Notes on Category Theory — Pieces

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1 Introduction to Topoi

1.1 Subobject classifiers

Throughout the current section, we assume \mathcal{E} is a category with initial object 1. That being the setting, we can give the following definition.

Definition 1. A *subobject classifier* for $\mathcal E$ is any morphism $t:1\to \Omega$ such that: for every monomorphism $f:a\to b$ of $\mathcal E$ there is one and only one morphism $\chi_f:b\to \Omega$ in $\mathcal E$ for which there is a pullback square

$$\begin{array}{ccc}
 a & \xrightarrow{f} & b \\
 \downarrow & & \downarrow \chi_f \\
 1 & \xrightarrow{t} & \Omega
\end{array}$$
(1.1)

That is we can assign to every monomorphism $f:a\to b$ the morphism $\chi_f:b\to\Omega$ satisfying the property of the definition. Let us introduce then some symbolism: for $b\in|\mathcal{E}|$ we write $\mathrm{Sub}_{\mathcal{E}}\,b$ for the class of all the monomorphisms of \mathcal{E} with codomain b. Hence we can introduce the function

$$\chi: \operatorname{Sub}_{\mathcal{E}} b \to \mathcal{E}(b, \Omega)$$

with χ_f defined to be that morphism $b \to \Omega$ for which there is a pullback square as the diagram (1.1).

It is worth to observe $\mathrm{Sub}_{\mathcal{E}}(b)$ has a natural structure of preorder: for



monomorphisms of \mathcal{E} , write $f_1 \leq f_2$ to say there is some $h: a_1 \to a_2$ in \mathcal{E} for which



commutes. Note that, being here f_1 and f_2 monomorphisms, there is at most one h as such and it is a monomorphism as well.

We show now the relation \simeq on $Sub_{\mathcal{E}} b$ defined by

$$f_1 \simeq f_2$$
 if and only if $f_1 \leq f_2$ and $f_2 \leq f_1$

for $f_1, f_2 \in \text{Sub}_{\mathcal{E}} b$ is an equivalence relation. [...]

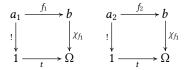
Yes, $\operatorname{Sub}_{\mathcal{E}} b$ is the full subcategory of $\mathcal{E} \downarrow b$ whose objects are all the monomorphisms of \mathcal{E} with codomain b, and whose isomorphism relation is \simeq .

Proposition 2. Let



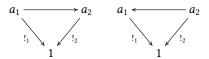
be monomorphisms. $\chi_{f_1} = \chi_{f_2}$ if and only if $f_1 \simeq f_2$.

Proof. Assume $\chi_{f_1} = \chi_{f_2}$. By definition of subobject classifiers, χ_{f_1} is the morphism for which



are pullback squares. Consequently, we must infer that there is one isomorphism $h: a_1 \to a_2$ such that $f_1 = f_2 h$. Hence $f_1 \le f_2$, and $f_2 \le f_1$ too, because $f_1 h^{-1} = f_2$.

For the remaining part of the proof, let us write $!_1$ the unique morphism $a_1 \to 1$ and $!_2$ the unique morphism $a_2 \to 1$. Also remember that triangles



always commute.

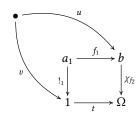
Now we suppose $f_1 \simeq f_2$. The plan for the proof is: if we show that



is a pullback square, then, being $\chi_{f_1}: b \to \Omega$ the one for which there is a pullback square like this, we can conclude $\chi_{f_1} = \chi_{f_2}$. First of all such square commutes: if we call h the morphism $a_1 \to a_2$ such that $f_1 = f_2 h$, then

$$\chi_{f_2}f_1 = \chi_{f_2}f_2h = t!_2h = t!_1.$$

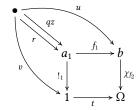
Consider



where $\chi_{f_2}u = tv$. Being



a pullback square we have one $z: \bullet \to a_2$ for which $f_2z=u$ and $!_2z=v$. From the assumption $f_1\simeq f_2$, we have $f_2\leq f_1$, that is $f_2=f_1q$ for some $q:a_2\to a_1$. Then $u=f_1qz$ and $v=!_1qz$. Let us see if qz is what we are looking for.



where we suppose $!_1r = v$ and $f_1r = u$. Being f_1 a monomorphism, the sole second identity is enough to conclude r = qz.