1 Analytical Tensor Fields

Given the symmetric second rank tensor

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix},$$

For this tensor field, degenerate points satisfy the condition

$$\begin{cases} \frac{T_{11}-T_{22}}{2} & = 0 \\ T_{12} & = 0 \end{cases} \tag{1}$$

Assume that the tensor components can be expressed in the vicinity of degenerate point $\vec{x}_0 = (x_0, y_0)$ as Taylor expansions that start with homogeneous polynomials for $m > 0$

$$\begin{cases} \frac{T_{11}-T_{22}}{2} & \approx P_m(x - x_0, y - y_0) + \cdots \\ T_{12} & \approx Q_m(x - x_0, y - y_0) + \cdots \end{cases} \tag{2}$$

1.1 Analytical Examples, see pages 134 - 139

$$T = \begin{bmatrix} \frac{x^2+4xy+y^2}{2} & -x^2 + y^2 \\ -x^2 + y^2 & \frac{-x^2-4xy-y^2}{2} \end{bmatrix},$$

which has a multiple degenerate point (see definition : 1) at the origin.

$\langle$ sepratrices of multiple degenerate points are described in page 131 - 133 $\rangle$

More such examples are detailed in the same section, with description of the sepratrices that occur.

2 Algorithm to Extract the Topology of a Tensor Field

In tensor fields different types of degenerate points can occur that correspond to different local patterns of neighboring hyperstreamlines. These patterns are determined by the tensor gradients at degenerate point positions.

Let $\vec{x}_0 = (x_0, y_0)$ be an isolated degenerate point. Assuming that the functions $T_{11}(\vec{x}) - T_{22}(\vec{x})$ and $T_{12}(\vec{x})$ are analytic, we can expand tensor components in Taylor series around $\vec{x}_0$.

$\vdots$

$\vdots$

$\langle$after some simplifications, we end up with the following - see pages 113-114$\rangle$

$$\begin{cases} \frac{T_{11}-T_{22}}{2} & \approx a(x - x_0) + b(y - y_0) + \cdots \\ T_{12} & \approx c(x - x_0) + d(y - y_0) + \cdots \end{cases} \tag{3}$$

where

$$a \equiv \frac{1}{2} \left. \frac{\partial (T_{11} - T_{22})}{\partial x} \right|_{x_0} \qquad b \equiv \frac{1}{2} \left. \frac{\partial (T_{11} - T_{22})}{\partial y} \right|_{y_0}$$

$$c \equiv \frac{1}{2} \left. \frac{\partial T_{12}}{\partial x} \right|_{x_0} \qquad d \equiv \frac{1}{2} \left. \frac{\partial T_{12}}{\partial y} \right|_{y_0}$$

An important quantity for characterizing degenerate points is

$$\delta = ad - bc \tag{4}$$

which is invariant under rotation.⟨See pages 114 - 115 for proof.⟩

Definition 1 (Simple and Multiple Degenerate Points)**.** Let $\vec{x}_0$ be an isolated degenerate point of a tensor field T $\in C^1(E)$, where E is an open subset of $\mathbf{R}^2$, and let $\delta = ad - bc$ be the corresponding third-order invariant. Then, $\vec{x}_0$ is

- a simple degenerate point iff $\delta \neq 0$, or
- a multiple degenerate point iff $\delta = 0$.

Theorem 1. The angle θ_k between the x-axis and the sepratrices s_k are obtained by computing the real roots z_k of the cubic equation

$$dz^3 + (c + 2b)z^2 + (2a - d)z - c = 0, \tag{5}$$

by inverting the relation $z_k = tan\theta_k$, and by keeping only those angles that lie along the boundary of a hyperbolic sector.

⟨for the whole theorem - see page 120⟩

2.1 The Algorithm - see page 154

The following algorithm handles simple degenerate points only :

1. locate degenerate points by searching for solutions to (1) in every grid cell;
2. classify each degenerate point as a trisector ($\delta < 0$) or a wedge point ($\delta > 0$) by evaluating a,b,c,d and computing $\delta = ad - bc$;
3. select an eigenvector field;
4. solve (5) to find the directions of the three sepratrices (s_1, s_2, s_3) at each trisector point; likewise, extract sepratrices (s_1, s_2) at wedge points where (5) admits three real roots and extract the unique sepratrix at wedge points where (5) has only one real root;
5. integrate hyperstreamlines along the sepratrices; terminate the trajectories wherever they leave the domain, or impinge on the parabolic sector of a wedge point.

References

[1] T. Delmarcelle, 1994. *The Visualization of Second-Order Tensor Fields.*