

Notes on Category Theory: Topoi

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Last revision: 30th November 2023.

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1 Subobject classifiers

Throughout the current section, we assume \mathcal{E} is a category with initial object 1. That being the setting, we can give the following definition.

Definition 1.1. A *subobject classifier* for \mathcal{E} is any morphism $t : 1 \rightarrow \Omega$ such that: for every monomorphism $f : a \rightarrow b$ of \mathcal{E} there is one and only one morphism $\chi_f : b \rightarrow \Omega$ in \mathcal{E} for which there is a pullback square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow ! & & \downarrow \chi_f \\ 1 & \xrightarrow{t} & \Omega \end{array} \quad (1.1)$$

That is we can assign to every monomorphism $f : a \rightarrow b$ the morphism $\chi_f : b \rightarrow \Omega$ satisfying the property of the definition. Let us introduce then some symbolism: for $b \in |\mathcal{E}|$ we write $\text{Sub}_{\mathcal{E}} b$ for the class of all the monomorphisms of \mathcal{E} with codomain b . Hence we can introduce the function

$$\chi : \text{Sub}_{\mathcal{E}} b \rightarrow \mathcal{E}(b, \Omega)$$

with χ_f defined to be that morphism $b \rightarrow \Omega$ for which there is a pullback square as the diagram (1.1).

It is worth to observe $\text{Sub}_{\mathcal{E}}(b)$ has a natural structure of preorder: for

$$\begin{array}{ccc} a_1 & & a_2 \\ & \searrow f_1 & \swarrow f_2 \\ & b & \end{array}$$

monomorphisms of \mathcal{E} , write $f_1 \leq f_2$ to say there is some $h : a_1 \rightarrow a_2$ in \mathcal{E} for which

$$\begin{array}{ccc} a_1 & \xrightarrow{h} & a_2 \\ & \searrow f_1 & \swarrow f_2 \\ & b & \end{array}$$

commutes. Note that, being here f_1 and f_2 monomorphisms, there is at most one h as such and it is a monomorphism as well.

We show now the relation \simeq on $\text{Sub}_{\mathcal{E}} b$ defined by

$$f_1 \simeq f_2 \text{ if and only if } f_1 \leq f_2 \text{ and } f_2 \leq f_1$$

for $f_1, f_2 \in \text{Sub}_{\mathcal{E}} b$ is an equivalence relation. [...]

Yes, $\text{Sub}_{\mathcal{E}} b$ is the full subcategory of $\mathcal{E} \downarrow b$ whose objects are all the monomorphisms of \mathcal{E} with codomain b , and whose isomorphism relation is \simeq .

Proposition 1.2. Let

$$\begin{array}{ccc} a_1 & & a_2 \\ & \searrow f_1 & \swarrow f_2 \\ & b & \end{array}$$

be monomorphisms. $\chi_{f_1} = \chi_{f_2}$ if and only if $f_1 \simeq f_2$.

Proof. Assume $\chi_{f_1} = \chi_{f_2}$. By definition of subobject classifiers, χ_{f_1} is the morphism for which

$$\begin{array}{ccc} a_1 & \xrightarrow{f_1} & b \\ \downarrow ! & & \downarrow \chi_{f_1} \\ 1 & \xrightarrow{t} & \Omega \end{array} \quad \begin{array}{ccc} a_2 & \xrightarrow{f_2} & b \\ \downarrow ! & & \downarrow \chi_{f_1} \\ 1 & \xrightarrow{t} & \Omega \end{array}$$

are pullback squares. Consequently, we must infer that there is one isomorphism $h : a_1 \rightarrow a_2$ such that $f_1 = f_2 h$. Hence $f_1 \leq f_2$, and $f_2 \leq f_1$ too, because $f_1 h^{-1} = f_2$.

For the remaining part of the proof, let us write $!_1$ the unique morphism $a_1 \rightarrow 1$ and $!_2$ the unique morphism $a_2 \rightarrow 1$. Also remember that triangles

$$\begin{array}{ccc} a_1 & \xrightarrow{\quad} & a_2 \\ & \searrow !_1 & \swarrow !_2 \\ & 1 & \end{array} \quad \begin{array}{ccc} a_1 & \xleftarrow{\quad} & a_2 \\ & \searrow !_1 & \swarrow !_2 \\ & 1 & \end{array}$$

always commute.

Now we suppose $f_1 \simeq f_2$. The plan for the proof is: if we show that

$$\begin{array}{ccc} a_1 & \xrightarrow{f_1} & b \\ \downarrow !_1 & & \downarrow \chi_{f_2} \\ 1 & \xrightarrow{t} & \Omega \end{array}$$

is a pullback square, then, being $\chi_{f_1} : b \rightarrow \Omega$ the one for which there is a pullback square like this, we can conclude $\chi_{f_1} = \chi_{f_2}$. First of all such square commutes: if we call h the morphism $a_1 \rightarrow a_2$ such that $f_1 = f_2 h$, then

$$\chi_{f_2} f_1 = \chi_{f_2} f_2 h = t !_2 h = t !_1.$$

Consider

$$\begin{array}{ccc} \bullet & \xrightarrow{u} & b \\ & \searrow v & \downarrow \chi_{f_2} \\ & & 1 \end{array} \quad \begin{array}{ccc} a_1 & \xrightarrow{f_1} & b \\ \downarrow !_1 & & \downarrow \chi_{f_2} \\ 1 & \xrightarrow{t} & \Omega \end{array}$$

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where $\chi_{f_2} u = tv$. Being

$$\begin{array}{ccc} a_2 & \xrightarrow{f_2} & b \\ !_2 \downarrow & & \downarrow \chi_{f_2} \\ 1 & \xrightarrow{t} & \Omega \end{array}$$

a pullback square we have one $z : \bullet \rightarrow a_2$ for which $f_2 z = u$ and $!_2 z = v$. From the assumption $f_1 \simeq f_2$, we have $f_2 \leq f_1$, that is $f_2 = f_1 q$ for some $q : a_2 \rightarrow a_1$. Then $u = f_1 q z$ and $v = !_1 q z$. Let us see if $q z$ is what we are looking for.

$$\begin{array}{ccc} \bullet & \xrightarrow{u} & b \\ & \searrow qz & \downarrow \chi_{f_2} \\ & \searrow r & a_1 \xrightarrow{f_1} b \\ & \searrow v & !_1 \downarrow \\ & & 1 \xrightarrow{t} \Omega \end{array}$$

where we suppose $!_1 r = v$ and $f_1 r = u$. Being f_1 a monomorphism, the sole second identity is enough to conclude $r = qz$. \square