

1 Visualization of Tensor Fields : Current State

Techniques for tensor field visualization can be viewed as two distinct methods : indirect and direct visualization methods. Direct visualization of tensor fields is accomplished with tensor glyphs; ellipsoids that are formed by the eigenvalues of the tensor field, or by drawing integral lines or surfaces. The purpose is to depict full information contained in the field [?], [?], [?], [?]. Indirect methods usually entail the extraction of of spesific structural features from the tensor field, like for example topological features [?], [?], [?]. In this thesis, our main focus will be direct visualization methods, with emphasis on integration methods.

1.1 Direct Visualization Methods

1.1.1 Hyperstreamlines

A second order symmetric tensor in \mathbf{R}^3 has six independent elements or components. We can reduce this number down to three by performing an eigendecomposition of the tensor, resulting in three real eigenvalues. Selecting one of these (usually either the major or the minor eigenvalue), we can use it's corresponding eigenvector as the direction for the so-called hyperstreamline trajectories, which in \mathbf{R}^3 are tube like structures where the other eigenvalues determine the thickness of the trajectories. They show, for example how forces are transmitted in a stress-tensor field and how momentum is transferred in a momentum-flux-density tensor field. Delmarcelle and Hesselberg, [?], were first to introduce the notion of hyperstreamlines. The issue with hyperstreamlines are the same as we find with streamlines for depiction of vector fields, or isosurfaces for scalar fields; occlusion and seeding. In addition, we need to handle degenerate points (similar to the notion of critical points in a vector field, where in the case of tensors, the degenerate points occur when at least two of the eigenvalues are equal.).

In [?], Delmarcelle and Hessellink suggest a simply way of handling degenerate points (by simply ignoring them). By assuming that the tensor field is smooth, we can then assume that the direction of the eigenvector is not likely to vary abruptly (i.e vary more than some user defined angle) in between grid points. If, however, the direction changes abruptly, then it is reasonable to assume that the hyperstreamline has crossed a degeneracy. To handle this, we can simply jump over this point and continue integrating along the direction of the eigenvector. Further, we can also keep a track over the transverse eigenvalues to make sure that, where they vanish, we can still include the singularities of the cross-section of the hyperstreamline.

1.1.2 Tensor Glyphs

The traditional approach to direct visualization of symmetric second order tensor fields in \mathbf{R}^2 and \mathbf{R}^3 have been so-called tensor glyphs. This technique has been employed heavily for medical imaging, depicting e.g. diffusion tensor. Diffusion tensor imaging (DTI) provides information about the diffusion properties of water molecules in brain tissue (which differs for gray matter and white matter). Within white matter the motion of water molecules is restricted in directions that are perpendicular to the fiber tracts. Because of the physical properties of white matter, DTI can be used to investigate white matter in the brain. It's main clinical application has been in the study and treatment of neurological disorders, such as schizophrenia. DTI makes it possible to evaluate the connectivity and coherence of white

matter fiber tract, which are thought to be abnormal in schizophrenia [?]. The diffusion tensor is a second order symmetric tensor in \mathbf{R}^3 . The tensor is visualized as an ellipsoid, with the longest axis pointing toward the principal direction, i.e the major eigenvector. The shape of the tensor depends upon the strength of the diffusion along the three eigenvectors.

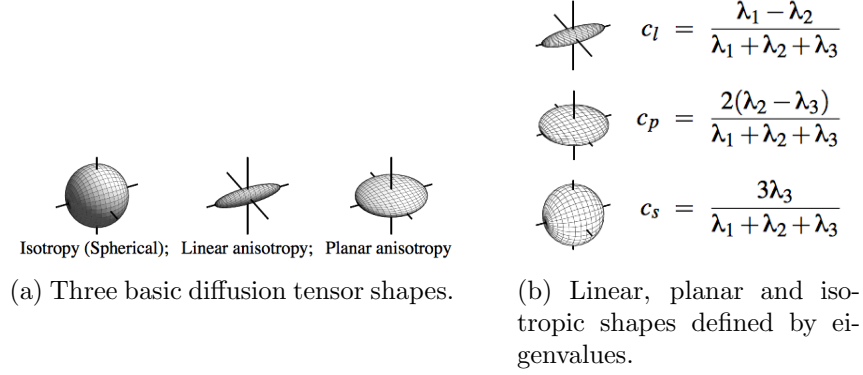


Figure 1: Linear, planar and isotropic shapes defined by eigenvalues. Picture taken from [?].

1.1.3 Asymmetric tensors fields

We can always describe any (of either covariant or contravariant type) second order tensor by the sum of their symmetric and anti-symmetric components. For a second-order tensor of type (2,0), we can write it as

$$S_{ij} = \frac{1}{2} (S_{ij} + S_{ji}) + \frac{1}{2} (S_{ij} - S_{ji}) \quad (1)$$

A physical example for this case is the gradient of a velocity field, $\nabla \mathbf{v}$. If we denote the components of \mathbf{v} as (v_i) , then we can write the gradient as

$$\nabla \mathbf{v} = \begin{bmatrix} \frac{\partial v_1}{\partial x^1} & \frac{\partial v_2}{\partial x^1} & \frac{\partial v_3}{\partial x^1} \\ \frac{\partial v_1}{\partial x^2} & \frac{\partial v_2}{\partial x^2} & \frac{\partial v_3}{\partial x^2} \\ \frac{\partial v_1}{\partial x^3} & \frac{\partial v_2}{\partial x^3} & \frac{\partial v_3}{\partial x^3} \end{bmatrix}, \quad (2)$$

or using index form we can write the components as $\partial v_i / \partial x^j$. Using the above notation (1), we can decompose (2) as

$$\frac{\partial v_i}{\partial x^j} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} \right) + \frac{1}{2} \left(\frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i} \right)$$

The symmetric part is the sum of the Jacobian and transpose of the Jacobian, also referred to as the *rate of strain tensor*. The diagonal entries are the normal strain rates and the off-diagonal entries are the shear strain rates.

We can associate a vector ω_i ,

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial v_3}{\partial x^2} - \frac{\partial v_2}{\partial x^3} \\ \frac{\partial v_1}{\partial x^3} - \frac{\partial v_3}{\partial x^1} \\ \frac{\partial v_2}{\partial x^1} - \frac{\partial v_1}{\partial x^2} \end{bmatrix}$$

with an anti-symmetric tensor defined by

$$R \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

where they are related as

$$R_{ij} = -e_{ijk}\omega_k, \\ \omega_k = -\frac{1}{2}e_{ijk}R_{ij}.$$

R is called the *Rotation tensor*.

We can therefore represent the anti-symmetric part as a vector and thereby employ techniques from vector field visualization. For the symmetric part of the tensor, we can perform an eigendecomposition. Which permits us to represent the tensor by it's eigenvectors and their corresponding eigenvalues. Delmarcelle and Hesselberg [?] were first to decompose 2D symmetric tensors of type (2,0) in such a manner.

1.1.4 The Geodesic Differential Equations

In Euclidean space, \mathbf{R}^3 , the distance between two points (x_1, x_2, x_3) and (y_1, y_2, y_3) is given as $d^2 = |x_i - y_i|^2$. In other words, a straight line that connects the two points. According to general relativity, particles in a gravitation field move along geodesics of space-time described by the metric g_{ij} . This is the shortest path between two points on curved space(e.g space-time continuum). In Euclidean geometry, initially parallel straight lines remain parallel. Initially parallel geodesics in spacetime do not necessarily remain parallel (as they follow the spacetime curvature).

In order to determine the geodesics, one must solve the following set of differential equations

$$\frac{d^2 x^j}{ds^2} + \left\{ \begin{matrix} j \\ h \ k \end{matrix} \right\} \frac{dx^h}{ds} \frac{dx^k}{ds} = 0, \quad (3)$$

where $\left\{ \begin{matrix} j \\ h \ k \end{matrix} \right\}$ is the Christoffel symbol of second kind. The geodesic solution $x^j(s)$ is a curve defined on the interval $s \in [s_0, s_1]$. Often, these equations can not be solved analytically, and we can only solve them numerically.

The solution is a way of depicting the metric tensor. We can extend this method to include any tensor T_{ij} . However, when solving the geodesic differential equations, one of main issues with this method is that the conjugate metric g^{ij} has to exist. If the metric g_{ij} is singular, then we can not solve the differential equations (3). So, if we are to extend the method of visualization of tensor fields T_{ij} to the concept of a metric, then a singular tensor will not work. The momentum flux density tensor $\rho v_i v_j$ is an important tensor which is singular. Hence, for such a tensor, we can not determine the geodesics.

1.2 Indirect Visualization Methods

[?] was first to introduce a topological approach for visualization of tensor fields, as noted by [?]. Delmarcelle introduced the notion of degenerate points, which correspond to critical points in vector fields. The following (based on [?]) describes an algorithm for tracking degenerate points.

Given the symmetric second rank tensor

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix},$$

For this tensor field, degenerate points satisfy the condition

$$\begin{cases} \frac{T_{11}-T_{22}}{2} = 0 \\ T_{12} = 0 \end{cases} \quad (4)$$

Assume that the tensor components can be expressed in the vicinity of degenerate point $\vec{x}_0 = (x_0, y_0)$ as Taylor expansions that start with homogeneous polynomials for $m > 0$

$$\begin{cases} \frac{T_{11}-T_{22}}{2} \approx P_m(x - x_0, y - y_0) + \dots \\ T_{12} \approx Q_m(x - x_0, y - y_0) + \dots \end{cases}$$

In tensor fields different types of degenerate points can occur that correspond to different local patterns of neighboring hyperstreamlines. These patterns are determined by the tensor gradients at degenerate point positions.

Let $\vec{x}_0 = (x_0, y_0)$ be an isolated degenerate point. Assuming that the functions $T_{11}(\vec{x}) - T_{22}(\vec{x})$ and $T_{12}(\vec{x})$ are analytic, we can expand tensor components in Taylor series around \vec{x}_0 . After some simplifications, we end up with the following expression

$$\begin{cases} \frac{T_{11}-T_{22}}{2} \approx a(x - x_0) + b(y - y_0) + \dots \\ T_{12} \approx c(x - x_0) + d(y - y_0) + \dots \end{cases}$$

where

$$\begin{aligned} a &\equiv \frac{1}{2} \frac{\partial(T_{11} - T_{22})}{\partial x} \Big|_{x_0} & b &\equiv \frac{1}{2} \frac{\partial(T_{11} - T_{22})}{\partial y} \Big|_{y_0} \\ c &\equiv \frac{1}{2} \frac{\partial T_{12}}{\partial x} \Big|_{x_0} & d &\equiv \frac{1}{2} \frac{\partial T_{12}}{\partial y} \Big|_{y_0} \end{aligned}$$

An important quantity for characterizing degenerate points is

$$\delta = ad - bc$$

which is invariant under rotation.

Definition (Simple and Multiple Degenerate Points). *Let \vec{x}_0 be an isolated degenerate point of a tensor field $T \in C^1(E)$, where E is an open subset of \mathbf{R}^2 , and let $\delta = ad - bc$ be the corresponding third-order invariant. Then, \vec{x}_0 is*

- a simple degenerate point iff $\delta \neq 0$, or
- a multiple degenerate point iff $\delta = 0$.

Theorem. The angle θ_k between the x -axis and the sepratrices s_k are obtained by computing the real roots z_k of the cubic equation

$$dz^3 + (c + 2b)z^2 + (2a - d)z - c = 0, \quad (5)$$

by inverting the relation $z_k = \tan\theta_k$, and by keeping only those angles that lie along the boundary of a hyperbolic sector.

The following algorithm handles simple degenerate points only :

1. locate degenerate points by searching for solutions to (4) in every grid cell;
2. classify each degenerate point as a trisector ($\delta < 0$) or a wedge point ($\delta > 0$) by evaluating a,b,c,d and computing $\delta = ad - bc$;
3. select an eigenvector field;
4. solve (5) to find the directions of the three sepratrices (s_1, s_2, s_3) at each trisector point; likewise, extract sepratrices (s_1, s_2) at wedge points where (5) admits three real roots and extract the unique sepratrix at wedge points where (5) has only one real root;
5. integrate hyperstreamlines along the sepratrices; terminate the trajectories wherever they leave the domain, or impinge on the parabolic sector of a wedge point.

1.3 Tensor Overview

Here we provide an overview of second order tensors.

The momentum flux density tensor, stems from the Navier Stokes equations, which we can split into three components

- advection of the i -th component in the j -th direction : $\rho v_i v_j$
- pressure : $p\delta_{ij}$
- stress tensor : σ_{ij}

The stress at a point can completely be specified [?] by nine components,

$$\tau = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix}.$$

Here, we differentiate between the diagonal components; which are normal stresses, and the off-diagonal components; which are shear stresses. The stress tensor describes stresses that act as reaction to external forces. Similarly, the strain tensor is related to the deformation of a body due to stress.

Second order tensor overview					
Tensor	Symmetric	Hyperstreamlines	Tensor Glyphs	Geodesics	Application
Velocity gradient	No				Fluid mechanics
Strain rate tensor	Yes	✓	✓		Fluid mechanics
Stress tensor	Yes	✓	✓		Continuum mechanics
Diffusion tensor	Yes	✓	✓	✓	Medical imaging
Reynolds stress	Yes	✓	✓		Computational fluid mechanics
Metric tensor	Yes	✓	✓	✓	Differential geometry/ General relativity
Momentum flux density	Yes	✓	✓		Fluid mechanics

Table 1: Tensor visualization options

The strain rate tensor is related to the velocity field \mathbf{v} by

$$S_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} \right)$$

Stress and strain along the eigenvectors are called principal stress and principal strain.

Another important tensor can be derived from the convective term of Reynolds averaged Navier Stokes equations, (the derivative of) the average of the products of the fluctuation velocity components, a second order tensor, with components $-\rho \overline{u_i u_j}$

$$\begin{bmatrix} -\rho u_1 u_1 & -\rho u_1 u_2 & -\rho u_1 u_3 \\ -\rho u_2 u_1 & -\rho u_2 u_2 & -\rho u_2 u_3 \\ -\rho u_3 u_1 & -\rho u_3 u_2 & -\rho u_3 u_3 \end{bmatrix}.$$

This is an important tensor in turbulence modeling. It is called the Reynolds stress tensor¹. The Reynolds stress tensor is symmetric. The diagonal components are normal stresses, and the off-diagonal components are shear stresses. It is a stress exerted by the turbulent fluctuations on the mean flow. Another way to interpret Reynolds stress is that it is the rate of mean momentum transfer by turbulent fluctuations [?].

In the Table 1 we have listed an overview of different tensors. Note that in order to determine geodesics for a tensor, the Christoffel symbol of second kind has to be invertible. If a tensor is singular, then this method can not be applied. Similarly, hyperstreamlines are found for symmetric tensors. By decomposing a tensor in symmetric and anti-symmetric components, we can find hyperstreamlines. For the anti-symmetric part of the tensor, we can apply common vector field visualization techniques.

¹Strictly speaking, the Reynolds stresses are not stresses, but the averaged effect of turbulent convection.