Polinòmi Taylor N variables

$f: \mathbb{R}^m \to \mathbb{R}$, punt $a = (a_1, \dots, a_m)$

$f(a) = f(a_1, \dots, a_m)$ "cap problema"

Grau 1: "$x-a$" tenim: $x_1 - a_1, x_2 - a_2, \dots, x_m - a_m$

$f'(a)$ el substituïm per $\nabla f(a)$

$f(a)(x-a)$ substituïm per $\underbrace{\nabla f(a)(x_1-a_1, \dots, x_m-a_m)}_{\text{Plà tangent en punt a una supe.}}$

Derivades d'ordre superior

$f: \mathbb{R}^m \to \mathbb{R}$ de classe C^1

$\frac{\partial f}{\partial x_i}: \mathbb{R}^m \to \mathbb{R}$ tmb són funcions de m variables i podem mirar si existeixen les seves derivades

Així obtenim derivades de segon grau $D_{ji} f = \frac{\partial^2 f}{\partial x_j \partial x_i} := \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$ "Parcial respecte x_j de la parcial respecte x_i"

Exemple: $f: \mathbb{R}^3 \to \mathbb{R}$ $f(x,y,z) = x^2 \cdot y \cdot \cos(xz) + xe^{yz} - 7z^3$

- $\frac{\partial f}{\partial x} = [x^2 \cdot a \cdot \cos(xb) + xe^{xa} - 7 \cdot b^3] = (2xy\cos(xz) + x^2 \cdot y \cdot (-\sin(xz) \cdot z) + e^{yz} = \underline{y(2x\cos(xz) - x^2yz\sin(xz))}$

- $\frac{\partial f}{\partial y} = [ya + xe^{y \cdot b}] = \underline{x^2 \cos(xz) + xe^{y \cdot z} \cdot z}$

- $\frac{\partial f}{\partial z} = [a \cdot \cos(bz) + xe^{zc} - 7z^d] = \underline{x^2 \cdot y \cdot (-\sin(xz)) \cdot x + xe^{zy} \cdot y - 21z^2}$

Ara tenim 3 noves funcions de 3 variables que pue tornar a fer derivades parcials per cadascuna

No surten EN AQUEST CAS 9 noves comb de derivades pq $\frac{\partial f^2}{\partial z \partial y} = \frac{\partial f^2}{\partial y \partial z}$

Es pot repetir procés per obtenir les m^K derivades parcials d'ordre K $\frac{\partial^K f}{\partial x_{i_1} \cdot \partial x_{i_2} \cdots x_{i_K}}$ on $i_1, i_2, \dots, i_K \in \{1, 2, \dots, m\}$

Una funció és de classe C^K si existeixen totes les derivades d'ordre K i són funcions contínues.

$f: \mathbb{R}^m \to \mathbb{R}$ és de classe $C^\infty(A)$ ($A \subseteq Dom(f)$) si existeixen les derivades parcials de tots els ordres i són contínues.

Teorema de Schwarz $f: \mathbb{R}^m \to \mathbb{R}$ $m \geq 2$.

$A \subseteq Dom(f)$ f de classe $C^2(A)$ $\boxed{\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a)}$ per tot $a \in A$

Matriu Hessiana "Matriu de les segones derivades"

Matriu Hessiana de f en $a \in \mathbb{R}^m$

$$H_f(a) = \begin{array}{cl} \begin{pmatrix} \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right)(a) & \cdots & \frac{\partial}{\partial x_m}\left(\frac{\partial f}{\partial x_m}\right)(a) \\ \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)(a) & \cdots & \frac{\partial}{\partial x_m}\left(\frac{\partial f}{\partial x_2}\right)(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_m}\right)(a) & \cdots & \frac{\partial}{\partial x_m}\left(\frac{\partial f}{\partial x_m}\right)(a) \end{pmatrix} & \begin{array}{l} \leftarrow x_1 \\ \leftarrow x_2 \\ \\ \leftarrow x_m \end{array} \end{array}$$

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$$P_{2,f,a}(x_1,\dots,x_n) = f(a) + \nabla f(a)\cdot(x_1-a_1,\dots,x_n-a_n) + \frac{1}{2!}(x_1-a_1,\dots,x_n-a_n)\,H_f(a)\begin{pmatrix} x_1-a_1 \\ \vdots \\ x_n-a_n\end{pmatrix}$$

$$\boxed{P_{2,f,a}(a_1,a_2) = f(a_1,a_2) + \underbrace{\nabla f(a)(x-a_1,\, y-a_2)}_{D_v f(a)} + \frac{1}{2!}\left[(x-a_1,\, y-a_2)\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a_1,a_2) & \frac{\partial^2 f}{\partial x\partial y}(a_1,a_2) \\ \frac{\partial^2 f}{\partial y\partial x}(a_1,a_2) & \frac{\partial^2 f}{\partial y^2}(a_1,a_2)\end{pmatrix}\begin{pmatrix} x-a_1 \\ y-a_2\end{pmatrix}\right]}$$

$$R_{2,f,a}(x,y) = \frac{1}{3!}\left(\frac{\partial f}{\partial x}(c_1,c_2)(x-a_1) + \frac{\partial f}{\partial y}(c_1,c_2)(y-a_2)\right)^{\text{"3"}}$$ on (c_1,c_2) està en el segment que uneix (x,y) i (a_1,a_2)

"3" ← NO és elevat al cub

$$\boxed{R_{k,f,a}(x_1,\dots,x_n) = \frac{1}{(k+1)!}\left[\frac{\partial f}{\partial x}(c)(x-a_1) + \dots + \frac{\partial f}{\partial x_n}(c)(x_n-a_n)\right]^{\text{"k+1"}}}$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \leadsto \frac{\partial^3 f}{\partial x^3}(x-a_1)^3 + 3\frac{\partial^3 f}{\partial x^2\partial y}(x-a_1)^2(y-a_2) + 3\frac{\partial^3 f}{\partial x\partial y^2}(x-a_1)(y-a_2)^2 + \frac{\partial^3 f}{\partial y^3}(y-a_2)^3$$

Extrems Relatius

$f:\mathbb{R}^2\to\mathbb{R}$,, $a=(a_1,\dots,a_n)\in \mathrm{Dom}(f)$,, f té un màxim relatiu (o local) en el punt a

si $f(x_1,\dots,x_n) \le f(a_1,\dots,a_n)$ per a tot entorn x "en una bola centre a".

$f(a)$

f té un mínim relatiu (o local) en el punt a si $\exists$ Bola centre a t.q. $f(x_1,\dots,x_n)\ge f(a_1,\dots,a_n)$

Pq ha de tindre derivada en entorn de a i un punt frontera no té tot.

$f:\mathbb{R}^n\to\mathbb{R}$,, punt (interior) Dom (f) ,, f de classe C^2 en entorn de a.

f té extrem relatiu (màx/mín. rel.) $\Rightarrow \nabla f(a) = (0,\dots,0)$ i a és punt crític

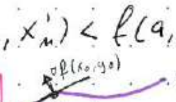
"Els candidats a extrems s'han de buscar entre punts crítics".

Punt sella: és un punt crític que no és màxim ni mínim relatiu

Ni max ni mín.

Té punts on $f(x_1,\dots,x_n) > f(a_1,\dots,a_n)$ i $f(x'_1,\dots,x'_n) < f(a_1,\dots,a_n)$

Recta tangent a la corba de nivell: $\boxed{\nabla f(x_0,y_0)(x-x_0,\, y-y_0) = 0}$



$z_0 = f(x,y)$ "La dir del gradient és sempre de creixement"

$\Rightarrow \nabla f(x_0,y_0)$ és perpendicular "a la corba".

$n=2$,, $f:\mathbb{R}^2\to\mathbb{R}$,, $P\in \mathrm{D\mathring{o}m}(f)$,, f de classe C^2 (entorn de P) ,, $P=(p_1,p_2)$,, $\nabla f(P) = 0$ ← és punt crític

$$H_{f,P} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(P) & \frac{\partial^2 f}{\partial x\partial y}(P) \\ \frac{\partial^2 f}{\partial y\partial x}(P) & \frac{\partial^2 f}{\partial y^2}(P)\end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ b & c\end{pmatrix}}_{\text{simètrica}} = H$$

H def ⊕ si té 2 val propis pos $\begin{pmatrix} + & \\ & + \end{pmatrix}$

H def ⊖ si té 2 val propis neg $\begin{pmatrix} - & \\ & - \end{pmatrix}$

H indef si té val pos i neg $\begin{pmatrix} + & \\ & - \end{pmatrix}$ o $\begin{pmatrix} - & \\ & + \end{pmatrix}$

Els valors propis són les arrels del polinomi $\boxed{x^2 - (a+c) + \det = (x-\lambda_1)(x-\lambda_2)}$

Sigui $\Delta = \det H_{f,P}$:
- → Si $\Delta < 0$ punt sella
- → Si $\Delta > 0$ → $a>0$ Mínim relatiu
  - → $a<0$ Màxim relatiu
- → Si $\Delta = 0$ s'ha de fer estudi local.

P és un punt crític

Camp de vetors gradients

En cada punt (x,y). Dibuixem el gradient $\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$ ↗ dir de creixements ⇒ podem "veure" màx i min.

Exemple d'estudi Local

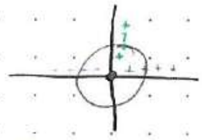
$f(x,y) = y^2 - x^3$,, Punts Crítics: $\left.\begin{array}{l}\frac{\partial f}{\partial x} = -3x^2 = 0 \\ \frac{\partial f}{\partial y} = 2y = 0\end{array}\right\} x = y = 0$ Llista Punts Crítics $= \{(0,0)\}$

Criteri del Hessià:

$$\left.\begin{array}{l}\cdot \frac{\partial^2 f}{\partial x^2} = -6x \\ \cdot \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0 \\ \cdot \frac{\partial^2 f}{\partial y^2} = 2\end{array}\right\} \Rightarrow H_{f(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

det = 0 NO decideix

· Es punt de sella?



▸ Mètode de les rectes:

$f(0,0) = 0$

▸ Si $y = 0 \Rightarrow f(x,0) = -x^3 \Rightarrow \begin{cases} f(x,0) < 0 \text{ si } x < 0 \\ f(x,0) \geq 0 \text{ si } x \geq 0 \end{cases}$

▸ Si $x = 0 \Rightarrow f(0,y) = y^2 \rightarrow f(0,y) \geq 0 \;\; \forall y$

Veiem que en l'entorn de $(0,0)$ hi ha "−" i "+" llavors $(0,0)$ és punt sella.

Exemple Extrem Relatiu

$f(x,y) = x^3 + y^3 - 9xy + 27$

1) Trobar tots els punts crítics

Solucions (x,y) de $\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases}$ ▽ Molta gent la def. està aquí.

$\cdot \frac{\partial f}{\partial x} = 3x^2 + 0 - 9y + 0 = 3x^2 - 9y = 0 \Rightarrow 3\left(\frac{y^2}{3}\right)^2 - 9y = 3 \cdot \frac{y^4}{9} - 9y = \frac{y^4}{3} - 9y = \frac{y^4 - 27y}{3} = 0 \Rightarrow$

$\cdot \frac{\partial f}{\partial y} = 3y^2 - 9x = 0 \Rightarrow 9x = 3y^2 \Rightarrow x = \frac{1}{3}y^2 \Rightarrow y^4 - 27y = 0 \Rightarrow y(y^3 - 27) = 0 \Rightarrow \begin{cases} \boxed{y = 0} \\ y^3 = 27 \Rightarrow \boxed{y = 3} \end{cases}$

Si $y = 0 \Rightarrow x = 0$,, $y = 3 \Rightarrow x = 3 \Rightarrow \boxed{(x,y) = (0,0)}$ i $\boxed{(x,y) = (3,3)}$

$P = (0,0)$, $Q = (3,3)$

2) Classificar els punts crítics.

Apliquem el critèri del Hessià.

$\cdot \frac{\partial f}{\partial x} = 3x^2 - 9y \dashrightarrow \cdot \frac{\partial^2 f}{\partial x^2} = 6x$

$\cdot \frac{\partial f}{\partial y} = 3y^2 - 9x \dashrightarrow \cdot \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -9$

$\cdot \frac{\partial^2 f}{\partial y^2} = 6y$

$H_f = \begin{pmatrix} 6x & -9 \\ -9 & 6y \end{pmatrix} \Rightarrow H_{f,(0,0)} \begin{pmatrix} 0 & -9 \\ -9 & 0 \end{pmatrix}$

$\Delta H_{f,(0,0)} = -81 < 0 \Rightarrow$ Punt de sella (P)

$\left.\begin{array}{l}\Delta H_{f(3,3)} = 648 > 0 \\ a_{11} = 27 > 0\end{array}\right\} \Rightarrow$ Mínim relatiu (Q).

Example Extrem Relatius

$f(x,y)=(x^2-2x+4y^2-8y)^2$ # Veiem que mai serà negatiu.
funció polinòmica de classe C^2 (i C^∞).

1) Punts crítics

$\frac{\partial f}{\partial x}=(2x-2)(2(x^2-2x+4y^2-8y)) \Rightarrow \overbrace{(x-1)}^{a}\overbrace{(x^2-2x+4y^2-8y)}^{b}$

$\frac{\partial f}{\partial y}=(8y-8)(2(x^2-2x+4y^2-8y)) \Rightarrow \underbrace{(y-1)}_{c}\underbrace{(x^2-2x+4y^2-8y)}_{d}$

1) $a=0$ i $c=0$,
2) $b=d=0$

Sense resoldre el que val b, d podem veure que és una el·lipse nomes positiva i que en el punt $(0,0)$ tindrà mínim pq al estar elevat $()^2$.

Altrament hem de resoldre $a=0$ i $c=0$ $\Rightarrow \begin{cases} x-1=0 \\ y-1=0 \end{cases} \Rightarrow P=(1,1)$

$\frac{\partial^2 f}{\partial x^2}=12x^2+16y^2-24x+32y$

$\frac{\partial^2 f}{\partial y^2}=16x^2+192y^2-32x-384y+128$

$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=32(y-1)(x-1)$

$H_{f,(1,1)}=\begin{pmatrix}-28 & 0\\ 0 & -80\end{pmatrix}$ $\Delta H_{f,(1,1)}=(-28)(-80)>0$
$a_{11}=-28<0$

El punt $P=(1,1)$, pel criteri del Hessià és màxim.

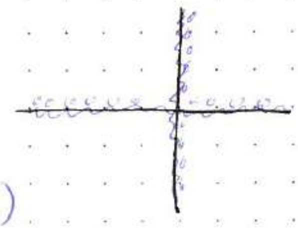
Example Extrem Relatius

$f(x,y)=x^2y^2(1-x-y)$

1) Punts crítics

$\frac{\partial f}{\partial x}=y^2(2x(1-x-y)+x^2(-1))=y^2(2x-2x^2-2xy-x^2)=\underline{-xy^2(3x+2y-2)=0}$

$\frac{\partial f}{\partial y}=x^2(2y(1-x-y)+y^2(-1))=\underline{-yx^2(3y+2x-2)=0}$



$\begin{rcases} -xy^2(3x+2y-2)=0 \\ -yx^2(3y+2x-2)=0 \end{rcases}$ $x=0$, $y=0$

$\begin{rcases} 3x+2y-2=0 \\ 3y+2x-2=0 \end{rcases} \Rightarrow \begin{rcases} 3x+2y=+2 \\ 3y+2x=2 \end{rcases}$ $P=\left(\frac{2}{5},\frac{2}{5}\right)$

$H_f(x,y)=\begin{pmatrix}-2y^2(3x+y-1) & -2xy(3x+3y-2)\\ -2xy(3x+3y-2) & -2x^2(x+3y-1)\end{pmatrix}$

$H_{f,(0,y)}=\begin{pmatrix}(y-1)2y^2 & 0\\ 0 & 0\end{pmatrix} \Rightarrow \det=0$ NO decideix

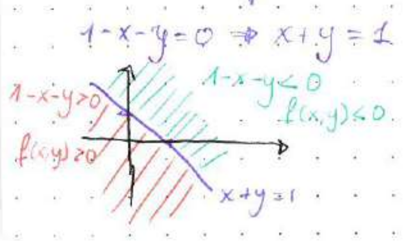
$H_f(x,0)=\begin{pmatrix}0 & 0\\ 0 & -2x^2(x-1)\end{pmatrix} \Rightarrow \det=0$ NO decideix

$H_f\left(\frac{2}{5},\frac{2}{5}\right)=\begin{pmatrix}-24/125 & -16/125\\ -16/125 & -24/125\end{pmatrix} \Rightarrow \det=\frac{24^2-16^2}{125^2}>0$

$P=\left(\frac{2}{5},\frac{2}{5}\right) \Rightarrow$ Màxim relatiu.

Signe $f(x,y)=x^2y^2(1-x-y)$? Fora dels eixos té el mateix signe $(1-x-y)$ pq eixos estan elevats al quadrat $\Rightarrow$ pos. signe $\Rightarrow$ no afecten.

$1-x-y=0 \Rightarrow x+y=1$



Agafem $P=(0,b)$
- Si $b>1 \Rightarrow$ Màxim
- Si $b=1 \Rightarrow$ Punt Sella
- Si $b<1 \Rightarrow$ Mínim

Sabem que $f(0,b)=0$ i $f(x,y)>0$

Mateix procés per $P=(a,0)$.

(Ex). $f(x,y)=\ln(1+2x+3y)$,, $P=(0,0)$,, Calcula Polinomi

$f(0,0)=\ln(1)=0$

- $\frac{\partial f}{\partial x}=\frac{1}{1+2x+3y}\cdot 2=\frac{2}{1+2x+3y} \Rightarrow \frac{\partial^2 f}{\partial x^2}=-2(1+2x+3)^{-2}\cdot 2=-4(1+2x+3y)^{-2}$
- $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=-3(1+2x+3y)^{-2}\cdot 2=-6(1+2x+3y)^{-2}$
- $\frac{\partial f}{\partial y}=\frac{3}{1+2x+3y}=3(1+2x+3y)^{-1} \Rightarrow \frac{\partial^2 f}{\partial y^2}=-3(1+2x+3y)^{-2}\cdot 3=-9(1+2x+3y)^{-2}$

$-4(1+2(0)+3(0))^{-2}=-4\cdot(1)^{-2}=-4\cdot 1=-4$

$H_f(a)=\begin{pmatrix}-4 & -6\\ -6 & -9\end{pmatrix}$ Polinomi: $0+(2,3)(x-0,y-0)+\frac{1}{2!}(-4x^2+2(-6)xy-9y^2)$ $\Pi\ \nabla f(c)$ $\Pi\ (x-0)\,y\text{-}0)\begin{pmatrix}-4 & -6\\ -6 & -9\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$

(1). $f(x,y)=\ln(1+2x+3y)$

a) Taylor de grau 2 en $(0,0)$.

$f(0,0)=\ln(1+2\cdot 0+3\cdot 0)=\ln(1)=0$

- $\frac{\partial f}{\partial x}=\frac{1}{1+2x+3y}\cdot 2=2(1+2x+3y)^{-1}$
- $\frac{\partial f}{\partial y}=\frac{1}{1+2x+3y}\cdot 3=3(1+2x+3y)^{-1}$

$\nabla f(0,0)=(2,3)$

- $\frac{\partial^2 f}{\partial x^2}=2\cdot(-1)\cdot(1+2x+3y)^{-2}\cdot(2)=-4(1+2x+3y)^{-2}$
- $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=2(-1)\cdot(1+2x+3y)^{-2}\cdot(3)=-6(1+2x+3y)^{-2}$
- $\frac{\partial^2 f}{\partial y^2}=3(-1)\cdot(1+2x+3y)^{-2}\cdot(3)=-9(1+2x+3y)^{-2}$

$-4(1+2(0)+3(0))^{-2}=-4(1)^{-2}=-4$

$H_{f(0,0)}=\begin{pmatrix}-4 & -6\\ -6 & -9\end{pmatrix}$

$P_{2,f,(0,0)}=f(0,0)+D_v f(0,0)+\frac{1}{2!}\left[(x-0,y-0)\cdot\begin{pmatrix}-4 & -6\\ -6 & -9\end{pmatrix}\begin{pmatrix}x-0\\ y-0\end{pmatrix}\right]= \ldots$

$=0+(2,3)(x,y)+\frac{1}{2!}\left((x,y)\begin{pmatrix}-4 & -6\\ -6 & -9\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\right)=0+2x+3y+\frac{1}{2!}\left[(-4x-6y,\,-6x-9y)\begin{pmatrix}x\\ y\end{pmatrix}\right]$

$=2x+3y+\frac{1}{2!}\left((-4x^2-6xy)+(-6xy-9y)\right)=\boxed{2x+3y+\frac{1}{2!}(-4x^2-12xy-9y^2)=P_{2,f,(0,0)}}$

b) Amb el polinomi calc valor aprox per $f(\frac{1}{10},\frac{1}{10})$

$f(\frac{1}{10},\frac{1}{10})\sim P_{2,f,(0,0)}=2(\frac{1}{10})+3(\frac{1}{10})+\frac{1}{2!}\left[-4(\frac{1}{10})^2-12(\frac{1}{10})(\frac{1}{10})-9(\frac{1}{10})^2\right]=\boxed{\frac{3}{8}\approx f(\frac{1}{10},\frac{1}{10})}$

c) Fita l'error obtingut.

- $\frac{\partial^3 f}{\partial x^3}=-4(-2)(1+2x+3y)^{-3}\cdot(2)=16(1+2x+3y)^{-3}$
- $\frac{\partial^3 f}{\partial^2 x\,\partial y}=-6(-2)(1+2x+3y)^{-3}\cdot(2)=24(1+2x+3y)^{-3}$
- $\frac{\partial^3 f}{\partial^2 y\,\partial x}=-6(-2)(1+2x+3y)^{-3}(3)=36(1+2x+3y)^{-3}$
- $\frac{\partial^3 f}{\partial y^3}=-9(-2)(1+2x+3y)^{-3}(3)=54(1+2x+3y)^{-3}$

$R_2(x,y)=\frac{1}{3!}[$

⑤. Troba primera i segona

b) $\ln(x^2+y^2)$;

- $\frac{\partial f}{\partial x} = \frac{1}{x^2+y^2}\cdot(2x+0) = 2x(x^2+y^2)^{-1}$
- $\frac{\partial f}{\partial y} = \frac{1}{x^2+y^2}(2y+0) = 2y(x^2+y^2)^{-1}$

- $\frac{\partial^2 f}{\partial x^2} = 2(x^2+y^2)^{-1} + 2x(-1)(x^2+y^2)^{-2}(2x) =$
- $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0 + (-1)(x^2+y^2)^{-2}\cdot(2y) = -2y(x^2+y^2)^{-2}$
- $\frac{\partial^2 f}{\partial y^2} = 2(x^2+y^2)^{-1} + 2y(x^2+y^2)^{-2}(-1)(2y)$

c) $f(x,y) = xy + \frac{x}{y} = xy + xy^{-1}$

- $\frac{\partial f}{\partial x} = y + y^{-1}$
- $\frac{\partial f}{\partial y} = x + x\cdot(-1)y^{-2} = x - xy^{-2}$

- $\frac{\partial^2 f}{\partial x^2} = 0$
- $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 1 - \frac{1}{y^2}$
- $\frac{\partial^2 f}{\partial y^2} = \frac{2x}{y^3}$

d) $f(x,y) = \operatorname{arctg}(xy^{-1})$ $\quad \operatorname{arctg}(f(x)) \Rightarrow \frac{1}{1+f(x)^2}\cdot f'(x)$

- $\frac{\partial f}{\partial x} = \frac{1}{1+(xy^{-1})^2}\cdot(y^{-1}) = \frac{1}{1+x^2y^{-2}}\cdot(y^{-1}) = \underline{y(x^2+y^2)^{-1}}$ vist en pissarra
- $\frac{\partial f}{\partial y} = \frac{1}{1+(xy^{-1})^2}\cdot(-xy^{-2}) = \frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{-x}{y^2} = \frac{1}{\frac{y^2+x^2}{\cancel{y^2}}}\cdot\frac{-x}{\cancel{y^2}} = -x(y^2+x^2)^{-1}$

- $\frac{\partial^2 f}{\partial x^2} = y(-1)(x^2+y^2)^{-2}\cdot(2x) = \underline{-2xy(x^2+y^2)^{-2}}$
- $\frac{\partial^2 f}{\partial y^2} = -x(-1)(y^2+x^2)^{-2}\cdot(2y) = \underline{2xy(y^2+x^2)^{-2}}$
- $\frac{\partial^2 f}{\partial x \partial y} = (x^2+y^2)^{-1} + y(-1)(x^2+y^2)^{-2}(2y) = \underline{(x^2+y^2)^{-1} - 2y^2(x^2+y^2)^{-2}}$

h) $h(x,y,z) = xyz\cdot e^{x+y+z}$ $\quad e^f \Rightarrow e^f\cdot f'$

- $\frac{\partial f}{\partial x} = yze^{x+y+z} + xyz\cdot e^{x+y+z}\cdot(1) = \underline{yze^{x+y+z} + xyze^{x+y+z}}$ $\rightarrow$ $\frac{\partial^2 f}{\partial x^2} = \left(yze^{x+y+z} + \left(yze^{x+y+z} + xyze^{x+y+z}\right)\right)$ $= \underline{2yze^{x+y+z} + xyze^{x+y+z}}$
- $\frac{\partial f}{\partial y} = \underline{xze^{x+y+z} + xyze^{x+y+z}}$ $\rightarrow$ $\frac{\partial^2 f}{\partial y^2} = \left(xze^{x+y+z} + \left(xze^{x+y+z} + xyze^{x+y+z}\right)\right) = \underline{2xze^{x+y+z} + xyze^{x+y+z}}$
- $\frac{\partial f}{\partial z} = \underline{xye^{x+y+z} + xyze^{x+y+z}}$ $\rightarrow$ $\frac{\partial^2 f}{\partial z^2} = \underline{2xye^{x+y+z} + xyze^{x+y+z}}$

⑥ $K=2$,, $f(x,y)=xy^2+\sin(xy)$,, $P=(1,\frac{\pi}{2})$

a $\cdot\ \frac{\partial f}{\partial x}=y^2+\cos(xy)\cdot y=\underline{y^2+y\cos(xy)}$ $\longrightarrow$ c $\cdot\ \frac{\partial^2 f}{\partial x^2}=y\cdot(-\sin(xy)\cdot y)=-y^2\sin(xy)$

b $\cdot\ \frac{\partial f}{\partial y}=x2y+\cos(xy)\cdot x=\underline{2xy+x\cos(xy)}$ $\longrightarrow$ d $\cdot\ \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=2y+\cos(xy)-xy\sin(xy)$

$\longrightarrow$ e $\cdot\ \frac{\partial^2 f}{\partial y^2}=2x+x(-\sin(xy)\cdot x)=2x-x^2\sin(xy)$

$a|_p=(\frac{\pi}{2})^2$ $\quad c|_p=(\frac{\pi}{2})^2$ $\quad e|_p=1$ $\quad\vdots\quad \nabla f(P)=((\frac{\pi}{2})^2,(\pi))$

$b|_p=\pi$ $\quad d|_p=\frac{\pi}{2}$ $\quad\vdots\quad f(P)=(\frac{\pi}{2})^2+1$

$P_2(f,(0,0))=\left[(\frac{\pi}{2})^2+1\right]+\left[((\frac{\pi}{2})^2(x-1)+\pi(y-\frac{\pi}{2})\right]+$

⑧

a) $\sqrt{1'03+2'98}$

$\cdot\ \underline{f(x,y)=\sqrt{x+y}}$ $\to x_0=1$,, $y_0=3\Rightarrow(x_0,y_0)=(1,3)$
$\to(x,y)=(1'03,2'98)$

Escollim alguna de les dues (per fer-ho més fàcil agafem f(x,y))

$\cdot\ \underline{g(x,y)=\sqrt{(1+x)(3+y)}}$ $\to x_0=0$,, $y_0=0\Rightarrow(x_0,y_0)=(0,0)$
$\to(x,y)=(0'03,-0'02)$.

a $\cdot\ \frac{\partial f}{\partial x}=\frac{1}{2}(x+y)^{\frac{1}{2}-1}\cdot(1)=\frac{1}{2}(x+y)^{\frac{-1}{2}}$ $\longrightarrow$ c $\cdot\ \frac{\partial^2 f}{\partial x^2}=\frac{1}{2}(\frac{-1}{2})(x+y)^{\frac{-1}{2}-1}\cdot(1)=\frac{-1}{4}(x+y)^{\frac{-3}{2}}$

d $\cdot\ \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=\frac{-1}{4}(x+y)^{\frac{-3}{2}}$

b $\cdot\ \frac{\partial f}{\partial y}=\frac{1}{2}(x+y)^{\frac{1}{2}-1}(1)=\frac{1}{2}(x+y)^{\frac{-1}{2}}$ $\longrightarrow$ e $\cdot\ \frac{\partial^2 f}{\partial y^2}=\frac{1}{2}(\frac{-1}{2})(x+y)^{\frac{-1}{2}-1}(1)=\frac{-1}{4}(x+y)^{\frac{-3}{2}}$

$a|_p=\frac{1}{2}(1+3)^{\frac{-1}{2}}=1/4$ $\quad c|_p=d|_p=e|_p=\frac{-1}{32}$!

$b|_p=1/4$! $f(P)=\sqrt{1+3}=2$! $\nabla f(P)=(1/4,1/4)$

$$P_{2,f,(1,3)}=[2]+[(1/4)(x-1)+(1/4)(y-3)]+\frac{1}{2!}\left[(x-1,y-3)\begin{pmatrix}-1/32 & -1/32\\ -1/32 & -1/32\end{pmatrix}\begin{pmatrix}x-1\\ y-3\end{pmatrix}\right]=$$

$$=[2]+\left[\frac{x-1}{4}+\frac{y-3}{4}\right]+\frac{1}{2}\left[\left(\frac{-8x-8y+32}{256},\frac{-8x-8y+32}{256}\right)\begin{pmatrix}x-1\\ y-3\end{pmatrix}\right]=$$

sub
$x=1'03$
$y=2'98$

$$=[2]+\left[\frac{1'03-1}{4}+\frac{2'98-3}{4}\right]+\frac{1}{2}\left[-3'125\times10^{-6}\right]=\boxed{2'0025=P_{2,f,(1,3)}(1'03,2'98)\approx f(1'03,2'98)}$$

9. Sigui $f:[-1,1]\times[-1,1]\to\mathbb{R}$ def. $f(x,y)=1+x^3+y^2+2\int_0^{3x}\sqrt{1+t^2}\,dt+x\int_0^{y^2}e^{t^2/2}\,dt$ Recorda que derivem respecte x i $\frac{d}{dx}y^2=0$

Calc polinòmi grau 2 en $P=(0,0)$. TFC

$\bullet\ \frac{\partial f}{\partial x}=0+3x^2+0+2\cdot\left[\left(\sqrt{1+(3x)^2}\cdot 3\right)-\left(\sqrt{1+0^2}\cdot 0\right)\right]+\left(1\int_0^{y^2}e^{t^2/2}dt+x\overbrace{\left[e^{y^{2^2}/2}\cdot 0-e^{0^2/2}\cdot 0\right]}^{TFC}\right)=$ NO $2y$

$=3x^2+6\sqrt{1+9x^2}+\int_0^{y^2}e^{t^2/2}dt\Rightarrow\frac{\partial f}{\partial x}\Big|_P=3\cdot(0)^2+6\sqrt{1+9\cdot0^2}+\int_0^0 e^{t^2/2}dt=6\sqrt{1}=\boxed{6}$

Pue veus que no depen de y, llavors tracte com constant.

$\bullet\ \frac{\partial f}{\partial y}=0+0+2y+0+x\overbrace{\left[e^{(y^2)^2/2}\cdot 2y+0\right]}^{TFC}=2y+2xye^{y^4/2}\Rightarrow\frac{\partial f}{\partial y}\Big|_P=0+0\cdot e^{0/2}=\boxed{0}$

$\bullet\ \frac{\partial^2 f}{\partial x^2}=6x+6\cdot\frac{1}{2}(1+9x^2)^{\frac{-1}{2}}\cdot(18x)+0=6x+\frac{3\cdot18x}{\sqrt{1+9x^2}}=6x+\frac{54x}{\sqrt{1+9x^2}}\Rightarrow\frac{\partial^2 f}{\partial x^2}\Big|_P=\boxed{0}$

$\bullet\ \frac{\partial^2 f}{\partial x\partial y}=0+0+\left[e^{(y^2)^2/2}\cdot 2y-0\right]=e^{y^4/2}\cdot 2y\Rightarrow\frac{\partial^2 f}{\partial x\partial y}\Big|_P=\boxed{0}$

$\bullet\ \frac{\partial^2 f}{\partial y^2}=2+\left(2xe^{y^4/2}+2xye^{y^4/2}\cdot 4y^3\right)=2+2xe^{y^4/2}+8xy^4e^{y^4/2}\Rightarrow\frac{\partial^2 f}{\partial y^2}\Big|_P=\boxed{2}$

$-\ f(0,0)=1+0+0+2\int_0^0+0\int_0^0=\boxed{1}$

$P_{2,f,(0,0)}=1+\left(6(x-0)+0(y-0)\right)+\frac{1}{2}\left((x-0,y-0)\begin{pmatrix}0&0\\0&2\end{pmatrix}\begin{pmatrix}x-0\\y-0\end{pmatrix}\right)=1+6x+\frac{1}{2}\left((0,2y)\begin{pmatrix}x-0\\y-0\end{pmatrix}\right)=$

$=\boxed{1+6x+y^2=P_{2,f,(0,0)}}$ Jodes men

(10.)

a) $f(x,y)=x^2+y^2+x+y+xy$

1) Trobem punts crítics.

$\bullet\ \frac{\partial f}{\partial x}=2x+0+1+0+y\Rightarrow\frac{\partial f}{\partial x}=2x+y+1$

$\bullet\ \frac{\partial f}{\partial y}=0+2y+0+1+x\Rightarrow\frac{\partial f}{\partial y}=2y+x+1$

$\nabla f=(2x+y+1,2y+x+1)=0\Rightarrow\begin{cases}2x+y+1=0\\2y+x+1=0\end{cases}\Rightarrow x=-1-2y$

$2(-1-2y)+y+1=0\Rightarrow-2-4y+y+1=-3y-1=0\Rightarrow\boxed{y=\frac{-1}{3}}$

$x=-1-2\left(\frac{-1}{3}\right)$

$x=\frac{-1}{3}$

$\boxed{\text{El punt crític és }\left(\frac{-1}{3},\frac{-1}{3}\right)=P}$

$\bullet\ \frac{\partial f}{\partial x^2}=2+0+0=2$

$\bullet\ \frac{\partial f}{\partial x\partial y}=0+1+0=1$

$\bullet\ \frac{\partial f}{\partial y^2}=2+0+0=2$

$H_{f,P}\begin{pmatrix}2&1\\1&2\end{pmatrix}$ on $\Delta H=4-1=3>0$ Pel criteri del Hessià sabem que $\Delta>0$ i $a_{11}>0\Rightarrow\boxed{\text{Mínim Relatiu }(P)}$

d) $f(x,y)=(x-1)^4+(x-y)^4$

1) Punts crítics

$\cdot \frac{\partial f}{\partial x}=4(x-1)^3+4(x-y)^3$

$\cdot \frac{\partial f}{\partial y}=4(x-y)^3$

$\nabla f=\left(4(x-1)^3+4(x-y)^3,\ 4(x-y)^3\right)$ i fem $\nabla f=0$ $\left.\begin{array}{r}4(x-1)^3+4(x-y)^3=0\\ 4(x-y)^3=0\end{array}\right\}$

Pq eq2 sigui $0 \Leftrightarrow (x-y)^3=0 \Leftrightarrow x-y=0 \Leftrightarrow x=y$ i fem sub en eq 1.

$4(x-1)^3+4(x-x)^3=4(x-1)^3=0 \Rightarrow$ i això només si $(x-1)^3 \Rightarrow x=1$ $\boxed{P=(1,1)}$

2) Classifiquem el punt.

$\cdot \frac{\partial^2 f}{\partial x^2}=12(x-1)^2+12(x-y)^2$

$\cdot \frac{\partial^2 f}{\partial y^2}=12(x-y)^2(-1)=-12(x-y)^2$

$\cdot \frac{\partial^2 f}{\partial x\partial y}=0+12\cdot 2(x-y)\cdot(-1)=-24(x-y)$

$\cdot \left.\frac{\partial^2 f}{\partial x^2}\right|_P=12(0)^2+0=0$

$\cdot \left.\frac{\partial^2 f}{\partial y^2}\right|_P=12(0)^2=0$

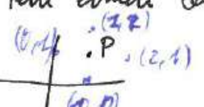
$\cdot \left.\frac{\partial^2 f}{\partial x\partial y}\right|_P=-24(0)=0$

$H_{f,P}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$.

Com que és la mat. nul·la el Hessià no diu res.

$(\Delta H_{f,P})=0$

3) Fem estudi local. de P



$\cdot f(1,2)=1$

$\cdot f(2,1)=2$

$\cdot f(0,1)=2$

$\cdot f(1,0)=1$

Veiem que tots son val (+) així que $\boxed{P(1,1) \text{ és mínim absolut}}$ Pq és

que podem pen

f) $f(x,y)=x^3-x^2y+3y^2$

1) Punts crítics.

$\cdot \frac{\partial f}{\partial x}=3x^2-2xy$

$\cdot \frac{\partial f}{\partial y}=-x^2+6y$

$\nabla f=(3x^2-2xy,\ -x^2+6y)$ i hem de veure x,y t.q. $\nabla f=0$. $\left.\begin{array}{r}3x^2-2xy=0\\-x^2+6y=0\end{array}\right\} \Rightarrow y=\frac{x^2}{6}$

$3x^2-2x\left(\frac{x^2}{6}\right)=3x^2-\frac{x^3}{3}=0 \Rightarrow 9x^2-x^3=0 \Rightarrow x^2(9-x)=0 \begin{cases}x=0\\-x+9=0 \Rightarrow x=9\end{cases}$

Si $x=0 \Rightarrow y=0$ $\boxed{P=(0,0)}$ $\cdot$ Si $x=9 \Rightarrow y=\frac{81}{6}=\frac{27}{2} \Rightarrow \boxed{Q=\left(9,\frac{27}{2}\right)}$

2) Classifiquem els punts.

$\cdot \frac{\partial^2 f}{\partial x^2}=6x-2y$

$\cdot \frac{\partial^2 f}{\partial y^2}=6$

$\cdot \frac{\partial^2 f}{\partial x\partial y}=-2x$

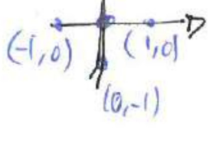
$\cdot \left.\frac{\partial^2 f}{\partial x^2}\right|_P=0$ $\cdot \left.\frac{\partial^2 f}{\partial y^2}\right|_P=6$ $\cdot \left.\frac{\partial^2 f}{\partial x\partial y}\right|_P=\left.\frac{\partial^2 f}{\partial y\partial x}\right|_P=0$ $H_{f,P}=\begin{pmatrix}0&0\\0&6\end{pmatrix}=0$

~~Com que $\Delta H_{f,P}<0 \Rightarrow$ P és punt sella~~ No ho podem saber.

$\cdot \left.\frac{\partial^2 f}{\partial x^2}\right|_Q=6\cdot(9)-2\left(\frac{27}{2}\right)=54-27=27$ $\cdot \left.\frac{\partial^2 f}{\partial y^2}\right|_Q=6$ $\cdot \left.\frac{\partial^2 f}{\partial x\partial y}\right|_Q=\left.\frac{\partial^2 f}{\partial y\partial x}\right|_Q=-18$

$H_{f,Q}=\begin{pmatrix}27&-18\\-18&6\end{pmatrix}=-162$

Com que $\Delta H_{f,Q}<0 \Rightarrow \boxed{Q \text{ és punt sella}}$



$f(0,1)=3$

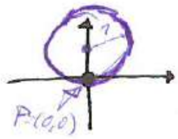
$f(1,0)=1-0=1$

$f(0,-1)=3$

$f(-1,0)=-1$

Com que hi ha + i − podem dir que $\boxed{\text{P és punt de sella}}$

⑨. Comprovem que en $(0,0)$ és punt sella. $f(x,y) = (x^2+(y-1)^2-1)(x^2-2y)$



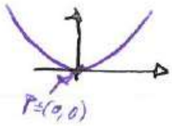
$\cdot \frac{\partial f}{\partial x} = \cancel{(x^2+(y-1)^2-1)'(x^2-2y) + (x^2+(y-1)^2-1)(x^2-2y)' =}$

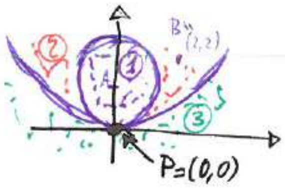
$f(0,0) = 0$ i ara fem estudi local.

f és un producte i si volem que $f=0$

→ $(x^2+(y-1)^2-1) = 0 \Rightarrow x^2+(y-1)^2 = 1$ → Circumf en $(0,1)$ i radi $\sqrt{1} = 1$

→ $(x^2-2y) = 0 \Rightarrow x^2 = 2y \Rightarrow y = \frac{x^2}{2}$





①. Agafem punt qualsevol $A=(0,1) \Rightarrow f(A) = 2$ ⊕

②. Agafem punt qualsevol $B=(0,3) \Rightarrow f(B) = -18$ ⊖

③. Agafem punt qualsevol $C=(0,1) \Rightarrow f(C) = 2$ ⊕

Veiem que hi ha dif en el signe així que $(0,0)=P$ sella.

⑪. $f(x,y) = e^{\lambda x+y^2} + \mu \sin(x^2+y^2)$ on $\lambda, \mu \in \mathbb{R}$.

Sabent $(0,0)$ és extrem relatiu i que pol. Taylor $n=2$ de f en $(0,0)$ val 6 en $p(1,2)$. Classifica.

$\cdot \frac{\partial f}{\partial x} = e^{\lambda x+y^2}\cdot\lambda + \mu\cos(x^2+y^2)\cdot 2x \Rightarrow \cdot \frac{\partial f}{\partial x}(0,0) = e^{0+0}\cdot\lambda + \mu\cos(0+0)\cdot 2\cdot 0 = \boxed{\lambda = 0}$ ← pq és extrem relatiu

Com que tant $\frac{\partial f}{\partial x}$ com $\frac{\partial f}{\partial y}$ hem de ser 0 pq $P(0,0)$ sigui extrem, podem dir que

$f(x,y) = e^{y^2} + \mu\sin(x^2+y^2)$ i no tindre en compte λ durant que farem la part més fàcil.

$\cdot \frac{\partial f}{\partial y} = e^{y^2}\cdot 2y + \mu\cos(x^2+y^2)\cdot 2y = 2y\left(e^{y^2}+\mu\cos(x^2+y^2)\right)$ i $\forall \mu$ $\cdot \frac{\partial f}{\partial y} = 0$ en $P=(0,0)$

$Hf =$