Notes on Category Theory — Pieces

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1 Preamble

1.1 Symbolism for functions

If f is the name of a function, we write $f(x)$ the image of x. However, we may find ourselves writing fx or f_x to avoid an overwhelming use of brackets. The expression $f : X \to Y$ means that f is a function from the domain X to the codomain Y.

A literary device we will quite often take advantage is that of *currying*: from a function out of a product

$$f : A \times B \to C$$

we have, for $x \in A$, the functions

$$f(x,\quad) : B \to C\,,\ f(x,\quad)(y) := f(x,y).$$

The idea is to 'hold' the first variable at some value and let the second one vary: this is done by leaving a blank space to be filled with values from B. Of course, for $y \in B$, we introduce the functions

$$f(\quad,y) : A \to C\,,\ f(\quad,y)(x) := f(x,y).$$

Symbols like $\bullet$, $\cdot$ and $-$ can be employed instead of leaving an empty space: in fact, you may find written $f(x,\bullet)$, $f(x,\cdot)$ or $f(x,-)$ for example.

While leaving blank spaces or using placeholders may be acceptable in prefix notation, it may be a pain if used in combination with infix notation: consider, for example, the function $\mathbb{N} \to \mathbb{N}$ that takes each natural number to the corresponding successor and writing it as follows

$$\bullet + 1\,,\ \cdot + 1\,,\ - + 1\,,\ \dots$$

or even

$$\underbrace{\quad}_{\text{blank space here, see?}} +1.$$

In all those situations, we may forget placeholders and use parentheses as in

$$(+1)$$

and $(+1)(n)$ would be a synonym of $n+1$.

Another way to introduce functions comes from Lambda Calculus. Suppose you are given some well-formed formula[1] Γ and some variable x: the symbol x may occur or not in Γ. Then we have

$$\lambda x.\Gamma$$

called *lambda abstraction*. When we write an expression like this one, x is a *dummy* or a *bound* variable; in contrast, the other variables in Γ are said to be *free*. It works like the more familiar expressions

$$\forall x : \phi,\ \exists x : \phi,\ \lim_{x\to c} f(x) \text{ and } \int_\Omega f(x)\mathrm{d}x :$$

instead of the symbol x you may use another symbol that does not occur freely Γ without changing the meaning of the expression.

For example, we all agree that $\lambda x.x+k$ and $\lambda y.y+k$ have the same exact meaning, thus they are equivalent. But, what if instead of x we use k? In the formula $\lambda x.x+k$ the letter k already appears free, and we would have $\lambda k : k+k$, which is not the same.

In fact, the most basic operation you can perform with formulas involving variables is *substitution*: if you are given a formula Γ and a variable x that occurs freely in Γ, then we write $\Gamma[x/a]$ the formula Γ with the occurrences of x replaced by a, after eventually renaming all the dummy occurrences of a throughout the formula to avoid the issues illustrated before.

The operation of substitution is the way you can 'pass values' to such things and have returned another values:

$$(\lambda x.\Gamma)(a) := \Gamma[x/a].$$

It is clear the practical use of this formal device: if a is an element of set X and $(\lambda x.\Gamma)(a)$ is a member of another set Y, then we could write the function doing that assignment as $\lambda x.\Gamma$.

If you want, instead of $\lambda x.\Gamma$ you may write

$$x \mapsto \Gamma.$$

Observe that, in general, lambda abstractions do not have incorporated the information of domain and codomain, or in general it might not be inferred without doubt from the context. For example, what is $\lambda x.x+1$? Without a hint from the context, it can be a function $\mathbb{R}\to\mathbb{R}$ or the successor function $\mathbb{N}\to\mathbb{N}$, and so on... To dispel any ambiguity, you can write explicitly something like this:

$$(\lambda n.n+1) : \mathbb{N}\to\mathbb{N}.$$

1.2 Functions and equivalences

If X is a set and $\sim$ is an equivalence relation on X, we write $X/\!\sim$ or $\frac{X}{\sim}$ to indicate the set whose members are the sets

$$[x]_\sim := \{a \in X \mid a \sim x\} \quad \text{for } x \in X,$$

1. Written in a sensible way, that is following some syntactic rules. [More precision?]

the equivalence classes under $\sim$. Sometimes we simply write $[x]$ when the name of the equivalence relation may be dropped without creating ambiguity.

Remember that equivalence classes form a partition, that is they are pairwise disjoint and their union is the entire set.

Annexed to that, there is the *canonical projection*

$$X \to X/\sim, \ x \to [x]_\sim.$$

Proposition 1.1 (Isomorphism Theorem of Set Theory)**.** Consider two sets X and Y, a function $f : X \to Y$ and an equivalence relation $\sim$ over X. Let also be $p : X \to X/\sim$ the canonical projection. If for every $a, b \in X$ such that $a \sim b$ we have $f(a) = f(b)$, then there exists one and only one function $\overline{f} : X/\sim \to Y$ such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & X/\sim \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

commutes. Moreover:

1. $\overline{f}$ is surjective if and only if so is f;
2. if also $a \sim b$ for every $a, b \in X$ such that $f(a) = f(b)$, then $\overline{f}$ is injective.

The name of this theorem will make you remember other isomorphism theorems. In Algebra, the *First Isomorphism Theorem* can be derived from this one. Consider, for example[2], two groups G and H a group homomorphism $f : G \to H$ and a normal subgroup N of G contained in $\ker f$. In that case, we have the equivalence relation $\sim_N$ on G defined by

$$x \sim_N y \quad \text{if and only if} \quad xa = y \text{ for some } a \in N$$

for $x, y \in G$. Further, being N normal in G, the set $G/N := G/\sim_N$ has as equivalence classes the lateral classes xN and has a group structure where the identity is N and the product of two lateral classes xN and yN is the lateral class $(xy)N$. In this case we have, if $x \sim_N y$, that is $xa = y$ for some $a \in N$,

$$f(y) = f(xa) = \underbrace{f(x)f(a) = f(x)}_{a \in N \subseteq \ker f}.$$

Hence, there is a unique function $\overline{f} : G/N \to H$ for which

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "p"'] & & H \\ & G/N \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

commutes. The function $\overline{f}$ is a group homomorphism and there are some other facts, but that takes us a bit away from the main topic of the section.

Exercise 1.2. Con you state something like that but for topological spaces and continuous functions? Remember the canonical projection $p : X \to X/\sim$ defines the topology for $X/\sim$ in which $U \subseteq X/\sim$ is open whenever so is $p^{-1}U$. (Here, $p^{-1}U$ is just the union of the equivalence classes in U.)

2. There are other First Isomorphism Theorems, one for rings and another one for modules.

Proof of Proposition 1.1. Consider the relation

$$\overline{f} := \{(u,v) \in (X/\sim) \times Y \mid p(x) = u \text{ and } f(x) = v \text{ for some } x \in X\}:$$

we will show that it is actually a function from $X/\sim$ to Y. Picked any $u \in X/\sim$ (it is not empty), there is some $x \in u$ and then we have the element $f(x) \in Y$; in this case, $(u, f(x)) \in \overline{f}$. Now, let (u,v) and (u,v') be two any pairs of $\overline{f}$. Then $u = p(x)$ and $v = f(x) = v'$ for some $x \in u$, and so we conclude $v = v'$. This function satisfies $\overline{f}p = f$, cause of its own definition.
Now, the uniqueness part comes. Assume you have a function $g : X/\sim \to Y$ such that $gp = f$: then for every $u \in X/\sim$ we have some $x \in u$ and

$$g(u) = g(p(x)) = f(x) = \overline{f}(p(x)) = \overline{f}(u),$$

that is $g = \overline{f}$. The most of the work is done now, whereas points (1) and (2) are immediate. □

Corollary 1.3. For X and Y sets, let $\sim_X$ and $\sim_Y$ be two equivalence relations on X and Y respectively and let $f : X \to Y$ be a function such that for every $a, b \in X$ such that $a \sim_X b$ we have $f(a) \sim_Y f(b)$. Then there exists one and only one function $\overline{f} : X/\sim_X \to Y/\sim_Y$ such that

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p_X"'] & Y \arrow[d, "p_Y"] \\
X/\sim_X \arrow[r, "\overline{f}"'] & Y/\sim_Y
\end{tikzcd}$$

commutes, where p_X and p_Y are the canonical projections. Moreover:

1. $\overline{f}$ is surjective if and only if so is f;
2. if also $a \sim_X b$ for every $a, b \in X$ such that $f(a) \sim_Y f(b)$, then $\overline{f}$ is injective.

Proof. Take the sets X and $Y/\sim_Y$ with the function $p_Y f : X \to Y/\sim_Y$ and use Proposition 1.1. □

[Functoriality here...]

Sometimes in Mathematics, generated equivalence relations are involved. Speaking in plain Set Theory terms: if you are given a set X and some $S \subseteq X \times X$, we define the equivalence relation *generated* by S as the smallest among the equivalence relations of X containing S. One can easily verify that such equivalence relation is the intersection of all the equivalence relations on X containing S. We write X/S or $\frac{X}{S}$ to mean the set X quotiented by the equivalence relation generated by R. One can use expressions like:

> On X consider the equivalence relation $\sim$ generated by (the family of statements)
>
> $$a_\lambda \sim b_\lambda \quad \text{for } \lambda \in I.$$
>
> (Of course, the a_λ-s the b_λ-s are elements of X.)

to say that:

> On X consider the equivalence relation $\sim$ generated by the set
>
> $$\{(a_\lambda, b_\lambda) \mid \lambda \in I\}.$$

That being said, it would be clear what we mean by writing

$$\frac{X}{a_\lambda \sim b_\lambda \text{ for } \lambda \in I}.$$

We will discuss about these constructions again in the chapter of limits and colimits.

Proposition 1.4. Let X and Y be two sets, $\sim$ an equivalence relation on X generated by $S \subseteq X \times X$ and $f : X \to Y$ any function. Then the following statements are equivalent:

1. for every $a, b \in X$, if $a \sim b$ then $f(a) = f(b)$
2. for every $a, b \in X$, if $(a, b) \in S$ then $f(a) = f(b)$.

Proof. The implication $(1) \Rightarrow (2)$ is trivially true. Consider on X the equivalence relation $\simeq$ define as: for all $a, b \in X$, $a \simeq b$ if and only if $f(a) = f(b)$. Here, $S \subseteq \simeq$, thus $\sim \subseteq \simeq$. $\square$

The proposition just proved permits to rewrite Proposition 1.1 for generated equivalence relations.

Corollary 1.5. Consider two sets X and Y, a function $f : X \to Y$ and an equivalence relation over X generated by $S \subseteq X \times X$. If for every $(a, b) \in S$ we have $f(a) = f(b)$, then there exists one and only one function $\overline{f} : X/S \to Y$ such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & X/S \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

commutes. Here, $p : X \to X/S$ is the canonical projection.