Notes on Category Theory: Adjointness

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Contents

1 Isolating the concept

Example 1.1 (Defining linear functions, part I)**.** Consider two vector spaces V and W (over the same field) and the problem:

> how can we define a linear function $f : V \to W$?

There is a well known theorem that says that prescribing the images of the elements of the base S determines uniquely a linear function $V \to W$. More precisely, the theorem sounds like this:

> Let V and W two vector spaces, both over a field k, and let S be a base of V. Let us write i for the inclusion $S \hookrightarrow V$. Then for every function $\phi : S \to W$ there exists one and only one linear function $f : V \to W$ such that
>
> $$\begin{tikzcd} S \arrow[r, hook, "i"] \arrow[dr, "\phi"'] & V \arrow[d, "f"] \\ & W \end{tikzcd} \tag{1.1}$$
>
> commutes.

The statement is equivalent to saying that the function

$$\mathbf{Vect}_k(V,W) \to \mathbf{Set}(S,W)\,,\ f \to fi \tag{1.2}$$

is a bijection.

Let us reason about the theorem. Fist of all, it is about a *function* $\phi : S \to W$, pointing to a vector space W: the morphisms of **Set** do not care whether the sets have an additional structure. Let us say that ϕ is a *function* from S to W 'downgraded' from the status of vector space to the one of set. On the other hand, from a set we construct an actual vector space, this is what being a base means. Indeed, behind the scenes two functors

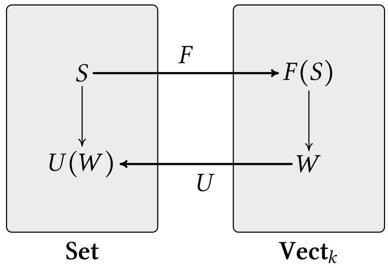

Figure 1. F upgrades sets to vector spaces; U does the opposite, that is downgrades.

$$\begin{tikzcd} \mathbf{Set} \arrow[r, shift left, "F"] & \mathbf{Vect}_k \arrow[l, shift left, "U"] \end{tikzcd}$$

are moving:

- F the functor that *constructs* a vector space $F(S)$ from set S it and extends a function of bases $\alpha : S \to T$ to a genuine linear functions $F(\alpha) : F(S) \to F(T)$, as we have already explained.
- U is the functor that *forgets*: that is takes a vector space and returns the set of vectors (thus no addition, no external product and no vector space axioms); it takes a linear function and returns the same function, but observe the homomorphism property cannot make sense any longer in **Set**.

Let us rewrite the quoted theorem to make it more aware of these functors. First, let us rewrite $i : S \to V$ as $i_S : S \to U(F(S))$ and $\phi : S \to W$ as $\phi : S \to U(W)$: we have just wrapped V and W with U. So far, the situation ca be depicted as

$$\begin{tikzcd} S \arrow[r, hook, "i_S"] \arrow[dr, "\phi"'] & U(F(S)) \\ & U(W) \end{tikzcd}$$

We are not loosing the commutative diagram (1.1)! The composition of a 'function' $S \to V$ with a 'linear function' $V \to W$ which gives as result a 'function' $S \to W$: in this composition we not care about linearity anymore, thus let us wrap the $f : V \to W$ with U, so that it can fit in a commutative diagram:

> Let V and W two vector spaces over a field k, and let S be a base of V. In this context, $V = F(S)$. Moreover, write i_S for the inclusion function $S \hookrightarrow U(F(S))$. Then for every function $\phi : S \to U(W)$ there exists one and only one linear function $f : V \to W$ for which
>
> $$\begin{tikzcd} S \arrow[r, hook, "i_S"] \arrow[dr, "\phi"'] & U(F(S)) \arrow[d, "U(f)"] \\ & U(W) \end{tikzcd} \tag{1.3}$$
>
> is a commutative diagram of **Set**.

The restyling touches also the bijections in (1.2):

$$\xi_{S,W} : \mathbf{Vect}_k(F(S), W) \to \mathbf{Set}(S, U(W)) \,,\ \xi_{S,W}(f) := U(f) i_S \tag{1.4}$$

We need to pause the example a bit now and resume it later.

Construction 1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be two locally small categories and two functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] & \mathcal{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

We have then the functor

$$\mathcal{C}(\ \ , R(\ \ )) : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$$

that maps objects (x, y) to $\mathcal{C}(x, R(y))$ and pairs of morphisms

$$\left(\begin{tikzcd} (x,y) \arrow[d, "(f,g)"'] \\ (x',y') \end{tikzcd} \right) = \left(\begin{tikzcd} x \\ x' \arrow[u, "f"] \end{tikzcd} , \begin{tikzcd} y \arrow[d, "g"] \\ y' \end{tikzcd} \right)$$

to functions

$$\begin{aligned} \mathcal{C}(x, R(y)) &\to \mathcal{C}(x', R(y')) \\ h &\to R(g)hf \end{aligned}$$

We have also the functor

$$\mathcal{D}(L(\ \ ), \ \ ) : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$$

that maps (x, y) to $\mathcal{C}(L(x), y)$ and pairs of morphisms

$$\left(\begin{tikzcd} (x,y) \arrow[d, "(f,g)"'] \\ (x',y') \end{tikzcd} \right) = \left(\begin{tikzcd} x \\ x' \arrow[u, "f"] \end{tikzcd} , \begin{tikzcd} y \arrow[d, "g"] \\ y' \end{tikzcd} \right)$$

to functions

$$\begin{aligned} \mathcal{D}(L(x), y) &\to \mathcal{D}(L(x'), y') \\ h &\to ghL(f). \end{aligned}$$

Example 1.3 (Defining linear functions, part II)**.** We return to the last example. The functions $\xi_{S,W}$ form a natural isomorphism

$$\xi : \mathbf{Vect}_k(F(\ \ ), \ \ ) \Rightarrow \mathbf{Set}(\ \ , U(\ \ ))$$

We know the components are bijections, hence it remains to check it is a natural transformation. Thus consider the diagram

$$\begin{tikzcd} \mathbf{Vect}_k(F(S), W) \arrow[r, "\xi_{S,W}"] \arrow[d, "\mathbf{Vect}_k(F(\alpha), f)"'] & \mathbf{Set}(S, U(W)) \arrow[d, "\mathbf{Set}(\alpha, U(f))"] \\ \mathbf{Vect}_k(F(S'), W') \arrow[r, "\xi_{S',W'}"'] & \mathbf{Set}(S', U(W')) \end{tikzcd}$$

and we show it is commutative. A linear function $h : F(S) \to W$ goes to the function $U(h)i_S : S \to U(W)$, which is sent to the function $U(f)(U(h)i_S)\alpha : S' \to U(W')$. By functoriality, $U(f)(U(h)i_S)\alpha = U(fh)i_S\alpha$. On the other way, h goes to the linear function $fhF(\alpha) : F(S') \to W'$ which goes to $U(fhF(\alpha))i_{S'} : S' \to U(W')$. Here, $U(fhF(\alpha))i_{S'} = U(fh)U(F(\alpha))i_{S'}$. Thus, to verify that the functions $S' \to U(W')$ here are equal, one could just verify that

$$\begin{tikzcd} S \arrow[r, "i_S"] & U(F(S)) \\ S' \arrow[u, "\alpha"] \arrow[r, "i_{S'}"'] & U(F(S')) \arrow[u, "U(F(\alpha))"'] \end{tikzcd}$$

commutes, which is immediate.

Example 1.1 and 1.3 of the introduction should have triggered your attention. If not, look at them closely now: after the initial restyling of one result of Linear Algebra, it is just a matter of categories and functors.

Construction 1.4. Let $\mathcal{C}$ and $\mathcal{J}$ two categories, a one of its objects and take a functor $F : \mathcal{J} \to \mathcal{C}$. We have the category $a{\downarrow}F$ made as follows:

- the objects are the morphisms $a \to F(x)$ of $\mathcal{C}$, with x being an object of $\mathcal{J}$;
- the morphisms from $f : a \to F(x)$ to $g : a \to F(y)$ are the morphisms $h : x \to y$ of $\mathcal{J}$ such that

$$\begin{tikzcd} & F(x) \arrow[dd, "F(h)"] \\ a \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & F(y) \end{tikzcd}$$

  commutes;
- the composition is that of $\mathcal{J}$.

Proposition 1.5. Suppose given two locally small categories $\mathcal{C}$ and $\mathcal{D}$, two functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] & \mathcal{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

and a natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow RL$ such that $\eta_x : x \to RL(x)$ is initial in $x{\downarrow}R$ for every $x \in |\mathcal{C}|$. Then, for $x \in |\mathcal{C}|$ and $y \in |\mathcal{D}|$, the functions

$$\alpha_{x,y} : \mathcal{D}(L(x), y) \to \mathcal{C}(x, R(y)), \ \alpha_{x,y}(f) := R(f)\eta_x$$

form an adjunction $\alpha : L \dashv R$.

Exercise 1.6. Look at Example 1.1 and 1.3: isolate what in the proposition is the natural transformation η. Can you prove the theorem by yourself?

Proof of Proposition 1.5. The fact that η_x is initial object implies that these function are all bijective. Now, we just need to verify the transformation is natural. Take $x, x' \in |\mathcal{C}|$, $y, y' \in |\mathcal{D}|$, $f \in \mathcal{C}(x', x)$ and $g \in \mathcal{D}(y, y')$ and examine the square

$$\begin{tikzcd} \mathcal{D}(L(x), y) \arrow[r, "\alpha_{x,y}"] \arrow[d, "\mathcal{D}(L(f),g)"'] & \mathcal{C}(x, R(y)) \arrow[d, "\mathcal{C}(f,R(g))"] \\ \mathcal{D}(L(x'), y') \arrow[r, "\alpha_{x',y'}"'] & \mathcal{C}(x', R(y')) \end{tikzcd}$$

For $h \in \mathcal{D}(L(x), y)$, we have

$$\mathcal{C}(f, R(g))(\alpha_{x,y}(h)) = R(g)R(h)\eta_x f = R(gh)\eta_x f$$
$$\alpha_{x',y'}(\mathcal{D}(L(f), g)(h)) = R(guL(f))\eta_{x'} = R(gu)RL(f)\eta_{x'}$$

By the naturality of η, we have $\eta_x f = RL(f)\eta_{x'}$, and the proof ends here. □

Here is another example in the spirit of the initial example about vector spaces and of the proposition just proved here.

Example 1.7 (Isomorphism Theorem for Set Theory)**.** We have defined **Eqv** earlier, recall it here. Let us introduce the functor

$$E : \mathbf{Set} \to \mathbf{Eqv}$$

that maps a set X to a setoid $(X, =_X)$, where $=_X$ is the equality relation over X, and a function $f : X \to Y$ to itself regarded as morphisms of setoids. Besides, Corollary 2.3 gives a functor

$$P : \mathbf{Eqv} \to \mathbf{Set}.$$

Consequently, Proposition 2.1 can be easily rephrased more concisely as:

the canonical projection $(X, \sim) \to E(P(X, \sim))$ is initial in $(X, \sim) \downarrow E$.

There is the dual of Proposition 1.5 as well.

Proposition 1.8. Suppose given two locally small categories $\mathcal{C}$ and $\mathcal{D}$, two functors

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

and a natural transformation $\theta : LR \Rightarrow 1_{\mathcal{D}}$ such that $\theta_y : LR(y) \to y$ is terminal in $L{\downarrow}y$ for every $y \in |\mathcal{C}|$. Then, for $x \in |\mathcal{C}|$ and $y \in |\mathcal{D}|$, the functions

$$\begin{aligned} \mathcal{C}(x, R(y)) &\to \mathcal{D}(L(x), y) \\ f &\to \theta_y L(f) \end{aligned}$$

form an adjunction $L \dashv R$.

Exercise 1.9. The proof is left to you.

Example 1.10 (Evaluation of polynomials)**.** [Yet to be TEXed...]

2 Definition, units and counits

Definition 2.1 (Adjunctions)**.** Consider two locally small categories and two functors as in

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

An *adjunction* from L to R any natural isomorphism

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, bend left, "\mathcal{D}(L(\ ),\ )"] \arrow[r, bend right, "\mathcal{C}(\ ,R(\ ))"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{Set} \end{tikzcd} \tag{2.5}$$

We say L is the *left adjoint* and R is the *right adjoint*: the reason behind the naming comes from when we write the bijection

$$\mathcal{D}(L(x), y) \cong \mathcal{C}(x, R(y))$$

L occurs in $\mathcal{D}(L(x), y)$ applied to the argument on the left, while R appears in $\mathcal{C}(x, R(y))$ applied on the right. Adjunctions are usually written as $\alpha : L \dashv R$. Sometimes $L \dashv R$ is written to mean that there is an adjunction in between without specifying which one. If on the the paper we are writing there is space, we can write something like

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd} \tag{2.6}$$

which has in addition shows the categories involved.

We will soon return to the introductory example later; as for now, let us indulge a bit more on the definition of adjunction.

Exercise 2.2 (Partial functions)**.** For A and B sets, a *partial function* from A to B is relation $f \subseteq A \times B$ with the property

for every $x \in A$ and $y_1, y_2 \in B$, if $(x, y_1) \in f$ and $(x, y_2) \in f$ then $y_1 = y_2$.

We want to compose partial functions as well: provided $f \in \mathbf{Par}(A, B)$ and $g \in \mathbf{Par}(B, C)$,

$$gf := \{(x, y) \in A \times C \mid (x, z) \in f \text{ and } (z, y) \in g \text{ for some } z \in B\}.$$

It is immediate to verify **Par** complies the rules that make it a category. Indeed, this is the *category of partial functions*, written as **Par**: here, the objects are sets and the morphisms are partial functions.
Suppose given a partial function $f : A \to B$. For every $x \in A$ the possibilities are two: there is one element of B bound to is, and we write it $f(x)$, or none. What if we considered *no value* as an output value? Provided two sets A and B and a partial function $f : A \to B$, we assign an actual function

$$\overline{f} : A \to B + 1\,,\ \overline{f}(x) := \begin{cases} f(x) & \text{if } x \text{ has an element of } B \text{ bound} \\ * & \text{otherwise} \end{cases}$$

where $1 := \{*\}$ with $*$ designating the absence of output. It is quite simple to show that

$$\mathbf{Par}(A, B) \to \mathbf{Set}(A, B + 1)\,,\ f \to \overline{f}$$

is a bijection for every couple of sets A and B. Now it's up to you to categorify this: find two functors that make an adjunction

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "I"] \arrow[r, phantom, "\perp"] & \mathbf{Par} \arrow[l, bend left, "J"] \end{tikzcd}$$

Proposition 2.3. Provided you have locally small categories and functors

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

and an adjunction

$$\alpha : \mathcal{D}(L(\ \ ),\ \ ) \to \mathcal{C}(\ \ , R(\ \ )).$$

Then:

1. For $x \in |\mathcal{C}|$ the morphisms

$$\eta_x : x \to RL(x)\,,\ \eta_x := \alpha_{x, L(x)}\left(1_{L(x)}\right)$$

form a natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow RL$ such that η_x is initial in $x{\downarrow}R$.

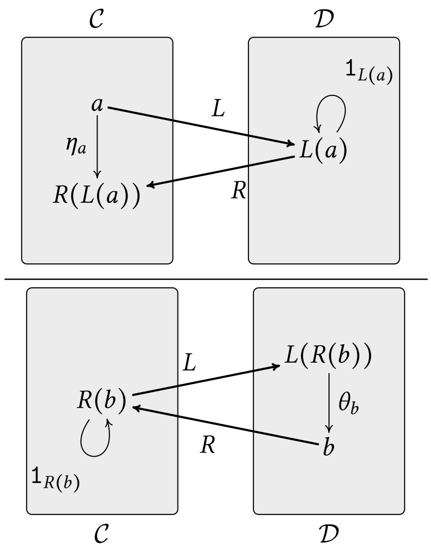

Figure 2. Units and co-units of an adjunction.

2. For $y \in |\mathcal{C}|$ the morphisms

$$\theta_y : LR(y) \to y\,,\ \theta_y := \alpha^{-1}_{R(y),y}\left(1_{R(y)}\right)$$

form a natural transformation $\theta : LR \Rightarrow 1_{\mathcal{D}}$ such that θ_y is terminal in $L{\downarrow}y$.

Proof. [Rewrite proof.] Let us write the adjunction of the statement above as

$$\overline{} : \mathcal{D}(L(\ \ ),\ \ ) \Rightarrow \mathcal{C}(\ \ , R(\ \ )).$$

We verify that

$$\begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[d, "f"'] & RL(x) \arrow[d, "RL(f)"] \\ y \arrow[r, "\eta_y"'] & RL(y) \end{tikzcd}$$

commutes for every f in $\mathcal{C}$. In fact,

$$RL(f)\eta_x = RL(f)\overline{1_{L(x)}}1_x = \overline{L(f)1_{L(x)}1_{L(x)}} = \overline{L(f)}$$
$$\eta_y f = R(1_{L(y)})\overline{1_{L(y)}}f = \overline{1_{L(y)}1_{L(y)}L(f)} = \overline{L(f)}.$$

It remains to show that the morphisms $\eta_x : x \to RL(x)$ are initial in $x{\downarrow}R$. In $\mathcal{C}$ we draw

$$\begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[dr, "g"'] & RL(x) \\ & R(y) \end{tikzcd}$$

We know that there is one and only one $h : L(x) \to y$ such that $g = \overline{h}$. Then

$$g = \overline{h1_{L(x)}L\left(1_x\right)} = R(h)\overline{1_{L(x)}}1_x = R(h)\eta_x.$$ □

Example 2.4 (Prescribing functions via currying). [Yet to be TEXed...]

[More examples of units and counits here...]

3 Adjunctions and limits

Let $\mathcal{I}$ and $\mathcal{C}$ be two categories. For every $v \in |\mathcal{C}|$ we have the *constant functor*

$$k_v : \mathcal{I} \to \mathcal{C}$$

where $k_v(i) := v$ for every $i \in |\mathcal{I}|$ and $k_v(f) := 1_v$ for every morphism f of $\mathcal{I}$. Recall that $\lambda : k_v \Rightarrow F$ being a limit of a functor $F : \mathcal{I} \to \mathcal{C}$ means:

> for every $\mu : k_v \Rightarrow F$ there exists one and only one $f : a \to v$ of $\mathcal{C}$ such that $\mu_i = \lambda_i f$ commutes for every object i of $\mathcal{I}$.

That is, if you put it in other words, it sounds like:

> there is a bijection
> $$\mathcal{C}(a,v) \to [\mathcal{I},\mathcal{C}](k_a,F)$$
> taking $f : a \to v$ to the natural transformation
> $$\lambda_\bullet f : k_a \Rightarrow F\,,\ \lambda_\bullet f(i) := \lambda_i f.$$

There is a smell of adjunction situation here. Let us start with finding an appropriate pair of functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left] & {[\mathcal{I},\mathcal{C}]} \arrow[l, shift left] \end{tikzcd}\ .$$

One functor is already suggested:

$$\Delta : \mathcal{C} \to [\mathcal{I},\mathcal{C}]$$

takes $x \in |\mathcal{C}|$ to the functor $\mathcal{I} \to \mathcal{C}$ that maps every object to x and every morphism to 1_x; then for $i \in |\mathcal{I}|$ define

$$\Delta\left(x \xrightarrow{f} y\right)$$

to be the natural transformation $\Delta(x) \Rightarrow \Delta(y)$ amounting uniquely of f.

From now on, assume $\mathcal{I}$ is small and every functor $\mathcal{I} \to \mathcal{C}$ has a limit. Now, in spite of not being strictly unique [‘strictly unique’... huh?], all the limits of a given functor are isomorphic, so are the vertices: let us indicate by $\lim F$ the vertex of any of the limits of F. Now, take a natural transformation

$$\begin{tikzcd} \mathcal{I} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{C} \arrow[from=U, to=D, Rightarrow, "\xi"] \end{tikzcd}\ ;$$

$\lim F$ is the vertex of some limit

$$\left\{ \lim F \xrightarrow{\lambda_i} F(i) \,\middle|\, i \in |\mathcal{I}| \right\}$$

and $\lim G$ is the vertex of a certain limit

$$\left\{ \lim G \xrightarrow{\mu_i} G(i) \,\middle|\, i \in |\mathcal{I}| \right\}.$$

If we display all the stuff we have gathered so far, we have for $i \in |\mathcal{I}|$

$$\begin{tikzcd} F(i) \arrow[r, "\xi_i"] & G(i) \\ \lim F \arrow[u, "\lambda_i"] & \lim G \arrow[u, "\mu_i"'] \end{tikzcd} \tag{3.7}$$

The universal property of limits ensures that there is one and only one morphism $\lim F \to \lim G$ making the above diagram a commuting square. Let us call this morphism $\lim \xi$. We have a functor

$$\lim : [\mathcal{I}, \mathcal{C}] \to \mathcal{C}$$

indeed. If you take $F = G$ and η the identity of the functor F in (3.7), then

$$\lim 1_F = 1_{\lim F},$$

obtained by uniquely employing the universal property of limit. Now take three functors $F, G, H : \mathcal{I} \to \mathcal{C}$ and two natural transformations $F \overset{\alpha}{\Longrightarrow} G \overset{\beta}{\Longrightarrow} H$. To these functors are associated the respective limits

$$\left\{ \lim F \xrightarrow{\lambda_i} F(i) \middle| i \in |\mathcal{I}| \right\}$$
$$\left\{ \lim G \xrightarrow{\mu_i} G(i) \middle| i \in |\mathcal{I}| \right\}$$
$$\left\{ \lim H \xrightarrow{\eta_i} H(i) \middle| i \in |\mathcal{I}| \right\}$$

so that we have commuting squares glued together:

$$\begin{tikzcd}
F(i) \arrow[r, "\alpha_i"] & G(i) \arrow[r, "\beta_i"] & H(i) \\
\lim F \arrow[u, "\lambda_i"] \arrow[r, "\lim \alpha"'] & \lim G \arrow[u, "\mu_i"] \arrow[r, "\lim \beta"'] & \lim H \arrow[u, "\eta_i"']
\end{tikzcd}$$

We have for every $i \in |\mathcal{I}|$

$$\eta_i \lim \beta \lim \alpha = \beta_i \mu_i \lim \alpha = \beta_i \alpha_i \lambda_i;$$

then, by how it is defined the limit of a natural transformation, it must be

$$\lim(\beta\alpha) = \lim \beta \lim \alpha.$$

The following proposition pushes all this discourse to a conclusion.

Proposition 3.1. There is an adjunction

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & {[\mathcal{I}, \mathcal{D}]} \arrow[l, bend left, "\lim"]
\end{tikzcd}$$

Proof. [Yet to be TEX-ed...] ☐

4 Triangle Identities

[Yet to be TEXed...]

5 Other exercises

Remark 5.1 (About the next exercise)**.** Let $\mathcal{C}$ be a category with binary products and consider one object c of $\mathcal{C}$. For a object in $\mathcal{C}$ choose a product of a and c

$$\begin{tikzcd}
& a \times c \arrow[dl, "p_a"'] \arrow[dr, "q_a"] & \\
a & & c
\end{tikzcd}$$

in $\mathcal{C}$. In that context, there is a sensible way to define the functor

$$(\times c) : \mathcal{C} \to \mathcal{C}.$$

(If you have not done the exercise of the chapter of limits that about this construction, do it now.)

Exercise 5.2 (Exponential objects)**.** In a category $\mathcal{C}$ that has binary products, the *exponential object* of two objects a and b of $\mathcal{C}$ is any

- object of $\mathcal{C}$, that we choose label as b^a
- a morphism $\mathrm{ev} : b^a \times a \to b$, that we call *evaluation map*

such that ev is a terminal object of $(\times a) \downarrow b$. A category $\mathcal{C}$ is said to 'have exponentials' whenever for every $a, b \in |\mathcal{C}|$ there is in $\mathcal{C}$ the corresponding exponential object.

1. Find an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "(\times c)"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "\square^c"] \end{tikzcd}$$

   This boils down to introducing an appropriate functor $\square^c$ using uniquely the definition of exponential object.
2. Assume $\mathcal{C}$ has also an initial object 0 and terminal object 1. Prove the following statements.
   (i) $a \times 0 \cong 0 \times a \cong a$ for every object a of $\mathcal{C}$.
   (ii) For every $a \in |\mathcal{C}|$, if there is some morphism $a \to 0$, then $a \cong 0$.
   (iii) Any morphism $0 \to a$ is monic for $a \in |\mathcal{C}|$.
   (iv) If $0 \cong 1$, then all the objects of $\mathcal{C}$ are isomorphic.
   (v) $a^1 \cong a$, $a^0 \cong 0$ and $1^a \cong 1$ for every $a \in |\mathcal{C}|$.