Geometric model of image formation in Scheimpflug cameras

Indranil Sinharoy, Prasanna Rangarajan, Marc P. Christensen
Department of Electrical Engineering, Southern Methodist University, 6251 Airline Road, Dallas, Texas 75275-0338
isinharoy@smu.edu, *prangara@mail.smu.edu*, *mpc@lyle.smu.edu*

Abstract
We present a geometric model of image formation in Scheimpflug cameras that is most general. Scheimpflug imaging is commonly used is scientific and medical imaging either to increase the depth of field of the imager or to focus on tilted object surfaces. Existing Scheimpflug imaging models do not take into account the effect of pupil magnification (i.e. the ratio of the exit pupil diameter to the entrance pupil diameter), which we have found to affect the type of distortions experienced by the image-field upon lens rotations. In this work, we have also derived the relationship between the object, lens and sensor planes in Scheimpflug configuration, which is very similar in form with the standard Gaussian imaging equation, but applicable for imaging systems in which the lens plane and the sensor plane are arbitrarily oriented with respect to each other. Since the conventional rigid camera, in which the sensor and lens planes are constrained to be parallel to each other, is a special case of the Scheimpflug camera, our model also applies to imaging with conventional cameras.

Keywords — Scheimpflug camera, Non-frontal camera, Image formation model.

1. Introduction

The Scheimpflug imaging is used in many scientific imaging applications such as optical microscopy, corneal topography and imaging, range-finding, etc. either to circumvent the problem of limited depth of field of cameras or to focus on object surface that are oriented obliquely with respect to the camera. In Scheimpflug cameras the sensor and lens are free to shift and tilt about a defined axis. While shifting (in-plane translation) of the lens/sensor doesn't change the orientation of the plane of sharp focus, rotating one with respect to the other results in the plane of sharp focus to rotate. The ability to adjust the orientation of the plane of sharp focus allows great flexibility for image composition; however, that flexibility is traded for complexity.

Accurate modeling of Scheimpflug imaging is quite involved, and its art of operation is often left to experts who frequently employ approximate methods. Existing models of Scheimpflug camera commonly employ overtly simple models that work quite well for documentary photography but are often restrictive and inaccurate for scientific purpose. A rich description of such cameras requires the development of a more general model. We aim to develop, in this work, a complete model for Scheimpflug imaging for first principles using the axioms of *geometric optics* (*ray optics*). The central piece of our model is the pupil magnification that critically affects the image distortion on lens rotation. We believe that this model will enable us to analyze Scheimpflug imaging systems—and thus to any existing imaging system that uses a lens and sensor—to a greater degree than which is possible now. Furthermore, we believe that this model will enable us to apply Scheimpflug imaging principles to a variety of new applications such as omnifocus image synthesis.

Assumptions are crucial and necessary for modeling that enable its expediency and limits its applicability. For the model described herein, we have assumed paraxial imaging, rotational symmetry and aberration-free optics in order to make the problem tractable. Additionally, we have assumed the refractive index of the lens elements and the interstitial medium to be isotropic (uniform along all directions) and homogeneous (uniform at all positions); this assumption imposes rectilinear propagation of light. Further, we have assumed the lens is surrounded by air of refractive index one. Consequently, the front and back focal lengths are equivalent, and the two nodal points coincide with the corresponding principal points.

2. Background

2.1 Relation between pupil magnification and chief ray angle

Optical imaging systems consist of several groups of elements; those elements endowed with optical power bends rays of light. The tiniest orifice in the system is called the *system aperture* or *stop*. Its interaction with the elements in the system gives rise to the pupils.

Pupils are the sine qua non of optical systems. They are indispensable in the design and specification of all optical systems, in both domains of *ray* and *wave optics*. The *entrance pupil* (E) is the "image" of the stop seen through the elements preceding it is. The *exit pupil* ($\acute{E}$) is the "image" of the stop seen through the elements following it is. That is, the pupils are the images of the stop produced by the elements on either side of it. The position of the aperture, the associated entrance and exit pupils, and their relation with the fundamental rays in a common Double Gauss lens system are illustrated in Figure 1. The figure also shows two types of rays that are fundamental to geometric analysis. The *marginal ray* originates from the axial object position and skirts the edges of the aperture and pupils (virtually); the *chief ray* starts at an off-axis object point and pierces the centers of the aperture and pupils (virtually) in an aberration-free optical imaging system. This pair of rays determines the location and size of the pupils, the position of the image, and the magnification. Furthermore, the bundle of chief rays from the object space converge at the center of the entrance pupil forming the vertex of the object-space perspective cone; in the image space, the bundle of chief rays diverge from the center of the exit pupil producing the vertex of the image-space perspective cone.

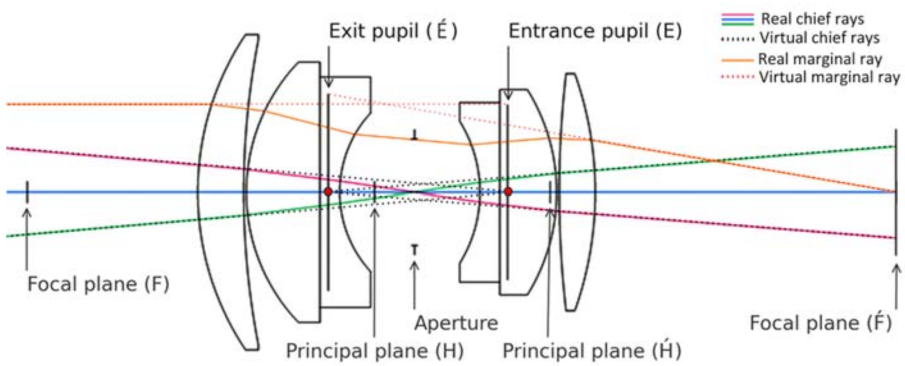

Figure 1 Aperture and pupils in a Double Gauss lens for an object at infinity. The chief rays—close to the optical axis (0°, ±5° in the object space at entrance pupil)—appear to converge at the center of entrance pupil and diverge from the center of exit pupil. The marginal ray, which is parallel to the optical axis since the object is at infinity, appear to skirt the edges of the two pupils. The red circles specify the vertices of the perspective cones (centers of the pupils).

The *pupil magnification* m_p is defined as the ratio of the paraxial exit-pupil diameter to the paraxial entrance-pupil diameter [1]. Figure 2 illustrates the meridional and sagittal planes associated with an arbitrarily located object of height y above the optical axis and its image of height $\acute{y}$ in a typical optical system. The figure also shows the chief ray from the object's edge further from the optical axis, the marginal ray from the axial point in the object, and the two pupils contained in the meridional plane.

Let the angles between the chief ray and the optical axis (called the *ray-angle*) in the object- and image-space be ω and $\acute{\omega}$ respectively. Also, let the angles produced by the marginal ray with the optical axis in the object- and image-space be Ω and $\acute{\Omega}$ respectively. Then, under paraxial assumption, we can obtain the relation between the chief ray ray-angles—ω and $\acute{\omega}$—and the pupil magnification m_p as follows (following the notations in Figure 2):

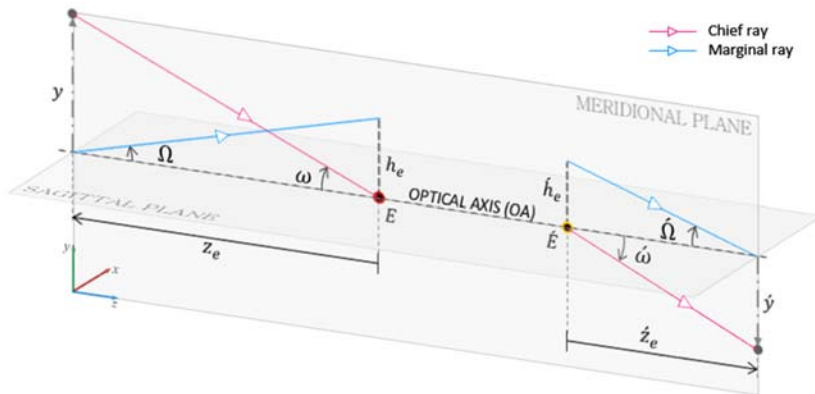

Figure 2 Schematic of chief and marginal rays. The ratio of the tangents of the chief ray angles in the object space to the image space yields the pupil magnification.

$$\frac{\tan(\omega)}{\tan(\acute{\omega})} = \frac{\acute{h}_e}{h_e}\frac{y\tan(\Omega)}{\acute{y}\tan(\acute{\Omega})} \approx \left(\frac{\acute{h}_e}{h_e}\right)\left(\frac{y}{\acute{y}}\right)\left(\frac{\Omega}{\acute{\Omega}}\right). \tag{1}$$

Where, $\acute{y}/y$ is the transverse magnification, $\acute{\Omega}/\Omega$ is the angular magnification, and $\acute{h}_e/h_e$ is the pupil magnification. According to the *Lagrange invariant* [2] property of the two rays (the chief ray and the marginal ray) $y/\acute{y} = \acute{\Omega}/\Omega$. Therefore, the relationship between the pupil magnification and the object- and image-space chief ray angles is:

$$\boxed{\frac{\tan(\omega)}{\tan(\acute{\omega})} = m_p} \tag{2}$$

The above relation (Eq.(2)) has been previously derived in [1] using a different formulation.

For a given optical system, the pupil magnification m_p is constant. This constancy of the ratio of the tangents of the chief ray angles for varying object (and image) heights is a necessary and sufficient condition for distortion-free imaging known as the *Airy's Tangent-Condition* [1,3]. Eq. (2) also suggests that when $m_p = 1$ the perspective cones in the object- and image-space are symmetric.

2.2 Transfer of chief ray's direction cosines between the pupils

The direction cosines, a unit vector of cardinality three, specify the direction of a ray. Its elements are the cosines of the angles the ray makes with the three coordinate axes. In other words, the elements of the direction cosine vector are the projections of the unit vector in the direction of the ray on the x-, y-, and z-axes. Given the direction cosine of the chief ray in the object space, we would like to know the direction cosine of the corresponding ray in the image space. Furthermore, what is the relation between the input and output chief ray's direction cosines if the lens is swiveled about a pivot point along the optical axis?

We begin by solving a specific problem of the *transfer* of the direction cosines between the pupils in which the optical axis coincides with the z-axis of the camera frame $\{C\}$, as show in Figure 3. Subsequently, we will deduce the general *transfer* expression in which the optical axis is free to swivel about the origin of $\{C\}$. Let $\boldsymbol{l} = [l, m, n]^T$ be the direction cosine of the chief ray from a world point $\mathbf{x}$ to the center of the entrance pupil, and let $\acute{\boldsymbol{l}} = [\acute{l}, \acute{m}, \acute{n}]^T$ be the corresponding direction cosine of the chief ray from the exit pupil. The parameters $\mathbf{x}$, $\boldsymbol{l}$, and $\acute{\boldsymbol{l}}$ are specified with respect to frame $\{C\}$.

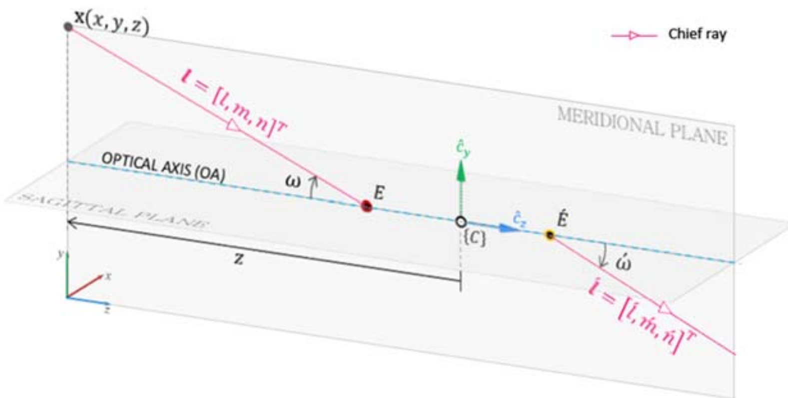

Figure 3 Specific problem—optical axis coincides with reference frame's z-axis. If θ and $\acute{\theta}$ are the angles of the CR with the OA in the object- and image-space respectively, then $\omega = \theta = \cos^{-1}(n)$ and $\acute{\omega} = \acute{\theta} = \cos^{-1}(\acute{n})$.

If θ and ϕ are the zenith and azimuthal angles of the chief ray in the object space, and $\acute{\theta}$ and $\acute{\phi}$ the corresponding angles in the image space, then the direction cosines, in $\{C\}$, are:

$$\begin{aligned} l &= \cos(\phi)\sin(\theta) & \acute{l} &= \cos(\acute{\phi})\sin(\acute{\theta}) \\ m &= \sin(\phi)\sin(\theta) & \acute{m} &= \sin(\acute{\phi})\sin(\acute{\theta}) \\ n &= \cos(\theta) & \acute{n} &= \cos(\acute{\theta}) \end{aligned} \tag{3}$$

Since the optical axis is aligned with the z-axis, $\omega = \theta$ and $\acute{\omega} = \acute{\theta}$. Substituting the expressions for $\sin(\theta)$, $\cos(\theta)$, $\sin(\acute{\theta})$ and $\cos(\acute{\theta})$ from Eq. (3) into Eq. (2) we obtain:

$$\frac{l}{\acute{l}}\frac{n}{\acute{n}}\cos(\acute{\phi}) = m_p\cos(\phi) \quad \text{and} \quad \frac{m}{\acute{m}}\frac{n}{\acute{n}}\sin(\acute{\phi}) = m_p\sin(\phi) \tag{4}$$

Further, since the input and out chief rays are confined to the same meridional plane [4], $\acute{\phi} = \phi$, yielding $\acute{l}$ and $\acute{m}$ in terms of l and m, the ratios of $\acute{n}$ to n, and m_p:

$$\acute{l} = \frac{1}{m_p}\frac{\acute{n}}{n}\,l \quad \text{and} \quad \acute{m} = \frac{1}{m_p}\frac{\acute{n}}{n}\,m \tag{5}$$

From Eq. (2) we have

$$m_p = \frac{\tan(\theta)}{\tan(\acute{\theta})} = \frac{\sin(\theta)\cos(\acute{\theta})}{\sin(\acute{\theta})\cos(\theta)} = \sqrt{\frac{1-n^2}{1-\acute{n}^2}} \times \frac{\acute{n}}{n} \tag{6}$$

which after simplification yields $\acute{n}$ in terms of the pupil magnification m_p and input n

$$\acute{n} = \pm \frac{m_p}{\sqrt{1 + (m_p^2 - 1)n^2}} n \tag{7}$$

Combining Eqs. (5) and (7) , we obtain the expression for output direction cosine of the chief ray in terms of the input direction cosines and the pupil magnification as:

$$\begin{bmatrix} \acute{l} \\ \acute{m} \\ \acute{n} \end{bmatrix} = \pm \frac{1}{\sqrt{1 + (m_p^2 - 1)n^2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m_p \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} \tag{8}$$

which we can write compactly as:

$$\acute{\boldsymbol{l}} = \pm \frac{1}{\sqrt{1 + (m_p^2 - 1)n^2}} M_p \boldsymbol{l} \tag{9}$$

Our objective is to derive the expression for the transfer of the chief ray's direction cosines from entrance pupil to exit pupil for arbitrary orientation of the optical axis as shown in Figure 4. Although a formal derivation is provided in Appendix A, we can readily infer the general *transfer* expression from Eq. (9) as follows:

Suppose we swivel the optical axis about the origin of the camera frame $\{C\}$. This rotation can be described by the matrix $R_\ell \in \mathbb{R}^{3\times3}$. As before, we designate the ray from the object-point $\mathbf{x}$ to the (new position of the) center of the entrance pupil as the chief ray. Let us also suppose that we have another coordinate frame, $\{\mathcal{L}\}$, sharing its origin with $\{C\}$ and its z-axis coincident with the optical axis. If $\boldsymbol{l}$ be the direction cosine of the chief ray from the object-point in the frame $\{C\}$, then the direction cosine in the frame $\{\mathcal{L}\}$ is ${R_\ell}^T \boldsymbol{l}$ and the third element of the direction cosine is $\boldsymbol{r}_{\ell,3}^T \boldsymbol{l}$, where $\boldsymbol{r}_{\ell,3}$ is the third column of R_ℓ. Representing $n_R = \boldsymbol{r}_{\ell,3}^T \boldsymbol{l}$, the direction cosine of the chief ray emerging from the exit pupil is obtained by substituting $\boldsymbol{r}_{\ell,3}^T \boldsymbol{l}$ for $\boldsymbol{l}$ and n_R for n in Eq. (9) :

$$\acute{\boldsymbol{l}} = \pm \frac{1}{\sqrt{1 + (m_p^2 - 1)n_R^2}} M_p {R_\ell}^T \boldsymbol{l}$$

The above expression represents the output direction cosine in the coordinate frame $\{\mathcal{L}\}$**.** In order to transform the output direction cosine from the coordinate frame $\{\mathcal{L}\}$ to the camera frame $\{C\}$ we need to multiply the direction cosine vector by R_ℓ to obtain:

$$\acute{\boldsymbol{l}} = \pm \frac{1}{\sqrt{1 + (m_p^2 - 1)n_R^2}} R_\ell M_p {R_\ell}^T \boldsymbol{l} \tag{10}$$

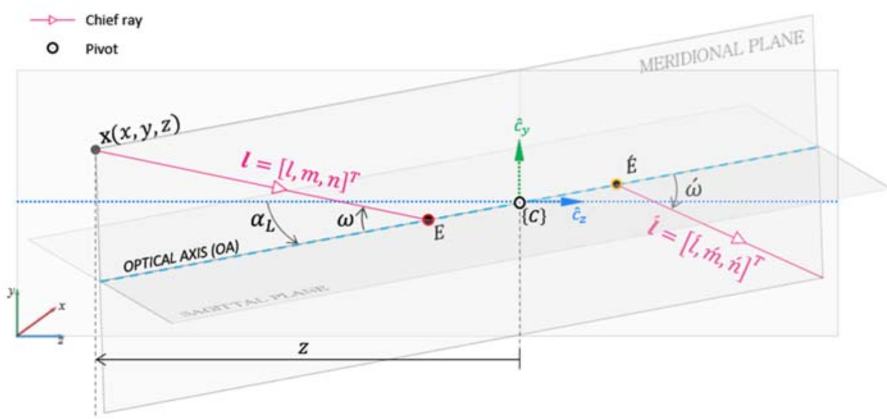

Figure 4 Configuration of the general problem—optical axis (OA) pivots freely about the origin of $\{C\}$.

Furthermore, the positive or negative sign of the direction cosine determines the forward or backward direction of light-travel along a rectilinear path. Under the assumptions of isotropy and homogeneity, the only condition under which a ray of light emerges in an antipodal path from an interface is if it encounters a mirror surface *normally*. This condition does not arise within the context of our problem. Therefore, without any loss of generality, we can drop the negative sign in Eq. (10) ; accordingly, the output direction cosines assume the sign of the corresponding input direction cosines. Therefore, the general expression for the direction cosines of the chief ray in the image space is obtained as:

$$\boxed{\acute{\boldsymbol{l}} = \frac{1}{\sqrt{1 + \left(m_p^2 - 1\right) n_R^2}} R_\ell M_p R_\ell{}^T \boldsymbol{l}} \tag{11}$$

where $n_R = \boldsymbol{r}_{\ell,3}^T \boldsymbol{l}$.

Note that Eq. (11) only describes the output chief ray's direction cosines—a free vector. The output chief ray is obtained from the knowledge of the direction cosine and the location of the exit pupil in the appropriate reference frame.

3. Geometric model of image formation

3.1 Image formation for arbitrary orientation of the lens and image plane

Geometric imaging is a mapping (*bijective* in projective space) between points in the three-dimensional world space to corresponding points on a mathematical surface that we call the *image*. Here we aim to study the nature of this mapping on a planar surface—the image plane—for arbitrary orientations of either the lens and image planes. To that effect, we will use the knowledge of the transfer of direction cosines of the chief ray derived previously.

An extended object emanates a multitude of chief-rays that reach the image space through the pupils and the stop. The locus of points formed by the intersection of these rays with the image plane constitutes the *projection* of the object in the image plane [5,6]. Further, we identify the projection of the world-point as an "image" if the pencil of rays (including the chief-ray) from the world-point, filling the pupils and stop, geometrically converge at a single point in the image space.

For simplicity, we assume that the lens is unencumbered by radial distortions and optical aberrations. Figure 5 represents a schematic of the problem in which we have introduced an image plane whose orientation is described by the unit surface normal $\hat{\boldsymbol{n}}_i$. Two local frames are also introduced: the frame $\{P\}$ is attached to the optical axis with its origin at entrance pupil (E), and the frame $\{I\}$ attached to the image plane with its origin at the image plane pivot. The image plane is free to swivel (tilt or swing about its local x-axis or y-axis respectively) about the image plane pivot.

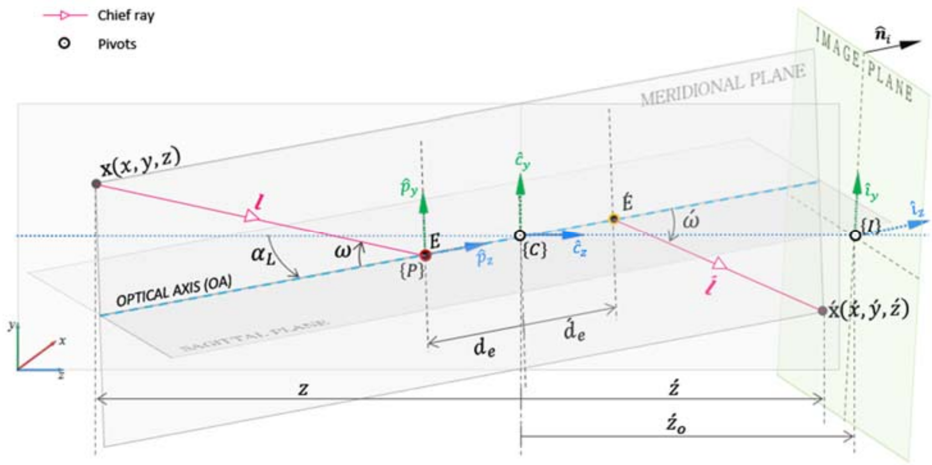

Figure 5 Schematic of geometric image formation. $\acute{\mathbf{x}}$ is the *central projection* of the world point $\mathbf{x}$ on image plane. The optical axis and image plane are free to swivel about the origins of coordinate frames $\{C\}$ and $\{I\}$ respectively.

Let the exit pupil ($\acute{E}$) center be located $\acute{d}_e$ units from the pivot point along the optical axis. Following the rotation of the optical axis, by applying the matrix $R_\ell \in \mathbb{R}^{3\times3}$, the position of the exit pupil in $\{C\}$ is given as $R_\ell\left[0,0,\acute{d}_e\right]^T = \acute{d}_e\boldsymbol{r}_{\ell,3}$.

We can represent the chief ray emerging from the exit pupil with direction cosine $\acute{\boldsymbol{l}}$ by the parametric equation:

$$\acute{\mathbf{x}} = \acute{d}_e\,\boldsymbol{r}_{\ell,3} + \frac{\lambda}{\sqrt{1+\left(m_p^2-1\right)n_R^2}}\,R_\ell M_p {R_\ell}^T \boldsymbol{l}\,, \tag{12}$$

where $\acute{\mathbf{x}}$ represents any point along the output chief ray in $\{C\}$. The first term on the R.H.S. of Eq. (12) is the initial position of the ray (at the center of $\acute{E}$) and λ is a real number that determines the length of the ray.

The equation of the image plane in Hessian normal form is

$$\hat{\boldsymbol{n}}_i^T \boldsymbol{\xi} = \acute{z}_{o\perp}\,, \tag{13}$$

where $\hat{\boldsymbol{n}}_i$ is the unit normal to the image plane, $\acute{z}_{o\perp}$ is the perpendicular distance from the origin of frame $\{C\}$ to the plane, and $\boldsymbol{\xi}$ is an arbitrary point on the image plane.

We obtain the expression for λ (in Eq.(12)) for which the ray intersects the image plane by equating $\boldsymbol{\xi}$ to $\acute{\mathbf{x}}$, multiplying Eq. (12) by $\hat{\boldsymbol{n}}_i^T$, and rearranging the terms:

$$\underbrace{\hat{\boldsymbol{n}}_i^T\acute{\mathbf{x}}}_{\acute{z}_{o\perp}} = \acute{d}_e\hat{\boldsymbol{n}}_i^T\boldsymbol{r}_{\ell,3} + \frac{\lambda}{\sqrt{1+\left(m_p^2-1\right)n_R^2}}\hat{\boldsymbol{n}}_i^T R_\ell M_p {R_\ell}^T\boldsymbol{l}\,,$$

$$\lambda = \frac{\left(\acute{z}_{o\perp} - \acute{d}_e\hat{\boldsymbol{n}}_i^T\,\boldsymbol{r}_{\ell,3}\right)\sqrt{1+\left(m_p^2-1\right)n_R^2}}{\hat{\boldsymbol{n}}_i^T\,R_\ell M_p {R_\ell}^T\boldsymbol{l}}\,. \tag{14}$$

Substituting Eq. (14) into Eq. (12) we get

$$\acute{\mathbf{x}} = \acute{d}_e\, \boldsymbol{r}_{\ell,3} + \frac{\left(\acute{z}_{o\perp} - \acute{d}_e \widehat{\boldsymbol{n}}_i^T\, \boldsymbol{r}_{\ell,3}\right)}{\widehat{\boldsymbol{n}}_i^T\, R_\ell M_p {R_\ell}^T \boldsymbol{l}}\, R_\ell M_p {R_\ell}^T \boldsymbol{l} \tag{15}$$

As previously stated, the origin of $\{I\}$, the image plane's local reference frame, is located at the intersection of the z-axis of camera frame $\{C\}$ with the image plane. The orientation of the image plane, which is fixed at the point $(0,0,\acute{z}_o)$ in $\{C\}$, can be described by applying a rotation matrix R_i to the unit vector $[0,0,1]^T$ about $(0,0,\acute{z}_o)$ as shown in Figure 6. The vector $[0,0,1]^T$ represents the normal of the un-rotated image plane. Therefore, in general, the normal to the image plane is:

$$\widehat{\boldsymbol{n}}_i = R_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{16}$$

Referring to Figure 6, the expression for $\acute{z}_{o\perp}$ is obtained as follows:
The equation of the image plane is

$$\widehat{\boldsymbol{n}}_i^T \boldsymbol{\xi} = \acute{z}_{o\perp}$$

Since $\boldsymbol{\xi} = [0, 0, \acute{z}_o]^T$ is a point on the plane,

$$\acute{z}_{o\perp} = \widehat{\boldsymbol{n}}_i^T \begin{bmatrix} 0 \\ 0 \\ \acute{z}_o \end{bmatrix} = \widehat{\boldsymbol{n}}_i(3)\acute{z}_o \tag{17}$$

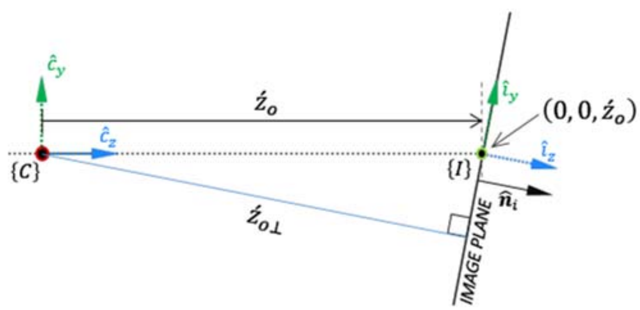

Figure 6 Schematic of the image plane. The image plane having surface normal $\widehat{\boldsymbol{n}}_i$ is located at $\acute{z}_o$ units from the origin of camera frame $\{C\}$ along the z-axis that intersects the plane at $(0, 0, \acute{z}_o)$. $\acute{z}_{o\perp}$ is the perpendicular distance from the origin to the plane. The local image coordinate frame with its origin at the intersection of the image plane and z-axis of the camera frame is represented by $\{I\}$.

Using the above result in Eq. (15) yields the expression for the point of intersection of the chief ray with the image plane in terms of the input direction cosines as

$$\acute{\mathbf{x}} = \acute{d}_e\, \boldsymbol{r}_{\ell,3} + \frac{\left(\widehat{\boldsymbol{n}}_i(3)\acute{z}_o - \acute{d}_e\, \widehat{\boldsymbol{n}}_i^T\, \boldsymbol{r}_{\ell,3}\right)}{\widehat{\boldsymbol{n}}_i^T\, R_\ell M_p {R_\ell}^T \boldsymbol{l}}\, R_\ell M_p {R_\ell}^T \boldsymbol{l} \tag{18}$$

Similar to the exit pupil, let the entrance pupil be located at a distance d_e from the pivot point along the optical axis in the camera frame $\{C\}$. Then, the location of the entrance pupil in $\{C\}$ is $\mathbf{x}_e = R_\ell\, [0,0,d_e]^T = d_e\, \boldsymbol{r}_{\ell,3}$. The direction cosines and the world point are related as

$$
\begin{aligned}
l &= \frac{x_e - x}{\sqrt{(x_e-x)^2+(y_e-y)^2+(z_e+z)^2}} = \frac{d_e\, R_\ell(1,3) - x}{\|\mathbf{x}_e - \mathbf{x}\|} \\
m &= \frac{y_e - y}{\sqrt{(x_e-x)^2+(y_e-y)^2+(z_e+z)^2}} = \frac{d_e\, R_\ell(2,3) - y}{\|\mathbf{x}_e - \mathbf{x}\|} \\
n &= \frac{z_e - (-z)}{\sqrt{(x_e-x)^2+(y_e-y)^2+(z_e+z)^2}} = \frac{d_e\, R_\ell(3,3) + z}{\|\mathbf{x}_e - \mathbf{x}\|}
\end{aligned} \tag{19}
$$

which can be written compactly as:

$$
\boldsymbol{l} = \frac{1}{\|\mathbf{x}_e - \mathbf{x}\|}\left\{\underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{E}} \mathbf{x} + d_e\, \boldsymbol{r}_{\ell,3}\right\}
$$

or

$$
\boldsymbol{l} = \frac{1}{\|\mathbf{x}_e - \mathbf{x}\|} E\mathbf{x} + d_e\, \boldsymbol{r}_{\ell,3} \tag{20}
$$

Substituting Eq. (20) into Eq. (18) , a general relation between the world point $\mathbf{x}$ and its corresponding image point $\acute{\mathbf{x}}$ is obtained as:

$$
\boxed{\begin{aligned} \acute{\boldsymbol{x}} &= \acute{d}_e\, \boldsymbol{r}_{\ell,3} \\ &\quad + \frac{\left(\widehat{\boldsymbol{n}}_i(3)\acute{z}_o - \acute{d}_e \widehat{\boldsymbol{n}}_i^T\, \boldsymbol{r}_{\ell,3}\right)}{\widehat{\boldsymbol{n}}_i^T\, R_\ell M_p R_\ell{}^T\left(\boldsymbol{E}\boldsymbol{x} + d_e\, \boldsymbol{r}_{\ell,3}\right)}\, R_\ell M_p R_\ell{}^T\left(\boldsymbol{E}\boldsymbol{x} + d_e\, \boldsymbol{r}_{\ell,3}\right) \end{aligned}} \tag{21}
$$

Eq. (21) represents the image point $\acute{\mathbf{x}}$ in the camera frame. Once an image is formed, we specify positions and dimensions within the image independent of the position and orientation of the sensor and lenses. We can transform the image coordinates in the camera frame $\{C\}$ to the image frame $\{I\}$ by observing that the origin of $\{I\}$ is displaced from $\{C\}$ by $\boldsymbol{t}_i = [0, 0, \acute{z}_o\,]^T$, and the standard basis vectors of $\{I\}$ are rotated by $R_i \in \mathbb{R}^{3\times 3}$. Therefore, a point ${}^I\acute{\mathbf{x}}$ in $\{I\}$ relative to $\{C\}$ may be expressed as:

$$
\acute{\mathbf{x}} = R_i\, {}^I\acute{\mathbf{x}} + \boldsymbol{t}_i \tag{22}
$$

Therefore, we obtain the coordinates of the image points in the image frame $\{I\}$ as (in metric space):

$$
\boxed{{}^I\acute{\mathbf{x}} = R_i^T(\acute{\mathbf{x}} - \boldsymbol{t}_i)} \tag{23}
$$

3.2 Object, lens and image plane relationships in Scheimpflug imaging

In this section, we will derive a *general expression* that relates the object, lens and image planes. In order to keep the problem tractable, we will impose the constraint that three pivots—for the object, lens and image planes—lie along the z-axis of the camera frame $\{C\}$, and the origin of $\{C\}$ is co-located with optical axis' pivot. We also bound the angle of rotations of object, lens, and image planes between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ about both x- and y- axes (in-plane rotations or rotations about the z-axis is irrelevant for our purpose). Provided we make no distinction between the faces (front or back) of the planes, this bound on the angles of rotations is not limiting in any way since we can uniquely describe all possible planer orientations in three dimensions. On the other hand, this bound warrants non-negative values for the z-component of the plane normal. This warranty permits us to unambiguously estimate the unknown plane normal.

We begin by deriving an expression for the chief ray joining an arbitrary point $\mathbf{x}$ in the object plane to a point $\acute{\mathbf{x}}$ in the image plane. In order for $\acute{\mathbf{x}}$ to be the geometric image of $\mathbf{x}$, the chief ray conjoining the conjugate points must satisfy the Gaussian imaging equation. This constraint allows us to uniquely determine the position (of the image plane along the z-axis of $\{C\}$) and orientation of the three planes. The setup is shown in Figure 7.

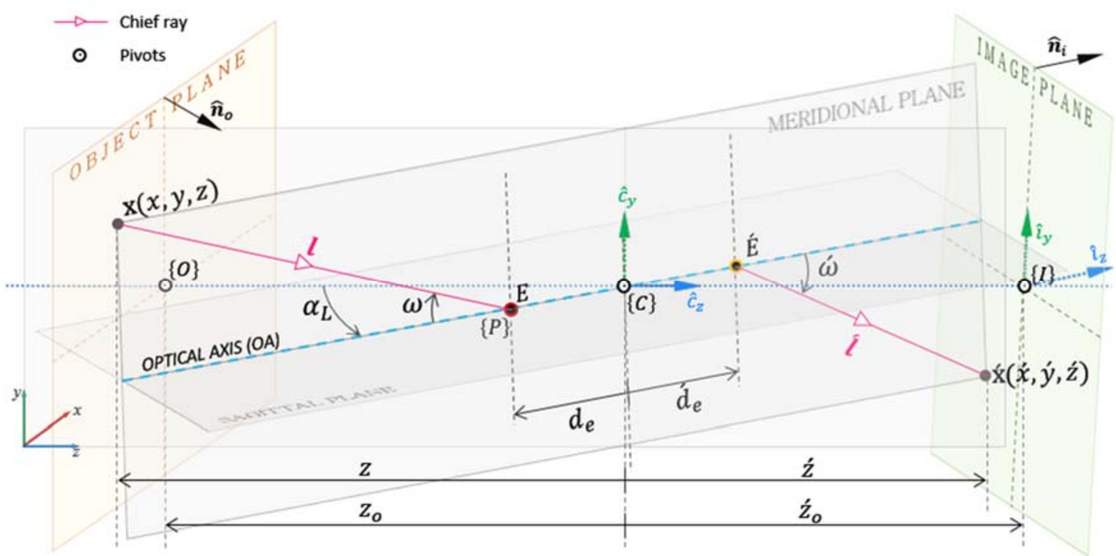

Figure 7 Schematic of Scheimpflug imaging. The figure shows the object plane, optical axis and image plane pivoted about points along the z-axis $\hat{\boldsymbol{c}}_z$ of the camera frame $\{C\}$. The local object plane and image plane coordinates ($\{O\}$ and $\{I\}$) are centered at the object- and image- plane pivots.

The object plane is located at a distance of z_o from the origin of camera frame $\{C\}$, along the z-axis. Pivoted about the point $(0, 0, -z_o)$ in the camera frame, it is completely described by the object plane pivot and the plane normal, $\hat{\boldsymbol{n}}_o$. We describe the normal itself as the product of the rotation matrix $R_o \in \mathbb{R}^{3\times3}$ and $\hat{\boldsymbol{c}}_z$ ($= [0,0,1]^T$). That is, the equation of the object plane normal is:

$$\hat{\boldsymbol{n}}_o = R_o \hat{\boldsymbol{c}}_z \,. \tag{24}$$

We can obtain the perpendicular distance to the object plane from the origin as

$$\hat{\boldsymbol{n}}_o^T \begin{bmatrix} 0 \\ 0 \\ -z_o \end{bmatrix} = d_\perp \,, \qquad \text{or} \quad d_\perp = -z_o \hat{\boldsymbol{n}}_o^T \hat{\boldsymbol{c}}_z \,. \tag{25}$$

Then, the equation of the object plane in Hessian Normal form is

$$\hat{\boldsymbol{n}}_o^T \boldsymbol{\xi} = -z_o (\hat{\boldsymbol{n}}_o^T \hat{\boldsymbol{c}}_z) \,, \tag{26}$$

where $\boldsymbol{\xi}$ is any point on the object plane.

Similar to the object plane normal, we describe the image plane normal as

$$\hat{\boldsymbol{n}}_i = R_i \hat{\boldsymbol{c}}_z \,, \tag{27}$$

where $R_i \in \mathbb{R}^{3\times3}$ is the rotation matrix applied to the image plane at the image plane pivot point $(0, 0, \acute{z}_o)$.

Repeating the steps used to derive the object plane equation, the equation for the image plane is

$$\hat{\boldsymbol{n}}_i^T \boldsymbol{\xi} = \acute{z}_o (\hat{\boldsymbol{n}}_i^T \hat{\boldsymbol{c}}_z) \,, \tag{28}$$

where $\boldsymbol{\xi}$ is any point on the image plane.

Suppose the centers of the entrance (E) and exit ($\acute{E}$) pupils are located at distances d_e and $\acute{d}_e$ from the lens plane pivot point (origin of $\{C\}$) respectively, along the optical axis. Also, let us describe the rotation of the optical axis by applying the matrix $R_\ell \in \mathbb{R}^{3\times3}$. Then, following the rotation, the positions of the pupils in $\{C\}$ are

$$\text{Entrance pupil position, } \mathbf{x}_e = R_\ell\,[0,0,d_e]^T = d_e\ \boldsymbol{r}_{\ell,3}\ , \tag{29}$$

$$\text{Exit pupil position, } \acute{\mathbf{x}}_e = R_\ell\left[0,0,\acute{d}_e\right]^T = \acute{d}_e\ \boldsymbol{r}_{\ell,3}\ . \tag{30}$$

Consider the chief ray from the object-point $\mathbf{x}$, to the corresponding image point $\acute{\mathbf{x}}$, passing through the center of the entrance and exit pupils. Let the direction cosines of ray in the object- and image- space be $\boldsymbol{l}$ and $\acute{\boldsymbol{l}}$ respectively. We can describe a line along the ray originating at the entrance pupil and extending backwards towards the object plane in parametric form as

$$\boldsymbol{\xi} = \mathbf{x}_e + \lambda(-\boldsymbol{l}) = d_e\,\boldsymbol{r}_{\ell,3} - \lambda\boldsymbol{l}\ , \tag{31}$$

where $\boldsymbol{\xi}$ is any point on the line (chief ray), $\mathbf{x}_e$ is the position of the entrance pupil, and the parameter λ determines the length of the ray.

Let the length of the ray between the object-point $\mathbf{x}$ and the entrance pupil be $\underset{\sim}{u}$. Then, in Eq. (31) , $\boldsymbol{\xi} = \mathbf{x}$ when $\lambda = \underset{\sim}{u}$, which implies that

$$\mathbf{x} = d_e\,\boldsymbol{r}_{\ell,3} - \underset{\sim}{u}\boldsymbol{l}\ . \tag{32}$$

Since $\mathbf{x}$ is a point on the object plane, it satisfies the object plane Eq. (26) . Substituting $\mathbf{x}$ into Eq. (26) we get

$$\underset{\sim}{u} = \frac{z_o(\widehat{\boldsymbol{n}}_o^T\widehat{\boldsymbol{c}}_z) + d_e(\widehat{\boldsymbol{n}}_o^T\boldsymbol{r}_{\ell,3})}{\widehat{\boldsymbol{n}}_o^T\boldsymbol{l}}\ . \tag{33}$$

The chief ray emerges from the exit pupil, having a direction cosine vector $\acute{\boldsymbol{l}}$, and intersecting the image plane at $\acute{\mathbf{x}}$ to the right (positive z direction) of the exit pupil. Therefore, the parametric equation of this chief ray is

$$\boldsymbol{\xi} = \acute{\mathbf{x}}_e + \lambda(\acute{\boldsymbol{l}}) = \acute{d}_e\,\boldsymbol{r}_{\ell,3} + \lambda\acute{\boldsymbol{l}}\ , \tag{34}$$

where $\boldsymbol{\xi}$ is any point on the ray, $\acute{\mathbf{x}}_e$ is the position of the exit pupil, and λ is the length of the ray.

If we let the length of the chief ray in the image space be $\underset{\sim}{\acute{u}}$, then $\boldsymbol{\xi} = \acute{\mathbf{x}}$ when $\lambda = \underset{\sim}{\acute{u}}$ in Eq. (34) , implying

$$\acute{\mathbf{x}} = \acute{d}_e\,\boldsymbol{r}_{\ell,3} + \underset{\sim}{\acute{u}}\acute{\boldsymbol{l}}\ . \tag{35}$$

Substituting $\acute{\mathbf{x}}$ in Eq. (28) we get

$$\underset{\sim}{\acute{u}} = \frac{\acute{z}_o(\widehat{\boldsymbol{n}}_i^T\widehat{\boldsymbol{c}}_z) - \acute{d}_e(\widehat{\boldsymbol{n}}_i^T\boldsymbol{r}_{\ell,3})}{\widehat{\boldsymbol{n}}_i^T\acute{\boldsymbol{l}}}\ . \tag{36}$$

Eqs. (33) and (36) are expressions for the lengths of the chief ray between points $\mathbf{x}$ and $\acute{\mathbf{x}}$ in the object and image space respectively. In order for $\acute{\mathbf{x}}$ to be the geometric image of $\mathbf{x}$, the expressions must satisfy the Gaussian imaging equation.

The well-known Gaussian imaging equation ($-1/u + 1/\acute{u} = 1/f$) relates the focal length and the conjugate plane distances measured from the principal planes. Instead, if the distances are measured from the pupils (entrance pupil to object plane and exit pupil to image plane) in lieu of the principal planes, then a variant of the Gaussian imaging equation is used. The updated imaging equation, which incorporates the pupil magnification m_p into the formula, is

$$-\frac{1}{m_p u}+\frac{m_p}{\acute{u}}=\frac{1}{f}\ , \tag{37}$$

where u and $\acute{u}$ are distances along the optical axis measured from the entrance pupil to the object plane and from the exit pupil to the image plane, m_p is the pupil magnification, and f is the focal length. We have provided a derivation and a brief exposition of the above formula in Appendix B.

The most common application of Eq. (37) is for frontoparallel imaging in which the conjugate planes are parallel to each other and perpendicular to the optical axis; moreover, all pairs of object-image conjugate points satisfy this relation even if the ensemble of object points and image points belong to planes on object and image spaces respectively that are *not parallel* to each other.

The ray vector of length $\underset{\sim}{u}$ and direction $\boldsymbol{l}$ in the object space is $\underset{\sim}{u}\boldsymbol{l}$. The projection of this ray vector on the optical axis $\hat{\boldsymbol{o}}$ $(= R_\ell\, \hat{\boldsymbol{c}}_z)$ is $u = \underset{\sim}{u}(\boldsymbol{l}\cdot\hat{\boldsymbol{o}})$. Similarly, the ray projection of the image space ray vector on the optical axis is $\acute{u} = \underset{\sim}{\acute{u}}\big(\acute{\boldsymbol{l}}\cdot\hat{\boldsymbol{o}}\big)$. Substituting u and $\acute{u}$ in to Eq. (37) we have:

$$-\frac{1}{m_p \underset{\sim}{u}(\boldsymbol{l}\cdot\hat{\boldsymbol{o}})}+\frac{m_p}{\underset{\sim}{\acute{u}}\big(\acute{\boldsymbol{l}}\cdot\hat{\boldsymbol{o}}\big)}=\frac{1}{f}\ . \tag{38}$$

Further, substituting the expressions for $\underset{\sim}{u}$ and $\underset{\sim}{\acute{u}}$ (Eqs. (33) and (36)) into the above equation, we have

$$-\frac{\hat{\boldsymbol{n}}_o^T\boldsymbol{l}}{m_p\big[z_o(\hat{\boldsymbol{n}}_o^T\hat{\boldsymbol{c}}_z)+d_e\big(\hat{\boldsymbol{n}}_o^T\boldsymbol{r}_{\ell,3}\big)\big](\boldsymbol{l}\cdot\hat{\boldsymbol{o}})}+\frac{m_p\hat{\boldsymbol{n}}_i^T\acute{\boldsymbol{l}}}{\big[\acute{z}_o(\hat{\boldsymbol{n}}_i^T\hat{\boldsymbol{c}}_z)-\acute{d}_e\big(\hat{\boldsymbol{n}}_i^T\boldsymbol{r}_{\ell,3}\big)\big]\big(\acute{\boldsymbol{l}}\cdot\hat{\boldsymbol{o}}\big)}=\frac{1}{f}\ . \tag{39}$$

The direction cosine of the chief-ray in the image space, $\acute{\boldsymbol{l}}$, is related to the direction cosine of the chief-ray in the object space as (Eq. (11))

$$\acute{\boldsymbol{l}}=\frac{1}{\sqrt{1+\big(m_p^2-1\big)n_R^2}}\,R_\ell M_p {R_\ell}^T\boldsymbol{l}\ .$$

Substituting $\acute{\boldsymbol{l}}$, into Eq. (39) we have

$$-\frac{\hat{\boldsymbol{n}}_o^T\boldsymbol{l}}{m_p\big[z_o(\hat{\boldsymbol{n}}_o^T\hat{\boldsymbol{c}}_z)+d_e\big(\hat{\boldsymbol{n}}_o^T\boldsymbol{r}_{\ell,3}\big)\big](\boldsymbol{l}\cdot\hat{\boldsymbol{o}})}+\frac{m_p\hat{\boldsymbol{n}}_i^T\big(R_\ell M_p {R_\ell}^T\boldsymbol{l}\big)}{\big[\acute{z}_o(\hat{\boldsymbol{n}}_i^T\hat{\boldsymbol{c}}_z)-\acute{d}_e\big(\hat{\boldsymbol{n}}_i^T\boldsymbol{r}_{\ell,3}\big)\big]\big(R_\ell M_p {R_\ell}^T\boldsymbol{l}\big)\cdot\hat{\boldsymbol{o}}}=\frac{1}{f}\ . \tag{40}$$

To simplify the above expression, let us consider $\big(R_\ell M_p {R_\ell}^T\hat{\boldsymbol{n}}_i\big)^T\boldsymbol{l}$ as

$$\begin{aligned}\big(R_\ell M_p {R_\ell}^T\hat{\boldsymbol{n}}_i\big)^T\boldsymbol{l} &= \hat{\boldsymbol{n}}_i^T\big({R_\ell}^T\big)^T\big(M_p\big)^T{R_\ell}^T\boldsymbol{l}\\ &= \hat{\boldsymbol{n}}_i^T{R_\ell}^T M_p {R_\ell}^T\boldsymbol{l}\ . \qquad (\because M_p \text{ is a diagonal matrix}\\ &\qquad\qquad\qquad\qquad\quad \text{and } R_\ell \text{ is a rotation matrix})\end{aligned} \tag{41}$$

Therefore, we can write $\hat{\boldsymbol{n}}_i^T\big(R_\ell M_p {R_\ell}^T\boldsymbol{l}\big)$ in Eq. (40) as $\big(R_\ell M_p {R_\ell}^T\hat{\boldsymbol{n}}_i\big)^T\boldsymbol{l}$. Similarly, we can also write $\big(R_\ell M_p {R_\ell}^T\boldsymbol{l}\big)\cdot\hat{\boldsymbol{o}}$ as $\big(R_\ell M_p {R_\ell}^T\boldsymbol{l}\big)^T\hat{\boldsymbol{o}}=\hat{\boldsymbol{o}}^T\big(R_\ell M_p {R_\ell}^T\boldsymbol{l}\big)=\big(R_\ell M_p {R_\ell}^T\hat{\boldsymbol{o}}\big)^T\boldsymbol{l}$. Then, Eq. (40) can be written as

$$-\frac{\hat{\boldsymbol{n}}_o^T\boldsymbol{l}}{m_p\big[z_o(\hat{\boldsymbol{n}}_o^T\hat{\boldsymbol{c}}_z)+d_e\big(\hat{\boldsymbol{n}}_o^T\boldsymbol{r}_{\ell,3}\big)\big]\hat{\boldsymbol{o}}^T\boldsymbol{l}}+\frac{m_p\big(R_\ell M_p {R_\ell}^T\hat{\boldsymbol{n}}_i\big)^T\boldsymbol{l}}{\big[\acute{z}_o(\hat{\boldsymbol{n}}_i^T\hat{\boldsymbol{c}}_z)-\acute{d}_e\big(\hat{\boldsymbol{n}}_i^T\boldsymbol{r}_{\ell,3}\big)\big]\big(R_\ell M_p {R_\ell}^T\hat{\boldsymbol{o}}\big)^T\boldsymbol{l}}=\frac{1}{f}\ . \tag{42}$$

We can further simplify the above equation by noting that:

1. $\hat{\boldsymbol{n}}_o^T\hat{\boldsymbol{c}}_z=\hat{\boldsymbol{n}}_o^T\begin{bmatrix}0\\0\\1\end{bmatrix}=\hat{\boldsymbol{n}}_o(3)$,
2. $\hat{\boldsymbol{o}}^T\boldsymbol{l}=(R_\ell\hat{\boldsymbol{c}}_z)^T\boldsymbol{l}=\boldsymbol{r}_{\ell,3}^T\boldsymbol{l}$,

3. $\hat{\boldsymbol{n}}_i^T \hat{\boldsymbol{c}}_z = \hat{\boldsymbol{n}}_i^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \hat{\boldsymbol{n}}_i(3)$,

4. $\left(R_\ell M_p {R_\ell}^T \hat{\boldsymbol{o}} \right)^T \boldsymbol{l} = \left(R_\ell M_p {R_\ell}^T R_\ell \hat{\boldsymbol{c}}_z \right)^T \boldsymbol{l} = \left(R_\ell M_p \hat{\boldsymbol{c}}_z \right)^T \boldsymbol{l} = m_p \boldsymbol{r}_{\ell,3}^T \boldsymbol{l}$.

Therefore, Eq. (42) reduces to

$$-\frac{\hat{\boldsymbol{n}}_o^T \boldsymbol{l}}{m_p \left[z_o \hat{\boldsymbol{n}}_o(3) + d_e \left(\hat{\boldsymbol{n}}_o^T \boldsymbol{r}_{\ell,3} \right) \right] \boldsymbol{r}_{\ell,3}^T \boldsymbol{l}} + \frac{\left(R_\ell M_p {R_\ell}^T \hat{\boldsymbol{n}}_i \right)^T \boldsymbol{l}}{\left[\acute{z}_o \hat{\boldsymbol{n}}_i(3) - \acute{d}_e \left(\hat{\boldsymbol{n}}_i^T \boldsymbol{r}_{\ell,3} \right) \right] \boldsymbol{r}_{\ell,3}^T \boldsymbol{l}} = \frac{1}{f} \ . \tag{43}$$

Further, following a trivial algebraic manipulation we get

$$\boldsymbol{l}^T \left[-\frac{\hat{\boldsymbol{n}}_o}{m_p \left[z_o \hat{\boldsymbol{n}}_o(3) + d_e \left(\hat{\boldsymbol{n}}_o^T \boldsymbol{r}_{\ell,3} \right) \right]} + \frac{R_\ell M_p {R_\ell}^T \hat{\boldsymbol{n}}_i}{\left[\acute{z}_o \hat{\boldsymbol{n}}_i(3) - \acute{d}_e \left(\hat{\boldsymbol{n}}_i^T \boldsymbol{r}_{\ell,3} \right) \right]} - \frac{\boldsymbol{r}_{\ell,3}}{f} \right] = 0 \ . \tag{44}$$

Note that in the above expressions $\boldsymbol{r}_{\ell,3}$, the third column of the rotation matrix R_ℓ, is the unit vector along the optical axis.

Since $\boldsymbol{l}$ is a direction cosine vector with ℓ^2-Norm equal to one, the above equation is satisfied if the second vector, $\left[-\frac{\hat{\boldsymbol{n}}_o}{m_p \left[z_o \hat{\boldsymbol{n}}_o(3) + d_e \left(\hat{\boldsymbol{n}}_o^T \boldsymbol{r}_{\ell,3} \right) \right]} + \frac{R_\ell M_p {R_\ell}^T \hat{\boldsymbol{n}}_i}{\left[\acute{z}_o \hat{\boldsymbol{n}}_i(3) - \acute{d}_e \left(\hat{\boldsymbol{n}}_i^T \boldsymbol{r}_{\ell,3} \right) \right]} - \frac{\boldsymbol{r}_{\ell,3}}{f} \right]$, is either perpendicular to $\boldsymbol{l}$ or identically equal to zero. Let us consider the possibility of the former case. *All* chief rays—an infinitude of $\boldsymbol{l}$ vectors within the object and image space perspective cones—must satisfy Eq. (44) . Are all possible $\boldsymbol{l}$ vectors perpendicular to the second vector? Since the second vector is a linear combination of $\hat{\boldsymbol{n}}_o$ (the object plane normal), $R_\ell M_p {R_\ell}^T \hat{\boldsymbol{n}}_i$ (the transformed image plane normal), and $\boldsymbol{r}_{\ell,3}$ (unit vector along the optical axis) whose weights are constant for a given system, we can conclude that $\boldsymbol{l}$ is not perpendicular to the second vector *in general*. Therefore,

$$-\frac{\hat{\boldsymbol{n}}_o}{m_p \left[z_o \hat{\boldsymbol{n}}_o(3) + d_e \left(\hat{\boldsymbol{n}}_o^T \boldsymbol{r}_{\ell,3} \right) \right]} + \frac{R_\ell M_p {R_\ell}^T \hat{\boldsymbol{n}}_i}{\left[\acute{z}_o \hat{\boldsymbol{n}}_i(3) - \acute{d}_e \left(\hat{\boldsymbol{n}}_i^T \boldsymbol{r}_{\ell,3} \right) \right]} = \frac{\boldsymbol{r}_{\ell,3}}{f} \ . \tag{45}$$

Further, we can simplify Eq. (45) if we let $\underset{\sim}{\hat{\boldsymbol{n}}}_o = \frac{\hat{\boldsymbol{n}}_o}{\hat{\boldsymbol{n}}_o(3)} = \left[\frac{\hat{\boldsymbol{n}}_o(1)}{\hat{\boldsymbol{n}}_o(3)}, \frac{\hat{\boldsymbol{n}}_o(2)}{\hat{\boldsymbol{n}}_o(3)}, 1 \right]^T$ and $\underset{\sim}{\hat{\boldsymbol{n}}}_i = \frac{\hat{\boldsymbol{n}}_i}{\hat{\boldsymbol{n}}_i(3)} = \left[\frac{\hat{\boldsymbol{n}}_i(1)}{\hat{\boldsymbol{n}}_i(3)}, \frac{\hat{\boldsymbol{n}}_i(2)}{\hat{\boldsymbol{n}}_i(3)}, 1 \right]^T$. Then, after factoring $\hat{\boldsymbol{n}}_o(3)$ and $\hat{\boldsymbol{n}}_i(3)$ out from the denominator terms, we can write Eq. (45) as

$$\boxed{-\frac{\underset{\sim}{\hat{\boldsymbol{n}}}_o}{m_p \left[z_o + d_e \left(\underset{\sim}{\hat{\boldsymbol{n}}}_o^T \boldsymbol{r}_{\ell,3} \right) \right]} + \frac{R_\ell M_p {R_\ell}^T \underset{\sim}{\hat{\boldsymbol{n}}}_i}{\left[\acute{z}_o - \acute{d}_e \left(\underset{\sim}{\hat{\boldsymbol{n}}}_i^T \boldsymbol{r}_{\ell,3} \right) \right]} = \frac{\boldsymbol{r}_{\ell,3}}{f} \ .} \tag{46}$$

This expedient simplification relies on our ability to describe the *unit* normal vectors $\hat{\boldsymbol{n}}_o$ and $\hat{\boldsymbol{n}}_i$ using only components along x- and y-axes. For example, if the object and lens plane orientations and distance are known, and we estimate the image plane distance $\acute{z}_o$ and orientation vector $\underset{\sim}{\hat{\boldsymbol{n}}}_i$ $(= \acute{p}, \acute{q}, 1)$ of the image plane using Eq. (46) , then we can determine the image plane normal as

$$\begin{aligned} \hat{\boldsymbol{n}}_i(3) &= \frac{1}{\sqrt{\acute{p}^2 + \acute{q}^2 + 1}} \ , \\ \hat{\boldsymbol{n}}_i(1) &= \hat{\boldsymbol{n}}_i(3)\, \acute{p} \ , \\ \hat{\boldsymbol{n}}_i(2) &= \hat{\boldsymbol{n}}_i(3)\, \acute{q} \ , \end{aligned} \tag{47}$$

where $\acute{p} = \frac{\hat{\boldsymbol{n}}_i(1)}{\hat{\boldsymbol{n}}_i(3)}$, $\acute{q} = \frac{\hat{\boldsymbol{n}}_i(2)}{\hat{\boldsymbol{n}}_i(3)}$, and we have dropped the negative sign from the expression for $\hat{\boldsymbol{n}}_i(3)$ since $\hat{\boldsymbol{n}}_i(3)$ is guaranteed to be positive as discussed under the assumptions at the beginning of this section.

4. Summary

In this work we have derived a general model of geometric image formation in for Scheimpflug cameras. Although, the current model ignores optical aberrations, it is rich enough to describe accurately the wrapping of image field experience under lens rotations.

Appendix A: Transfer of direction cosine vector from entrance to exit pupil

We will apply the method of induction to yield the solution of the general *transfer* problem—in which the optical axis is free to swivel about the origin of $\{C\}$.

Eq. (8) accurately represents the *transfer* for the specific problem; however, we will cast the expression in a slightly different form whose raison d'être is to enable generalization—through direct application of the result. Specifically, we can express the output chief ray as a linear combination of the input chief ray and the optical axis since the two rays and the optical axis span the same (meridional) plane. Let $\hat{\boldsymbol{c}}_z$, the standard basis vector along z-axis of $\{C\}$, represent the optical axis since the optical axis is coincident with the z-axis. Then,

$$\acute{\boldsymbol{l}} = w_1 \boldsymbol{l} + w_2 \hat{\boldsymbol{c}}_z \tag{48}$$

where w_1 and w_2 are the weights, and $\hat{\boldsymbol{c}}_z = [0, 0, 1]^T$. Rewriting the above equation as

$$\begin{bmatrix} \acute{l} \\ \acute{m} \\ \acute{n} \end{bmatrix} = w_1 \begin{bmatrix} l \\ m \\ n \end{bmatrix} + w_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{49}$$

the weight w_1 is readily obtained by comparing Eqs. (8) and (49) :

$$w_1 = \pm \frac{1}{\sqrt{1 + \left(m_p^2 - 1\right) n^2}} \tag{50}$$

Substituting the expression for w_1 into $\acute{n} = w_1 n + w_2$ and comparing with (7) yields w_2:

$$w_2 = \pm \frac{\left(m_p - 1\right) n}{\sqrt{1 + \left(m_p^2 - 1\right) n^2}} \tag{51}$$

We are now ready to apply the result of the specific problem to the general problem. Figure 3 shows the schematic of the general problem—the optical axis pivots about the origin of $\{C\}$. Let us describe the general orientation of the optical axis by the action of the rotation matrix ${}^cR_\ell \in \mathbb{R}^{3\times3}$ on $\hat{\boldsymbol{c}}_z$. The matrix ${}^cR_\ell$ may be a composition of two or more matrices that denotes a sequence of rotations about the x-axis and/or y-axis. Then, $\hat{\boldsymbol{o}}$, the unit vector representing the new orientation of the optical axis, is obtained as: $\hat{\boldsymbol{o}} = {}^cR_\ell\, \hat{\boldsymbol{c}}_z$ or $\hat{\boldsymbol{c}}_z = ({}^cR_\ell)^T \hat{\boldsymbol{o}}$.

As the output direction cosine $\acute{\boldsymbol{l}}$, the input direction cosine $\boldsymbol{l}$, and the optical axis $\hat{\boldsymbol{o}}$ lie on the same plane,

$$\acute{\boldsymbol{l}} = w_1 \boldsymbol{l} + w_2 \hat{\boldsymbol{o}} \tag{52}$$

Note that the input direction cosine $\boldsymbol{l}$ in Eq. (52) is different from the corresponding $\boldsymbol{l}$ in Eq. (48) even for the same object-point $\mathbf{x}$. This difference is due to the displacement of entrance pupil (E) following the rotation of the optical axis; in fact, the designation of a ray as the chief ray (from $\mathbf{x}$ to E) keeps altering as we keep displacing the entrance pupil. Multiplying Eq. (52) by $({}^cR_\ell)^T$:

$$({}^cR_\ell)^T \acute{\boldsymbol{l}} = w_1 ({}^cR_\ell)^T \boldsymbol{l} + w_2 ({}^cR_\ell)^T \hat{\boldsymbol{o}} \tag{53}$$

Letting $({}^cR_\ell)^T\, \acute{\boldsymbol{l}} = \acute{\boldsymbol{l}}_R$ and $({}^cR_\ell)^T \boldsymbol{l} = \boldsymbol{l}_R$,

$$\acute{\boldsymbol{l}}_R = w_1 \boldsymbol{l}_R + w_2 \hat{\boldsymbol{c}}_z \tag{54}$$

Comparing Eqs. (48) and (54) the expressions for the weights w_1 and w_2 are obtained as:

$$w_1 = \pm \frac{1}{\sqrt{1 + (m_p^2 - 1)n_R^2}}$$

and

$$w_2 = \pm \frac{(m_p - 1)n_R}{\sqrt{1 + (m_p^2 - 1)n_R^2}} \tag{55}$$

Where n_R represents the projection of the direction cosine vector, $\boldsymbol{l}$**,** on the rotated optical axis. If we write the matrix ${}^cR_\ell = [{}^c\boldsymbol{r}_{\ell,1} \quad {}^c\boldsymbol{r}_{\ell,2} \quad {}^c\boldsymbol{r}_{\ell,3}]$ where ${}^c\boldsymbol{r}_{\ell,i}$ for $i = 1, 2, 3$ are the columns of ${}^cR_\ell$. Then,

$$({}^cR_\ell)^T = \left[({}^c\boldsymbol{r}_{\ell,1})^T \quad ({}^c\boldsymbol{r}_{\ell,2})^T \quad ({}^c\boldsymbol{r}_{\ell,3})^T\right]^T$$

and

$$\begin{aligned}
\boldsymbol{l}_R &= ({}^cR_\ell)^T\boldsymbol{l} \\
&= \left[({}^c\boldsymbol{r}_{\ell,1})^T \quad ({}^c\boldsymbol{r}_{\ell,2})^T \quad ({}^c\boldsymbol{r}_{\ell,3})^T\right]^T\boldsymbol{l} \\
&= \left[({}^c\boldsymbol{r}_{\ell,1})^T\boldsymbol{l} \quad ({}^c\boldsymbol{r}_{\ell,2})^T\boldsymbol{l} \quad ({}^c\boldsymbol{r}_{\ell,3})^T\boldsymbol{l}\right]^T
\end{aligned}$$

Therefore, $n_R = ({}^c\boldsymbol{r}_{\ell,3})^T\boldsymbol{l}$ since n_R is the third element of $\boldsymbol{l}_R$.

Rewriting Eq. (54) as:

$$\begin{aligned}
\acute{\boldsymbol{l}}_R &= \pm \frac{1}{\sqrt{1 + (m_p^2 - 1)n_R^2}}\boldsymbol{l}_R \pm \frac{(m_p - 1)n_R}{\sqrt{1 + (m_p^2 - 1)n_R^2}}\hat{\boldsymbol{c}}_z \\
&= \pm \frac{1}{\sqrt{1 + (m_p^2 - 1)n_R^2}}\left(\begin{bmatrix} l_R \\ m_R \\ n_R \end{bmatrix} \pm (m_p - 1)n_R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\
&= \pm \frac{1}{\sqrt{1 + (m_p^2 - 1)n_R^2}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m_p \end{bmatrix}}_{\boldsymbol{M}_p} \begin{bmatrix} l_R \\ m_R \\ n_R \end{bmatrix}
\end{aligned}$$

which can be compactly written as:

$$\acute{\boldsymbol{l}}_R = \pm \frac{1}{\sqrt{1 + (m_p^2 - 1)n_R^2}} M_p\boldsymbol{l}_R \tag{56}$$

Finally, substituting $({}^cR_\ell)^T\acute{\boldsymbol{l}} = \acute{\boldsymbol{l}}_R$ and $({}^cR_\ell)^T\boldsymbol{l} = \boldsymbol{l}_R$ yields the general expression for the direction cosines of the chief ray in the image space in terms of the pupil magnification and direction cosines in the object space as:

$$\acute{\boldsymbol{l}} = \pm \frac{1}{\sqrt{1 + (m_p^2 - 1)n_R^2}} {}^cR_\ell M_p ({}^cR_\ell)^T\boldsymbol{l} \tag{57}$$

where $n_R = ({}^c\boldsymbol{r}_{\ell,3})^T\boldsymbol{l}$.

Appendix B: Gaussian imaging equation with pupil magnification

The familiar Gaussian imaging equation, $-1/u + 1/\acute{u} = 1/f$, relates the directed object and image plane distances with the focal length f. u is the directed distance (numerically negative) between the object plane (perpendicular to the optical axis) and the principal plane (H) in the object space, $\acute{u}$ is the directed distance (numerically positive for *real* images) between the in-focus image plane and the principal plane ($\acute{H}$) in the image space. The distances being measured along the optical axis.

If the distances of the object and image planes are specified from the entrance (E) and exit pupil ($\acute{E}$) instead of the principal planes, then the Gaussian lens formula needs to be slightly modified to incorporate the pupil magnification (m_p). Here we derive the modified formula starting from the Gaussian lens formula. The same result was derived in [1] using a slightly different approach.

Figure 8 shows a schematic of the entrance and exit pupils, the object and image space principal planes, and the object and image points.

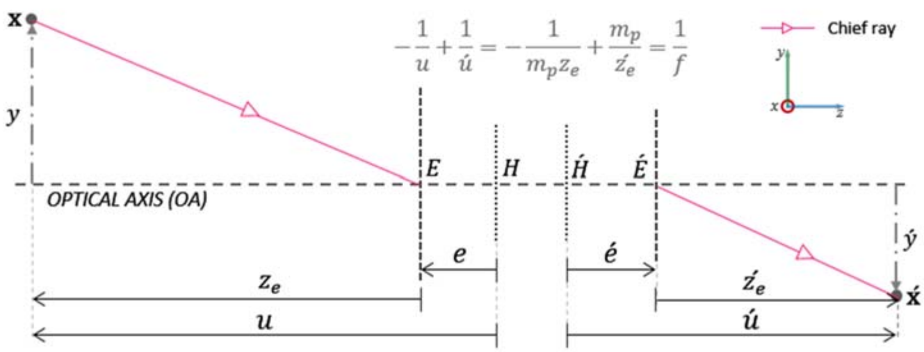

Figure 8 Schematic of imaging through a lens. The figure shows the object (y) and its image ($\acute{y}$), the object space principal plane (H) and the image side principal plane ($\acute{H}$), the entrance (E) and exit ($\acute{E}$) pupils, and the associated distances along the optical axis.

In the figure, u and $\acute{u}$ are the distances from the principal planes to the object and image planes, e and $\acute{e}$ are distances from the principal planes to the entrance- and exit-pupils and z_e and $\acute{z}_e$ are the distances from the entrance- and exit-pupils to the object and image planes. Since the entrance- and exit-pupil planes are conjugates, like the object and image planes, the Gaussian lens formula holds as follows:

$$-\frac{1}{u}+\frac{1}{\acute{u}}=\frac{1}{f}\ , \tag{58}$$

and

$$-\frac{1}{e}+\frac{1}{\acute{e}}=\frac{1}{f}\ . \tag{59}$$

The transverse magnification m_t between the object and image planes is

$$m_t=\frac{\acute{u}}{u}\ . \tag{60}$$

For images that are *real* and inverted, the transverse magnification m is numerically negative since the directed distance u is numerically negative, and $\acute{u}$ is numerically positive.

The pupil magnification m_p is defined as the ratio of the exit pupil diameter to the entrance pupil diameter. It is also the ratio between the exit pupil and entrance pupil distances (measured from the principal planes) just like the transverse magnification between any conjugate planes:

$$m_p = \frac{\acute{e}}{e} \ . \tag{61}$$

Equating Eqs. (58) and (59) , we obtain

$$\frac{u-e}{eu} = \frac{\acute{u}-\acute{e}}{\acute{e}\acute{u}} \ .$$

Substituting $z_e = u - e$ and $\acute{z_e} = \acute{u} - \acute{e}$ in the above equation, and using Eqs. (60) and (61) , we get

$$\boxed{\frac{\acute{z_e}}{z_e} = \left(\frac{\acute{e}}{e}\right)\left(\frac{\acute{u}}{u}\right) = m_p m_t} \tag{62}$$

Further, we can also substitute $u = z_e + e$ and $\acute{u} = \acute{z_e} + \acute{e}$ in Eq. (58) and equate with Eq. (59) as

$$\frac{1}{\acute{z_e}+\acute{e}} - \frac{1}{z_e + e} = \frac{1}{\acute{e}} - \frac{1}{e} \ ,$$

which after cross-multiplication and cancellations of common terms produces

$$z_e \acute{z_e}(e - \acute{e}) + e^2 \acute{z_e} - \acute{e}^2 z_e = 0 \ .$$

Dividing throughout by $z_e \acute{z_e} e \acute{e}$, and substituting $\frac{\acute{e}}{e}$ by the pupil magnification m_p, and $\left(\frac{e-\acute{e}}{e\acute{e}}\right)$ by $\frac{1}{f}$ we obtain:

$$\boxed{-\frac{1}{m_p z_e} + \frac{m_p}{\acute{z_e}} = \frac{1}{f}} \tag{63}$$

Eq. (63) is valid even if the z_e and $\acute{z_e}$ denote distances from the principal planes provided we let $m_p = 1$. This outcome is indeed consistent with geometric optics theory, according to which the magnification between the principal planes is unity. In fact, Eq. (37) is more general than the Gaussian Lens formula in that it relates a pair of conjugate planes with any other pair of conjugate planes for which the transverse magnification (between the planes) is known. When one of the pairs happen to be the principal planes (H and $\acute{H}$), between which the magnification is one, we obtain the Gaussian Lens formula.

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