Similarly, $(X+Y)+Z = X+(\ldots$
and $X+0 = X$ is dual of $X.1 = X$

In proving the theorems or rules of boolean algebra, it is then necessary to prove only one theorem, and the dual of the theorem follows necessarily.

In effect, *all boolean algebra is predicated on this two-for-one basis.*

In proving the theorems or rules of boolean algebra, it is then necessary to prove only one theorem, and the dual of the theorem follows necessarily.

EXAMPLE 12.5. *Give the dual of the following result in Boolean algebra :*

$$X . X' = 0 \text{ for each } X.$$

Solution. Using duality principle, dual of $X . X' = 0$ is $X + X' = 1$ (By changing (.) to (+) and vice versa and by replacing 1's by 0's and vice versa).

EXAMPLE 12.6. *Give the dual of $X + 0 = X$ for each X.*

Solution. Using duality principle, dual of $X + 0 = X$ is $X.1 = X$

EXAMPLE 12.7. *State the principle of duality in boolean algebra and give the dual of the boolean expression :*

$$(X+Y).(\overline{X}+\overline{Z}).(Y+Z)$$

Solution. Principle of duality states that from every boolean relation, another boolean relation can be derived by

(*i*) changing each OR sign (+) to an AND (.) sign
(*ii*) changing each AND (.) sign to an OR (+) sign
(*iii*) replacing each 1 by 0 and each 0 by 1.

The new derived relation is known as the dual of the original relation.

Dual of $(X+Y).(\overline{X}+\overline{Z}).(Y+Z)$ will be

$$(X.Y)+(\overline{X}.\overline{Z})+(Y.Z) = XY + \overline{X}\,\overline{Z} + YZ$$

12.9 Derivation of Boolean Expression

Boolean expressions which consist of a single variable or its complement *e.g.*, X or $\overline{Y}$ or $\overline{Z}$ are known as *literals*.

Now before starting derivation of boolean expression, first we will talk about two very important terms. These are (*i*) Minterms (*ii*) Maxterms

12.9.1 Minterms

One of the most powerful theorems within boolean algebra states that any boolean function can be expressed as the sum of products of all the variables within the system. For example, $X + Y$ can be expressed as the *sum of several products*, each of the product containing letters X and Y. These products are called *minterms* and each contains all the *literals* with or without the bar.

…n A minterm is a …f all the literals …without the bar) …logic system.

Also when values are given for different variables, minterm can easily be formed. *e.g.*, if $X=0, Y=1, Z=0$ then minterm will be $\overline{X}Y\overline{Z}$ *i.e.*, for variable with a value 0, take its complement and the one with value 1, multiply it as it i… Similarly for $X=1, Y=0, Z=0$, minterm will be $X\overline{Y}\,\overline{Z}$.

Steps involved in minterm expansion of expression

1. First convert the given expression in sum of products form.

2. In each term, if any variable is missing (e.g., in the following example Y is missing in first term and X is missing in second term), multiply that term with (missing term + $\overline{\text{missing term}}$) factor, (e.g., if Y is missing, multiply with $Y + \overline{Y}$).
3. Expand the expression.
4. Remove all duplicate terms and we will have minterm form of an expression.

EXAMPLE 12.8. *Convert X + Y to minterms.*

Solution. $X + Y = X.1 + Y.1$

($\because$ in 1st term Y is missing and in 2nd term X is missing)

$= X.(Y + \overline{Y}) + Y(X + \overline{X})$ ($X + \overline{X} = 1$ complementarity law)

$= XY + X\overline{Y} + XY + \overline{X}Y$

$= XY + XY + X\overline{Y} + \overline{X}Y$

$= XY + X\overline{Y} + \overline{X}Y$ ($XY + XY = XY$ Idempotent law)

Note that each term in the above example contains all the letters used : X and Y. The terms XY, $X\overline{Y}$ and $\overline{X}Y$ are therefore minterms. This process is called *expansion of expression.*

Other procedure for expansion could be

1. Write down all the terms
2. Put X's where letters much be inserted to convert the term to a product term
3. Use all combinations of X's in each term to generate minterms
4. Drop out duplicate terms.

EXAMPLE 12.9. *Find the minterms for AB + C.*

Solution. It is a 3 variable expression, so a product term must have all three letters A, B and C

1. Write down all the terms $AB + C$
2. Insert X's where letters are missing $ABX + XXC$
3. Write all the combinations of X's in first term $AB\overline{C}$, ABC

   Write all the combinations of X's in second term $\overline{A}\,\overline{B}C$, $A\overline{B}C$, ABC, $\overline{A}BC$
4. Add all of them.

   Therefore, $AB + C = AB\overline{C} + ABC + \overline{A}\,\overline{B}C + A\overline{B}C + ABC + \overline{A}BC$
5. Now remove all duplicate terms

$$= AB\overline{C} + ABC + \overline{A}\,\overline{B}C + A\overline{B}C + \overline{A}BC$$

Now, to verify we will prove vice versa

i.e., $AB\overline{C} + ABC + \overline{A}\,\overline{B}C + A\overline{B}C + \overline{A}BC = AB + C$

L.H.S. $= AB\overline{C} + ABC + \overline{A}\,\overline{B}C + A\overline{B}C + \overline{A}BC$

$= \overline{A}\,\overline{B}C + \overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$ (*rearranging the terms*

$= \overline{A}C(\overline{B} + B) + A\overline{B}C + AB(\overline{C} + C)$

$= \overline{A}C1 + A\overline{B}C + AB.1$ ($\overline{B} + B = 1$ and $\overline{C} + C =$

$= \overline{A}C + AB + A\overline{B}C$

$= \overline{A}C + A(B + \overline{B}C)$

$= \overline{A}C + A(B+C)$ $(X + \overline{X}Y = X + Y$ Rule 10, Table 12...

$= \overline{A}C + AB + AC = \overline{A}C + AC + AB$

$= C(\overline{A} + A) + AB$

$= C.1 + AB$ $(\overline{A} + A...$

$= C + AB$

$= AB + C = \text{R.H.S.}$

Shorthand minterm notation

Since all the letters (*2 in case of 2 variable expression, 3 in case of 3 variable expression*) must appear in every product, a shorthand notation has been developed that saves actually writing down the letters themselves. To form this notation, following steps are to be followed :

1. First of all, copy original terms.
2. Substitute 0's for barred letters and 1's for nonbarred letters.
3. Express the decimal equivalent of binary word as a subscript of m.

Example 12.10. *To find the minterm designation of $X\overline{Y}\,\overline{Z}$.*

Solution. 1. Copy Original form $= X\overline{Y}\,\overline{Z}$

2. Substitute 1's for non barred and 0's for barred letters

Thus, Binary equivalent will be 100.

Decimal equivalent of $100 = 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = 4 + 0 + 0 = 4$

3. Express as decimal subscript of $m = m_4$

Thus $X\overline{Y}\,\overline{Z} = m_4$

Similarly, minterm designation of $A\overline{B}C\overline{D}$ would be

Copy Original Term $A\overline{B}C\overline{D}$

Binary equivalent $= 1\,0\,1\,0$

Decimal equivalent $= 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 8 + 0 + 2 + 0 = 10$

Express as subscript of $m = m_{10}$

12.9.2 Maxterms

Trying to be logical about logic, if there is something called *minterm*, there surely must be one called *Maxterm* and there is.

If the value of a variable is 1, then its complement is added otherwise the variable is added as it is *e.g.*,

If the values of variables are $X = 0, Y = 1$ and $Z = 1$ then its Maxterm will be

$$X + \overline{Y} + \overline{Z}$$

(Y and Z are 1's, so their complements are taken ; X = 0, so it is taken as it is)

Similarly, if given values are $X = 1, Y = 0, Z = 0$ and $W = 1$ then its Maxterm is

$$\overline{X} + Y + Z + \overline{W}$$

> **Maxterm** A *Maxterm* is a sum of all the literals (with or without the bar) within the logic system.

Maxterms can also be written as M (capital M) with a subscript which is decimal equivalent of given input combination *e.g.*, above mentioned Maxterm $\overline{X} + Y + Z + \overline{W}$ whose input combination is 1001 can be written as M_9 as decimal equivalent of 1001 is 9.

... expression composed entirely either of Minterms or Maxterms is referred to as Canonical Expression.

Canonical expression can be represented in following *two* forms :

(*i*) Sum-of-Products (S-O-P) form (*ii*) Product-of-sums (P-O-S) form

...nical Expression A boolean Expression composed entirely either of Minterms or Maxterms is referred to as Canonical Expression.

(*i*) Sum-of-Products (S-O-P)

A logical expression is derived from *two* sets of known values :

1. various possible input values
2. the desired output values for each of the input combinations.

Let us consider a specific problem.

A logical network has two inputs X and Y and an output Z. The relationship between inputs and outputs is to be as follows :

(*i*) When $X=0$ and $Y=0$ then $Z=1$

(*ii*) When $X=0$ and $Y=1$ then $Z=0$

(*iii*) When $X=1$ and $Y=0$, then $Z=1$

(*iv*) When $X=1$ and $Y=1$, then $Z=1$

We can prepare a truth table from the above relations which is as follows :

Table 12.30 Truth table for product terms (2-input).

X	Y	Z	Product Terms
0	0	1	$\overline{X}\,\overline{Y}$
0	1	0	$\overline{X}Y$
1	0	1	$X\overline{Y}$
1	1	1	XY

Here, we have added one more column to the table consisting list of *product terms* or *minterms.*

...onical Sum-of-Products When a boolean ...ssion is represented as sum of minterms, it ... to be in Canonical ...-Products Form.

Adding all the terms for which the output is 1 *i.e.*, $Z=1$ we get following expression :

$$\overline{X}\,\overline{Y}+X\overline{Y}+XY=Z$$

Now see, it is an expression containing only minterms. This type of expression is called *minterm canonical form of boolean expression* or *canonical sum-of-products form of* expression.

EXAMPLE 12.11 *A boolean function F defined on three input variables X, Y and Z is 1 if and only if number of 1 (one) inputs is odd (e.g., F is 1 if $X=1, Y=0, Z=0$). Draw the truth table for the above function and express it in canonical sum-of-Products form.* (C.B.S.E. 1994)

Solution. The output is 1, only if one of the inputs is odd. All the possible combinations when one of inputs is odd are

$$X=1,\quad Y=0,\quad Z=0$$

$$X=0,\quad Y=1,\quad Z=0$$

$$X=0,\quad Y=0,\quad Z=1$$

for these combinations output is 1, otherwise output is 0. Preparing the truth table for it, we get :

Table 12.31
Truth table for product terms (3-input)

X	Y	Z	F	Product Terms / Minterms
0	0	0	0	$\overline{X}\,\overline{Y}\,\overline{Z}$
0	0	1	1	$\overline{X}\,\overline{Y}Z$
0	1	0	1	$\overline{X}Y\overline{Z}$
0	1	1	0	$\overline{X}YZ$
1	0	0	1	$X\overline{Y}\,\overline{Z}$
1	0	1	0	$X\overline{Y}Z$
1	1	0	0	$XY\overline{Z}$
1	1	1	0	XYZ

Adding all the minterms (product terms) for which output is 1, we get

$$\overline{X}\,\overline{Y}Z + \overline{X}Y\overline{Z} + X\overline{Y}\,\overline{Z} = F$$

This is the desired *Canonical Sum-of-Products form.*

So, ***deriving S-O-P expression from Truth Table*** can be summarised as follows :

1. For a given expression, prepare a truth table for all possible combinations of inputs.
2. Add a new column for minterms and list the minterms for all the combinations.
3. Add all the minterms for which there is output as 1. This gives you the desired canonical *S-O-P* expression.

Another method of deriving canonical *S-O-P* expression is **Algebraic Method**. This is just the same as we have covered in section 12.9.1. We will take another example here.

EXAMPLE 12.12. *Convert* $\overline{(\overline{X}Y)+(\overline{X}\;\overline{Z})}$ *into canonical sum of products form.*

Solution. Rule 1. Simplify the given expression using appropriate theorems/rules.

$$\overline{(\overline{X}Y)+(\overline{X}\;\overline{Z})}=(X+\overline{Y})(X+Z)$$

(using DeMorgan's 2nd theorem *i.e.*, $\overline{AB}=\overline{A}+\overline{B}$)

$$=X+\overline{Y}Z$$ (using Rule 18(*ii*) of Table 12.29)

Since it is a 3 variable expression, a product term must have all 3 variables.

Rule 2. Wherever a literal is missing, multiply that term with

(missing variable + $\overline{\textbf{missing variable}}$)

$$=X+\overline{Y}Z$$

$$=X(Y+\overline{Y})(Z+\overline{Z})+(X+\overline{X})\overline{Y}Z$$

(Y, Z are missing in first term, X is missing in second term)

$$=(XY+X\overline{Y})(Z+\overline{Z})+X\overline{Y}Z+\overline{X}\,\overline{Y}Z$$

$$=Z(XY+X\overline{Y})+\overline{Z}(XY+X\overline{Y})+X\overline{Y}Z+\overline{X}\,\overline{Y}Z$$

$$=XYZ+X\overline{Y}Z+XY\overline{Z}+X\overline{Y}\,\overline{Z}+X\overline{Y}\,\overline{Z}+\overline{X}\,\overline{Y}Z$$

Rule 3. Remove duplicate terms *i.e.*,

$$= XYZ + X\bar{Y}Z + XY\bar{Z} + X\bar{Y}\bar{Z} + \bar{X}\bar{Y}Z$$

This is the desired Canonical Sum-of-Products form.

Above Canonical Sum-of-Products expression can also be represented by following *short hand notation e.g.*,

$$F = \Sigma(1, 4, 5, 6, 7), \quad \text{where } F \text{ is a variable function.}$$

This specifies that output F is sum of 1^{st}, 4^{th}, 5^{th}, 6^{th} and 7^{th} minterms *i.e.*,

$$F = m_1 + m_4 + m_5 + m_6 + m_7$$

Converting Shorthand Notation to Minterms

We already have learnt how to represent minterm into short hand notation. Now we will learn how to convert vice versa.

Rule 1 Find binary equivalent of decimal subscript *e.g.*, for m_6 subscript is 6, binary equivalent of 6 is 110.

Rule 2 For every 1's write the variable as it is and for 0's write variable's complemented form *i.e.*, for 110 it is $XY\bar{Z}$, $XY\bar{Z}$ is the required minterm for m_6.

Example 12.13. *Convert the following three input function F denoted by the expression :*

$F = \Sigma(0, 1, 2, 5)$ *into its canonical Sum-of-Products form.*

Solution. If three inputs we take as X, Y and Z then

$$F = m_0 + m_1 + m_2 + m_5$$

$m_0 = 000 \Rightarrow \bar{X}\bar{Y}\bar{Z}$; $\quad m_1 = 001 \Rightarrow \bar{X}\bar{Y}Z$

$m_2 = 010 \Rightarrow \bar{X}Y\bar{Z}$; $\quad m_5 = 101 \Rightarrow X\bar{Y}Z$

Canonical *S-O-P* form of the expression is

$$\bar{X}\bar{Y}\bar{Z} + \bar{X}\bar{Y}Z + \bar{X}Y\bar{Z} + X\bar{Y}Z$$

...onical Product-of-Sum ...n a boolean expression ...presented purely as ...ict of Maxterms, it is ... to be in Canonical ...ct-of-Sum form of ...ssion.

(ii) Product-of-Sum form

When a boolean expression is represented purely as product of Maxterms, it is said to be in canonical Product-of-Sum form of expression.

This form of expression is also referred to as *Maxterm canonical form of boolean expression.*

Just as any boolean expression can be transformed into a sum of minterms, it can also be represented as a product of Maxterms.

Truth Table Method

The truth table method for arriving at the desired expression is as follows :

1. Prepare a table of inputs and outputs
2. Add one additional column of sum terms. For each row of the table, a sum term is formed by adding all the variables in complemented or uncomplemented form *i.e.*, if input value for a given variable is 1, variable is complemented and if 0, not complemented.

   for $X = 0, Y = 1, Z = 1$, sum term will be $X + \bar{Y} + \bar{Z}$.

Now ***the desired expression is product of the sums from the rows in which the output is 0.***

EXAMPLE 12.14. *Express in the product of sums form, the boolean function F(x, y, z), and the truth table for which is given below :*

X	Y	Z	F
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Solution. 1. Add a new column containing Maxterms. Now the table is as follows :

X	Y	Z	F	Maxterms
0	0	0	1	$X + Y + Z$
0	0	1	**0**	$X + Y + \overline{Z}$
0	1	0	1	$X + \overline{Y} + Z$
0	1	1	**0**	$X + \overline{Y} + \overline{Z}$
1	0	0	1	$\overline{X} + Y + Z$
1	0	1	**0**	$\overline{X} + Y + \overline{Z}$
1	1	0	1	$\overline{X} + \overline{Y} + Z$
1	1	1	1	$\overline{X} + \overline{Y} + \overline{Z}$

2. Now by multiplying Maxterms for the output 0s, we get the desired product of sums expression which is

$$(X + Y + \overline{Z})(X + \overline{Y} + \overline{Z})(\overline{X} + Y + \overline{Z})$$

Algebraic Method

We will explain this method with the help of an example.

EXAMPLE 12.15. *Express $\overline{X}Y + Y(\overline{Z}(\overline{Z} + Y))$ into canonical product-of-sums form.*

Solution. Rule 1. Simplify the given expression using appropriate theorems/rules :

$$\overline{X}Y + Y(\overline{Z}(\overline{Z} + Y)) = \overline{X}Y + Y(\overline{Z}\,\overline{Z} + Y\overline{Z})$$

(*using distributive law i.e.,* $X(Y + Z) = XY + XZ$)

$$= \overline{X}Y + Y(\overline{Z} + Y\overline{Z}) \qquad (\overline{Z}.\overline{Z} = \overline{Z} \text{ as } X.X = X)$$

$$= \overline{X}Y + Y.\overline{Z}(1 + Y)$$

$$= \overline{X}Y + Y\overline{Z}.1 \qquad (1 + Y = 1)$$

$$= \overline{X}\,Y + Y\,\overline{Z}$$

Rule 2. To convert into product of sums form, apply the boolean algebra rule which states that

$$X + YZ = (X + Y)(X + Z)$$

Now applying this rule we get,

$$\overline{X}Y + Y\overline{Z} = (\overline{X}Y + Y)(\overline{X}Y + \overline{Z})$$
$$= (Y + \overline{X}Y)(\overline{Z} + \overline{X}Y)$$
$$= (Y + \overline{X})(Y + Y)(\overline{Z} + \overline{X})(\overline{Z} + Y) \qquad (\because X + Y = Y + X)$$
$$= (\overline{X} + Y)(Y)(\overline{X} + \overline{Z})(Y + \overline{Z}) \qquad (\because Y + Y = Y)$$

Now, this is in product of sums form but not in canonical product of sums form (In canonical expression all the sum terms are Maxterms).

Rule 3. After converting into product of sum terms, in a sum term for a missing variable add (**missing variable . missing variable** e.g., if variable Y is missing add $Y\overline{Y}$.

i.e.,

$$(\overline{X} + Y)\ (Y)\ (\overline{X} + \overline{Z})\ (Y + \overline{Z})$$

Terms : 1 2 3 4

$$= (\overline{X} + Y + Z\overline{Z})(X\overline{X} + Y + Z\overline{Z})(\overline{X} + Y\overline{Y} + \overline{Z})(X\overline{X} + Y + \overline{Z})$$

Rule 4. Keep on simplifying the expression (using the rule, $X + YZ = (X + Y)(X + Z)$) until you get product of sum terms which are Maxterms.

$$\Rightarrow\ (\overline{X} + Y + Z\overline{Z})(X\overline{X} + Y + Z\overline{Z})(\overline{X} + Y\overline{Y} + \overline{Z})(X\overline{X} + Y + \overline{Z})$$
$$= (\overline{X} + Y + Z)(\overline{X} + Y + \overline{Z})(X\overline{X} + Y + Z)(X\overline{X} + Y + \overline{Z})(\overline{X} + Y + \overline{Z})$$
$$(\overline{X} + \overline{Y} + \overline{Z})(X + Y + \overline{Z})(\overline{X} + Y + \overline{Z})$$
$$= (\overline{X} + Y + Z)(\overline{X} + Y + \overline{Z})(X + Y + Z)(\overline{X} + Y + Z)(X + Y + \overline{Z})(\overline{X} + Y + \overline{Z})$$
$$(\overline{X} + Y + \overline{Z})(\overline{X} + \overline{Y} + \overline{Z})(X + Y + \overline{Z})(\overline{X} + Y + \overline{Z})$$

Rule 5. Removing all the duplicate terms, we get

$$(\overline{X} + Y + Z)(\overline{X} + Y + \overline{Z})(X + Y + Z)(X + Y + \overline{Z})(\overline{X} + \overline{Y} + \overline{Z})$$

This is the desired canonical product of sums form of expression.

Shorthand Maxterm Notation

Shorthand notation for the above given canonical product of sums expression is

$$F = \Pi(0, 1, 4, 5, 7)$$

This specifies that output F is product of 0^{th}, 1^{st}, 4^{th}, 5^{th} and 7^{th} Maxterms

i.e. $$F = M_0 . M_1 . M_4 . M_5 . M_7$$

Here M_0 means Maxterm for Binary equivalent of 0 *i.e.*, 0 0 0

$$\Rightarrow \qquad X = 0,\ Y = 0,\ Z = 0$$

And Maxterm will be $(X + Y + Z)$

(*Complement the variable if input is 1 otherwise not*)

Similarly, M_1 means 0 0 1 $\Rightarrow$ $X + Y + \overline{Z}$

As $F = M_0 . M_1 . M_4 . M_5 . M_7$

and

$$M_0 = 000 \qquad X + Y + Z$$
$$M_1 = 001 \qquad X + Y + \overline{Z}$$
$$M_4 = 100 \qquad \overline{X} + Y + Z$$
$$M_5 = 101 \qquad \overline{X} + Y + \overline{Z}$$
$$M_7 = 111 \qquad \overline{X} + \overline{Y} + \overline{Z}$$

$$\Rightarrow \quad F = (X + Y + Z)(X + Y + \overline{Z})(\overline{X} + Y + Z)(\overline{X} + Y + \overline{Z})(\overline{X} + \overline{Y} + \overline{Z})$$

EXAMPLE 12.16. *Convert the following function into canonical product of sums form*

$$F(X, Y, Z) = \Pi(0, 2, 4, 5)$$

Solution. $F(X, Y, Z) = \Pi(0, 2, 4, 5) = M_0 \cdot M_2 \cdot M_4 \cdot M_5$

$M_0 = 000 \quad X + Y + Z$

$M_2 = 010 \quad X + \overline{Y} + Z$

$M_4 = 100 \quad \overline{X} + Y + Z$

$M_5 = 101 \quad \overline{X} + Y + \overline{Z}$

$\Rightarrow \quad F = (X + Y + Z)(X + \overline{Y} + Z)(\overline{X} + Y + Z)(\overline{X} + Y + \overline{Z})$

Sum-term Vs Maxterm and Product-term Vs minterm

Sum term means sum of the variables. It does not necessarily mean that all the variables must be included whereas *Maxterm* means sum term having all the variables.

For example, for a 3 variables $F(X, Y, Z)$, functions $X + Y, X + Z, \overline{Y} + Z$ *etc.* are sum terms whereas $X + Y + Z, \overline{X} + Y + \overline{Z}, \overline{X} + \overline{Y} + Z$ *etc.* are Maxterms.

Similarly, *product term* means product of the variables, not necessarily all the variables whereas *minterm* means product of all the variables.

For a 3 variable (a, b, c) function $ab\overline{c}, abc, \overline{a}\,\overline{b}c$ etc. are minterms whereas $ab, \overline{b}\,\overline{c}, \overline{b}c, ac$ *etc.* are product terms only.

Same is the difference between *Canonical S-O-P or P-O-S expression* and *S-O-P expression* or *P-O-S expression*. A Canonical SOP or POS expression must have all the Maxterms or Minterms respectively whereas a simple *S-O-P* or *P-O-S* expression can just have product terms or sum terms.

Check Point 12.5

1. What is a logical product having all the variables of a function called ?
2. What is a logical sum having all the variables of a function called ?
. What do you understand by a Minterm and a Maxterm ?
Write the minterm and Maxterm for a function $F(x, y, z)$ when
$x = 0, y = 1, z = 0$.
Write the minterm and maxterm for a function $F(x, y, z)$ when
$x = 1, y = 0, z = 0$.
Write short hand notations for the ollowing minterms :
$X Y Z, \overline{X} Y \overline{Z}, \overline{X} Y Z$
rite short hand notation for the llowing maxterms :
$X + Y + Z, \; X + \overline{Y} + Z,$
$\overline{X} + Y + \overline{Z}, X + \overline{Y} + \overline{Z}.$
at is the boolean expression, taining only the sum of minterms, ed ?
t is the boolean expression, aining only the product of erms, called ?

12.10 Minimization of Boolean Expression

After obtaining an S-O-P or P-O-S expression, the next thing to do is to simplify the boolean expression because boolean operations are practically implemented in the form of gates. A minimized boolean expression means less number of gates which means simplified circuitary. This section deals with two methods of simplification of boolean expressions.

12.10.1 Algebraic Method

This method makes use of boolean postulates, rules and theorems to simplify the expressions.

Example 12.17. *Simplify* $A\overline{B}C\overline{D} + A\overline{B}CD + ABC\overline{D} + ABCD$.

Solution. $A\overline{B}C\overline{D} + A\overline{B}CD + ABC\overline{D} + ABCD$

$= A\overline{B}C(\overline{D} + D) + ABC(\overline{D} + D) = A\overline{B}\,C.1 + ABC.1 \qquad (\overline{D} + D = 1)$

$= AC(\overline{B} + B) = AC.1 = AC \qquad (\overline{B} + B = 1)$

EXAMPLE ... Reduce the expression $\overline{XY} + \overline{X} + XY$.

Solution. $\overline{XY} + \overline{X} + XY$

$= (\overline{X} + \overline{Y}) + \overline{X} + XY$

$= \overline{X} + \overline{X} + \overline{Y} + XY$ (*using DeMorgan's 2nd theorem i.e.,* $\overline{XY} = \overline{X} + \overline{Y}$)

$= \overline{X} + \overline{Y} + XY$

$= \overline{X} + XY + \overline{Y}$ ($\because \overline{X} + \overline{X} = \overline{X}$ as $X + X = X$)

$= (\overline{X} + \overline{\overline{X}}Y) + \overline{Y} = (\overline{X} + XY) + \overline{Y}$

$= \overline{X} + Y + \overline{Y}$ (*putting* $X = \overline{\overline{X}}$)

$= \overline{X} + 1$ ($X + \overline{X}Y = X + Y$)

$= 1$ (*putting* $Y + \overline{Y} = 1$)

(*putting* $\overline{X} + 1 = 1$ as $1 + X = 1$)

EXAMPLE 12.19. *Minimise* $AB + \overline{AC} + A\overline{B}C(AB + C)$.

Solution. $AB + \overline{AC} + A\overline{B}C(AB + C) = AB + \overline{AC} + A\overline{B}CAB + A\overline{B}CC$

$= AB + \overline{AC} + AAB\overline{B}C + A\overline{B}CC$

$= AB + \overline{AC} + 0 + A\overline{B}CC$ (*putting* $B\overline{B} = 0$)

$= AB + \overline{AC} + A\overline{B}.C$ (*putting* $C.C = C$)

$= AB + \overline{A} + \overline{C} + A\overline{B}C$ (*putting* $\overline{AC} = \overline{A} + \overline{C}$ *DeMorgan's 2nd theorem*)

$= \overline{A} + AB + \overline{C} + A\overline{B}C$ (*rearranging the terms*)

$= \overline{A} + B + \overline{C} + A\overline{B}C$ (*putting* $\overline{A} + AB = A + B$ *because* $X + \overline{X}Y = X + Y$)

$= \overline{A} + \overline{C} + B + A\overline{B}C = \overline{A} + \overline{C} + B + \overline{B}AC$

$= \overline{A} + \overline{C} + B + AC$ (*putting* $B + \overline{B}AC = B + AC$ *because* $X + \overline{X}Y = X + Y$)

$= \overline{A} + B + \overline{C} + CA$

$= \overline{A} + B + \overline{C} + A$ ($\because \overline{C} + CA = \overline{C} + A$)

$= A + \overline{A} + B + \overline{C}$

$= 1 + B + \overline{C}$ (*putting* $A + \overline{A} = 1$)

$= 1$ (*as* $1 + X = 1$ *i.e., anything added to 1 results in 1*)

EXAMPLE 12.20. *Reduce* $\overline{X}\,\overline{Y}\,\overline{Z} + \overline{X}Y\overline{Z} + X\overline{Y}\,\overline{Z} + XY\overline{Z}$.

Solution. $\overline{X}\,\overline{Y}\,\overline{Z} + \overline{X}Y\overline{Z} + X\overline{Y}\,\overline{Z} + XY\overline{Z} = \overline{X}(\overline{Y}\,\overline{Z} + Y\overline{Z}) + X(\overline{Y}\,\overline{Z} + Y\overline{Z})$

$= \overline{X}\,(\overline{Z}\,(\overline{Y} + Y)) + X\,(\overline{Z}\,(\overline{Y} + Y))$

$= \overline{X}\,(\overline{Z}.1) + X\,(\overline{Z}.1)$ ($\overline{Y} + Y = 1$)

$= \overline{X}\,\overline{Z} + X\overline{Z}$

$= \overline{Z}(\overline{X} + X)$ ($\overline{X} + X = 1$

$= \overline{Z}.1$

$= \overline{Z}$

12.10.2 Simplification Using Karnaugh Maps

Truth Tables provide a nice, natural way to list all values of a function. There a[re] several other ways to represent function values. One of them is *Karnaugh Map* ([in] short *K-Map*) named after its originator *Maurice Karnaugh*. These maps a[re] also called *Veitch diagrams*.

What is Karnaugh Map ?

Karnaugh map or K-map is a graphical display of the fundamental products in a truth table.

Karnaugh map is nothing but a rectangle made up of certain number of squares, each square representing a *Maxterm* or *minterm*.

12.10.3 Sum-of-Products Reduction using Karnaugh Map

In *S-O-P* reduction each square of K-map represents a minterm of the given function. Thus, for a function of *n* variables, there would be a map of 2^n squares, each representing a minterm (refer to Fig. 12.7). Given a K-map, for *S-O-P* reduction the map is filled in by placing 1s in squares whose minterms lead to a 1 output.

Figure 12.7 shows 2, 3, 4 variable K-maps for *S-O-P* reduction.

(a)

X \ Y	[0] $\bar{Y}$	[1] Y
[0] $\bar{X}$	$\bar{X}\bar{Y}$ 0	$\bar{X}Y$ 1
[1] X	$X\bar{Y}$ 2	XY 3

(b)

X \ Y	[0] $\bar{Y}$	[1] Y
[0] $\bar{X}$	0	1
[1] X	2	3

2-variable K-map representing minterms.

(c)

X \ YZ	[00] $\bar{Y}\bar{Z}$	[01] $\bar{Y}Z$	[11] YZ	[10] $Y\bar{Z}$
[0] $\bar{X}$	$\bar{X}\bar{Y}\bar{Z}$ 0	$\bar{X}\bar{Y}Z$ 1	$\bar{X}YZ$ 3	$\bar{X}Y\bar{Z}$ 2
[1] X	$X\bar{Y}\bar{Z}$ 4	$X\bar{Y}Z$ 5	XYZ 7	$XY\bar{Z}$ 6

(d)

X \ YZ	[00] $\bar{Y}\bar{Z}$	[01] $\bar{Y}Z$	[11] YZ	[10] $Y\bar{Z}$
[0] $\bar{X}$	0	1	3	2
[1] X	4	5	7	6

3-variable K-map representing minterms

(e)

WX \ YZ	[00] $\bar{Y}\bar{Z}$	[01] $\bar{Y}Z$	[11] YZ	[10] $Y\bar{Z}$
[00] $\bar{W}\bar{X}$	$\bar{W}\bar{X}\bar{Y}\bar{Z}$ 0	$\bar{W}\bar{X}\bar{Y}Z$ 1	$\bar{W}\bar{X}YZ$ 3	$\bar{W}\bar{X}Y\bar{Z}$ 2
[01] $\bar{W}X$	$\bar{W}X\bar{Y}\bar{Z}$ 4	$\bar{W}X\bar{Y}Z$ 5	$\bar{W}XYZ$ 7	$\bar{W}XY\bar{Z}$ 6
[11] WX	$WX\bar{Y}\bar{Z}$ 12	$WX\bar{Y}Z$ 13	$WXYZ$ 15	$WXY\bar{Z}$ 14
[10] $W\bar{X}$	$W\bar{X}\bar{Y}\bar{Z}$ 8	$W\bar{X}\bar{Y}Z$ 9	$W\bar{X}YZ$ 11	$W\bar{X}Y\bar{Z}$ 10

(f)

WX \ YZ	[00] $\bar{Y}\bar{Z}$	[01] $\bar{Y}Z$	[11] YZ	[10] $Y\bar{Z}$
[00] $\bar{W}\bar{X}$	0	1	3	
[01] $\bar{W}X$	4	5	7	
[11] WX	12	13	15	
[10] $W\bar{X}$	8	9	11	

4-variable K-map representing minterms

Figure 12.7 2, 3, 4 variable K-maps for S-O-P expression.

Note in every square a number is written. These subscripted numbers denote that this square corresponds to that number's minterm. For example, in 3 variable map $X\,Y\,\bar{Z}$ box has been given number 2 which means this square corresponds to m_2. Similarly, box number 7 means it corresponds to m_7 and so on.

Please notice the numbering scheme here, it is 0, 1, 3, 2 then 4, 5, 7, 6 and so on. Always squares are marked using this scheme while making a K-map.

Gray Code is the binary code in which each successive number differs only in one place.

Observe carefully above given K-map. See the binary numbers at the top of K-map. These do not follow binary progression, instead they differ by only one place when moving from left to right : 00, 01, 11, 10. *It is done so that only one variable changes from complemented to uncomplemented form or vice versa*. See $\bar{A}\bar{B}$, $\bar{A}B$, AB, $A\bar{B}$.

This binary code 00, 01, 11, 10 is called **Gray Code**. *Gray Code is the binary code in which each successive number differs only in one place*. That is why box numbering scheme follows above order only.

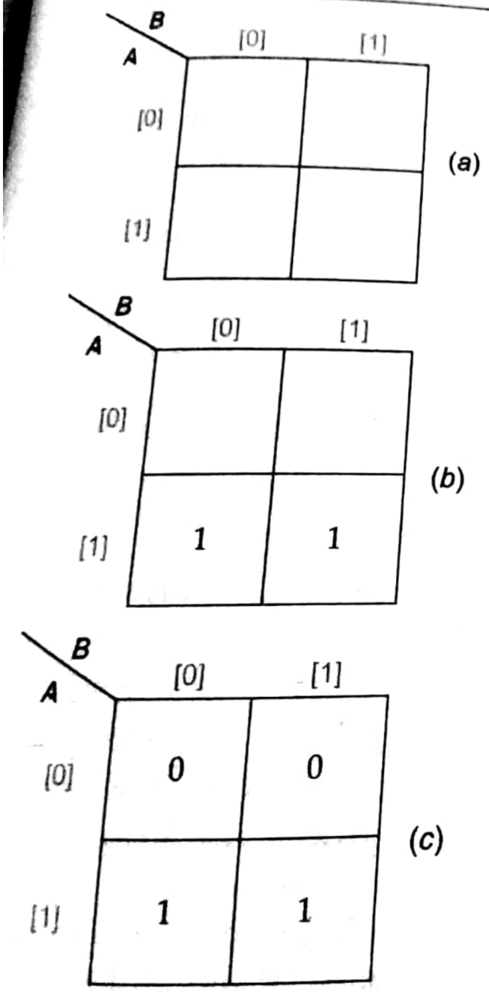
How to map in K-map ?

We'll take an example of 2-variable map to illustrate this :

Suppose, we have been given with the following truth table for mapping (Table 12.32).

Table 12.32

A	B	F
0	0	0
0	1	0
1	0	1
1	1	1



.8 **How to fill 2-variable K-map for a given truth table.**

Canonical *S-O-P* expression for this table is

$$F = A\bar{B} + AB \quad \text{or} \quad F = \Sigma\,(2, 3)\,.$$

To map this function first we'll draw an empty 2-variable K-map as shown in Fig. 12.8(*a*).

Now look for output 1 in the given truth table (12.32) for a given truth table.

For minterms m_2 and m_3 the output is 1. Thus mark 1 in the squares for m_2 and m_3 *i.e.*, square numbered as 2 and the one numbered as 3. Now our K-map will look like Fig. 12.8(*b*).

After entering 1's for all 1 outputs, enter 0's in all blank squares. K-map will now look like Fig 12.8(*c*). Same is the method for mapping 3-variable and 4-variable maps *i.e.*, enter 1's for all 1 outputs in the corresponding squares and then enter 0's in the rest of the squares.

How to reduce boolean expression in S-O-P form using K-map ?

For reducing the expression, first we have to mark *pairs, quads and octets.*

To reduce an expression, adjacent 1's are encircled. If *two* adjacent 1's are encircled, it makes a *pair* ; *if four adjacent 1's* are encircled, it makes a *quad* ; and if *eight adjacent 1's* are encircled, it makes an *octet*.

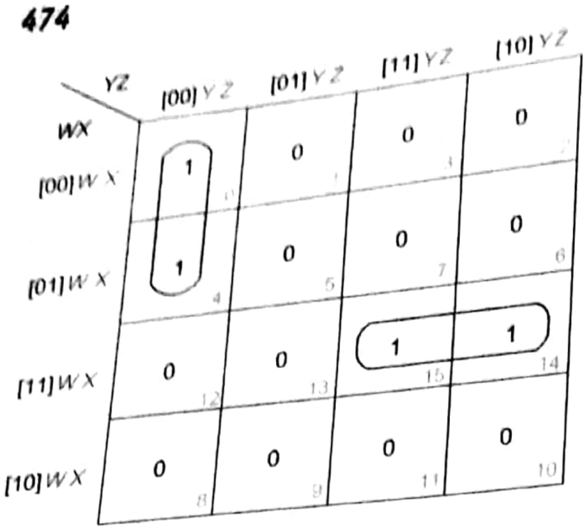


Figure 12.9 Pairs in a given K-map.

While encircling groups of 1's, firstly search octets and mark them, then for *quads* and lastly for *pairs*. Thus is because a bigger group removes more variables thereby making the resultant expression simpler.

Reduction of a pair. In the K-map in Fig. 12.9 after mapping a given function F (W, X, Y, Z) two pairs have been marked. Pair-1 is $m_0 + m_4$ (grouping 0^{th} minterm and 4^{th} minterm as these numbers tell us minterm's subscript). Pair-2 is $m_{14} + m_{15}$.

Observe that Pair-1 is a vertical pair. Moving vertically in pair-1, see one variable X is changing its state from $\overline{X}$ to X as m_0 is $\overline{W}\ \overline{X}\ \overline{Y}\ \overline{Z}$ and m_4 $\overline{W}\ X\ \overline{Y}\ \overline{Z}$. Compare the two and we see $\overline{W}\ \overline{X}\ \overline{Y}$ changes to $\overline{W}\ X\ \overline{Y}\ \overline{Z}$. So, the variable X can be removed.

Pair Reduction Rule

Remove the variable which changes its state from complemented to uncomplemented or vice versa. Pair removes one variable only.

> In a K-map, the variable(s) changing state (bar to no bar or vice versa) get(s) removed from the resultant expression.

Thus reduced expression for Pair-1 is $\overline{W}\ \overline{Y}\ \overline{Z}$ as $\overline{W}\ \overline{X}\ \overline{Y}\ \overline{Z}\ (m_0)$ changes to $\overline{W}\ X\overline{Y}\ \overline{Z}\ (m_4)$. We can prove the same algebraically also as follows :

$$\text{Pair-1} = m_0 + m_4 = \overline{W}\ \overline{X}\ \overline{Y}\ \overline{Z} + \overline{W}\ X\overline{Y}\ \overline{Z}$$
$$= \overline{W}\ \overline{Y}\ \overline{Z}\,(\overline{X} + X)$$
$$= \overline{W}\ \overline{Y}\ \overline{Z}\,.1 \qquad (\overline{X} + X = 1)$$
$$= \overline{W}\ \overline{Y}\ \overline{Z}$$

Similarly, reduced expression for Pair-2 $(m_{14} + m_{15})$ will be WXY as $W\,X\,Y\,\overline{Z}\,(m_{14})$ changes to $WXYZ\ (m_{15})$. Z will be removed as it is changing its state from $\overline{Z}$ to Z.

Reduction of a quad

If we are given with the K-map shown in Fig. 12.10 in which two quads have been marked.

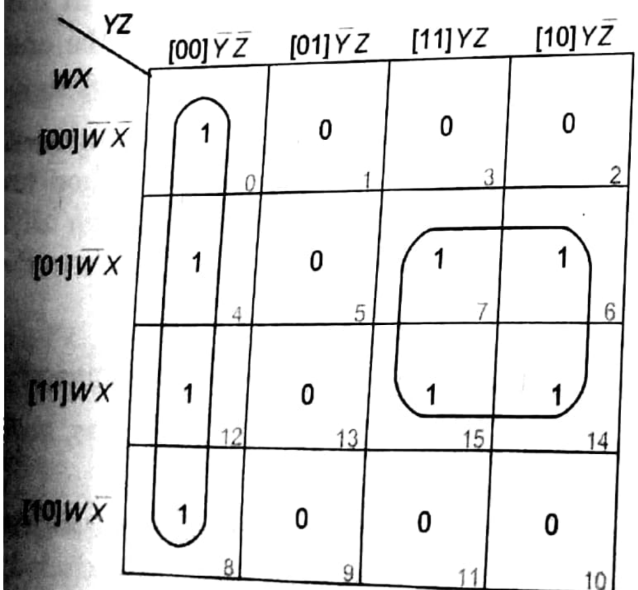


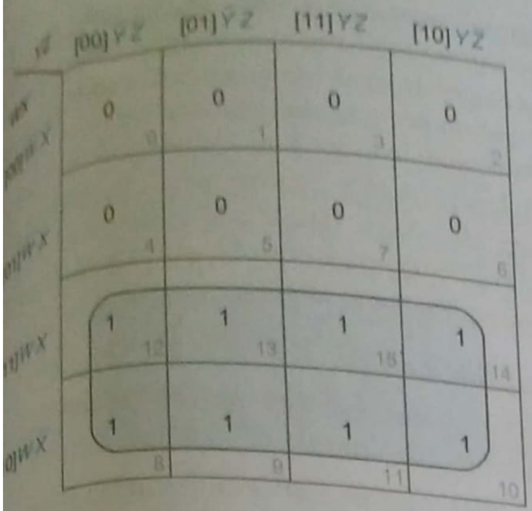
Figure 12.9 Quads in a given K-map

Quad-1 is $m_0 + m_4 + m_{12} + m_8$ and Quad-2 $m_7 + m_6 + m_{15} + m_{14}$. When we move across quad-1 two variables change their states *i.e.*, W and X are changing their states, so these two variables will be removed.

Quad. Reduction Rule

Remove the two variables which change their states. Quad removes two variables. Thus reduced expression for Quad-1 is $\overline{Y}\ \overline{Z}$ as W and X (both) are removed.

Similarly, in Quad-2 $(m_7 + m_6 + m_{15} + m_{14})$ horizontally moving, variable Z is removed as $\overline{W}\ XY\ Z(m_7)$ changes to $\overline{W}\ X\ Y\ \overline{Z}\ (m_6)$ and vertically moving, variable W is removed as $\overline{W}\,X\,Y\,Z$ changes to $WXYZ$. Thus reduced expression for quad-2 is (by removing W and Z) XY.



12.11 Octets in a given K-map.

Reduction of an octet

Suppose, we have K-map with an octet marked as shown in Fig. 12.11.

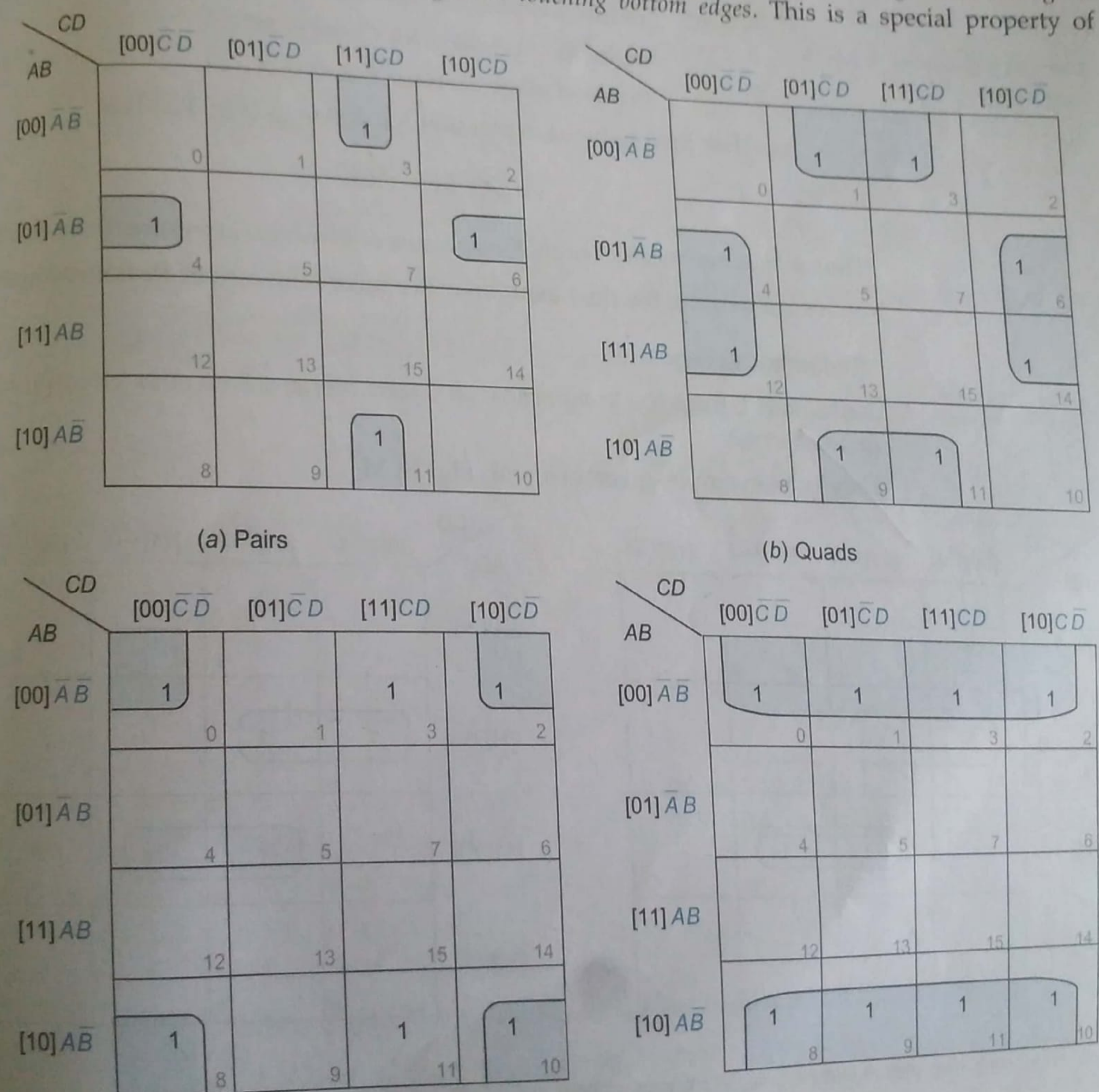
While moving horizontally in the octet two variables Y and Z are removed and moving vertically one variable X is removed. Thus eliminating X, Y and Z, the reduced expression for the octet is *W* only.

Octet Reduction Rule

Remove the three variables which change their states. An octet *removes 3-variables.* After reducing pairs, quads and octets, there are certain other things to be taken care of before arriving at the final expression. These are map rolling, overlapping groups and reduntant groups.

Map Rolling

Map Rolling means roll the map *i.e., consider the map as if its left edges are touching the right edges and top edges are touching bottom edges.* This is a special property of



(*a*) Pairs (*b*) Quads (*c*) quad (*d*) octet

12.12 Map rolling

Karnaugh maps that its opposite edges squares and corner squares are considered contiguous (Just as the world map is treated contiguous at its opposite ends). *As in opposite edges squares and in corner squares only one variable changes its state from complemented to uncomplemented state or vice versa.* Therefore, while marking the pairs, quads and octets, map must be rolled. Following pairs, quads and octets are marked after rolling the map.

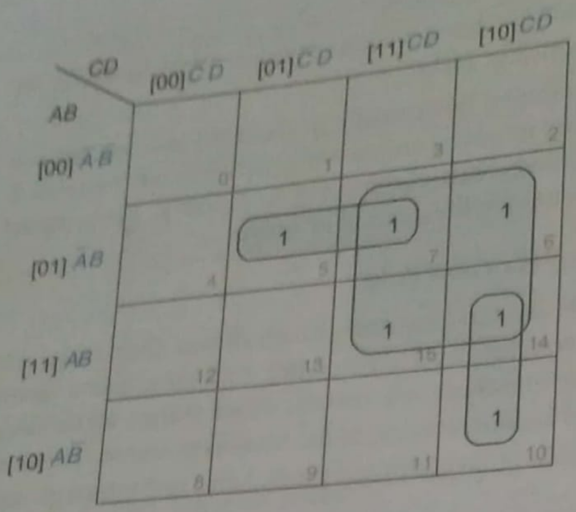


Figure 12.13 Overlapping Groups.

Overlapping Groups

Overlapping means same 1 can be encircled more than once. For example, consider the K-map given in Fig. 12.13. Observe that 1 for m_7 has been encircled twice. Once for Pair-1 $(m_5 + m_7)$ and again for Quad $(m_7 + m_6 + m_{15} + m_{14})$. Also 1 for m_{14} has been encircled twice. For the Quad and for Pair-2 $(m_{14} + m_{10})$.

>>> NOTE

Overlapping always leads to simpler expressions.

Here, reduced expression for Pair-1 is $\bar{A}BD$

reduced expression for Quad is BC

reduced expression for Pair-2 is $AC\bar{D}$

Thus final reduced expression for this map (Fig. 12.13) is

$$\bar{A}BD + BC + AC\bar{D}$$

Thus *reduced expression for entire K-map is sum of all reduced expressions in the very K-map.* But before writing the final expression we must take care of ***Reduntant Groups.***

Reduntant Group

Reduntant Group *is a group whose all 1's are overlapped by other groups* (i.e., pairs, quads, octets).

Here is an example, given below in Fig. 12.14.

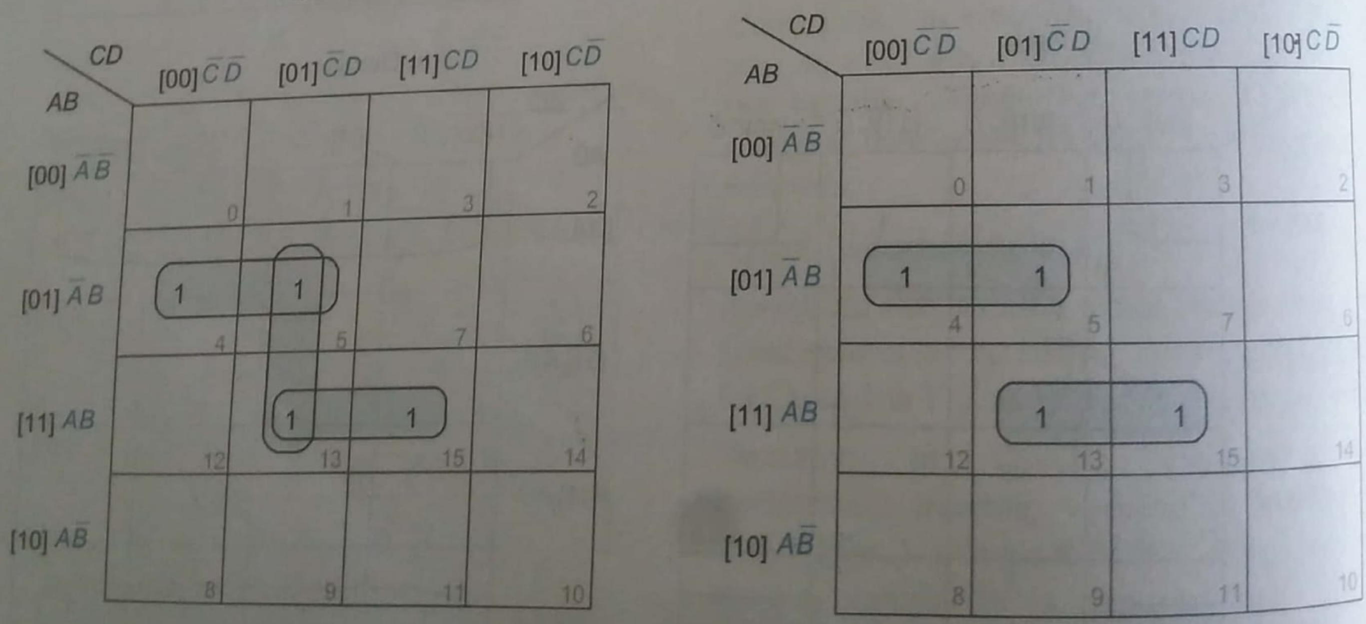


Figure 12.14 (*a*) K-map with reduntant group (*b*) K-map without reduntant group.

Summary of all the rules for S-O-P reduction using K-map

1. Prepare the truth table for given function.
2. Draw an empty K-map for the given function (*i.e.*, 2 variable K-map for 2 variable function ; 3 variable K-map for 3 variable function, and so on).
3. Map the given function by entering 1's for the outputs as 1 in the corresponding squares.
4. Enter 0's in all left out empty squares.
5. Encircle adjacent 1's in form of octets, quads and pairs. Do not forget to roll the map and overlap.
6. Remove reduntant groups, if any.
7. Write the reduced expressions for all the groups and OR (+) them.

Figure 12.14(*a*) has a reduntant group. There are three pairs : Pair-1 ($m_4 + m_5$), Pair-2 ($m_5 + m_{13}$), Pair-3 ($m_{13} + m_{15}$). Bur Pair 2 is a reduntant group as its all 1's are marked by other groups.

With this reduntant group, the reduced expression will be

$$\overline{A}B\overline{C} + B\overline{C}D + ABD$$

For a simpler expression, ***Reduntant Groups must be removed***. After removing the reduntant group, we get the K-map shown in Fig. 12.14(*b*).

The reduced expression, for K-map in Fig. 12.14(*b*), will be

$$\overline{A}B\overline{C} + ABD$$

which is much simpler expression.

Thus ***removal of reduntant group leads to much simpler expression.***

Example 12.21. *Reduce F (a, b, c, d) = Σ (0, 2, 7, 8, 10, 15) using Karnaugh map.*

Solution. Given $F(a, b, c, d) = \Sigma(0, 2, 7, 8, 10, 15)$

$$= m_0 + m_2 + m_7 + m_8 + m_{10} + m_{15}$$

$m_0 = 0000 = \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$ $\quad m_2 = 0010 = \overline{A}\,\overline{B}\,C\,\overline{D}$

$m_7 = 0111 = \overline{A}BCD$ $\quad m_8 = 1000 = A\overline{B}\,\overline{C}\,\overline{D}$

$m_{10} = 1010 = A\overline{B}C\overline{D}$ $\quad m_{15} = 1111 = ABCD$

Truth Table for the given function is as follows :

A	B	C	D	F
0	0	0	0	1
0	0	0	1	
0	0	1	0	1
0	0	1	1	
0	1	0	0	
0	1	0	1	
0	1	1	0	
0	1	1	1	1
1	0	0	0	1
1	0	0	1	
1	0	1	0	1
1	0	1	1	
1	1	0	0	
1	1	0	1	
1	1	1	0	
1	1	1	1	1

Mapping the given function in a K-map, we get

AB \ CD	[00] $\overline{C}\,\overline{D}$	[01] $\overline{C}D$	[11] CD	[10] $C\overline{D}$
[00] $\overline{A}\,\overline{B}$	1 0	0 1	0 3	1 2
[01] $\overline{A}B$	0 4	0 5	1 7	0 6
[11] AB	0 12	0 13	1 15	0 14
[10] $A\overline{B}$	1 8	0 9	0 11	1 10

In the above K-map two groups have been marked, One Quad and One Pair.

Quad is $m_0 + m_2 + m_8 + m_{10}$ and Pair is $m_7 + m_{15}$

Reduced expression for quad $(m_0 + m_2 + m_8 + m_{10})$ is $\bar{B}\bar{D}$ as for horizontal corners is removed and for vertical corners A is removed. Reduced expression for $(m_7 + m_{15})$ is BCD as A is removed.

Thus final reduced expression is $\bar{B}\bar{D} + BCD$.

EXAMPLE 12.22. *What is the simplified boolean equation for the function :*

$$F(A, B, C, D) = \Sigma(7, 9, 10, 11, 12, 13, 14, 15)\ ?$$

Solution. *Completing the given Karnaugh map by entering 0's in the empty squares, numbering the squares with their minterm's subscripts and then by encircling all possible groups, we get the following K-map.*

CD / AB	[00] $\bar{C}\bar{D}$	[01] $\bar{C}D$	[11] CD	[10] $C\bar{D}$
[00] $\bar{A}\bar{B}$	0 0	0 1	0 3	0 2
[01] $\bar{A}B$	0 4	0 5	1 7	0 6
[11] AB	1 12	1 13	1 15	1 14
[10] $A\bar{B}$	0 8	1 9	1 11	1 10

There is *three* quads, *one* pair

Quad-1 $= m_{12} + m_{13} + m_{15} + m_{14}$

Quad-2 $= m_{13} + m_{15} + m_9 + m_{11}$;

Quad-3 $= m_{15} + m_{11} + m_{14} + m_{10}$

Pair-1 $= m_7 + m_{15}$;

Reduced expression for *Quad-1* $(m_{12} + m_{13} + m_{15} + m_{14})$ is AB, as while moving across the Quad, C and D both are removed because both are changing their state from complemented to uncomplemented or vice-versa.

Reduced expression for *Quad-2* $(m_{13} + m_{15} + m_9 + m_{11})$ is AD, as moving horizontally, C is removed and moving vertically, B is removed.

Reduced expression of *Quad-3* $(m_{15} + m_{11} + 14 + m_{10})$ is AC as horizontal movement removes D and vertical movement removes B.

Reduced expression for pair-1 $(m_7 + m_{15})$ is BCD, as $\bar{A}BCD(m_7)$ changes to $ABCD$ (m_{15}) eliminating A.

Thus, Quad-1 $= AB$, Quad-2 $= AD$, Quad-3 $= AC$, Pair-1 $= BCD$

Hence final reduced expression will be $AB + AD + AC + BCD$.

EXAMPLE 12.23. *Obtain a simplified expression for a Boolean function F (X, Y, Z), the Karnaugh map for which is given below :*

Y / X	[00]	[01]	[11]	[10]
[0]		1	1	
[1]		1	1	

YZ / X	[00] $\bar{Y}\bar{Z}$	[01] $\bar{Y}Z$	[11] YZ	[10] $Y\bar{Z}$
[0] $\bar{X}$	0 0	1 1	1 3	0 2
[1] X	0 4	1 5	1 7	0 6

Solution. Completing the given K-map,

We have 1 group which is a Quad *i.e.*, $m_1 + m_3 + \ldots$

Reduced expression for this Quad is Z, as moving horizontally from $\bar{X}\bar{Y}Z(m_1)$ to $\bar{X}YZ(m_3)$, Y is removed (Y changing from $\bar{Y}$ to Y) and moving vertically from ... to m_5 or m_3 to m_7, $\bar{X}$ changes to X, thus X is removed.

Simplified boolean expression for given K-map is $F(X, \ldots$

EXAMPLE 12.24. *Minimise the following function using a Karnaugh map:*

$$F(W, X, Y, Z) = \Sigma(0, 4, 8, 12).$$

Solution. Given function $F(W, X, Y, Z) = \Sigma(0, 4, 8, 12)$

WX \ YZ	[00] $\overline{Y}\,\overline{Z}$	[01] $\overline{Y}Z$	[11] YZ	[10] $Y\overline{Z}$
[00] $\overline{W}\,\overline{X}$	1 0	0 1	0 3	0 2
[01] $\overline{W}X$	1 4	0 5	0 7	0 6
[11] WX	1 12	0 13	0 15	0 14
[10] $W\overline{X}$	1 8	0 9	0 11	0 10

$$F = m_0 + m_4 + m_8 + m_{12}$$

$$m_0 = 0000 = \overline{W}\,\overline{X}\,\overline{Y}\,\overline{Z},$$

$$m_4 = 0100 = \overline{W}X\overline{Y}\,\overline{Z},$$

$$m_8 = 1000 = W\overline{X}\,\overline{Y}\,\overline{Z},$$

$$m_{12} = 1100 = WX\overline{Y}\,\overline{Z}$$

Mapping the given function on a K-map, we get

Only 1 group is here, a Quad

$$(m_0 + m_4 + m_{12} + m_8)$$

Reduced expression for this quad is $\overline{Y}\,\overline{Z}$, as while moving across the Quad W and X are removed because these are changing their states from complemented to uncomplemented or vice versa.

Thus, final reduced expression is $\overline{Y}\,\overline{Z}$.

EXAMPLE 12.25. *Using the Karnaugh technique obtain the simplified expression as sum of products for the following map :*

X \ YZ	[00]	[01]	[11]	[10]
[0]			1	1
[1]			1	1

Solution. Completing the given K-map, we get

X \ YZ	[00] $\overline{Y}\,\overline{Z}$	[01] $\overline{Y}Z$	[11] YZ	[10] $Y\overline{Z}$
[0] $\overline{X}$	0 0	0 1	1 3	1 2
[1] X	0 4	0 5	1 7	1 6

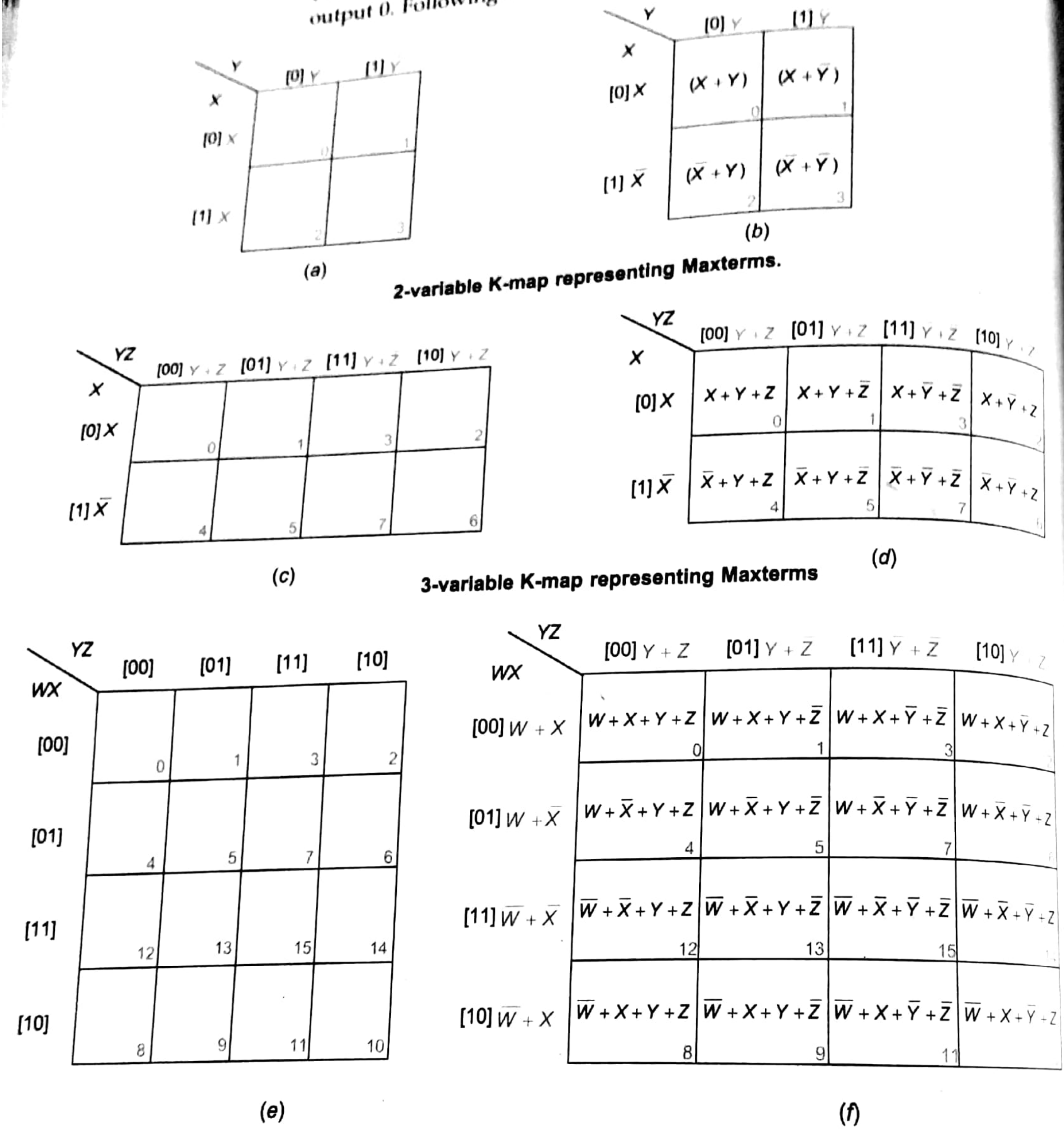
One group which is a Quad has been marked.

Quad reduces *two variables*. Moving horizontally, Z is removed as it changes from Z to $\overline{Z}$ and moving vertically, X is removed as it changes from $\overline{X}$ to X. Thus only one variable Y is left. Hence Reduced *S-O-P* expression is Y. Thus $F = Y$ assuming F is the given function.

12.10.4 Product-of-Sum Reduction using Karnaugh Map

In *P-O-S* reduction each square of K-map represents a Maxterm. Karnaugh map is just the same as that of the used in *S-O-P* reduction. For a function of n variables, map would represent 2^n squares, each representing a Maxterm.

For *P-O-S* reduction map is filled by placing 0's in squares whose Maxterms le… output 0. Following are 2, 3, 4 variable K-maps for *P-O-S* reduction.



2-variable K-map representing Maxterms.

3-variable K-map representing Maxterms

4-variable K-map representing minterms

gure 12.15 2,3,4 variable K-Maps of P-O-S expression.

Again, the numbers in the squares represent *Maxterm subscripts*. Box with number 1 represe… M_1, number 6 box represents M_6, and so on. Also notice *box numbering scheme is the same* i.e., 0, 1, 3, 4, 5, 7, 6 ; 12, 13, 15, 14 ; 8, 9, 11, 10.

One more similarity in *S-O-P* K-map and *P-O-S* K-map is that they are binary progression in Gr… code only. So, here also same Gray Code appears at the top.

But one major difference is that in P-O-S K-map, complemented letters represent 1's and uncom… letters represent 0's whereas it is just the opposite in S-O-P K-map. Thus in the Fig. 12.15 (*b*), (*d*), (*f*) for … uncomplemented letters appear and for 1's complemented letters appear

How to derive P-O-S boolean expression using K-map

Rules for deriving expression are the same except for one thing *i.e., for P-O-S expression adjacent 0's are encircled in the form of pairs, quads and octets*. Therefore, rules for deriving *P-O-S* boolean expression can be summarized as follows :

1. Prepare the truth table for a given function.
2. Draw an empty K-map for given function (*i.e.*, 2-variable K-map for 2-variable function, 3-variable K-map for 3-variable function and so on).
3. Map the given function by entering 0's for the outputs as 0 in the corresponding squares. (*i.e.*, if M_5 and M_{13} are 0's then squares numbered 5 and 13 will be having 0's).
4. Enter 1's in all left out empty squares.
5. Encircle adjacent 0's in the form of octets, quads, and pairs. Do not forget to roll the map and overlap.
6. Remove reduntant groups if any.
7. Write the reduced expressions for all the groups and AND (.) them.

EXAMPLE 12.26. *Reduce the following Karnaugh map in Product-of-sums form:*

A \ BC	[00]	[01]	[11]	[10]
[0]	0	0	0	1
[1]	0	1	1	1

Solution. To reach at *P-O-S* expression, we'll have to encircle all possible groups of adjacent 0's. Encircling we get the following K-map.

A \ BC	[00] $B+C$	[01] $B+\overline{C}$	[11] $\overline{B}+\overline{C}$	[10] $\overline{B}+C$
[0] A	0 $_0$	0 $_1$	0 $_3$	1 $_2$
[1] $\overline{A}$	0 $_4$	1 $_5$	1 $_7$	1 $_6$

There are 3 pairs which are :

Pair-1 = $M_0 . M_1$;

Pair-2 = $M_0 . M_4$;

Pair-3 = $M_1 . M_3$.

But there is one reduntant group also *i.e.*, Pair-1 (its all 0's are encircled by other groups). Thus removing this reduntant pair-1, we have only two groups now.

Reduced *P-O-S* expression for Pair-2 is $(B+C)$, as while moving across pair-2, A changes its state from A to $\overline{A}$, thus A is removed.

Reduced *P-O-S* expression for Pair-3 is $(A+\overline{C})$, as while moving across Pair-3 B changes to $\overline{B}$, hence eliminated.

Final *P-O-S* expression will be $(B+C).(A+\overline{C})$

Check Point 12.6

hat is the other name of Karnaugh ap ? Who invented Karnaugh maps ?

ow is gray code different from normal nary code ?

ow many variables are reduced by a ir, quad and octet respectively ?

EXAMPLE 12.27. *Find the minimum P-O-S expression of*

$$Y(A, B, C, D) = \Pi(0, 1, 3, 5, 6, 7, 10, 14, 15).$$

Solution. As the given function is 4-variable function, we'll draw 4-variable K-map and then put 0's for the given Maxterms *i.e.*, in the squares whose numbers are

0, 1, 3, 5, 6, 7, 10, 14, 15

As each square number represents its Maxterm.
So, K-map will be

AB \ CD	[00] $C + D$	[01] $C + \overline{D}$	[11] $\overline{C} + \overline{D}$	[10] $\overline{C} + D$
[00] $A + B$	0 (0)	0 (1)	0 (3)	1 (2)
[01] $A + \overline{B}$	1 (4)	0 (5)	0 (7)	0 (6)
[11] $\overline{A} + \overline{B}$	1 (12)	1 (13)	0 (15)	0 (14)
[10] $\overline{A} + B$	1 (8)	1 (9)	1 (11)	0 (10)

Encircling adjacent 0's we have following groups :

Pair-1 $= M_0 \cdot M_1$; Pair-2 $= M_{14} \cdot M_{10}$;

Quad-1 $= M_1 \cdot M_3 \cdot M_5 \cdot M_7$; Quad-2 $= M_7 \cdot M_6 \cdot M_{15} \cdot M_{14}$

Reduced expressions are as follows :

For Pair-1, $(A + B + C)$ (*as D is eliminated : D changes to* $\overline{D}$

For Pair-2, $(\overline{A} + \overline{C} + D)$ ($\overline{B}$ *changes to B ; hence eliminated*

For Quad-1, $(A + \overline{D})$ (horizontally C and vertically B is eliminated as C, are changing their state

For Quad-2, $(\overline{B} + \overline{C})$ (horizontally D and vertically A is eliminated

Hence final *P-O-S* expression will be

$$Y(A, B, C, D) = (A + B + C)\ (\overline{A} + \overline{C} + D)\ (A + \overline{D})\ (\overline{B} + \overline{C})$$