# Chapter 3

⬀⬀⬀**GEOMETRIC MODEL OF SCHEIMPFLUG IMAGING**

We would like to investigate the capabilities and limitations of Scheimpflug imaging for extending the imaging volume of iris-image acquisition for biometrics.

Imaging systems primarily consists of a lens, a light sensitive surface called the sensor, and a surface in sharp focus called the object. The object and its image formed by the lens are called *conjugates*. In this work, we restrict the object and image surfaces to planes. Further, we define the plane containing the sensor as the *image plane*, the object surface (in focus) as the *object plane*, and a plane representing the lens as the *lens plane*. Amongst the several possible planes that pass through a lens, we cherry-pick a plane that provides some advantage in our model to be the lens plane.

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| **Figure 3.1** Scheimpflug camera movements. Insets (a), (b), (c), and (d) depict few key camera movements—lens and sensor plane tilts about a horizontal axis—amongst several possible, in a Sinar P3 camera. Labels indicate the lens (1), the lens standard (2), bellows (3), sensor standard (4), and the sensor (5). The cyan lines on the two standards accentuates the orientations. Inset (e) is a superimposed sequence of images of the camera with the two standards in a multitude of orientations. The physical locations of the two pivots emerge at the intersection of the superimposed cyan lines. |

In the *fronto-parallel* configuration used in common camera design, the lens and the sensor planes are parallel to each other and perpendicular to the optical axis. In such designs, the physics of optical imaging ("Gaussian Imaging equation") dictates that the object plane is also parallel to the lens and image planes. In contrast, the lens- and image- planes in a Scheimpflug camera are free to swivel about their pivots (as shown in Figure 3.1) resulting in a corresponding swivel of the object plane. We would like to exploit this feature—the freedom to arbitrarily orient the object plane—of Scheimpflug configuration to improve the depth of field of the iris acquisition devices.

Scheimpflug cameras provide the greatest flexibility for image composition; however, that flexibility is traded for complexity. Accurate modeling of Scheimpflug imaging is quite involved, and its art of operation is often left to experts who frequently employ approximate methods. While these cameras have been used in few scientific imaging applications, but the vast majority of them contemporarily are used for landscape and studio photography.

Existing models of Scheimpflug camera commonly employ overtly simple models that work quite well for documentary photography but are often restrictive and inaccurate for scientific purpose. A rich description of such cameras requires the development of a more general model. We aim to develop, in this chapter, a model for Scheimpflug imaging using the axioms of *geometric optics* (*ray optics*).

Ray optics does not provide a complete picture of imaging; yet, the definitions and postulates therein provide useful tools for the analysis and synthesis of optical systems.

Assumptions are crucial and necessary for modeling that enable its expediency and limits its applicability. For the model described herein, we have assumed paraxial imaging, rotational symmetry and aberration-free optics in order to make the problem tractable. Additionally, justified by environment in which …., we have assumed the refractive index of the lens elements and the interstitial medium to be isotropic (uniform along all directions) and homogeneous (uniform at all positions); this assumption imposes rectilinear propagation of light. Further, we have assumed the lens is surrounded by air of refractive index one. Consequently, the front and back focal lengths are equivalent, and the two nodal points coincide with the corresponding principal points.

TODO: Review what is out there. Type of models that are there, their limitations. Also, comment on the existing process of “focus-transfer” why that is erroneous. Point out, without explicitly stating, that this method has several advantages (and explicitly point out the advantages), the new insights that it provides and not a re-engineering of existing knowledge just for the sake of being different.

TO DO: State the novelty of this approach, and why needed to develop this model. Is there any relation to eikonal equations?

TODO: Preview of what is coming in the following sections.

### 3.1 Introduction

Optical imaging systems consist of several groups of elements; those elements endowed with optical power bends rays of light. The tiniest orifice in the system is called the *system aperture* or *stop*. Its interaction with the elements in the system gives rise to the pupils.

*Pupils* are the sine qua non of optical systems. They are indispensable in the design and specification of all optical systems, in both domains of *ray* and *wave optics*. The *entrance pupil* () is the “image” of the stop seen through the elements preceding it is. The *exit pupil* () is the “image” of the stop seen through the elements following it is. That is, the pupils are the images of the stop produced by the elements on either side of it. The region preceding the entrance pupil, which includes the objects and light sources, is called the object space; and the region following the exit pupil, which includes the image plane, is called the image space. The size and position of the stop (and hence the pupils) affect image resolution, aberration, brightness, and geometry.

Rotationally symmetric lenses have an axis of symmetry—the optical axis. A ray coincident with the optical axis traverses undeviated through the lens. Planes passing through the axis of such lenses are the meridional planes. Rays restricted to the meridional planes are *meridional rays*. Patterns formed by the meridional rays on either side of the optical axis are mirror-reversed, exhibiting bilateral symmetry. Figure 3.2 shows two types of meridional rays, traced in Zemax, that are fundamental to geometric analysis. The *marginal ray* (MR) originates from the axial object position and skirts the edges of the aperture and pupils (virtually); the *chief ray* (CR) starts at an off-axis object point and pierces the centers of the aperture and pupils[[1]](#footnote-1) (virtually). This pair of rays determines the location and size of the pupils, the position of the image, and the magnification. Furthermore, the bundle of chief rays from the object space converge at the center of the entrance pupil—thus *homocentric*—forming the vertex of the object-space perspective cone; in the image space, the bundle of chief rays diverge from the center of the exit pupil producing the vertex of the image-space perspective cone.

Imagine a film projector working backwards. Imagine the stream of light rays flowing from the illuminated portion of the scene towards a small circular hole in the projector. This pencil of rays forms a conical volume of light—the perspective cone—with its vertex at the hole and its base towards the scene. The “illuminated portion” is the angular extent of the scene visible in the image, confined by the circumferential chief rays. These extreme chief rays determine the opening angle of the cone. The “small hole” represents the entrance pupil of a camera or the pupil at the center of the iris in an eye. In the image space (behind the hole), the ray-pencil form another cone with the vertex at the center of the exit pupil. This image-space perspective cone projects the light from the scene onto the film surface or the retina in the eye. This process of image formation, known as the *central projection*, is fundamental to all imaging systems—inanimate and animate—including the camera and the eye. While the opening angle of the object-space perspective cone determines the field-of-view, its counterpart in the image space determines the angular dimension of the image. The ratio of the pupil sizes (pupil magnification) determines the relationship between the image and object-space opening angles of the two perspective cones.

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| **Figure 3.2** Fundamental rays (contained within the meridional plane) and pupils in a Double Gauss lens for an object at infinity. The chief rays—close to the optical axis (0°, ±5° in the object space at entrance pupil)—appear to converge at the center of entrance pupil and diverge from the center of exit pupil. The marginal ray, which is parallel to the optical axis since the object is at infinity, appear to skirt the edges of the two pupils. The red circles specify the vertices of the perspective cones (centers of the pupils). The rays were traced in Zemax. |

### 3.2 Notation (this section is currently a placeholder for the notation, skip reading this for now. I will write it appropriately after completion of the major math)

* Right-handed coordinate system with the +z along the direction of travel of light, (left to right in the plane of drawing)
* In general, non-primed quantities are used to indicate input or object space (e.g.) and primed quantities are used to indicate output or image space (e.g.).
* Unit vectors are represented using a hat (), with the exception of the direction cosine vectors (e.g. although the norm of the direction cosine vectors are unity.
* A left superscript indicates the frame of reference. For example, indicates that the variable is w.r.t. the world coordinate frame. If no reference is explicitly stated it implies that the variable is w.r.t. the world coordinate frame (or the camera coordinate frame if the camera coordinate frame and the world coordinate frame are the same.
* A subscript is used to associate a variable with a particular xxx like entrance pupil position (), image plane (), for example is used to represent the 3D rotation matrix applied to the entrance pupil plane in the camera frame. The same notation is also used to indicate a transformed variable, for example is used to represent under the rotational transformation by in the camera coordinate frame. As also mentioned earlier, if the camera coordinate frame is the same as the world coordinate frame, then the notation shall be used.
* represents the pose of frame w.r.t. frame TO DO: mention how a point in one frame is represented in another frame.
* The one-based indexing of matrices and vectors. Also a matrix is also represented as where are the columns of .
* Describe what I mean by lens plane.
* Overloading of the term “direction cosine(s)” and “direction cosine vector”. It should be clear from the context
* Point out notation for angles , , ,

### 3.3 Relation between pupil magnification and chief ray angle

The *pupil magnification* is defined as the ratio of the paraxial exit-pupil diameter to the entrance-pupil diameter [refref].

Figure 3.3 illustrates the meridional and sagittal planes associated with an arbitrarily located object of height above the optical axis and its image of height in a typical optical system. The figure also shows the chief ray from the object’s edge further from the optical axis, the marginal ray from the axial point in the object, and the two pupils contained in the meridional plane. The schematic, although simple, is quite general as a (meridional) plane always exist for a given object point irrespective of its position in the three-dimensional space, if the lens is rotationally symmetric.

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| **Figure 3.3** Schematic of chief and marginal rays. The ratio of the tangents of the chief ray angles in the object space to the image space yields the pupil magnification. |

Let the angles between the chief ray and the optical axis (called the *ray-angle*) in the object- and image-space be and respectively. Also, let the angles produced by the marginal ray with the optical axis in the object- and image-space be and respectively. Then, we can obtain the relation between the chief ray ray-angles— and —and the pupil magnification as follows:

From the Figure 3.3,

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Eliminating and after dividing by, we have

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A common observation in imaging is that we can increase the transverse magnification () by increasing the lens-to-image-plane distance while correspondingly decreasing the lens-to-object-plane distance in order to maintain focus on the object. However, increasing (decreasing) the image plane distance proportionally decreases (increases) the marginal ray angle (see Figure 3.3). Consequently, the angular magnification () decreases with increase in lens-to-image-plane distance. Therefore, a large transverse magnification is associated with a correspondingly small angular magnification. This result follows from a more general theory called the *Lagrange invariant* [ref] property of the two rays (the chief ray and the marginal ray) when applied between conjugate locations. As per the invariant property, the product of the transverse magnification and the angular magnification equals to one, i.e., or. Cancelling the corresponding terms in Eq. (3.2) yields the relationship between the pupil magnification and the object- and image-space chief ray angles:

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The above relation (Eq. (3.3)) has been previously derived in [ref] using a different formulation.

For a given optical system, the pupil magnification is constant. This constancy of the ratio of the tangents of the chief ray angles for varying object (and image) heights is a necessary and sufficient condition for distortion-free imaging known as the *Airy’s Tangent-Condition* [ref]. Eq. (3.3) also suggests that when the perspective cones in the object- and image-space are symmetric. In the following section, we will use Eq. (3.3) to derive the relationship between the direction cosines of the object-space (input) chief rays and direction cosines of the image-space (output) chief rays.

### 3.4 Transfer of chief ray’s direction cosines between the pupils

The direction cosines, a unit vector of cardinality three, specify the direction of a ray. Its elements are the cosines of the angles the ray makes with the three coordinate axes. In other words, the elements of the direction cosine vector are the projections of the unit vector in the direction of the ray on the x-, y-, and z-axes. Given the direction cosine of the chief ray in the object space, we would like to know the direction cosine of the corresponding ray in the image space. Furthermore, what is the relation between the input and output chief ray’s direction cosines if the lens is swiveled about a pivot point along the optical axis?

We begin by solving a specific problem of the *transfer* of the direction cosines between the pupils in which the optical axis coincides with the z-axis of the camera frame, as show in Figure 3.4. Subsequently, we will deduce the general *transfer* expression in which the optical axis is free to swivel about the origin of. Let be the direction cosine of the chief ray from a world point to the center of the entrance pupil, and let be the corresponding direction cosine of the chief ray from the exit pupil. The parameters , ,and are specified with respect to frame .

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| **Figure 3.4** Specific problem—optical axis coincides with reference frame’s z-axis. If and are the angles of the CR with the OA in the object- and image-space respectively, then and. |

If and are the zenith and azimuthal angles of the chief ray in the object space, and and the corresponding angles in the image space, then the direction cosines, in , are:

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Since the optical axis is aligned with the z-axis, and. Substituting the expressions for from Eq. (3.4) into Eq. (3.3) we obtain:

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Further, since the input and out chief rays are confined to the same meridional plane [ref], , yielding and in terms of and , the ratios of to , and :

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From (3.3) we have

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which after simplification yields in terms of the pupil magnification and input

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Combining Eqs. (3.6) and (3.8), we obtain the expression for output direction cosine of the chief ray in terms of the input direction cosines and the pupil magnification as:

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which we can write compactly as:

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Our objective is to derive the expression for the transfer of the chief ray’s direction cosines from entrance pupil to exit pupil for arbitrary orientation of the optical axis as shown in Figure 3.5. Although a formal derivation is provided in [Appendix], we can readily infer the general *transfer* expression from Eq. (3.10) as follows:

Suppose we swivel the optical axis about the origin of the camera frame. This rotation can be described by the matrix. As before, we designate the ray from the object-point to the (new position of the) center of the entrance pupil as the chief ray. Let us also suppose that we have another coordinate frame,, sharing its origin with and its z-axis coincident with the optical axis. If be the direction cosine of the chief ray from the object-point in the frame , then the direction cosine in the frame is and the third element of the direction cosine is , where is the third column of . Representing, the direction cosine of the chief ray emerging from the exit pupil is obtained by substituting for and for in Eq. (3.10):

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The above expression represents the output direction cosine in the coordinate frame In order to transform the output direction cosine from the coordinate frame to the camera frame we need to multiply the direction cosine vector by to obtain:

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| **Figure 3.5** Configuration of the general problem—optical axis (OA) pivots freely about the origin of. |

The positive or negative sign of the direction cosine determines the forward or backward direction of light-travel along a rectilinear path. Under the assumptions of isotropy and homogeneity, the only condition under which a ray of light emerges in an antipodal path from an interface is if it encounters a mirror surface *normally*. This condition does not arise within the context of our problem. Therefore, without any loss of generality, we can drop the negative sign in Eq. (3.11); accordingly, the output direction cosines assume the sign of the corresponding input direction cosines. Therefore, the general expression for the direction cosines of the chief ray in the image space is obtained as:

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where.

Note that Eq. (3.12) only describes the output chief ray’s direction cosines—a free vector. The output chief ray is obtained from the knowledge of the direction cosine and the location of the exit pupil in the appropriate reference frame.

Although it is not obvious from the expression (3.12), we expect to have unit magnitude. We have provided a proof in [Appendix] that shows the (magnitude) of is indeed equal to one, and is the normalizing term.

We can draw the following inferences about from the Eq. (3.12):

1. If the pupil magnification, , then , which implies that the opening angles of the image- and object-space perspective cones are equal, irrespective of the orientation of the optical axis. Then, the lens is symmetric about a plane perpendicular to the optical axis (in addition to the symmetry about the optical axis). Such symmetric lenses are can be reversed without affecting system properties [ref].
2. If we let, such that, then we can write , where is the scalar normalization term. Furthermore, as is a diagonal matrix, and is orthonormal, we can immediately recognize the form as the Eigen value decomposition of the symmetric matrix, with —the columns of —as the eigenvectors and the corresponding eigenvalues. As is a diagonal matrix,

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In Eq. (3.13) the terms are the projections of the input direction cosine along the eigenvectors. Also,, which is the third column of the rotation matrix , is the direction of the optical axis. The effect of the transformation is a *shearing* of the input direction cosine along the optical axis.

### 3.5 Image formation for arbitrary orientation of the lens and image plane

Geometric imaging is a mapping (*bijective* in projective space) between points in the three-dimensional world space to corresponding points on a mathematical surface that we call the *image*. Here we aim to study the nature of this mapping on a planar surface—the image plane—for arbitrary orientations of either the lens and image planes. To that effect, we will use the knowledge of the transfer of direction cosines of the chief ray derived previously.

An extended object emanates a multitude of chief-rays that reach the image space through the pupils and the stop. The locus of points formed by the intersection of these rays with the image plane constitutes the *projection* of the object in the image plane [ref]. Further, we identify the projection of the world-point as an “image” if the pencil of rays (including the chief-ray) from the world-point, filling the pupils and stop, geometrically converge at a single point in the image space.

For simplicity, we assume that the lens is unencumbered by radial distortions and optical aberrations. Figure 3.6 represents a schematic of the problem in which we have introduced an image plane whose orientation is described by the unit surface normal. Two local frames are also introduced: the frame is attached to the optical axis with its origin at entrance pupil (), and the frame attached to the image plane with its origin at the image plane pivot. The image plane is free to swivel (tilt or swing about its local x-axis or y-axis respectively) about the image plane pivot.

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| **Figure 3.6** Schematic of geometric image formation. is the *central* *projection* of the world point on image plane. The optical axis and image plane are free to swivel about the origins of coordinate frames and respectively. |

Let the exit pupil () be located units from the pivot point along the optical axis. Following the rotation of the optical axis, by applying the matrix, the position of the exit pupil in is given as.

We can represent the chief ray emerging from the exit pupil with direction cosine by the parametric equation:

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where represent points along the output chief ray in. The first term on the R.H.S. of Eq. (3.14) is the initial position of the ray (at the center of) and is a real number that determines the length of the ray.

The equation of the image plane in Hessian normal form is

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where is the unit normal to the image plane, is the perpendicular distance from the origin of frame to the plane, and is an arbitrary point on the image plane.

We obtain the expression for (in Eq. (3.14)) for which the ray intersects the image plane by equating to, multiplying Eq. (3.14) by, and rearranging the terms:

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Substituting Eq. (3.16) into Eq. (3.14) we get

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Now we proceed to find the expression for in terms of the known parameters. The origin of the image plane’s local reference frame,, is located at the intersection of the z-axis of camera frame with the image plane (see Figure 3.7). Given the point , we can describe the orientation of the image plane using the surface normal, which is obtained by applying to the unit vector :

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The equation of the image plane is

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Since is a point on the plane,

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| **Figure 3.7** Schematic of the image plane. The image plane having surface normal is located at units from the origin of camera frame along the z-axis that intersects the plane at . is the perpendicular distance from the origin to the plane. The local image coordinate frame with its origin at the intersection of the image plane and z-axis of the camera frame is represented by . |

Using the above result in Eq. (3.17) yields the expression for the point of intersection of the chief ray with the image plane in terms of the input direction cosines as

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Similar to the exit pupil, let the entrance pupil be located at a distance from the pivot point along the optical axis in the camera frame. Then, the location of the entrance pupil in is.

The direction cosines and the world point are related as

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which can be written compactly as:

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Substituting Eq. (3.22) into Eq. (3.20), a general relation between the world point and its corresponding image point is obtained as:

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Eq. (3.23) represents the image point in the camera frame. Once an image is formed, we specify positions and dimensions within the it independent of the position and orientation of the sensor and lenses (e.g. in terms of pixels in a digital image). We can transform the image coordinates in the camera frame to the image frame by observing that the origin of is displaced from by , and the standard basis vectors of are rotated by . Consequently, a point in relative to may be expressed as:

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Therefore, we can write the expression for the image point coordinates with respect to the image plane’s reference frame as

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Substituting from Eq. (3.23) into Eq. (3.25) we obtain the expression of the 2D image point in the image frame as

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Where,

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|  | 3D Cartesian coordinates (in physical units) of the world point in camera frame . In the adopted coordinate convention, the numerical value of z-component of is negative. |
|  | 2D Cartesian coordinates (in physical units) of the image point in the image frame . Note that Eq. (3.26) produces a 3x1 vector with the third element (i.e. the   component) identically equal to zero. |
|  | Equal to , where is the pupil magnification. |
|  | Location of the entrance pupil from the pivot (origin of ) along the optical axis. This scalar quantity physical units, usually in millimeters. |
|  | Location of the exit pupil from the pivot (origin of ) along the optical axis. |
|  | Location of the image plane from the pivot (origin of ) along the z-axis of . |
|  | Rotation matrix used to describe the orientation of the lens plane. |
|  | The third column of . |
|  | Rotation matrix used to describe the orientation of the image plane. |
|  | The image plane normal. denotes the z-component of the normal. |
|  | Origin of the image frame with respect to camera frame; . |

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| **Figure 3.8** Ray tracing for verifying Eq. (3.26). Chief rays traced from a grid of points in the object plane through an ideal lens tilted about a point away from the entrance pupil along the optical axis to the tilted image plane. |

We derived the Eq. (3.26), which relates a 3D object point to its projection in the 2D image plane of a Scheimpflug camera, analytically. Now we show that the relationship is accurate by comparing the numerically computed values of image points (intersection of chief ray with the image plane) using Eq. (3.26) with corresponding image points obtained by tracing chief rays from a grid of points in an object plane. Figure 3.8 is a layout plot of an optical system created in Zemax, showing the object plane; an ideal lens made from two paraxial surfaces and pivoted about a point away from the entrance pupil (); and an image plane pivoted about the image plane pivot along the z-axis. Both the lens and image planes are arbitrarily rotated (about their local frames) with respect to both x- and y-axis. The orientations of both the planes are represented using intrinsic rotations (elemental rotations first about the x-axis and then about the new y-axis) matrices. and represent the angles of rotation of the lens plane about the x- and y- axes while and represent the angles of rotation of the image plane about the x- and y- axes. The set of object points, the numerically computed image points, the Zemax ray traced image points, and the absolute difference between the numerically computed and ray traced image points are tabulated in Table 3.1. It shows that the difference between the numerically computed and ray traced values of the image points are in the order of XXX. This comparison demonstrates that the algebraic expression that expresses the geometric relation between a 3D world point and its image point in the absence of optical aberrations is accurate.

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| **Table 3.1** Comparison of numerically computed image points with ray-traced image points for the optical system shown in Figure 3.8. | | | |
| World point | Computed image point | Ray-traced image point | Absolute difference |
| (0.0, 0.0, -509.0) | (-0.3108, -0.6291, 0.0) | (-0.3108, -0.6291, 0.0) | (1.8e-09, 3.1e-09, 7.5e-15) |
| (10.0, -10.0, -509.0) | (-0.8003, -0.0863, 0.0) | (-0.8003, -0.0863, 0.0) | (2.1e-09, 2.7e-09, 3.0e-15) |
| (-50.0, 50.0, -509.0) | (2.1291, -3.3352, 0.0) | (2.1291, -3.3352, 0.0) | (1.2e-09, 3.2e-09, 2.9e-15) |
| (70.71, 70.71, -509.0) | (-4.2013, -5.0221, 0.0) | (-4.2013, -5.0221, 0.0) | (2.6e-09, 5.1e-09, 4.7e-15) |
| (100.0, 0.0, -509.0) | (-5.5251, -1.0101, 0.0) | (-5.5251, -1.0101, 0.0) | (1.3e-09, 8.4e-09, 3.1e-15) |
| (0.0, 100.0, -509.0) | (-0.6031, -6.4387, 0.0) | (-0.6031, -6.4387, 0.0) | (2.2e-09, 4.0e-09, 2.2e-16) |
| (100.0, 100.0, -509.0) | (-5.8238, -6.8542, 0.0) | (-5.8238, -6.8542, 0.0) | (5.6e-10, 2.5e-10, 2.2e-15) |

Following the verification of the analytic expression, we use the expression to qualitatively study the effects of lens and sensor rotations on the geometric image shape. Figures 3.9 – 3.14 show the basic nature of geometric distortions in images two object planes for different lens and sensor orientations and the role of the pupil magnification and lens pivot point. In all these figures, the basic setup is similar to that shown in Figure 3.8 except that the object space consists of two planes—the near plane, a square of 88.15 mm on each side, and the far plane that is a square of 178.3 mm on each side placed at twice the distance of the near plane from the entrance pupil. The exact distances vary depending upon the pupil magnification, such that the images of the two planes are 4.5 mm on each side in the image plane. Also, since the z-axis of the camera frame passes through the center of both object planes, the two images are coincident in the frontoparallel configuration (i.e. when the object planes are parallel to lens and image planes).

The image points are the points of intersection of the chief-rays emanating from a 7x7 square grid in the object planes with the image plane. The lighter shaded orange “Y” markers represent the group of image points from the near object plane in frontoparallel configuration. The lighter shaded blue “inverted Y” markers represent the image points from the far object plane in the frontoparallel configuration. Note that in frontoparallel configuration the two images of the two object planes coincide; however, for the sake of visual clarity, we displaced the two set of image points horizontally by 5 *mm* on either side of the center. The darker shaded markers of either color represent the image points following the rotation of the sensor or lens. Rotation of either the lens or the sensor induces a geometric distortion of the image field in which the image points across the image field translates by different amounts and directions. These translations are shown by the gray-to-white arrows between the original and shifted positions (drawn if the magnitude of the shift is greater than a certain threshold). The white level of the arrows specifies the normalized magnitude of translation—brighter indicates relatively larger translation. The figures also display information about the standard deviation (SD) of the arrow lengths. This statistic gives a sense of the non-uniform translation of the image points across the image field. If all image points shift by the same amount, then the standard deviation will be zero. A larger value of the standard deviation indicates greater diversity in shifts, and hence greater distortion. In addition to the standard deviation, we also measure how much the centroid of the set of points from the two images shifts. The translation of the centroid gives a measure of how the total image field shifts. Note that in all cases shown here, the *image points* were not determined using a “best focus” criterion, but rather by the point of intersection of the chief rays with the image plane. However, this definition of the *image* adopted for the current discussion does not limit the study of geometric properties, such as the kind of transformations induced by the rotations of the sensor and lens planes.

Based on the study of the figures, we can make the following key observations:

*Properties of image field induced by sensor rotation ():*

1. Due to the rotation of the sensor, the image plane ceases to be frontoparallel with the object plane imparting a varying transverse magnification across the field. As a result, the image points undergo a field dependent, asymmetrical, geometric distortion.
2. The amount of perspective distortion depends on the pupil magnification.
3. Since the location of both the entrance pupil nor the exit pupil remain fixed, the on-axis image point continues to remain on-axis subsequent to the rotations. Therefore, there is no net shift of the image field due to sensor rotation.
4. The distortion is independent of the object distance.
5. Since the distortion is independent of object distance, rotation of the sensor does not introduce parallax between images obtained under different rotations of the sensor.
6. If we capture multiple images under several rotations of the sensor planes, then the inter-image homography (the mapping between corresponding points of two images) is a perspective mapping of the following form:

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*Properties of image field induced by lens rotation away from ENPP (; ):*

1. The dominant effect of rotating the lens about a point along the optical axis is, in general, a *non-uniform shift* of the image field.
2. The amount of shift of the image field depends on the pupil magnification.
3. The points in the image field does not undergo equal amount of translation. As a result, the standard deviation of the translation vector lengths is non-zero. Also, because the amount of shift of the image field is dependent on the object distance, the value of the standard deviation of the translation vector lengths is different for the images of the two object planes.
4. The amount by which points in the image field translates also depend on the object distance. As a result, images obtained under several rotations of the lens exhibit parallax.
5. If the pupil magnification is different from one, then the inter-image homography is a depth (object distance) dependent perspective mapping. In other words, for every object plane at a certain distance from the lens, the inter-image homography is of the form:

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1. However, if the pupil magnification equals unity, then the inter-image homography is a depth dependent scaling transformation of the form:

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*Properties of image field induced by lens rotation about the ENPP (; ):*

1. Just like the previous case, rotation of the lens about the ENPP induces a shift of the image field.
2. However, the shift of the image field is independent of the object distance. We can see that the standard deviation of the translation vector lengths is equal for both the group of points form the two object planes.
3. Since, the shift of the image field is independent of the object distance, there is no parallax between several images obtained while rotating the lens about the entrance pupil.
4. If the pupil magnification is not equal to one, then the inter-image homography is a depth independent perspective transformation of the form:

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1. If the pupil magnification is equal to one, then the inter-image homography is a depth independent perspective transformation of the form:

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| **Figure 3.9** Geometric image under image plane rotation for varying pupil magnifications. (a) , (b) , (c) . The distortion is dependent on the pupil magnification and field height, but independent of object distance. |

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| **Figure 3.10** Translation of centroids. Since the distortion of image field |

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| **Figure 3.11** Geometric image under lens rotation for varying pupil magnifications. (a) , (b) , (c) . The amount of image field shift is dependent on the pupil magnification. |

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| **Figure 3.12** Geometric I |

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| **Figure 3.13** Geometric image under lens rotation for varying pupil magnifications. (a) , (b) , (c) . The lens is rotated about the x-axis by 5° about a point -5 *mm* away from the ENPP along the optical axis. |

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| **Figure 3.14** Centroid shift. (a) The shift is independent of object distance () if and only the lens is rotated about the ENPP, (b) . |

### 3.6 Object, lens and image plane relationships in Scheimpflug imaging

In this section, we will derive a *general expression* that relates the object, lens and image planes. We will further show that this general expression yields relationships that are more specific to various types of Scheimpflug configurations. In order to keep the problem tractable, we will impose the constraint that three pivots lie along the z-axis of the camera frame, and the origin of is co-located with optical axis’ pivot. We also restrict the angle of rotations of object, lens, and image planes between and about both x- and y- axes (in-plane rotations or rotations about the z-axis is irrelevant for our purpose). Provided we make no distinction between the faces (front or back) of the planes, this bound on the angles of rotations is not limiting in any way since we can uniquely describe all possible planer orientations in three dimensions. On the other hand, this bound warrants non-negative values for the z-component of the plane normal. As we will see later, this warranty permits us to unambiguously estimate the unknown plane normal.

We begin by deriving an expression for the chief ray joining an arbitrary point in the object plane to a point in the image plane. In order for to be the geometric image of , the chief ray conjoining the conjugate points must satisfy the Gaussian imaging equation. This constraint allows us to uniquely determine the position (of the image plane along the z-axis of) and orientation of the three planes in Scheimpflug configuration. The setup is shown in Figure 3.15.

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| **Figure 3.15** Schematic of Scheimpflug imaging. The figure shows the object plane, optical axis and image plane pivoted about points along the z-axis, , of the camera frame. The local object plane and image plane coordinates ( and) are centered at the object- and image- plane pivots. |

The object plane is located at a distance of (the numerical value of is negative in our convention) from the origin of camera frame, along the z-axis. Pivoted about the point in the camera frame, it is completely described by the object plane pivot and the object plane normal,. We describe the normal itself as the product of the rotation matrix and (). The rotation matrix is typically composed of elementary rotation matrices that represent rotations about the x- and y-axis.

That is, the equation of the object plane normal is

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We can obtain the perpendicular distance to the object plane from the origin as

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Then, the equation of the object plane in Hessian Normal form is

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where is any point on the object plane.

Similar to the object plane normal, we describe the image plane normal as

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where is the rotation matrix applied to the image plane at the image plane pivot point .

Repeating the steps used to derive the object plane equation, the equation for the image plane is

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where is any point on the image plane.

Suppose the entrance () and exit () pupils are located at distances and from the lens’ pivot (origin of) respectively, along the optical axis. Also, let if we describe the rotation of the optical axis by applying the matrix. Then, following the rotation, the positions of the pupils in are

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Consider the chief ray from the object-point to the corresponding image point , passing through the center of the entrance and exit pupils. Let the direction cosines of the ray in the object- and image- space be and respectively. Since is the direction vector of the ray from to , we can write

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where, is the position of the entrance pupil, and the is the length of the ray. Substituting Eq. (3.37) in to Eq. (3.39), we get

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Since is a point on the object plane, it satisfies the object plane Eq. (3.34). Substituting into Eq. (3.34) we get

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The chief ray emerges from the exit pupil, having a direction cosine vector, and intersecting the image plane at . Therefore, we can write the parametric equation of this ray as

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where is any point on the ray, is the position of the exit pupil, and is the length of the ray.

If we let the length of the chief ray in the image space be, then when in Eq. (3.42), implying

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Substituting in Eq. (3.36) we get

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Eqs. (3.41) and (3.44) are expressions for the lengths of the chief ray between points and in the object and image space respectively. In order for to be the geometric, in focus, image of , the expressions must satisfy the Gaussian imaging equation.

The well-known Gaussian imaging equation () relates the focal length and the conjugate plane distances (*directed*) measured from the principal planes. Instead, if the *directed* distances are measured from the pupils (from entrance pupil to object plane; and from exit pupil to image plane) in lieu of the principal planes, then a variant of the Gaussian imaging equation is used. The updated imaging equation, which incorporates the pupil magnification into the formula, is

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where and are directed distances along the optical axis measured from the entrance pupil to the object plane; and from the exit pupil to the image plane, is the pupil magnification, and is the focal length. We have provided a derivation and a brief exposition of the above formula in [Appendix].

The most common application of Eq. (3.45) is for fronto-parallel imaging in which the conjugate planes are parallel to each other and perpendicular to the optical axis; moreover, all pairs of object-image conjugate points satisfy this relation even if the ensemble of object- and image- points belong to planes on object and image spaces respectively that are *not parallel* to each other.

The ray vector of length and direction in the object space is. The projection of this ray vector on the optical axis () is. Similarly, the ray projection of the image space ray vector on the optical axis is. In order to substitute and in to Eq. (3.45) we need to ensure that they are the directed distances. In our adopted sign convention, the directed distance from the entrance pupil to the object point is . Substituting and in to Eq. (3.45) we have:

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Further, substituting the expressions for and (Eqs. (3.41) and (3.44)) into the above equation, we have

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The direction cosine of the chief-ray in the image space,, is related to the direction cosine of the chief-ray in the object space as (Eq. (3.12))

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Substituting, into Eq. (3.47) we have

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To simplify the above expression, let us consider as

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Therefore, we can write in Eq. (3.48) as. Similarly, we can also write as. Then, Eq. (3.48) can be written as

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We can further simplify the above equation by noting that:

1. ,
2. ,
3. ,
4. .

Therefore, Eq. (3.50) reduces to

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Further, multiplying the above equation by the scalar , and using the commutative and distributive properties of dot product, we get

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Note that in the above expressions, the third column of the rotation matrix, is the unit vector along the optical axis.

Since is a direction cosine vector with -Norm equal to one, the above equation is satisfied if the second vector,, is either perpendicular to or identically equal to zero. Is it possible for the second vector to be perpendicular to , which represents a chief ray? Since we did not make any specific assumption about in the derivation of Eq. (3.52), *all* chief rays—an infinitude of vectors within the object- and image-space perspective cones—must satisfy Eq. (3.52). Are all possible vectors perpendicular to the second vector? Since the second vector is a linear combination of (the object plane normal), (the transformed image plane normal), and (unit vector along the optical axis) whose weights are constant for a given system, we can conclude that , *in general*, is not perpendicular to the second vector. Therefore,

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Further, we can simplify Eq. (3.53) if we let and. Then, after factoring and out of the denominator terms, we can write Eq. (3.53) as

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Where,

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| --- | --- |
|  | Directed distance of the object plane’s pivot from the origin of the camera frame, along the z-axis of the camera frame. (In most of the problems that we are interested, the value of is negative.) |
|  | Directed distance of the image plane’s pivot from the camera frame, along the z-axis of the camera frame. |
|  | Focal length of the lens. |
|  | Equal to , where is the pupil magnification. |
|  | Location of the entrance pupil from the pivot (origin of ) along the optical axis. This scalar quantity physical units, usually in millimeters. |
|  | Location of the exit pupil from the pivot (origin of ) along the optical axis. |
|  | Rotation matrix used to describe the orientation of the lens plane. |
|  | The third column of . |
|  | Equals , where is the object plane normal. |
|  | Equals , where is the image plane normal. |

The expedient simplification from Eq. (3.53) to Eq. (3.54) relies on our ability to describe the *unit* normal vectors and using only the components along x- and y- axes. If we know the x- and y- components of the normal vector, we can determine the z- component uniquely because we have restricted the angle of rotations of the planes between and about both x- and y- axes (one of the starting assumptions). For example, if the object- and lens- plane orientations and distances are known, and we estimate the image plane distance and orientation vector () of the image plane using Eq. (3.54), then we can determine the image plane normal as

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where , , and we have dropped the negative sign from the expression for since is guaranteed to be positive as discussed under the assumptions at the beginning of this section.

Comment about why only the chief ray and the fact that we have fixed the z-axis between the object’s, lens’ and image’s pivots is sufficient to determine the image plane location and orientation, intuitively. ….

Eq. (3.54) is most general in the sense that it readily yields the various expressions for specific cases of object, lens and image plane orientations as we now demonstrate in the following set of examples.

### 3.7 Examples of typical scheimpflug imaging configurations

**Example 3.1:** Suppose the object plane is frontoparallel with a thick lens, and units from the origin of camera frame (along the z-axis of the ), what should be the orientation and position of the image plane for forming a geometrically focused image of the object plane?

Based on the given data, we have the following:

1. Since the lens is not tilted, , .
2. Since the object plane is also not tilted, and .
3. Let and. Then, , and , and

Substituting the above parameters in Eq. (3.54) we have

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which yields the following relations from each row of Eq. (3.56):

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Writing and where, is the directed distance from the entrance pupil to the object plane and is the directed distance from the exit pupil the image plane, we can rewrite Eq. (3.59) as

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For example, if , , ,  , and , then we can calculate the image plane distance from the camera frame using Eq. (3.59) equal to .

Further, if, then the sign of is positive, which is a condition for forming real and inverted images. For example, if, and, then is positive for. If the sign of is negative, then a virtual and upright image is formed in front of the lens. In a survey of 120 imaging lenses (see Appendix xxx) from a database (Zemax Zebase) of well-designed lenses, we found over 90% of all lenses to have pupil magnification greater than 0.5 and no lens having pupil magnification less than 0.2. Thus, in the common imaging scenarios, the sign of is positive.

From Eqs. (3.57) and (3.58) we see that the image plane normal is equal to. That is, the image plane is parallel to the lens and object plane and perpendicular to the optical axis if the object plane is parallel to the lens plane.

For thin lens model (,), Eq. (3.59) reduces to the Gaussian lens equation for thin lenses:

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Furthermore, if the distances to the object and image plane are specified from the object- and image- space principal planes instead of the pupils, Eq. (3.59) reduces to ()

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in which, is the distance of the object plane from the object-space principal point, and and represent the locations of the object- and image-space principal points in the camera frame . While is numerically negative, the signs of and depend on the position of the principal points with respect to the origin of.

**Example 3.2:** Suppose the object plane, pivoted at, is tilted by an angle about the x-axis in the camera frame (Figure 3.16), what is the conjugate orientation and position of the image plane for achieving a geometrically focused image assuming a thick lens model?

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| **Figure 3.16** Object and image plane tilt. In the above cross-sectional (y-z plane) view, the object plane is tilted by an angle of about the x-axis at. We would like to find the position and orientation of the image plane in order to focus on the object. |

1. Since the lens is not tilted, , .
2. Let and. Then, and .
3. Let and. Then, , and .

Substituting the above parameters in Eq. (3.54) we have

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As in the Example 3.1, the distance of the image plane pivot in the camera frame, obtained from the third of Eq. (3.63), is

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Also, as in Example 1, writing and where, is the directed distance from the entrance pupil to the origin of the object plane and is the directed distance from the exit pupil to the origin of the image plane, we get

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in which is numerically negative and is positive for macroscopic imaging.

From the first and second rows of Eq. (3.63), we have

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and

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Since the object plane is rotated by about the x-axis, the rotation matrix is

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Note that if the direction of rotation of the object plane about the x-axis is from +z-axis to +y-axis (as depicted in Figure 3.16), then is numerically negative.

The object plane normal is (Eq. (3.32))

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from which, we can get (by dividing by ) as:

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Substituting into Eqs. (3.66) and (3.67) we get

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Eq. (3.71) in conjunction with Eq. (3.55) suggest that, which implies that the image plane normal is confined to the y-z plane. That is, if the object plane is rotated only about the x-axis, then the image plane must also be rotated only about the x-axis in order to achieve geometric focus on the tilted object plane. The exact expression for the components of the image plane normal is obtained by substituting and into Eq. (3.55). The angle of the image plane normal with respect to the three axes is further obtained as the cosine inverses of the corresponding components of the normal vector.

Furthermore, there remains a desideratum to have a more direct relationship between the rotation angles of the object and image planes that can be readily used in lieu of computing the plane normals. To that end, let us suppose that the required image plane tilt (rotation about the x-axis) is. If we represent the rotation matrix of the image plane as

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it yields the image plane normal as

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which produces as

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Substituting (from Eq. (3.74)) into Eq. (3.71) we obtain

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where, for common macroscopic imaging, is numerically negative, and is positive. Therefore, the sign of is opposite to the sign of . This result implies that the image plane must be rotated in the opposite direction of rotation of the object plane.

We can further modify the above relation if the distances are specified with respect to principal planes (Figure 3.17). Then, represents the magnification between the image and object side principal planes and therefore, is equal to one; the ratio represents the ratio of the distances of the principal-plane-to-object plane in the object side and principal-plane-to-image plane in the image side along the z-axis, and therefore is equal to the magnification along the z-axis. Letting,, and, we can rewrite Eq. (3.75) for distances measured from the principal planes as

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| **Figure 3.17** Object and image plane tilt (distances measured from principal planes). In the above cross-sectional (y-z plane) view, the object plane is tilted by an angle of about the x-axis. The distances to the object and image plane pivots are specified from the respective principal planes. |

**Example 3.3:** Determine the orientation of a thin lens required to focus on a tilted object if the lens plane is not tilted.

1. Since we have a thin lens, , , .
2. Let and. Then, .
3. Since the image plane is not rotated, .

The unknowns in this problem are the image plane distance and the lens plane angle. In general, a rotation matrix has three degrees of freedom, however, since we have restricted the rotation of the lens plane to only about the x- and y- axes, the rotation matrix in our problem has only two degrees of freedom. Further, we can describe the orientation of the lens plane using just the third column of (the normal to the lens plane). Therefore, in total, we have three unknowns—the image plane distance and the two rotation angles of the lens plane. Another way to think about the number of knowns and unknowns is that we are required to determine the normal vector to the lens plane and the distance of the image plane. Since only two components of the normal vector are essential to determine the orientation of the plane (when the angles of rotation of the plane is restricted between ), we have three unknowns.

Substituting the known parameters in Eq. (3.54) we have

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The three rows of the above equation yields

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and

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In order to solve for the image plane distance in Eq. (3.80) we need . Can we uniquely determine from and ? A property of any rotation matrix is that each column (or row) has unit length. Therefore, . Furthermore, if the rotation matrix is composed of elementary rotations only about the x- and y- axes, then , which is a product of the cosine of the angles of rotations about the two axes, is guaranteed to be positive. Hence,

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For a more concrete example, suppose the object plane is tilted only about the x-axis by an angle . Then, (as in the previous example), and . Therefore, , , and . The image plane distance is given by .

The dependence of on (and implicitly on the amount of lens tilt as shown below in Eq. (3.83)) implies that we need to also shift the image plane location along the z-axis of , in order to focus on a tilted plane. Instead, if we chose to focus on the tilted plane employing just sensor rotation, the distance of the sensor’s pivot from the camera center remains fixed.

Furthermore, implies that the lens is rotated only about the x-axis. If we let (the third column of ), then the relationship between the lens plane rotation angle () and object plane rotation angle () is given as (and shown in Figure 3.18)

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In the above equation is numerically negative, therefore the sign of is same as the sign of , which implies that the direction of rotation of the lens and object planes are congruent.

The image plane distance along the z-axis of is given as:

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The sign of is positive, forming real and inverted image behind the lens, if .

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| **Figure 3.18** Object and lens (thin lens model) plane tilt. In the above cross-sectional (y-z plane) view, the object plane is tilted by an angle of about the x-axis at. We would like to find the position of the image plane and orientation of the lens plane in order to focus on the tilted object surface. |

**Example 3.4:** If the image plane is not rotated, what is the required angle of rotation of the lens in order to focus on an object surface that is titled about the x-axis by an angle using a thick-lens model?

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| **Figure 3.19** Object and lens (thick lens model) plane tilt. In the above cross-sectional (y-z plane) view, the object plane is tilted by an angle of about the x-axis at. We would like to find the position of the image plane   and orientation of the lens plane in order to focus on the tilted object surface. |

We can represent the orientation of the object plane that is tilted about the x-axis using the rotation matrix:

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such that

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In the previous problem (using thin lens model) we observed that the direction of rotation of the object- and lens- planes are congruent. Therefore, the structure of the rotation matrix representing the lens plane’s orientation is similar to that of the object plane:

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such that,

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As the image plane is not tilted, we have

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Therefore, , , and

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Substituting the above parameters into Eq. (3.54) we have

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The third row, following simple algebraic steps, yields the formula for the image plane distance in terms of the angles of the object and lens planes:

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From the second row we have

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Substituting from Eq. (3.90) into Eq. (3.91) and writing , , and , we get

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Multiplying by we have

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Following few algebraic steps, we get the finite conjugate imaging relationship between the object plane tilt angle and lens plane tilt angle when the lens is rotated about a pivot (offset from the entrance pupil) as:

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In the discussion that follows we will use the notation to represent the right-hand-side of the above equation. Eq. (3.94) is an implicit relationship between the angles and . Comparing Eq. (3.94) with Eq. (3.82) immediately shows that for a given object plane tilt angle the lens tilt angles obtained by the thick-lens (more accurate) model deviates from that obtained using a thin lens model. Further, the object plane angle (in focus) obtained via Eq. (3.94) for a given lens tilt angle depends on the location of the lens pivot point along the optical axis. The variation of with respect to the pivot position (offset from entrance pupil, ) for an object plane pivoted at from the entrance pupil of a lens with pupil magnification is show in figure 3.20. The thin-lens model doesn’t account for such deviations.

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| **Figure 3.20** Variation of (y-axis) with respect to lens pivot position for (a) , (b) , and (c) . The x-axis is the offset of the pivot position from the entrance pupil. The plots show that the range of is large for larger values of . In each graph the two other plots are also plotted in lighter values for comparison of the slopes. |

As the object plane distance increases, the effective object-to-entrance-pupil distance () in Eq. (3.94) tends to a constant value for relatively small changes in the entrance-pupil-to-pivot-point distance . Therefore, for relatively large object plane distances the variation of is expected to be negligible for small changes in .

Before we begin to examine the consequences of the Scheimpflug imaging equations Eq. (3.90) and Eq. (3.94) that relates the lens plane orientation and image plane distance from the origin of camera frame if the lens is tilted about a point away from the entrance pupil, it is imperative to ensure that the equations are verified. Table 3.2 enumerates the results of our test. In order to test the equations, we created a thick lens model in Zemax using two paraxial surface—to ensure aberration-free geometric imaging model—with pupil magnification . The lens surfaces were grouped within two coordinate break surfaces that allowed the lens to be tilted about a point away from the entrance pupil. The object plane surface was placed at from . Then, for every object plane orientation (col. 1), the appropriate lens tilt angle (col. 2) and image plane distance (col. 3) were obtained using Zemax’s optimization function, to minimize spot radius across the field. Following optimization for every , the value of obtained from Zemax (along with the values of , , ) was used to numerically compute (col. 4) and (col. 5) using the derived equations Eq. (3.90) and Eq. (3.94). We can observe that the values of and obtained numerically using the derived equations are very closely matched. It must be emphasized that the values ( and ) obtained in Zemax were using ray-tracing and optimization, rather than numerical evaluation of analytic expressions.

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| **Table 3.2** Verification of imaging equations Eq. (3.90) and Eq. (3.94) for focusing on a tilted object plane by tilting a lens about a point away from the entrance pupil. | | | | |
| (Zemax)1 | (Zemax)2 | (Zemax)3 | (numerical)4 | (numerical)5 |
| 0.0° | 0.0° | 24.17073 *mm* | 2.07042e-13° | 24.17073 *mm* |
| -10.0° | -0.46989° | 24.17163 *mm* | -9.99973° | 24.17163 *mm* |
| 25.0° | 1.24260° | 24.17701 *mm* | 24.995702° | 24.17701 *mm* |
| -40.0° | -2.23573° | 24.19107 *mm* | -39.98214° | 24.19107 *mm* |
| 65.0° | 5.70827° | 24.30378 *mm* | 64.91024° | 24.30377 *mm* |
| -80.0° | -14.99585° | 25.11194 *mm* | -79.74010° | 25.11146 *mm* |
| 1. *Object plane tilt set in Zemax.* 2. *Lens plane tilt obtained using optimization using ray-tracing in Zemax.* 3. *Image plane distance obtained using optimization using ray-tracing in Zemax.* 4. *Object plane tilt computed numerically using the value of in column 2.* 5. *Image plane distance computed numerically using the value of in column 2.* | | | | |

Eq. (3.94) suggests that for a given lens tilt angle , we can determine the orientation of the plane-of-sharp focus . But what if we need to compute given ? Quite often we need the inverse relationship; that is, we need to determine the lens tilt angle required to focus on a tilted object plane. We will return to this question a few times in this section—first qualitatively in the imminent discussion followed by a detailed analysis near the end of this section. Another related question that we may ask at this point is whether the relationship between and , as depicted using Eq. (3.94), always one-to-one? In other words, is the function , which represents the right-hand-side of Eq. (3.94), *monotonic* and *invertible* within the interval ? If is monotonic, then it follows that will be monotonic within and we will have a one-to-one relationship between and . A test to determine the monotonicity of within an interval is to examining if the first derivative, , changes sign within the interval. Later in this section, when we examine the case of rotating the lens about the entrance pupil, we will present a more detailed analysis of the first derivative of . But first we carry out a qualitative analysis of Eq. (3.94) which relates the orientations of the object and lens planes if lens is pivoted about a point away from the entrance pupil. Although we will seldom rotate a lens about a point away from its entrance pupil because of the unwieldy distortions induced to the image field (section xxx), the methods and insights developed in its study, nevertheless, will carry over to our examination of the latter case.

In Figure 3.21 we have plotted values of — or —and the object tilt angle versus the lens tilt angle for four different values of while keeping the parameters , and fixed. We can observe that while the function is monotonic for and , it becomes non-monotonic for and as evidenced by the presence of stationary points (the exact locations are not important) in the corresponding plots. Consequently, for example, when we obtain the same value of ( 72.3°) for two different values of (17.65° and 45.0°). These plots suggest that is not always monotonic. In our simulations we have found that tends to become non-monotonic as the value of becomes very small.

Apart from the monotonicity of , we also need to consider the sign of the image plane distance from the exit pupil (), equal to , obtained through Eq. (3.90). If in Eq. (3.90), then is negative which implies that a *virtual image* is formed in front of the lens. Therefore, in order to form a *real image* on a sensor, the condition must be satisfied.

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| **Figure 3.21** Object plane angle and versus lens tilt angle if a lens is rotated about a point away from the entrance pupil. In these plots , and . The plots show that the function is not always monotonic, especially for small values of . For e.g., in the plots corresponding to the function yields the same value of for and . |

Once the two independent conditions—the monotonicity of and the formation of real image—are satisfied, we can return to the question regarding how to determine the lens tilt for a known object plane tilt angle . Obtaining an expression for as a function of from Eq. (3.94) is not straightforward. However, we can develop some insights into the problem if we substitute and into Eq. (3.94), which yields the implicit equation:

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We can recognize Eq. (3.95) as a *quartic plane curve* of the form . The object plane tilt angle along with the parameters , , and form the coefficients of this fourth degree plane curve. Further, since and , we also obtain a second curve—a unit circle with equation . The lens tilt angle (more precisely and ) satisfies both these equations, therefore it must be at the point of intersection of the two curves. In Figure 3.22 we have plotted several quartic curves corresponding to different pupil magnifications and two choices of lens rotations and . Note that the corresponding value of for each curve in both groups will be different. The other parameters— and —are same for all curves. In order to keep the discussion simple, these parameters

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| **Figure 3.22** Determination of lens tilt angle for known object tilt angle using point of intersection of *quartic plane curve* with the unit circle. If a lens is rotated about a point away from the entrance pupil, the equation for as a function of , , , and can be expressed as a quartic plane curve in Cartesian coordinates with , . Furthermore, the intersection of the quartic curve with the unit circle yields the unknown lens tilt angle . The figure plots two groups of quartic curves—the blue curves correspond to equations for different values of (, , and ) and but the same while the green curves corresponds to . For all curves , and . |

along with value the for each curve used in the figure satisfy the conditions of monotonicity of and real image. The curves in blue correspond to the that satisfy Eq. (3.95), and the green curves correspond to . Since the constant in Eq. (3.95)—corresponding to in the general quartic equation—is zero, all curves pass through the origin. Additionally, since the coefficient of the term is zero while the coefficient of the term is non-zero, the curves are either above the x-axis (for positive values of ) or below the y-axis (for negative values of ). As shown in the figure, all such quartic curves (for which the parameters satisfy the monotonicity condition) intersect the unit circle at two points—in the first and second quadrants if , or in the third and fourth quadrants if . However, since in Eq. (3.95) the abscissa of the point of intersection must always be positive for . Consequently, we can a determine a unique point of intersection of the quartic curve (given , , , and ) with the unit circle that correspond to the unknown lens tilt angle .

Based on the above discussion we see that it is possible to implement an algorithm to find the point of intersection of a quartic curve and the unit circle—for example, using Newton’s method—to determine given . Alternatively, if a good initial estimate of as a starting point is known, an iterative algorithm that converges towards the true point of intersection along the unit circle may be used to find the . We demonstrate such an algorithm at the end of this section.

We have seen in section xxxx that rotating the lens about a point away from the entrance pupil distorts the image field in complex ways and induces parallax between corresponding scene points in sequence of images obtained under lens rotations. Therefore, in several imaging applications that require multiple image captures under lens rotation it is imperative to rotate the lens about the entrance pupil. In the rest of this section we will undertake a closer examination of this scenario.

We can obtain the equations for image plane distance and plane of sharp focus orientation for the case when the lens is pivoted at the entrance pupil by substituting and (where, is the distance of the exit pupil from the entrance pupil) in Eqs. (3.90) and (3.94) respectively yielding:

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and

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Table 3.3 enumerates the results of verifying the above equations against Zemax. The optical system used in the test is similar to that used to verify the Scheimpflug imaging equations for lens rotation about a point away from the entrance pupil, except that since the pivot was shifted to the entrance pupil, the object distance changed from to . The table shows that the numerically computed value of object plane tilt angle using the analytic expressions matches exactly to that used in Zemax to tilt the object plane. Also, the value of obtained in Zemax using ray tracing and optimization is in exact match with that obtained using the analytic expression we derived, proving the accuracy of the above expressions.

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| **Table 3.3** Verification of imaging equations Eq. (3.96) and Eq. (3.97) for focusing on a tilted object plane by tilting a lens about the entrance pupil. | | | | |
| (Zemax)1 | (Zemax)2 | (Zemax)3 | (numerical)4 | (numerical)5 |
| 0.0° | 0.0° | 29.17073 *mm* | -2.24301e-15° | 29.17073 *mm* |
| -10.0° | -0.46989° | 29.17145 *mm* | -10.0° | 29.17145 *mm* |
| 25.0° | 1.24249° | 29.17572 *mm* | 25.0° | 29.17572 *mm* |
| -40.0° | -2.23504° | 29.18687 *mm* | -40.0° | 29.18687 *mm* |
| 65.0° | 5.69682° | 29.27607 *mm* | 65.0° | 29.27607 *mm* |
| -80.0° | -14.79587° | 29.90304 *mm* | -80.0° | 29.90304 *mm* |
| 1. *Object plane tilt set in Zemax.* 2. *Lens plane tilt obtained using optimization using ray-tracing in Zemax.* 3. *Image plane distance obtained using optimization using ray-tracing in Zemax.* 4. *Object plane tilt computed numerically using the value of in column 2.* 5. *Image plane distance computed numerically using the value of in column 2.* | | | | |

Comparing Eqs.(3.96) and (3.97) with Eqs. (3.90) and (3.94) respectively, we can immediately observe that the equations are far less complex when the lens is rotated about the entrance pupil. However, just like before, the function that represents the right-hand-side of Eq. (3.97) is not always monotonic. As shown in Figure 3.23, is monotonic for and but non-monotonic for and .

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| **Figure 3.23** Object plane angle and versus lens tilt angle if a lens is rotated about the entrance pupil. The plots show that the function is not always monotonic, especially for small values of . For e.g., in the plots corresponding to the function yields the same value of for and . and for all plots. |

The following two subsections are devoted to a deeper examination of the condition of monotonicity of and the development of an iterative algorithm for determining the lens tilt angle .

#### Condition for monotonicity of

The first derivative of with respect to is:

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| where, . |  |

We observe that the first derivative is a cubic equation in . A cubic equation has at least one real root which implies that the graph of must cross the real x-axis at least once. However, in our problem, the lens rotation angle is restricted to within . Consequently, we are only concerned with those roots of that are in the open interval , because . More concretely, if has a real positive root in the interval implying that it changes sign, then is non-monotonic within . In such a case we cannot find a unique for a given value of . In Figure 3.24 we have plotted the first derivative for varying values of . We can see that the plots of for and have at least two roots as they cross the real x-axis twice with the interval . Of course, this result was expected as we have already seen in Figure 3.24 that is non-monotonic for and .

Additionally, we can easily examine the first derivative when , which is give as

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For , the first derivative is a linear function of and crosses the x-axis at the origin as can be seen in Figure 3.24. Therefore, it has no real roots in the open interval , implying that is monotonic. Furthermore, since is numerically negative (directed distance), is a monotonically increasing function implying that , and consequently , increases with .

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| **Figure 3.24** Plots of the first derivative of . The plots first derivative for and cross the real x-axis twice within the interval implying real roots in the interval . Consequently, is non-monotonic in . |

Heretofore we have used visualizations to analyze the conditions under which it is possible to invert the function . We now proceed to find an analytic expression that can be used to test the monotonicity of . We can, of course, use any numerical computation tool to find the cubic roots of the first derivative and verify if the first derivative has real roots in the open interval . In fact, our derivation is based on the algebraic method for solving cubic roots published by Gerolamo Cardano in his treatise *Ars Magna* in 1545 [ref WolframMathWorld “cubic Formula”].

The roots of a cubic polynomial is given as:

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where,

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and

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We can determine the nature of the roots from —if , then is real resulting in one real root () and two roots that are complex conjugates (, ); if , then the imaginary terms of and vanishes resulting in all three roots being real and at least two equal; and if , then and becomes complex conjugates resulting in all three roots being real.

Since in first derivative represented by Eq. (3.98) is positive in our problem, the roots of the first derivative are same as the roots of the scaled cubic equation

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with coefficients , , , and and .

Eq. (3.103) is not suitable if ; however, we have already seen that when , the first derivative represents a line passing through the origin. For , the coefficient of the linear term—— is larger than both the constant and quadratic terms. In fact, the magnitude of tends to two, starting from a very large magnitude, as increases from one. At the same time the magnitude of tends towards zero. Therefore, for the curve represented by Eq. (3.103) never crosses the real x-axis in the interval ; where is a very small number whose exact value depends on and corresponds to an angle that is very close to . Therefore, we only need to test for the monotonicity of if .

Based on the discussion of nature of the cubic roots, we would expect that if all three roots are real, then there is a high chance that one or more of these real valued roots would lie within the interval . Indeed, based on tens of thousands of randomly generated combinations of parameters , and we have found that the only instances in which we obtain real roots of Eq. (3.103) in the interval is when . Finally, substituting the expressions for the coefficients , , , and from Eq. (3.103) into Eq. (3.102) and Eq. (3.101) we get the *sufficient condition* for to be monotonic and invertible if as:

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#### Algorithm for finding for known

Eq. (3.97) can be used to easily determine the tilted object plane orientation that is brought to focus if the lens is tiled by an angle about the center of the entrance pupil. However, it is often required to find the lens tilt angle as a function of a known object plane tilt angle . Deriving a closed-form inverse relation of the Eq. (3.97) is not easy. Therefore, we will develop an iterative algorithm for determining . The central idea behind the algorithm was introduced earlier when we saw that the general relationship between and , if the lens is pivoted about a point away from the entrance pupil, lead to a quartic plane curve equation parameterized by , , and . The point of intersection of the quartic curve with the unit-circle (and having a positive value of abscissa) yielded . In a similar vein, by substituting and in Eq. (3.97), we obtain the following implicit equation if the lens is rotated about the entrance pupil:

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Similar to our earlier observation on the reduction of complexity of the equation relating and , the implicit equation too reduces from a fourth degree *quartic plane curve* (Eq. (3.95)) to a second degree *quadratic plane curve*. We have plotted several such curves in two groups in figure 3.25. The curves in each group, distinguished by the green and blue lines, belong to and respectively. In each group the various curves correspond to different values of (and ). For all curves in the figure the focal length , and the object plane distance . The shapes of these curves are almost always elliptic, although for small values of we have also observed parabolic and hyperbolic shapes. At the curve, irrespective of other parameters, is a circle. The point of intersection of each quadratic curve with the unit circle in the first or third quadrants yields the lens tilt angle for positive or negative sign of object tilt angle respectively. The red cross in the figure depicts the second possible solution for (equal to ) when , implying that the system does not meet the *sufficient condition* for determining .

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| **Figure 3.25** Determination of lens tilt angle for known object plane tilt angle using point of intersection of *quadratic plane curve* with the unit circle. If a lens is rotated about the center of its entrance pupil, the equation relating as a function of , , , and can be expressed as a quadratic plane curve in Cartesian coordinates with , . Furthermore, the intersection of the quadratic plane curve with the unit circle yields the unknown lens tilt angle . The figure plots two groups of quadratic curves—the blue curves correspond to equations for different values of (, , , and ) and but the same while the green curves corresponds to . The red cross indicates a second point of intersection of the quadratic curve corresponding to with the unit circle implying that there exist two values of (equal to and ) that evaluates to the same value of using Eq. (3.97). Therefore, the relationship between and is not unique for with the particular choice of object distance () and focal length (). |

As discussed previously, one possible method to find the lens tilt angle for known , , , and would be to numerically compute the points of intersection of the corresponding quadratic curve with the unit circle, for example using Newton’s method, followed by selecting the point of intersection from the appropriate quadrant depending on the sign of . In our problem, since the desired lens tilt angle is constrained to reside on the unit circle, instead of fining the roots of a cubic equation, we can use an iterative algorithm to find , provided we can find a good starting point that is relatively close to the desired . In fact, we do have a good starting point—the value of for . If and the lens is rotated about the entrance pupil, the equation relating and is equivalent to rotating a thin lens represented by Eq.(3.82).

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| **Table 3.4** Algorithm for determining lens tilt angle required to focus on an object plane tilted by an angle . |
| Objective  Given object plane tilt angle , determine , the lens tilt angle about the entrance pupil, required to focus on object plane that is at a distance from the lens pivot. The lens has focal length and pupil magnification . The iteration quits if either the specified accuracy for the result, , or the specified maximum steps, , is reached.  Algorithm   1. Define function: 2. Compute initial estimate of to provide a starting point: 3. Compute initial error: 4. Compute initial value of step to increment/ decrement : 5. Set iteration variables for error: , 6. Set iteration variable for angle : 7. Set step counter: 8. repeat:    1. Increment step counter:    2. Update estimate of lens tilt angle :    3. Store current error:    4. Compute new error:    5. Modify step direction and size if error changes sign: :       * Set new step size: |

The iterative algorithm used for determining the lens tilt angle for a known value of is shown in table 3.4. In addition, figures 3.26 and 3.27 aim to provide a visual explanation of how the algorithm functions. Figure 3.26 describes how , , and evolves over iteration number as the algorithm converges to the value of starting from an initial estimate given by thin-lens approximation. The figure also shows how the step size is modified when the error changes sign. While Figure 3.26 illustrates how the algorithm converges to the right values of , Figure 3.27 shows how evolves along the unit circle starting from the initial estimate as a function of the iteration number for two different choices of . Both figures show that for typical optical systems the algorithm converges quite fast very close to the true value of the unknown . Furthermore, large number of iterations improves the precision of the estimate.

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| **Figure 3.26** Example determination of lens tilt angle . The plot shows the iteration variables (red), (cyan), and (blue) against the iteration number. While the range of values of and are shown on the vertical axis on the left, the range of values for are shown on the right vertical axis. The ground true value of the lens tilt angle is . This true value of corresponds to with , and . The algorithm starts with an initial estimate of (obtained using thin lens approximation) and converges to approx. in the first 20 iterations. The final value of , after 48 iterations, is . The plot also shows that the step size remains constant as the error in each iteration decreases until the value of the error changes sign; at which point the step’s value and sign are modified. |

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| **Figure 3.27** Visualization of the iterative algorithm for determining given . The figure shows the unit circle (magenta) and two quadratic curves. The ellipse in blue correspond to and , and the hyperbola in green correspond to and . The corresponding ground truth values of the lens tilt angle for the blue and green quadratic curves are and respectively, shown by the blue and green spheres on the unit circle. The initial estimates for as starting points for the iterative algorithm are shown by the smaller red spheres. The transparency, position, and numeric tags of white spheres depict the direction of convergence, the value of the angle in the iteration number and the iteration number for some intermediate steps. The iteration for converges to the true value of at a faster rate than the iteration for as indicated by the number of steps and precision of the estimated in the two cases. It is important to note that the system with (green) is only used to demonstrate the algorithm. Although the parameters meet the sufficient condition test, the image plane of this hypothetical system is located several meters behind the exit pupil. On the other hand, an optical system with is quite common. For both curves, and . |

# Appendix⬀

### Derivation 3.1

Later, we will apply the method of induction to yield the solution of the general *transfer* problem—in which the optical axis is free to swivel about the origin of.

Eq. (3.9) accurately represents the *transfer* for the specific problem; however, we will cast the expression in a slightly different form whose raison d'être is to enable generalization—through direct application of the result. Specifically, we can express the output chief ray as a linear combination of the input chief ray and the optical axis since the two rays and the optical axis span the same (meridional) plane. Let, the standard basis vector along z-axis of, represent the optical axis since the optical axis is coincident with the z-axis. Then,

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where and are the weights, and.

Rewriting the above equation as

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the weight is readily obtained by comparing Eqs. (3.9) and (4.2):

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Substituting the expression for into and comparing with (3.8) yields:

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We are now ready to apply the result of the specific problem to the general problem. Figure 3.4 shows the schematic of the general problem—the optical axis pivots about the origin of. Let us describe the general orientation of the optical axis by the action of the rotation matrix on. The matrix may be a composition of two or more matrices that denotes a sequence of rotations about the x-axis and/or y-axis. Then,, the unit vector representing the new orientation of the optical axis, is obtained as: or .

As the output direction cosine, the input direction cosine, and the optical axis lie on the same plane,

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Note that the input direction cosine in Eq. (4.5) is different from the corresponding in Eq. (4.1) even for the same object-point . This difference is due to the displacement of entrance pupil () following the rotation of the optical axis; in fact, the designation of a ray as the chief ray (from to) keeps altering as we keep displacing the entrance pupil. Multiplying Eq. (4.5) by:

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Letting and,

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Comparing Eqs. (4.1) and (4.7) the expressions for the weights and are obtained as:

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Where represents the projection of the direction cosine vector, **,** on the rotated optical axis. If we write the matrix where are the columns of . Then,

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and

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Therefore, since is the third element of.

Rewriting Eq. (4.7) as:

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which can be compactly written as:

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Finally, substituting and yields the general expression for the direction cosines of the chief ray in the image space in terms of the pupil magnification and direction cosines in the object space as:

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where.

### Claim 3.1 The output direction cosine, originating from exit pupil, has unit -Norm

The direction cosine in the image space, obtained by the linear transformation of the direction cosinein the object space, has unit , and is the normalization term.

*Proof*.

The expression for the direction cosine in the image space is

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where , is the column of the rotation matrix applied to the optical axis, , and is the pupil magnification.

Our objective is to prove.

For the convenience of notation within the proof, let

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where, and , the diagonal matrix with non-negative real values.

Also, let us represent the columns of as

Then, , and

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Now, since is a rotation matrix, it is orthonormal (the column of, having unit length, are orthogonal to each other). Therefore, .

Then,

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where

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where and .

As is a diagonal matrix, we can rewrite as

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Now,

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Also

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Substituting in Eq. (4.15),

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Substituting into Eq. (4.14) we have

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It follows that the scalar quantity is the normalization term. ☐

### Derivation 3.2 Gaussian imaging equation with pupil magnification

The familiar Gaussian imaging equation, , relates the object and image plane distances with the focal length . is the *directed distance* (numerically negative as per our sign convention) between the object plane (perpendicular to the optical axis) and the principal plane () in the object space, is the directed distance (numerically positive for *real* images) between the in-focus image plane and the principal plane () in the image space. The distances being measured along the optical axis.

If the distances of the object and image planes are specified from the entrance () and exit pupil () instead of the principal planes, then the Gaussian lens formula needs to be slightly modified to incorporate the pupil magnification (). Here we derive the modified formula starting from the Gaussian lens formula. The same result was derived in [Ref] using a slightly different approach.

Figure x.x shows a schematic of the entrance and exit pupils, the object and image space principal planes, and the object and image points.

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| **Figure x.x** Schematic of imaging through a lens. The figure shows the object () and its image (), the object space principal plane () and the image side principal plane (), the entrance () and exit () pupils, and the associated distances along the optical axis. |

In the figure, and are the distances from the principal planes to the object and image planes, and are distances from the principal planes to the entrance- and exit-pupils and and are the distances from the entrance- and exit-pupils to the object and image planes. Since the entrance- and exit-pupil planes are conjugates, like the object and image planes, the Gaussian lens formula holds as follows:

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and

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The transverse magnification between the object and image planes is

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For images that are *real* and inverted, the transverse magnification is numerically negative since the directed distance is numerically negative, and is numerically positive.

The pupil magnification is defined as the ratio of the exit pupil diameter to the entrance pupil diameter. It is also the ratio between the exit pupil and entrance pupil distances (measured from the principal planes) just like the transverse magnification between any conjugate planes:

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Equating Eqs. (4.20) and (4.21), we obtain

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Substituting and in the above equation, and using Eqs. (4.22) and (4.23), we get

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Further, we can also substitute and in Eq. (4.20) and equate with Eq. (4.21) as

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which after cross-multiplication and cancellations of common terms produces

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Dividing throughout by, and substituting by the pupil magnification, and by we obtain:

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Where,

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| --- | --- |
|  | Pupil magnification. |
|  | Directed distance from the entrance pupil to the object plane. |
|  | Directed distance from the exit pupil to the image plane. |
|  | Focal length. |

Eq. (4.25) is valid even if the and denote distances from the principal planes provided we let. This outcome is indeed consistent with geometric optics theory, according to which the magnification between the principal planes is unity. In fact, Eq. (3.45) is more general than the Gaussian Lens formula in that it relates a pair of conjugate planes with any other pair of conjugate planes for which the transverse magnification (between the planes) is known. When one of the pairs happen to be the principal planes ( and), between which the magnification is one, we obtain the Gaussian Lens formula.

Finally, we can also derive the equations for computing the entrance- and exit-pupil distances from the respective principal planes by substituting (4.23) into (4.21)

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### A brief account on the significance of pupil magnification

Although a pupil magnification close to one is a desirable property from the point of view of distortion in the presence of orientation misalignment, it seems to be hardly a critical design choice for most practical lenses except for those used for Scheimpflug photography as evident in plot of pupil magnifications in Figure x.x. In addition, the figure shows that telephoto lenses have pupil magnification less one and retrofocus wide-angle lenses have pupil magnification greater than one. This is because the telephoto lenses employ a negative focal length group near the sensor plane to make a long focal length lens in a compact body, which also results in a relatively smaller image of the stop in the image side compared to the image of the stop in the object side. That is, the entrance pupil is larger compared to the exit pupil in telephoto lenses resulting in pupil magnification less than one. On the other hand, a negative focal length group is placed at the front in short focal length retrofocus lenses to create space between the lens and sensor which results in larger exit pupil compared to the entrance pupil. Thus, the pupil magnification of retrofocus lenses are greater than one.

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| **Figure x.x** Pupil magnification in a wide variety of lenses that form *real* images. The figure demonstrates the absence of any correlation between pupil magnification and focal length. In addition, only 20 in the sample 120 (or one in six) lenses have pupil magnification in the range. Over 90% of all lenses have pupil magnification greater than 0.5. We obtained the samples from the Zemax Zebase library, which is a comprehensive catalogue of well-designed professional lenses. |

The F-number (or F/#) is the ratio of the effective focal length (distance between the image side principal plane to the image plane with the object at infinity) to the paraxial entrance-pupil diameter [ref]

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This infinite conjugate F-number is commonly specified by lens manufacturers and marked on lens bodies.

The pupil magnification is the ratio of the exit-pupil diameter to the entrance-pupil diameter

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Substituting from Eq. (4.28) into Eq. (4.27), we get

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When the object is at infinity, the distance between the image plane and the exit-pupil is obtained from Eq. (4.25) as

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Substituting in place of in Eq. (4.29) yields the alternative and equivalent definition for F-number—*as the ratio of the exit-pupil-to-image-plane distance to the exit-pupil diameter*:

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where is the distance from the exit-pupil to the image plane and is the diameter of the exit-pupil.

The F-number along with the wavelength determines the diffraction limited spatial resolution of optical imaging systems on the image plane as given by the equation [ref]

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For finite conjugate imaging, the object-plane-to-entrance-pupil distance decreases concomitant with an increase in exit-pupil-to-image-plane distance. This increase in the image plane distance effectively increases the F-number. The expression for the effective F-number is obtained as follows:

Substituting (where is the transverse magnification) from Eq. (4.24) into Eq. (4.25), followed by simple algebraic steps yields

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We have established that the F-number (for infinite conjugate) is the ratio of the exit-pupil-to-image-plane distance to the exit-pupil diameter. To obtain the effective F-number at finite conjugates, we substitute Eq. (4.33), the expression for the image-plane distance for finite conjugate imaging, into Eq. (4.31) :

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Further, substituting (Eq. (4.28)) and replacing with we get

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where is the transverse magnification (numerically negative for *real* images).

Now, we can obtain a more accurate equation for diffraction limited spatial resolution that is equally valid for both finite and infinite conjugate imaging by substituting Eq. (4.35) into Eq. (4.32)

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where is the wavelength, is the standard F-number defined for infinite conjugate imaging, is the pupil magnification, and is the transverse magnification (numerically negative for *real* images). When the object is at infinity, and Eq. (4.36) reduces to the optical resolution expression for infinite conjugate imaging.

To Investigate.

### Estimation of pupil locations, and pupil magnification

Describe why you would like to estimate these parameters

The entrance-pupil is the

Eliminating by substituting into Eq. (4.25) following simple algebraic manipulation we get

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where is the transverse magnification, is the distance of the object plane from the entrance-pupil, is the focal length. and are numerically negative for *real* images.

1. In the presence of spherical aberrations, the chief ray goes through the center of the aperture but may not exactly go through the center of the pupils [Ref Mirrors, Prisms … Southall, Lens design by Kingslake]. [↑](#footnote-ref-1)