1 Tensor Fields

Our focus in this thesis is visualization of sceond order tensors in $\mathbf{R}^2$ and $\mathbf{R}^3$. However, the theory we present in the following subsections will not be curtailed to second order tensors, but will be applicable to any order tensor field. In order to achieve this, we require a general notation. In this chapter, we first introduce the definition of a tensor of type (M, N). Using this definition we show how tensors can be differentiated. We require this in order to introduce the Riemann Christoffel tensor; also known as Riemann curvature tensor, which quantifies the relative variation of initially parallel geodesics [**?**]. Since such geodesics only variate relative to each other in a curved space, the Riemann tensor will be zero everywhere in flat space.

1.1 Notation

Here we provide a short introduction to Einstein notation, which we will employ throughout this thesis.

In an orthogonal coordinate system we can write a vector $\mathbf{A}$ in component form

$$\mathbf{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3,$$

where $\hat{e}_i,\ i = 1, 2, 3$ are the orthogonal unit vectors. The short hand notation we just employed for the orthogonal vectors uses a dummy subscript i. Using this subscript, we can achieve a shorter notation, also called the *index* notation :

$$A_i,\ i = 1, 2, 3.$$

Here A_i refers to all the components of the vector[1] $\mathbf{A}$.

To avoid confusion, a system of any order when raised to a power will be enclosed in paranthesis

$$(B^2)^3, ({y_1}^1)^2, ({T_{ij}}^{kl})^{1/2}.$$

Hence, the component B^2 is raised to the power of 3, the component ${y_1}^1$ is raised to the power of 2, and for the tensor ${T_{ij}}^{kl}$ we take the square root[2].

When a system has indices that occur unrepeated, it is implicitly understood that each of the subscripts and superscripts can take any of the integer values $1, \ldots, N$. For example the *Kronecker delta* symbol δ_{ij}, defined by

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j, \end{cases} \tag{1}$$

[1] As a single subscript/superscript denotes the components of a vector, we can extend this to higher(or lower) order systems. No subscript or superscript denote scalars, which we refer to as zeroth-order systems. While two indices imply a second order system, which in the language of linear algebra is called a matrix (if the indices are not mixed). We will, however, refer to such systems as tensors. In fact, nomenclature tensor is the unified reference to all such systems, and those of higher order.

[2] Which is not as simple or straight forward as it sounds. In order for any mathematical operation to be performed on a tensor, the resulting tensor must obey the transformation law. We will discuss this later when we introduce the general definition of tensors.

with $i, j = 1, 2, 3$, represents nine values. The indices i and j are called *free indices* and can take on any of the values specified by a given range.

The summation convention states that when an index which is repeated twice on the same side of an equation, it is understood to represent a summation of the repeated indices. Hence, a repeated index is called a *summation index*, while an unrepeated index is called a *free index*. To sum the diagonal entries of the Kroncker delta[1], we simply repeat the indices

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3.$$

Note that we implicitely understand, when we write δ_{ii}, to mean that we are performing the summation

$$\sum_{i}^{3} \delta_{ii}.$$

Often when performing certain operations, the alternating tensor becomes a handy tool to possess. It is a short-hand notation, just like the Kronecker delta.

Definition. *The e-permutation sysmbol is defined as*

$$e_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation,} \\ -1 & \text{if } ijk \text{ is an odd permutation,} \\ 0 & \text{when indices overlap.} \end{cases}$$

The definition is also applicable for larger sets as well. Another identity which is useful, is the e-δ identity. Given e_{ijk} the e-permutation symbol, and δ_{ij} the Kronecker delta, then for $i, j, k, m, n = 1, 2, 3$,

$$e_{ijk}e_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km},$$

where i is the summation index and j, k, m, n are the free indices. A mixed superscript and subscript identity, is defined as

$$e^{ijk}e_{imn} = \delta^{j}{}_{m}\delta^{k}{}_{n} - \delta^{j}{}_{n}\delta^{k}{}_{m}$$

Using the e-permutation symbol we can describe the determinant of a tensor A with the components a_{ij}, where $i, j = 1, \ldots, n$. Writing the determinant in index notation, it becomes

$$detA = \|A\| = e_{i_1 \ldots i_n} a_{1i_1} \ldots a_{ni_n}$$

For a 3×3 matrix,

$$\begin{aligned} detA &= e_{ijk}a_{1i}a_{1j}a_{1k} \\ &= e_{1jk}a_{11}a_{1j}a_{1k} + e_{2jk}a_{12}a_{1j}a_{1k} + e_{3jk}a_{13}a_{1j}a_{1k} \\ &= e_{123}a_{11}a_{12}a_{13} + e_{132}a_{11}a_{13}a_{12} + e_{213}a_{12}a_{11}a_{13} + e_{231}a_{12}a_{13}a_{11} + e_{312}a_{13}a_{11}a_{12} + e_{321}a_{13}a_{12}a_{11} \\ &= a_{11}a_{12}a_{13} - a_{11}a_{13}a_{12} - a_{12}a_{11}a_{13} + a_{12}a_{13}a_{11} + a_{13}a_{11}a_{12} - a_{13}a_{12}a_{11} \end{aligned}$$

This is a simple illustration for the need of using short hand notation.

[1] In linear algebra this is referred to as taking the *trace* of an $n \times n$ square matrix.

1.2 A General Definition of Tensor

We now introduce a definition of a tensor field, evaluated at a point $x^j = x^j(\bar{x}^j)$. The bar quantity are coordinates in different coordinate system. We assume that a mapping exists, such that $\bar{x}^j = \bar{x}^j(x^j)$. We define the transformation between the barred coordinate system and the barless coordinates as

$$\bar{x}^j = \bar{x}^j(x^j). \tag{2}$$

Definition. *A tensor of rank* $(M+N)$, $\mathcal{T}_{i_1 \dots i_M}{}^{j_1 \dots j_N}(x^j)$, *defined on a point P of a differential manifold* X_n *exists, if, under the coordinate transformation* (2), *the components transform according to the law*

$$\overline{\mathcal{T}}_{m_1 \dots m_M}{}^{n_1 \dots n_N} = \frac{\partial x^{i_1}}{\partial \bar{x}^{m_1}} \cdot \ldots \cdot \frac{\partial x^{i_M}}{\partial \bar{x}^{m_M}} \cdot \frac{\partial \bar{x}^{n_1}}{\partial x^{j_1}} \cdot \ldots \cdot \frac{\partial \bar{x}^{n_N}}{\partial x^{j_N}} \mathcal{T}_{i_1 \dots i_M}{}^{j_1 \dots j_N}. \tag{3}$$

The tensor is said to be covariant *of order M and* contravariant *of order N, if it obeys the* transformation law.

Example. Let $\phi = \phi(x^1, \dots, x^N)$ denote a tensor of type $(0,0)$. Then from Equation (3) it follows that

$$\bar{\phi}(\bar{x}^1, \dots, \bar{x}^N) = \phi(x^1, \dots, x^N).$$

This is simply a scalar. Hence scalars are zero order tensors. The gradient of the scalar is another tensorial quantity. By chain rule,

$$\frac{\partial \bar{\phi}}{\partial \bar{x}^j} = \frac{\partial x^i}{\partial \bar{x}^j} \frac{\partial \phi}{\partial x^i}$$

which we see is a covariant vector of type (1,0). □

In general, we can write such vectors by the transformation law.

$$\bar{A}_j = \frac{\partial x^i}{\partial \bar{x}^j} A_i.$$

Similarly, a contravariant vector is a type (0,1) tensor, and by the transformation law given as

$$\bar{A}^j = \frac{\partial \bar{x}^j}{\partial x^i} A^i.$$

The importance of tensors is made clear due to the fact that they are invariant under coordinate transformations. Hence, if all the components of the tensor vanish in one coordinate system, they will do so in any other system. Similarly, if the tensor is symmetric in one coordinate system, the property will perpetuate on to another coordinate system.

Proof. Given the tensor R_{ijk}, it is said to be symmetric in two of its indices if the components are unchanged when the indices are interchanged. For example, the third order tensor R_{ijk} is symmetric in the indices i and k, if

$$R_{ijk} = R_{kji} \tag{4}$$

for all values of i, j and k. By the transformation law (3),

$$\overline{R}_{lmn} = \frac{\partial x^i}{\partial \bar{x}^l}\frac{\partial x^j}{\partial \bar{x}^m}\frac{\partial x^k}{\partial \bar{x}^n}R_{ijk}, \tag{5}$$

and

$$\overline{R}_{nml} = \frac{\partial x^k}{\partial \bar{x}^n}\frac{\partial x^j}{\partial \bar{x}^m}\frac{\partial x^i}{\partial \bar{x}^l}R_{kji}.$$

By Equation (4) and (5) it follows that

$$\overline{R}_{nml} = \frac{\partial x^k}{\partial \bar{x}^n}\frac{\partial x^j}{\partial \bar{x}^m}\frac{\partial x^i}{\partial \bar{x}^l}\left(\frac{\partial \bar{x}^l}{\partial x^i}\frac{\partial \bar{x}^m}{\partial x^j}\frac{\partial \bar{x}^n}{\partial x^k}\overline{R}_{lmn}\right) = \overline{R}_{lmn},$$

where we used the inverse relation of Equation (5). □

A tensor is skew-symmetric in two of its indices if the components are transformed to their negative values when the indices are interchanged. Using R_{ijk} again to illustrate this,

$$R_{ijk} = -R_{kji}$$

for all values of i, j and k. Using the proof above, we can readily show for a skew-symmetric tensor the property of invariance from one coordinate system to another.

1.3 Tensor Operations

In order for an operation to make sense when performed on a tensor, the resulting tensor must satisfy the transformation law (3).

Example. We can add or subtract similar components for different tensors. The following operation is permitted only when the indices match

$$C^i{}_{jk} = A^i{}_{jk} + B^i{}_{jk}. \tag{6}$$

This is invalid

$$A^i + B_j. \tag{7}$$

For (7), the system B is a covariant of order 1, while system A is contravariant of order 1.

We can show that the operation performed in (6) is correct by employing the transformation law. It is a good way to show that a certain tensor operation is valid. Let A and B be expressed by the transformation law (3)

$$\bar{A}^l{}_{mn} = \frac{\partial \bar{x}^l}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^m}\frac{\partial x^k}{\partial \bar{x}^n}A^i{}_{jk}, \tag{8}$$

and,

$$\overline{B}^{l}{}_{mn} = \frac{\partial \bar{x}^l}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^m} \frac{\partial x^k}{\partial \bar{x}^n} B^{i}{}_{jk}. \tag{9}$$

Then,

$$\overline{C}^{l}{}_{mn} = A^{l}{}_{mn} + B^{l}{}_{mn} = \frac{\partial \bar{x}^l}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^m} \frac{\partial x^k}{\partial \bar{x}^n} \left(A^{i}{}_{jk} + B^{i}{}_{jk}\right) = \frac{\partial \bar{x}^l}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^m} \frac{\partial x^k}{\partial \bar{x}^n} C^{i}{}_{jk}.$$

Which clearly satisfies the transformation law.

For the case in (7), we can use same procedure to demonstrate that the quantity is not tensorial. In general, we can say that only tensors of same type (r, s) can be added together.

Multiplication (outer product) on the other hand does not require tensors have the exact same type. For example the outer product of the systems $A_{m}{}^{i}$ and B^{jkl} result in the new system $C_{m}{}^{ijkl}$,

$$C_{m}{}^{ijkl} = A_{m}{}^{i} B^{jkl}.$$

The newly constructed system from the outer product is a fifth order system, consisting of all the possible products from the components of $A_{m}{}^{i}$ with B^{jkl}.

The operation of contraction occurs when the lower and upper index are set equal to each other and the summation convention is thereby applied. Using the system $C_{m}{}^{ijkl}$ from the previous example, we can perform contraction on the lower index n with the upper index i as following

$$C_{m}{}^{mjkl} = C_{1}{}^{1jkl} + \cdots + C_{N}{}^{Njkl} = D^{jkl}$$

where we have summed the same indices through the summation convention. As a result of contracting the system C, the resulting system is now a third order system. In fact, performing contraction on a system always lowers the order of the system by two.

An *inner product* between two tensors is performed as following. Firstly, take the outer product of the tensor, thereafter perform a contraction on two of the indices. As contraction requires both super and sub-script, the tensors involved in such an operation must be at least of rank 1. Given two vectors A^i and B_j, their inner product is found by first performing an outer product

$$C^{i}{}_{j} = A^i B_j.$$

Thereupon, we perform a contraction by setting the indices equal to each other, and sum all the terms. Using the transformation law for the contravariant tensor A^i and covariant tensor B_j, they take the form

$$\bar{A}^i = \frac{\partial \bar{x}^j}{\partial x^m} A^m,$$
$$\overline{B}_j = \frac{\partial x^n}{\partial \bar{x}^j} B_n.$$

There upon we perform the product

$$\bar{A}^i\overline{B}_j = \left(\frac{\partial \bar{x}^j}{\partial x^m}A^m\right)\left(\frac{\partial x^n}{\partial \bar{x}^j}B_n\right),$$
$$\bar{A}^i\overline{B}_j = \frac{\partial \bar{x}^j}{\partial x^m}\frac{\partial x^n}{\partial \bar{x}^j}A^m B_n.$$

Let $\overline{C} = \bar{A}^i\overline{B}_i$, then the contraction by setting $i = j$ and summing all the terms, becomes

$$\begin{aligned}\overline{C} &= \bar{A}^i\overline{B}_i,\\ &= \delta^n{}_m A^m B_n,\\ &= A^n B_n = C.\end{aligned}$$

As we observe, the end result becomes a scalar (as implied by the terminology - scalar product).

□

1.4 Reciprocal Basis and the Metric Tensor

Definition. *Two bases* $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ *and* $(\mathbf{E}^1, \mathbf{E}^2, \mathbf{E}^3)$ *are said to be reciprocral if they satisfy the condition*

$$E_i E^j = \delta_i{}^j = \begin{cases} 1 & \textit{for } i = j \\ 0 & \textit{for } i \neq j \end{cases}$$

One such basis that satisfies this is

$$E_i = \frac{\partial \mathbf{r}}{\partial u^i}$$

where $\mathbf{r} = x(u,v,w)\mathbf{e}_1 + y(u,v,w)\mathbf{e}_2 + z(u,v,w)\mathbf{e}_3$. The unit vectors e_i are defined as $\frac{E_i}{|E_i|}$. Given the basis $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$, we can always determine $(\mathbf{E}^1, \mathbf{E}^2, \mathbf{E}^3)$ by the following triple scalar products

$$E^i = \frac{e_{ijk}E_j E_k}{e_{ijk}E_i E_j E_k}.$$

Given these new basis vectors, we can now represent any vector, $\mathbf{A}$, by either of the baseis. Given the basis $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$, we can represent $\mathbf{A}$ in the form

$$A^i E_i.$$

The components of $\mathbf{A}$, A^i, relative to basis E_i are called *contravariant components* of $\mathbf{A}$. Similarly, the components A_i relative to the basis E^i are called *covariant components* of $\mathbf{A}$:

$$A_i E^i.$$

The contra and co-variant are different ways of representing $\mathbf{A}$ with respect to a set of reciprocal basis vectors. The inter-relationship between these components is given by the *metric* and the *cojugate metric* of space

$$g_{ij} = E_i E_j = \frac{\partial \mathbf{r}}{\partial u^i}\cdot\frac{\partial \mathbf{r}}{\partial u^j},$$
$$g^{ij} = E^i E^j = (g_{ij})^{-1},$$

where the last identity follows from the reciprocity. In index notation, the relationship expressed with the metric becomes

$$A_i = g_{ij} A^j,$$
$$A^i = g^{ij} A_j.$$

Hence, we can both raise the indices, and lower them, by applying either the conjugate metric g^{ij} or the metric g_{ij}. This is an useful operation (e.g. when applied to higher-order tensors like the Riemann curvature tensor.)

Let $x = x(u, v, w), y = y(u, v, w), z = z(u, v, w)$, then the curve element ds is given as

$$ds^2 = g_{ij} du_i du_j.$$

An example of this is illustrated by transforming the Cartesian coordinates (x, y, z) to cylindrical (r, θ, z). The relationship between these coordinate systems is as following

$$x = r\cos(\theta),$$
$$y = r\sin(\theta),$$
$$z = z.$$

Hence, x, y, and z are functions of $u_i = (r, \theta, z)$, and the line vector $\mathbf{r}$ is given as $\mathbf{r} = x(r, \theta, z)\mathbf{e}_1 + y(r, \theta, z)\mathbf{e}_2 + z(r, \theta, z)\mathbf{e}_3$. The the non-zero entries of the metric are the diagonal entries

$$g_{11} = \frac{\partial \mathbf{r}}{\partial r} \cdot \frac{\partial \mathbf{r}}{\partial r}, \qquad g_{22} = \frac{\partial \mathbf{r}}{\partial \theta} \cdot \frac{\partial \mathbf{r}}{\partial \theta}, \qquad g_{33} = \frac{\partial \mathbf{r}}{\partial z} \cdot \frac{\partial \mathbf{r}}{\partial z}.$$

The non-zero calcuated entries of the metric are given as

$$g_{11} = [\cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2] \cdot [\cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2] = 1,$$
$$g_{22} = [-r\sin(\theta)\mathbf{e}_1 + r\cos(\theta)\mathbf{e}_2] \cdot [-r\sin(\theta)\mathbf{e}_1 + r\cos(\theta)\mathbf{e}_2] = r^2,$$
$$g_{33} = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1.$$

The line element ds thus becomes

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2.$$

1.5 Tensor Differentiation

Say that we want to differentiate a tensor field $T_{ij}(x^k)$, where $x^k = x^k(\bar{x}^k)$, as following

$$\frac{\partial T_{ij}}{\partial x^k}.$$

We know now that this quantity has to satisfy the transformation law (3). The second order tensor of type $(2, 0)$ satisfies the following transformation law

$$\overline{T}_{pq} = \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} T_{ij}$$

If we now perform differentiation on this expression, we get

$$\frac{\partial \overline{T}_{pq}}{\partial x^k} = \frac{\partial^2 x^i}{\partial x^k \partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^q}T_{ij} + \frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial^2 x^j}{\partial x^k \partial \bar{x}^q}T_{ij} + \frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^q}\frac{\partial T_{ij}}{\partial x^k}.$$

This quantity is certainly not a tensor. In order to derivate a tensor, we need to introduce the Christoffel symbols.

Consider the metric tensor g_{ab} which satifies the transformation law

$$\overline{g}_{kl} = g_{ab}\frac{\partial x^a}{\partial \bar{x}^k}\frac{\partial x^b}{\partial \bar{x}^l}. \tag{10}$$

We define a quantity,

$$(k, l, m) = \frac{\partial \overline{g}_{kl}}{\partial \bar{x}^m}. \tag{11}$$

We insert (10) in (11), and simply apply chain rule

$$(k, l, m) = \frac{\partial g_{ab}}{\partial x^c}\frac{\partial x^c}{\partial x^m}\frac{\partial x^a}{\partial \bar{x}^k}\frac{\partial x^b}{\partial \bar{x}^l} + g_{ab}\frac{\partial^2 x^a}{\partial x^m \partial \bar{x}^k}\frac{\partial x^b}{\partial \bar{x}^l} + g_{ab}\frac{\partial x^a}{\partial \bar{x}^k}\frac{\partial^2 x^b}{\partial x^m \partial \bar{x}^l}.$$

By combining the terms, [**?**],

$$\frac{1}{2}\left[(k, l, m) + (l, m, k) - (m, k, l)\right]$$

we can extract Christoffel symbol of *first kind*, which is defined as

$$\frac{1}{2}\left[\frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ac}}{\partial x^b}\right]. \tag{12}$$

By introducing a shorter notation [**?**], we can write (12) as

$$[ac, b] = [ca, b].$$

This implies that we have symmetry about the variables a and c. This new quantity is not a tensor, as it does not satisfy the transformation law.

Christoffel symbol of *second kind* is defined as

$$g^{mb}\,[ac, b] = \frac{1}{2}g^{mb}\left[\frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ac}}{\partial x^b}\right].$$

We can write this in a shorter notation by using brackets

$$\begin{Bmatrix} m \\ b\ c \end{Bmatrix} = \begin{Bmatrix} m \\ c\ b \end{Bmatrix} = g^{mb}\,[ac, b]\,, \tag{13}$$

where we have symmetry between b and c. This quantity is not a tensor either; agian, it does not obey the transformation law. We can interchange between the second kind and first kind, by multiplying (13) with the metric g_{mb}. As $g_{mb}\,g^{mb} = \delta_m{}^b$ (see Section 1.4), we end up with the Christoffel symbol of first kind, (12).

So far we have introduced new notation for derivating tensors, yet the operators themselves are not tensors[1]. The purpose of Christoffel symbols becomes quite apparent, when we use these operators to define a covariant derivative of a tensor.

The covariant derivative of a covariant tensor, A_m, is given as

$$A_{b,c} = \frac{\partial A_b}{\partial x^c} - \begin{Bmatrix} m \\ b\ c \end{Bmatrix} A_m.$$

which becomes a second order tensor, satisfying the transformation law

$$\bar{A}_{i,j} = A_{b,c} \frac{\partial x^b}{\partial \bar{x}^i} \frac{\partial x^c}{\partial \bar{x}^j}.$$

We have sucessfully managed to derivate a tensor, and the resulting operation adheres to the transformation law. In other words, the derivative of the tensor results in a another tensor.

Similary, we can show that the covariant derivative of the contravariant tensor A^m,

$$A^m{}_{,n} = \frac{\partial A^m}{\partial x^n} + \begin{Bmatrix} m \\ l\ n \end{Bmatrix} A^l$$

obeys the transformation law. Here, the first term on the right hand side of the equation is the rate of the tensor field as we move along a coordinate curve, while the second term is the change in local basis vectors as we move along the coordinate curves, [**?**].

For second order tensors $A_{ij}, A^i{}_j, A^{ij}$, their covariant derivatives are given as

$$\begin{aligned}
A_{ij,k} &= \frac{\partial A_{ij}}{\partial x^k} - A_{mj} \begin{Bmatrix} m \\ i\ k \end{Bmatrix} - A_{im} \begin{Bmatrix} m \\ j\ k \end{Bmatrix}, \\
A^i{}_{j,k} &= \frac{\partial A^i{}_j}{\partial x^k} + A^m{}_j \begin{Bmatrix} i \\ m\ k \end{Bmatrix} - A^i{}_m \begin{Bmatrix} m \\ j\ k \end{Bmatrix}, \\
A^{ij}{}_{,k} &= \frac{\partial A^{ij}}{\partial x^k} + A^{mj} \begin{Bmatrix} i \\ m\ k \end{Bmatrix} + A^{im} \begin{Bmatrix} j \\ m\ k \end{Bmatrix}.
\end{aligned}$$

Given two tensors, say A_{ij} and B_{ij}, the covariant derivation is same as ordinary derivation, where

1. $(A_{ij} + B_{ij})_{,k} = A_{ij,k} + B_{ij,k}$ (derivative of sum is the sum of derivatives)
2. $(A_{ij} B_{ij})_{,k} = A_{ij,k} B_{ij} + A_{ij} B_{ij,k}$ (product rule)
3. $(A_{ij,k})_{,l} = A_{ij,kl}$ (higher-order derivatives are defined as derivatives) [2]

[1] [**?**] purposely introduced the above notation as to clearly avoid any confusion between a tensor and Christoffel symbols. Another oft-used symbols for Christoffel symbol of second kind is $\left\{ {i \atop j\ k} \right\} = \Gamma^i_{jk}$. This notation is used for instance by [**?**], and many other authors.

[2] It is worth noting that unlike partial derivatives, where for example the second order derivative of a function f, $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$. This does not necessarily apply for higher-order covariate derivatives of tensors, where in general $A_{i,jk} \neq A_{i,kj}$.

We can now finally introduce the Riemann-Christoffel Tensor. The tensor is found by the following identity

$$A_{i,jk} - A_{i,kj} = A_m R^m{}_{ijk}, \tag{14}$$

where the fourth-order tensor

$$R^m{}_{ijk} = \frac{\partial}{\partial x^j}\begin{Bmatrix} m \\ i\ k \end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix} m \\ i\ j \end{Bmatrix} + \begin{Bmatrix} n \\ i\ k \end{Bmatrix}\begin{Bmatrix} m \\ n\ j \end{Bmatrix} - \begin{Bmatrix} n \\ i\ j \end{Bmatrix}\begin{Bmatrix} m \\ n\ k \end{Bmatrix}, \tag{15}$$

is called the Riemann-Christoffel tensor[1,2], Riemann curvature tensor, or simply Riemann tensor. The covariant form of the tensor can be found by multiplying the Riemann curvature tensor with the metric g_{pm}

$$R_{pijk} = g_{pm} R^m{}_{ijk}.$$

This tensor is skew-symmetric in two of it's indices

$$R_{pijk} = -R_{pikj}$$
$$R_{jkpi} = -R_{jkip}$$

From the covariant Riemann-Christoffel tensor it follows that there are $N = \frac{1}{12}n^2(n^2-1)$ independent components. For two-dimensions there is only one independent component, while for three-dimensions there are 6 independent components

The tensor therefore provides a way for determining if the space is flat or curved. If the Riemann tensor for a given spacetime is zero every where, initially any parallel geodesics remain parallel (the spacetime is flat, otherwise the spacetime is curved).

We may also define the Ricci tensor R_{jk} by contracting the curvature tensor (15),

$$R_{jk} = R^m{}_{mjk} = g^{pi} R_{pijk}.$$

When expressed with Christoffel symbols, we can show that the Ricci tensor is symmetric. Performing another contraction, we get the scalar curvature

$$R = R^j{}_k = g^{kj} R_{jk}.$$

Since this quantity is invariant in any coordinate system, it gives a coordinate independent measure of a space curvature. However, even though both the Ricci tensor and the scalar curvature can be zero in *curved space*, a non-zero value clearly indicates that the space is curved. Which in it self is useful. Only by evaluating the Riemann tensor can one conclusively distinguish flat and curved space [**?**].

[1] Named after Bernhard Riemann and Elwin Bruno Christoffel.

[2] Another way to introduce the Riemann curvature tensor is by using the geodesic differential equations. We will later introduce these when we discuss techniques for visualization of second order tensor fields.

1.6 Example: Riemann Curvature Tensor for Toroidal Coordinates

This coordinate system (η, θ, ψ) results from rotating a two-dimensional bipolar coordinate system about the axis that separates it's two foci. The coordinate relationship between toroidal and cartesian coordinate system is given as

$$x = \frac{a \sinh(\eta) \cos(\psi)}{\cosh(\eta) - \cos(\theta)} \tag{16}$$

$$y = \frac{a \sinh(\eta) \sin(\psi)}{\cosh(\eta) - \cos(\theta)} \tag{17}$$

$$z = \frac{a \sin(\theta)}{\cosh(\eta) - \cos(\theta)} \tag{18}$$

where θ coordinate of a point P equals the angle F_1PF_2, where F_1 and F_2 are the two foci as displayed below.

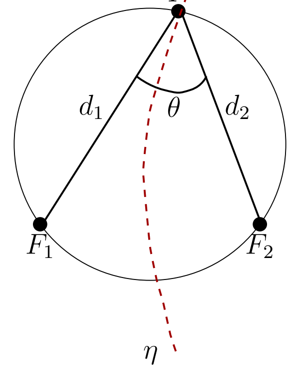

Figure 1: Reference circle on a toroid.

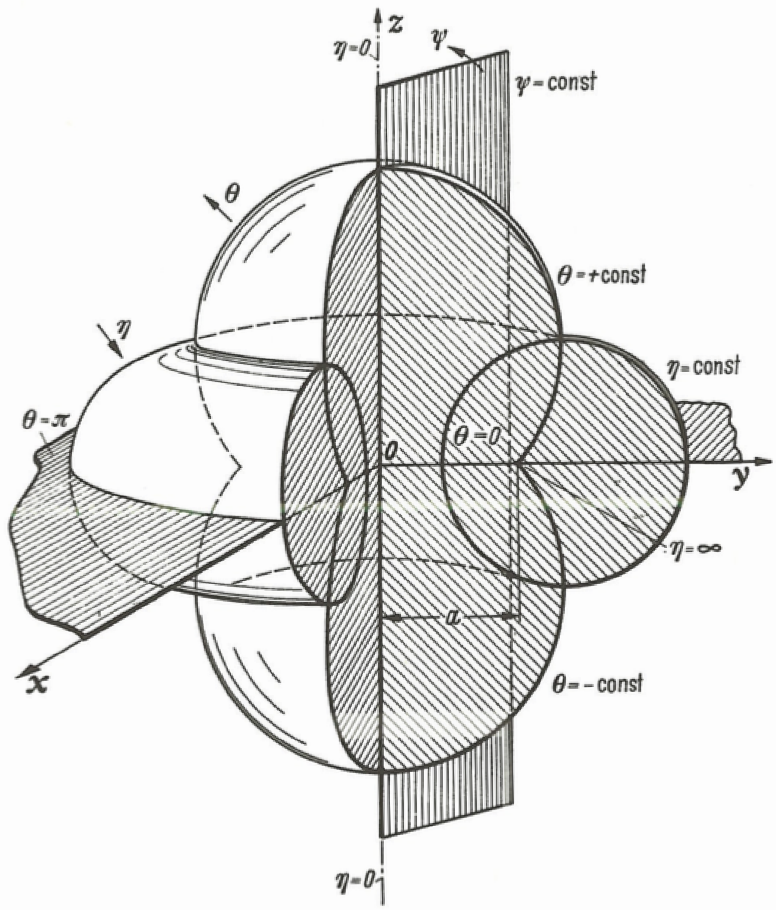

Fig. 4.04. Toroidal coordinates (η, θ, ψ). Coordinate surfaces are toroids ($\eta = \text{const}$), spherical bowls ($\theta = \text{const}$), and half-planes ($\psi = \text{const}$)

Figure 2: Coordinate surfaces for toriodal coordinates. Picture scanned from [**?**].

In the xy-plane the focal ring(also known as the *reference circle*) has radius a. The η coordinate is given as the following ratio

$$\eta = \ln \frac{d_1}{d_2}.$$

The coordinates vary as following :

$$\begin{cases} -\pi & < & \theta & < & \pi, \\ 0 & \leq & \psi & < & 2\pi, \\ & & \eta & \geq & 0. \end{cases}$$

Our first task is to determine the metric (found from the transformation between cartesian and toroidal coordinates), which is given as

$$g_{ij} = \frac{\partial \mathbf{r}}{\partial u_i} \cdot \frac{\partial \mathbf{r}}{\partial u_j},$$

where $u_i = (\eta, \theta, \psi)$, for $i = 1, 2, 3$, and $\mathbf{r}$ is the vector

$$\mathbf{r} = x\hat{\mathbf{e}}_1 + y\hat{\mathbf{e}}_2 + z\hat{\mathbf{e}}_3 = x(\eta, \theta, \psi)\hat{\mathbf{e}}_1 + y(\eta, \theta, \psi)\hat{\mathbf{e}}_2 + z(\eta, \theta, \psi)\hat{\mathbf{e}}_3$$

spanned by cartesian orthogonal basis[1]. We can find all the components of the metric by hand calculations, or through Python using the symbolic package SymPy (see Appendix **??**).

The entries of the metric g_{ij} are found from change of coordinates when applying the equations (16)-(18). The only non-zero entries are found to be the diagonal entries, i.e $i = j$. Which are given as

$$\begin{aligned} g_{11} &= \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_1} = \frac{\partial \mathbf{r}}{\partial \eta} \cdot \frac{\partial \mathbf{r}}{\partial \eta} \\ &= \frac{\partial(x\hat{\mathbf{e}}_1 + y\hat{\mathbf{e}}_2 + z\hat{\mathbf{e}}_3)}{\partial \eta} \cdot \frac{\partial(x\hat{\mathbf{e}}_1 + y\hat{\mathbf{e}}_2 + z\hat{\mathbf{e}}_3)}{\partial \eta} \\ &= \left[\frac{\partial x}{\partial \eta}\right]^2 + \left[\frac{\partial y}{\partial \eta}\right]^2 + \left[\frac{\partial z}{\partial \eta}\right]^2. \end{aligned}$$

Similarly, the other non-zero entries of the metric are

$$\begin{aligned} g_{22} &= \left[\frac{\partial x}{\partial \theta}\right]^2 + \left[\frac{\partial y}{\partial \theta}\right]^2 + \left[\frac{\partial z}{\partial \theta}\right]^2, \\ g_{33} &= \left[\frac{\partial x}{\partial \psi}\right]^2 + \left[\frac{\partial y}{\partial \psi}\right]^2 + \left[\frac{\partial z}{\partial \psi}\right]^2. \end{aligned}$$

Here we list the calculations necessary to find g_{11}. The other entries of g_{ij} are found by using the same techniques as demonstrated here.

$$\begin{aligned} \frac{\partial x}{\partial \eta} &= \frac{a\cos(\psi)[1 - \cosh(\eta)\cos(\theta)]}{[\cosh(\eta) - \cos(\theta)]^2} \\ &\Downarrow \\ \left[\frac{\partial x}{\partial \eta}\right]^2 &= \frac{a^2\cos^2(\psi)[1 - \cosh(\eta)\cos(\theta)]^2}{[\cosh(\eta) - \cos(\theta)]^4} \end{aligned}$$

Similary, we find the other expressions

$$\begin{aligned} \left[\frac{\partial y}{\partial \eta}\right]^2 &= \frac{a^2\sin^2(\psi)[1 - \cosh(\eta)\cos(\theta)]^2}{[\cosh(\eta) - \cos(\theta)]^4}, \\ \left[\frac{\partial z}{\partial \eta}\right]^2 &= \frac{a^2\sinh^2(\eta)\sin^2(\theta)}{[\cosh(\eta) - \cos(\theta)]^4}. \end{aligned}$$

[1] Which has the property $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$, and any derivation of the basis is identically zero. Which becomes quite practical when attempting to find the metric.

Adding all the terms together,

$$\begin{aligned} g_{11} &= \frac{a^2\cos^2(\psi)[1-\cosh(\eta)\cos(\theta)]^2 + a^2\sin^2(\psi)[1-\cosh(\eta)\cos(\theta)]^2 + a^2\sinh^2(\eta)\sin^2(\theta)}{[\cosh(\eta)-\cos(\theta)]^4} \\ &= a^2\left(\frac{1-2\cosh(\eta)\cos(\theta)+\cosh^2(\eta)\cos^2(\theta)+[1-\cos^2(\theta)]\sinh^2(\eta)}{[\cosh(\eta)-\cos(\theta)]^4}\right) \\ &= a^2\left(\frac{\cosh^2(\eta)-2\cosh(\eta)\cos(\theta)+\cos^2(\theta)}{[\cosh(\eta)-\cos(\theta)]^4}\right) \end{aligned}$$

where we have used the trignometric identities $\sin^2(\psi)+\cos^2(\psi)=\sin^2(\theta)+\cos^2(\theta)=1$, and $\cosh^2(\eta)-\sinh^2(\eta)=1$. We can simplify this expression further, by expanding the factor $[\cosh(\eta)-\cos(\theta)]^2 = \cosh^2(\eta)-2\cosh(\eta)\cos(\theta)+\cos^2(\theta)$. Hence, we end up with the first entry of the metric,

$$g_{11} = \frac{a^2}{[\cosh(\eta)-\cos(\theta)]^2}. \tag{19}$$

The other diagonal entries (i.e the non-zero entries of g_{ij}) are found to be

$$g_{22} = g_{11} = \frac{a^2}{[\cosh(\eta)-\cos(\theta)]^2}, \tag{20}$$

$$g_{33} = \frac{a^2\sinh^2(\eta)}{[\cosh(\eta)-\cos(\theta)]^2}. \tag{21}$$

Now we can find the Christoffel symbols for our metric g_{ij}, and thereby be able to determine the Riemann curvature tensor. The process is as following :

1. Determine the non-zero contributions from Equation (12)
2. Determine the non-zero contributions from Equation (13)[1]
3. Finally determine all the derivatives in Equation (15)

Here are points (1) and (2) : In order determine all the terms of the Riemann curvature tensor we must determine the Christoffel symbols of second kind, which again implies that we must determine the corresponding Christoffel symbols of first kind. For the sake of clarity, we restate the Riemann curvature tensor

$$R^m{}_{ijk} = \frac{\partial}{\partial x^j}\begin{Bmatrix} m \\ i\ k \end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix} m \\ i\ j \end{Bmatrix} + \begin{Bmatrix} n \\ i\ k \end{Bmatrix}\begin{Bmatrix} m \\ n\ j \end{Bmatrix} - \begin{Bmatrix} n \\ i\ j \end{Bmatrix}\begin{Bmatrix} m \\ n\ k \end{Bmatrix}, \tag{22}$$

Let us initially focus on the first term here

$$\frac{\partial}{\partial x^j}\begin{Bmatrix} m \\ i\ k \end{Bmatrix}. \tag{23}$$

[1]Notice that the conjugate metric g^{ij} is relatively easy to determine when the metric is diagonal. The entries of the conjugate metric become the inverse of the entries of g_{ij}.

Using the definition for Christoffel symbol of second kind (13), and inserting for the first kind (12), we get the following expression

$$\frac{\partial}{\partial x^j}\begin{Bmatrix} m \\ i\ k \end{Bmatrix} = \frac{\partial}{\partial x^j}\, g^{mk}\, [ai, k] \tag{24}$$

$$= \frac{1}{2}\frac{\partial}{\partial x^j}\, g^{mk}\left(\frac{\partial g_{ak}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^a} - \frac{\partial g_{ai}}{\partial x^k}\right) \tag{25}$$

for $a, i, k, m = 1, 2, 3$, and where the coordinates are $x^i = \eta, \theta, \psi$, for $i = 1, 2, 3$.

There are three cases where we get contribution from the Equation (25).

- Case 1, where $m = k$, and $a = k$,
- Case 2, where $m = k$, and $i = k$,
- Case 3, where $m = k$, and $i = a$.

For every case we require that $m = k$. This is because the conjugate metric g^{mk} is zero every where except on the diagonal. As $g^{mk} = (g_{mk})^{-1}$, we find conjugate metric by inverting the entries of g_{mk}.

For Case 1, we get the following expression

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial x^j}\, g^{mk}\left(\frac{\partial g_{ak}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^a} - \frac{\partial g_{ai}}{\partial x^k}\right) &= \frac{1}{2}\frac{\partial}{\partial x^j}\, g^{kk}\left(\frac{\partial g_{kk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^k}\right) \\ &= \frac{1}{2}\frac{\partial}{\partial x^j}\left(g^{kk}\frac{\partial g_{kk}}{\partial x^i}\right)\end{aligned}$$

Similarly, for the other two cases we get the expressions

$$\begin{aligned}&\text{Case 2}: \ m = k, \qquad i = k \\ \frac{1}{2}\frac{\partial}{\partial x^j}\, g^{mk}\left(\frac{\partial g_{ak}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^a} - \frac{\partial g_{ai}}{\partial x^k}\right) &= \frac{1}{2}\frac{\partial}{\partial x^j}\, g^{kk}\left(\frac{\partial g_{ak}}{\partial x^k} + \frac{\partial g_{kk}}{\partial x^a} - \frac{\partial g_{ak}}{\partial x^k}\right) \\ &= \frac{1}{2}\frac{\partial}{\partial x^j}\left(g^{kk}\frac{\partial g_{kk}}{\partial x^a}\right) \\ &\text{Case 3}: \ m = k, \qquad i = a \\ \frac{1}{2}\frac{\partial}{\partial x^j}\, g^{mk}\left(\frac{\partial g_{ak}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^a} - \frac{\partial g_{ai}}{\partial x^k}\right) &= \frac{1}{2}\frac{\partial}{\partial x^j}\, g^{kk}\left(\frac{\partial g_{ik}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^k}\right) \\ &= \frac{1}{2}\frac{\partial}{\partial x^j}\left(g^{kk}\frac{\partial g_{kk}}{\partial x^i}\right)\end{aligned}$$

Adding the results from all three of the cases, we get the result

$$\frac{\partial}{\partial x^j}\begin{Bmatrix} m \\ i\ k \end{Bmatrix} = \frac{\partial}{\partial x^j}\left(g^{kk}\frac{\partial g_{kk}}{\partial x^i} + \frac{1}{2}g^{kk}\frac{\partial g_{kk}}{\partial x^a}\right) \tag{26}$$

The other terms of the Riemann tensor (22) give corresponding terms

$$-\frac{\partial}{\partial x^k}\begin{Bmatrix} m \\ i\ j \end{Bmatrix} = -\frac{\partial}{\partial x^k}\left(g^{jj}\frac{\partial g_{jj}}{\partial x^i} - \frac{1}{2}g^{jj}\frac{\partial g_{jj}}{\partial x^a}\right) \tag{27}$$

$$\begin{Bmatrix} n \\ i\ k \end{Bmatrix}\begin{Bmatrix} m \\ n\ j \end{Bmatrix} = \left(g^{kk}\frac{\partial g_{kk}}{\partial x^i} + \frac{1}{2}g^{kk}\frac{\partial g_{kk}}{\partial x^a}\right)\left(g^{jj}\frac{\partial g_{jj}}{\partial x^a} + \frac{1}{2}g^{jj}\frac{\partial g_{jj}}{\partial x^i}\right) \tag{28}$$

$$-\begin{Bmatrix} n \\ i\ j \end{Bmatrix}\begin{Bmatrix} m \\ n\ k \end{Bmatrix} = -\left(g^{jj}\frac{\partial g_{jj}}{\partial x^i} + \frac{1}{2}g^{jj}\frac{\partial g_{jj}}{\partial x^a}\right)\left(g^{kk}\frac{\partial g_{kk}}{\partial x^a} + \frac{1}{2}g^{kk}\frac{\partial g_{kk}}{\partial x^j}\right) \tag{29}$$

We see that, for $i = j = k = a = 3$, every term becomes zero (as the respective derivatives are zero). For every other permutation, each term cancels the other, resulting in that every element of the Riemann curvature tensor is zero for the toroidal coordinates. Implying that this is a flat space.