty, \ldots$
Corresponding to each number of discontinuities allowed in the interval [0, 1] there exists a solution which meets the boundary conditions. We have therefore an infinite sequence of solutions having respectively 0, 1, 2, 3, ... discontinuities in [0, 1] and associated with each solution we have a corresponding characteristic initial velocity and associated total energy.
If the above model, where a particle disappears at $+\infty$ and reappears at $-\infty$, seems somewhat artificial one may consider the motion to be that of a particle constrained to move clockwise on a circle of radius r (see fig. 1). The arc length y is given by

$$y = r\{\theta - 2k\pi\} \quad \text{if } 2k\pi \leq \theta < (2k+1)\pi,$$

and

$$y = r\{\theta - 2(k+1)\pi\} \quad \text{if } (2k+1)\pi \leq \theta < 2(k+1)\pi$$

where $k = 0, 1, 2, \ldots$

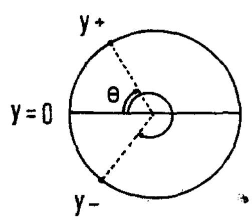

Fig. 1.

Here we have y switching from a positive to a negative value as the particle moves through odd multiples of π while the sign of $\dot{y}$ remains unchanged. The trajectories described above then simply correspond to the limiting case as the radius r becomes large.
The numerical method we have already described for obtaining the continuous solution may readily be adapted to give the discontinuous solutions as well. The only additional problem is the determination of t_∞ which computationally simply reduces to finding that point at which the velocity exceeds 10^{38}. The computed values of $\dot{y}(0)$ for n = 10 are shown in table III. The continuous solution, and the first three discontinuous solutions having respectively 1, 2 and 3 discontinuities in [0, 1] are depicted in fig. 2. Only three discontinuous solutions are given, although, as has been pointed out, an infinite many more are possible, of which any one may readily be calculated.

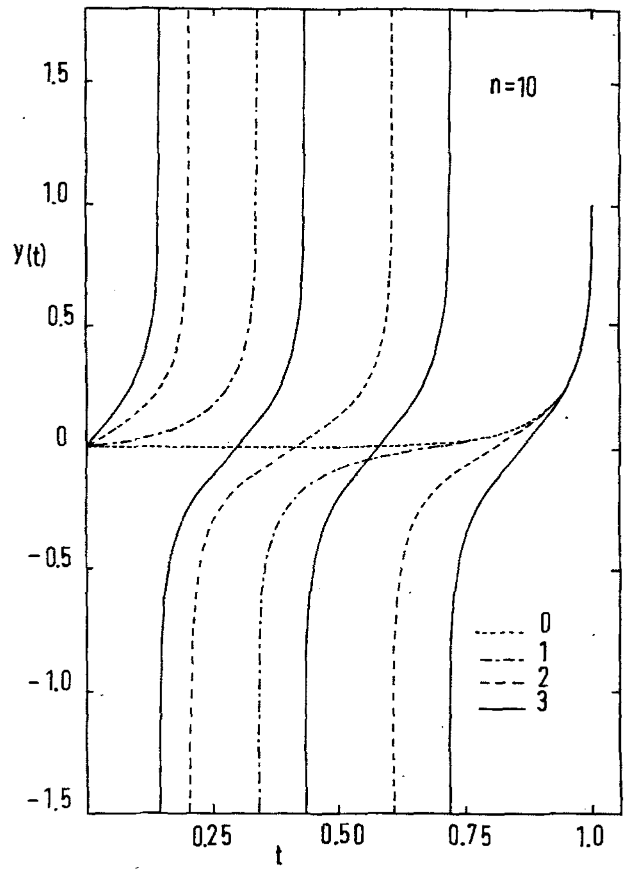

Fig. 2.

Table III

Number of discontinuities	$\dot{y}(0)$ for n = 10: This method	Roberts and Shipman [8]
0	0.358 337 9(10^{-3})	0.358 337 7(10^{-3})
1	0.287 565	0.287 587
2	1.189 75	1.189 88
3	2.401 06	—

6. CONCLUSION

It has been shown that by taking full advantage of an intimate knowledge of the problem at hand it is possible to develop a simple numerical method which yields accurate results with ease. Not only is high accuracy obtained in the case of the continuous solution over a wide range of n but the existence of an infinite number of discontinuous solutions, for any value of n, is also demonstrated. Roberts and Shipman allowed for only a finite number of solutions for a given n, for example for n = 10 only two discontinuous solutions are allowed. This happens because in their method of solution in terms of Jacobian elliptic functions the method of solution restricts the value of m to the interval [0, 1] and therefore by (2.7) places a corresponding limitation on the allowable values of $\dot{y}(0)$.

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