Notes on Category Theory — Pieces

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Contents

1 Limits and Colimits

To read this chapter you need to know what categories and a functors are, that is the first two chapters. About natural transformations, only the definition of natural transformation is required to formalize the definition of (co)limit.

1.1 Definition

[Expand this section with details about duality and TEX some initial exercises.]

Definition 1.1 (Limits & colimits)**.** Let $\mathcal{I}$ and $\mathcal{C}$ be two categories. For every object v of $\mathcal{C}$ we have the *constant functor*

$$k_v : \mathcal{I} \to \mathcal{C}$$

where $k_v(i) := v$ for every object i and $k_v(f) := 1_v$ for every morphism f. [Later, in the chapter of the adjunctions, we will introduce one functor $\Delta : \mathcal{C} \to [\mathcal{I}, \mathcal{C}]$.] A *limit* of a functor $F : \mathcal{I} \to \mathcal{C}$ is some object v of $\mathcal{C}$ with a natural transformation $\lambda : k_v \Rightarrow F$ such that: for any object a of $\mathcal{C}$ and $\mu : k_a \Rightarrow F$ there is one and only one $f : a \to v$ of $\mathcal{C}$ such that

$$\begin{tikzcd} a \arrow[dr, "\mu_i"] \arrow[dd, "f"'] & \\ & F(i) \\ v \arrow[ur, "\lambda_i"'] & \end{tikzcd}$$

commutes for every i in $\mathcal{I}$. A *colimit*, instead, is an object u of $\mathcal{C}$ together with a $\chi : F \Rightarrow k_u$ that has the property: for every object b of $\mathcal{C}$ and $\xi : F \Rightarrow k_b$ there exists one and only one $g : u \to b$ of $\mathcal{C}$ that makes

$$\begin{tikzcd} u \arrow[d, "g"'] & \\ & F(i) \arrow[ul, "\chi_i"'] \arrow[dl, "\xi_i"] \\ b & \end{tikzcd}$$

commute for every i in $\mathcal{I}$.

Example 1.2. We have already seen how a preordered set is a category; in this example let us employ $\mathbb{N}$ with the usual ordering $\leq$. First of all, let us figure out what cones and cocones of functors $H : \mathbb{N} \to \mathcal{C}$ are. Such functors, in other words, are sequence of objects and morphisms of $\mathcal{C}$ so arranged

$$H_0 \xrightarrow{\partial_0} H_1 \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_{n-1}} H_n \xrightarrow{\partial_n} H_{n+1} \xrightarrow{\partial_{n+1}} \cdots$$

In this case, a cone on H is a collection $\{\alpha_i : A \to H_i \mid i \in \mathbb{N}\}$ such that $\alpha_j = \partial_{j-1} \cdots \partial_i \alpha_i$ for every $i, j \in \mathbb{N}$ such that $i < j$; remember that if $j > i$ there is no morphism $H_j \to H_i$ and, for $i \in |\mathcal{I}|$, the morphism $H_i \to H_i$ is the identity. [Continue after you have fixed some parts before.]

Example 1.3. [Rewrite.] Let $\mathcal{C}$ be a category and $\mathbf{1}$ a category that has one object and one morphism, and take a functor $f : \mathbf{1} \to \mathcal{C}$, some $v \in \mathcal{C}$ and the corresponding constant functor $k_v : \mathbf{1} \to \mathcal{C}$. A natural transformation $\zeta : k_v \Rightarrow f$ amounts of a single morphism $v \to \widetilde{f}$ of $\mathcal{C}$, where $\widetilde{f}$ indicates the image of the unique object of $\mathbf{1}$ via f. Thus, a limit of f is some $v \in |\mathcal{C}|$ and a morphism $\lambda : v \to \widetilde{f}$ of $\mathcal{C}$ such that: for every object u and morphism $\mu : u \to \widetilde{f}$ in $\mathcal{C}$, there is a unique morphism $u \to v$ of $\mathcal{C}$ that makes

$$\begin{tikzcd} u \arrow[d] \arrow[dr, "\mu"] & \\ v \arrow[r, "\lambda"'] & \widetilde{f} \end{tikzcd}$$

commute.

Exercise 1.4. What are colimts of functors $\mathbf{1} \to \mathcal{C}$?

Example 1.5. [Rewrite.] Consider a monoid (viz a single object category) $\mathcal{G}$: for the scope of this example we write G for the set of the morphisms of $\mathcal{G}$. Let $F : \mathcal{G} \to \mathbf{Set}$ be a functor, and let $\widehat{F}$ indicate the F-image of the unique object of $\mathcal{G}$ whilst, for $f \in G$, $\widehat{f}$ the function $F(f) : \widehat{F} \to \widehat{F}$. Now, being $k_X : \mathcal{G} \to \mathbf{Set}$ the functor constant at X, with X a set, a natural transformation $\lambda : F \Rightarrow k_X$ is a morphism $\lambda : \widehat{F} \to X$ such that $\lambda = \lambda \widehat{f}$ for every $f \in G$. These two things, the set X and the function λ, together are a colimit of F whenever

> for every set Y and function $\mu : \widehat{F} \to Y$ such that $\mu = \mu \widehat{f}$ for every $f \in G$ there exists one and only one function $h : X \to Y$ such that $\mu = h\lambda$.

[Is that thing even interesting?] [Write about functors $\mathcal{G} \to$ **Set**...]

[Write about duality here. Explain how limits and colimits are dual...]

The following is very basic property: limits of a same functor are are essentially the same.

Proposition 1.6. Let $F : \mathcal{I} \to \mathcal{C}$ be a functor. If $\{\eta_i : a \to F(i) \mid i \in |\mathcal{I}|\}$ and $\{\theta_i : b \to F(i) \mid i \in |\mathcal{I}|\}$ are limits of F, then $a \cong b$.

Proof. By definition of limit, we a have a unique $f : a \to b$ and a unique $g : b \to a$ making the triangles in

$$\begin{tikzcd} a \arrow[dr, "\eta_i"] \arrow[dd, bend left, "f"] & \\ & F(i) \\ b \arrow[ur, "\theta_i"'] \arrow[uu, bend left, "g"] & \end{tikzcd}$$

commute for every object i of $\mathcal{I}$. In this case,

$$\eta_i = \theta_i f = \eta_i(gf)$$
$$\theta_i = \eta_i g = \theta_i(fg)$$

Invoking again the universal property of limits, $gf = 1_a$ and $fg = 1_b$. □

Fortunately, there are few shapes that are both ubiquitous and simple. This section is dedicated to them, while in the successive one we will prove (Proposition 1.63) that if some simple functors have limits, then all the functors do have limits.

1.2 Terminal and initial objects

Definition 1.7 (Terminal & initial objects)**.** For $\mathcal{C}$ category, the limits of the empty functor $\varnothing \to \mathcal{C}$ are called *terminal objects* of $\mathcal{C}$, whereas the colimits *initial objects*.

Let us expand the definition above so that we can can look inside. A cone over the empty functor $\varnothing \to \mathcal{C}$ with vertex a is a natural transformation

$$\begin{tikzcd} \varnothing \arrow[r, bend left, "k_a", ""{name=U, below}] \arrow[r, bend right, ""{name=D}] & \mathcal{C} \arrow[Rightarrow, from=U, to=D] \end{tikzcd} .$$

Here, the empty functor is k_a because there is at most one functor $\varnothing \to \mathcal{C}$. Again, because there must be a unique one, our natural transformation is the empty transformation, viz the one devoid of morphisms. A similar reasoning leads us to the following explicit definition of terminal and initial object.

Definition 1.8 (Terminal and initial objects, explicit)**.** Let $\mathcal{C}$ be a category.

- A *terminal object* of $\mathcal{C}$ is an object 1 of $\mathcal{C}$ such that for every object x of $\mathcal{C}$ there exists one and only one $x \to 1$ in $\mathcal{C}$.
- An *initial object* of $\mathcal{C}$ is an object 0 in $\mathcal{C}$ such that for every object x in $\mathcal{C}$ there exists one and only one morphism $0 \to x$ in $\mathcal{C}$.

Example 1.9 (Empty set and singletons)**.** It may sound weird, but for every set X there does exist a function $\varnothing \to X$; moreover, it is the unique one. To get this, think set-theoretically: a function is any subset of $\varnothing \times X$ that has the property we know. But $\varnothing \times X = \varnothing$, so its unique subset is $\varnothing$. This set is a function from $\varnothing$ to X since the statement

for every $a \in \varnothing$ there is one and only one $b \in X$ such that $(a, b) \in \varnothing$

is a *vacuous truth*. So $\varnothing$ is an initial object of **Set**. This case is quite particular, since the initial objects of **Set** are actually equal to $\varnothing$.
Now let us look for terminal objects in **Set**. Take an arbitrary set X: there is exactly one function from X to any singleton, that is singletons are terminal object of **Set**. Conversely, by Proposition 1.6, the terminal objects of **Set** must be singletons.

Exercise 1.10. Trivial groups — there is a unique way a singleton can be a group — are either terminal and initial objects of **Grp**.

Construction 1.11. Let us introduce a nice category that allows us to express some nice and simple facts in Mathematics. Let $\mathcal{C}$ and $\mathcal{J}$ two categories, a one of its objects and take a functor $F : \mathcal{J} \to \mathcal{C}$. We have the category $(a \downarrow F)$, this made:

- the objects are the morphisms $a \to F(x)$ of $\mathcal{C}$, with x being an object of $\mathcal{J}$;
- the morphisms from $f : a \to F(x)$ to $g : a \to F(y)$ are the morphisms $h : x \to y$ of $\mathcal{J}$ such that

$$\begin{tikzcd} & F(x) \arrow[d, "F(h)"] \\ a \arrow[ur, "f"] \arrow[dr, "g"'] & F(y) \end{tikzcd}$$

  commutes;
- the composition is that of $\mathcal{J}$.

Example 1.12. [Rewrite this example in $\mathbf{Mod}_R$, for R ring? Yes...] In Linear Algebra we have a nice theorem:

> Let V be a vector space over a field k and $S \subseteq V$ a base. For every vector space W over k and function $\phi : S \to W$ there exists a unique linear function $f : V \to W$ such that
>
> $$\begin{tikzcd} S \arrow[r, hook, "i"] \arrow[dr, "\phi"'] & V \arrow[d, "f"] \\ & W \end{tikzcd}$$
>
> commutes.

In other words, this statement says that a linear function is completely determined by what it does with the vectors of S. We will consider now two functors

$$\mathbf{Set} \xrightarrow{\langle\cdot\rangle} \mathbf{Vect}_k \xrightarrow{U} \mathbf{Set}.$$

The first one takes a set S and produces the vector space on k

$$\langle S \rangle := \left\{ \sum_{x \in S} \lambda_x x \,\middle|\, \lambda : S \to k,\ \lambda_x \neq 0 \text{ for finitely many times} \right\}$$

(considered with two obvious operations). Furthermore, a function of sets $f : S \to T$ induces a linear function $\langle f \rangle : \langle S \rangle \to \langle T \rangle$ defined by

$$\langle f \rangle \left(\sum_{x \in S} \lambda_x x \right) := \sum_{x \in S} \lambda_x f(x)$$

where $\lambda : S \to k$ is almost always null. The functoriality of $\langle \cdot \rangle$ is just a matter of quick controls. [Do we really need all that machinery?] The functor U instead takes vector spaces and returns the correspondent set of vectors; we write $U(V) := V$, but observe that in **Set** we don't care anymore of the vector structure of V. Similarly, it takes linear functions and the return them: but, since U lands onto **Set**, who cares about linearity there? (We may say that U is the 'inclusion' of $\mathbf{Vect}_k$ into **Set**.) [Talk about forgetful functors elsewhere...] All this words allow us restate the aforementioned theorem as:

> For if S is a set, the inclusion $S \hookrightarrow \langle S \rangle$ is an initial object of $(S \downarrow U)$.

Exercise 1.13. In the previous example some details are omitted: you can be more talkative, though. However, it is really worth to think about such examples — not only because we will meet such pattern later under the vest of adjunctions. You may also look for another examples of similar kind, I'm sure you will find some.

Example 1.14 (Isomorphism Theorem for Set Theory)**.** We have defined **Eqv** earlier, recall it here. We have the functor

$$j : \mathbf{Set} \to \mathbf{Eqv}$$

that maps sets X to setoids X together with the equality relation, and functions $f : X \to Y$ to themselves. To get the mood for this example, sets *are* setoids where the equivalence relation is equality and functions *are* functoids between such setoids. In this case, the classical theorem

> Let X and Y be two sets and $\sim$ an equivalence relation on X. For every function $f : X \to Y$ such that $f(a) = f(b)$ for every $a, b \in X$ with $a \sim b$, there exists one and only one function $\overline{f} : X/\sim \to Y$ such that
>
> $$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & X/\sim \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$
>
> commutes, where $p : X \to X/\sim$ is the canonical projection.

can be restated as follows

> the canonical projection $p : (X, \sim) \to X/\sim$ is initial in $(X, \sim) \downarrow j$.

Example 1.15 (Recursion)**.** In Set Theory, there is a nice theorem, the *Recursion Theorem*:

> Let $(\mathbb{N}, 0, s)$ be a Peano Model, where $0 \in \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$ is its successor function. For every pointed set X, $a \in X$ and $f : X \to X$ there exists one and only one function $x : \mathbb{N} \to X$ such that $x_0 = a$ and $x_{s(n)} = f(x_n)$ for every $n \in \mathbb{N}$.

Here, by Peano Model we mean a set $\mathbb{N}$ that has one element, we write 0, stood out and a function $s : \mathbb{N} \to \mathbb{N}$ such that, all this complying some rules:

1. s is injective;
2. $s(x) \neq 0$ for every $x \in \mathbb{N}$;
3. for if $A \subseteq \mathbb{N}$ has 0 and $s(n) \in A$ for every $n \in A$, then $A = \mathbb{N}$.

We show now how we can involve Category Theory in this case. First of all, we need a category where to work.
The statement is about things made as follows:

> a set X, one distinguished $x \in X$ and one function $f : X \to X$.

[Is there a name for these things?] We may refer to such new things by barely a triple (X, a, f), but we prefer something like this:

$$1 \xrightarrow{x} X \xrightarrow{f} X,$$

where 1 is any singleton, as usual. Peano Models are such things, with some additional properties. It is told about the existence and the uniqueness of a certain function. We do not want mere functions, of course: given

$$1 \xrightarrow{x} X \xrightarrow{f} X \text{ and } 1 \xrightarrow{y} Y \xrightarrow{g} Y,$$

we take the functions $r : X \to Y$ such that

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, "r"] & X \arrow[d, "r"] \\
1 \arrow[ur, "x"] \arrow[r, "y"'] & Y \arrow[r, "g"'] & Y
\end{tikzcd}$$

commutes and nothing else. [Is there a name for such functions?] These ones are the things we want to be morphisms. Suppose given

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, "p"] & X \arrow[d, "p"] \\
1 \arrow[ur, "x"] \arrow[r, "y"] \arrow[dr, "z"'] & Y \arrow[r, "g"] \arrow[d, "q"] & Y \arrow[d, "q"] \\
 & Z \arrow[r, "h"'] & Z
\end{tikzcd}$$

where all the squares and triangles commute: thus we obtain the commuting

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, "qp"] & X \arrow[d, "qp"] \\
1 \arrow[ur, "x"] \arrow[r, "z"'] & Z \arrow[r, "h"'] & Z
\end{tikzcd}$$

This means that composing two morphisms as functions in **Set** produces a morphism. This is how we want composition to defined in this context. This choice makes the categorial axioms automatically respected. We call this category **Peano**. [Unless there is a better naming, of course.]
Being the environment set now, the Recursion Theorem becomes more concise:

> Peano Models are initial objects of **Peano**.

By Proposition 1.6, any other initial object of **Peano** are isomorphic to some Peano Model: does this mean its initial objects are Peano Models? (Exercise.)

Exercise 1.16 (Induction $\Leftrightarrow$ Recursion)**.** In **Set**, suppose you have $1 \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$, where s is injective and $s(n) \neq 0$ for every $n \in \mathbb{N}$. Demonstrate that the following statements are equivalent:

1. for if $A \subseteq \mathbb{N}$ has 0 and $s(n) \in A$ for every $n \in A$, then $A = \mathbb{N}$;
2. $1 \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$ is an initial object of **Peano**.

(1) ⇒ (2) proves the Recursion Theorem, whereas (2) ⇒ (1) requires you to codify a proof by induction into a recursion. Try it, it could be nice. [Prepare hints...]

Limits (colimits) are terminal (initial) objects of appropriate categories. Definition 1.8 does not make reference to limits and colimits as stated in Definition 1.1: thus, you can say what terminal and initial objects are and then tell what limits and colimits are in terms of terminal and initial objects.

Construction 1.17 (Category of cones)**.** For $\mathcal{C}$ category, let $F : \mathcal{I} \to \mathcal{C}$ be a functor. Then we define the *category of cones* over F as follows.

- The objects are the cones over F.
- For $\alpha := \left\{a \xrightarrow{\alpha_i} F(i)\right\}_{i \in |\mathcal{I}|}$ and $\beta := \left\{b \xrightarrow{\beta_i} F(i)\right\}_{i \in |\mathcal{I}|}$ two cones, the morphisms from α to β are the morphisms $f : a \to b$ of $\mathcal{C}$ such that

$$\begin{tikzcd} a \arrow[dr, "\alpha_i"] \arrow[dd, "f"'] & \\ & F(i) \\ b \arrow[ur, "\beta_i"'] & \end{tikzcd}$$

  commutes for every $i \in |\mathcal{I}|$.
- The composition of morphisms here is the same as that of $\mathcal{C}$.

We write such category as $\mathbf{Cn}_F$. We define also the *category of cocones* over F, written as $\mathbf{CoCn}_F$.

- The objects are the cocones over F.
- For $\alpha := \left\{F(i) \xrightarrow{\alpha_i} a\right\}_{i \in |\mathcal{I}|}$ and $\beta := \left\{F(i) \xrightarrow{\beta_i} b\right\}_{i \in |\mathcal{I}|}$ cocones, the morphisms from α to β are the morphisms $f : a \to b$ of $\mathcal{C}$ such that

$$\begin{tikzcd} & a \arrow[dl, "\alpha_i"'] \arrow[dd, "f"] \\ F(i) & \\ & b \arrow[ul, "\beta_i"] \end{tikzcd}$$

  commutes for every $i \in |\mathcal{I}|$.
- The composition of morphisms here is the same as that of $\mathcal{C}$.

It is quite immediate in either of the cases to show that categorial axioms are verified.

Proposition 1.18. For $\mathcal{C}$ category and $F : \mathcal{I} \to \mathcal{C}$ functor,

- limits of F are terminal objects of $\mathbf{Cn}_F$ and viceversa.
- colimits of F are initial objects of $\mathbf{CoCn}_F$ and viceversa.

Proof. This is **exercise 1.19**. □

1.3 Products and coproducts

Let $\mathcal{C}$ be a category and I a discrete category (that is a class). We have seen how functors $x : I \to \mathcal{C}$ are exactly families $\{x_i \mid i \in I\}$ of objects of $\mathcal{C}$. We call *(co)products* of $\{x_i \mid i \in I\}$ the (co)limits of $\{x_i \mid i \in I\}$. Let us put this definition into more explicit terms.

First of all, let us make clear what cones over a collection $\{x_i \mid i \in I\}$ are. For $p \in |\mathcal{C}|$ and $k_p : I \to \mathcal{C}$ the functor constant at p, a natural transformation

$$\begin{tikzcd} I \arrow[r, bend left, "k_p"{name=U}] \arrow[r, bend right, "x"'{name=D}] & \mathcal{C} \arrow[from=U, to=D, Rightarrow, shorten=2pt] \end{tikzcd}$$

is just a family $\{p \to x_i \mid i \in I\}$ of morphisms of $\mathcal{C}$. In this fortunate case, the naturality condition automatically holds because I has no morphisms other than identities. Similarly, one can easily make explicit what cocones are.

Definition 1.20 (Products & coproducts)**.** Let $\mathcal{C}$ be a category. A *product* of a family $\{x_i \mid i \in I\}$ of objects in $\mathcal{C}$ is any family $\{\mathrm{pr}_i : p \to x_i \mid i \in I\}$ of morphisms of $\mathcal{C}$, usually called *projections*, respecting the following property: for every family $\{f_i : a \to x_i \mid i \in I\}$ of morphisms of $\mathcal{C}$ there exists one and only one $h : a \to p$ of $\mathcal{C}$ such that

$$\begin{tikzcd} a \arrow[d, "h"'] \arrow[dr, "f_i"] & \\ p \arrow[r, "\mathrm{pr}_i"'] & x_i \end{tikzcd}$$

commutes for every $i \in I$. A *coproduct* of $\{x_i \mid i \in I\}$ of objects of $\mathcal{C}$ is any family $\{\mathrm{in}_i : x_i \to q \mid i \in I\}$ of morphisms of $\mathcal{C}$, often referred to as *injections*, having the property: for every family $\{g_i : x_i \to b \mid i \in I\}$ of morphisms of $\mathcal{C}$ there exists one and only one $k : q \to b$ of $\mathcal{C}$ such that

$$\begin{tikzcd} b & \\ q \arrow[u, "k"] & x_i \arrow[ul, "g_i"'] \arrow[l, "\mathrm{in}_i"] \end{tikzcd}$$

commutes for every $i \in I$.

Example 1.21 (Infima and suprema in prosets)**.** Consider a proset $(\mathbb{P}, \leq)$ and a subset S of $\mathbb{P}$. In this instance, a product of S is some $p \in \mathbb{P}$ such that:

1. $p \leq x$ for every $x \in S$;
2. for every $p' \in \mathbb{P}$ such that $p' \leq x$ for every $x \in S$ we have $p' \leq p$.

If we have quick look to some existing mathematics, cones over S are what are called *lower bounds* of S. An *infimum* of S is any of the greatest lower bounds for S.
On the other hand, a coproduct of S is some $q \in \mathbb{P}$ such that:

1. $x \leq q$ for every $x \in S$;
2. for every $q' \in \mathbb{P}$ such that $x \leq q'$ for every $x \in S$ we have $q \leq q'$.

In other words, the cones over S are precisely the *upper bounds* of S. A *supremum* of S is any of the lowest upper bounds for S.

There is a dedicated notations for such elements, if $(\mathbb{P}, \leq)$ is a poset: the infimum of S is written as $\inf S$, whereas $\sup S$ is the supremum of S. If the elements of S are indexed, that is $S = \{x_i \mid i \in I\}$, then it is customary to write $\inf_{i \in I} x_i$ and $\sup_{i \in I} x_i$.

Exercise 1.22. Prosets provide some examples in which some subsets does not have infima or suprema.

Now let us turn our attention to a pair of quite ubiquitous constructs.

Example 1.23 (Cartesian product)**.** Given a family of sets $\{X_\alpha \mid \alpha \in \Gamma\}$, we have the corresponding *Cartesian product*

$$\prod_{\alpha \in \Gamma} X_\alpha := \left\{ f : \Gamma \to \bigcup_{\alpha \in \Gamma} X_\alpha \,\middle|\, f(\lambda) \in X_\lambda \text{ for every } \lambda \in \Gamma \right\},$$

whose elements are the *choices* from $\{X_\alpha \mid \alpha \in \Gamma\}$. As the name indicates, a choice f for every $\lambda \in \Gamma$ indicates one element of X_λ. Our product comes with the *projections*, one for each $\mu \in \Gamma$,

$$\begin{aligned} \mathrm{pr}_\mu &: \prod_{\alpha \in \Gamma} X_\alpha \to X_\mu \\ \mathrm{pr}_\mu(f) &:= f(\mu). \end{aligned}$$

Now, any family of functions $\{f : A \to X_\alpha \mid \alpha\}$ ca be compressed into one function

$$\prod_{\alpha \in \Gamma} f_\alpha : A \to \prod_{\alpha \in I} X_\alpha.$$

by defining $\left(\prod_{\alpha \in \Gamma} f_\alpha\right)(a)$ to be the function $\Gamma \to \bigcup_{\alpha \in \Gamma} X_\alpha$ mapping $\mu \in \Gamma$ to $f_\mu(a)$. It is simple to show that

$$\begin{tikzcd} A \arrow[dr, "f_\mu"] \arrow[dd, "\prod_{\alpha\in\Gamma} f_\alpha"'] & \\ & X_\mu \\ \prod_{\alpha\in\Gamma} X_\alpha \arrow[ur, "\mathrm{pr}_\mu"'] & \end{tikzcd}$$

commutes for every $\mu \in \Gamma$. Moreover, $\prod_{\alpha \in \Gamma} f_\alpha$ is the only one that does this. Consider any function $g : A \to \prod_{\alpha \in \Gamma} X_\alpha$ with $f_\mu = \mathrm{pr}_\mu g$ for every $\mu \in \Gamma$: then for every $x \in A$ we have

$$\begin{aligned} \big(g(x)\big)(\mu) &= \mathrm{pr}_\mu\big(g(x)\big) = f_\mu(x) = \\ &= p_\mu\left(\left(\prod_{\alpha \in \Gamma} f_\alpha\right)(x)\right) = \left(\left(\prod_{\alpha \in \Gamma} f_\alpha\right)(x)\right)(\mu), \end{aligned}$$

that is $g = \prod_{\alpha \in \Gamma} f_\alpha$.

Exercise 1.24. It may be simple to reason about the Cartesian product of only two sets X_1 and X_2. In this case, the product is written as $X_1 \times X_2$ and its elements are represented as pairs $(a, b) \in X_1 \times X_2$ rather than functions $f : \{1, 2\} \to X_1 \cup X_2$ with $f(i) \in X_i$ for $i \in \{1, 2\}$. By setting things like this, the 'compression' of two functions $f_1 : A \to X_1$ and $f_2 : A \to X_2$ into a function $A \to X_1 \times X_2$ becomes more obvious.

Example 1.25 (Coproduct of sets)**.** For if $\{X_\alpha \mid \alpha \in \Lambda\}$ is a family of sets, we introduce the *disjoint union*

$$\sum_{\alpha\in\Lambda} X_\alpha := \bigcup_{\alpha\in\Lambda} X_\alpha \times \{\alpha\} = \{(x,\alpha) \mid \alpha \in \Lambda,\ x \in X_\alpha\}.$$

While the elements of every member of $\{X_\alpha \mid \alpha \in \Lambda\}$ are amalgamated in $\bigcup_{\alpha\in\Lambda} X_\alpha$, in the disjoint union $\sum_{\alpha\in\Lambda} X_\alpha$ the elements have attached a record of their provenience — in this case, the index of the set they come from. Because of this feature, the elements of $\sum_{\alpha\in\Lambda} X_\alpha$ are called *dependent pairs*. The disjoint union of $\{X_\alpha \mid \alpha \in \Lambda\}$ has one *injection* for each $\alpha \in \Lambda$:

$$\mathrm{in}_\mu : X_\mu \to \sum_{\alpha\in\Lambda} X_\alpha\,,\ \mathrm{in}_\mu(x) := (x,\mu).$$

Similarly to what we have done in the previous example, a family of functions $\{f_\alpha : X_\alpha \to A \mid \alpha \in \Lambda\}$ can be compressed into this one

$$\sum_{\alpha\in\Lambda} f_\alpha : \sum_{\alpha\in\Lambda} X_\alpha \to A$$
$$\left(\sum_{\alpha\in\Lambda} f_\alpha\right)(x,\mu) := f_\mu(x),$$

which, in other words, checks the provenience of an element and give it to an appropriate function f_α. This new function makes the diagram

$$\begin{tikzcd}
A & \\
& X_\mu \arrow[ul, "f_\mu"'] \arrow[dl, "\mathrm{in}_\mu"] \\
\sum_{\alpha\in\Lambda} X_\alpha \arrow[uu, "\sum_{\alpha\in\Lambda} f_\alpha"] &
\end{tikzcd}$$

commute for every $\mu \in \Lambda$, and it is the unique to do this.

Exercise 1.26. Prove $\bigcup_{\alpha\in\Lambda} X_\alpha$ is a coproduct if the X_α-s are pairwise disjoint. By the way, $\bigcup_{\alpha\in\Lambda} X_\alpha$ is isomorphic to an appropriate quotient of $\sum_{\alpha\in\Lambda} X_\alpha$. Part of the exercise is to find an equivalence relation $\sim$ on $\sum_{\alpha\in\Lambda} X_\alpha$ and a function

$$\sum_{\alpha\in\Lambda} X_\alpha \to \bigcup_{\alpha\in\Lambda} X_\alpha$$

which maps $\sim$-equivalent elements to the same element.

Exercise 1.27. Haskell natively offers the function

```
either :: (a -> c) -> (b -> c) -> Either a b -> c
```

How do `Either a b` and this function fit in the current topic? If you accept this little exercise, remember `Either a b` is defined to be either `Left a` or `Right b`. We haven't talked about the category of types, but it is not be that unseen.

Example 1.28 (Product of topological spaces)**.** Consider now a family of topological spaces $\{X_i \mid i \in I\}$ and let us see if we can have a product of topological space in the sense of the Definition above.
In order to talk about product topological space we shall determine a topology over the set $\prod_{i\in I} X_i$. From the Example 1.23, we have a nice machinery, but it is

all about sets and functions! We define the *product topology* — sometimes called 'Tychonoff topology' — as the smallest among the topologies for $\prod_{i \in I} X_i$ for which all the projections $\mathrm{pr}_j : \prod_{i \in I} X_i \to X_j$ of the Example 1.23 are continuous. The question is now: do these continuous functions form a product in **Top**? Taking a family of continuous functions $\{f_i : A \to X_i \mid i \in I\}$ and looking at the 'underground' **Set**, there does exist one function $\widehat{f} : A \to \prod_{i \in I} X_i$ such that $f_i = \mathrm{pr}_i \widehat{f}$ for every $i \in I$, but we do not know if it is continuous! To give an answer, let us consider the family

$$\mathcal{T} := \left\{ U \subseteq \prod_{i \in I} X_i \text{ open} \,\middle|\, \widehat{f}^{-1} U \text{ is open in } A \right\} :$$

the idea is that if we demonstrate $\mathcal{T}$ is a topology for $\prod_{i \in I} X_i$ and $\mathcal{T}$ makes all the pr_i's continuous, then we can conclude the continuity of $\widehat{f}$. The first part is immediate, so let us focus on the remaining part. If we take an open subset V of X_j, the open subset $\mathrm{pr}_j^{-1} V$ of the product is in $\mathcal{T}$, because $f_j^{-1} V = \widehat{f}^{-1} \left(\mathrm{pr}_j^{-1} V\right)$ is open in A.

Example 1.29 (Coproduct of topological spaces)**.** As in the previous example, we move from Example 1.25. In Topology, it is maybe more customary to use

$$\coprod_{i \in I} X_i \text{ instead of } \sum_{i \in I} X_i$$

when $\{X_i \mid i \in I\}$ is a family of topological spaces. However, if we do not give a topology to $\coprod_{i \in I} X_i$, this object remains a bare set. In analogy to what happened with the Cartesian product, we prescribe the open subsets of $\coprod_{i \in I} X_i$ by making reference to the injections $\mathrm{in}_j : X_j \to \coprod_{i \in I} X_i$:

> we define a subset A of $\coprod_{i \in I} X_i$ to be open if and only if $\mathrm{in}_j^{-1} A$ is an open subset of X_j for every $j \in I$.

Let us recycle the universal property enjoyed by the family of the injections, that is for every family $\{g_i : X_i \to A \mid i \in I\}$ of continuous functions there exists one function $\widetilde{g} : \coprod_{i \in I} X_i \to A$ such that $g_j = \widetilde{g}\, \mathrm{in}_j$ for every $j \in I$. Furthermore, $\widetilde{g}$ is continuous: if $U \subseteq A$ is open, then so are the subsets $g_j^{-1} U \subseteq X_j$; consequently $g_j^{-1} U = \mathrm{in}_j^{-1} \left(\widetilde{g}^{-1} U\right)$ for every $j \in I$, which implies $\widetilde{g}^{-1} U$ is open.

Sometimes, products and coproducts can be isomorphic, as in the following example.

Example 1.30 (Product and coproduct of modules)**.** [Yet to be TEX-ed...]

Let us talk about *finite* products, that is products of a finite set of objects. The following arguments will be useful when we will deal with finite completeness of categories. Keep an eye on Figure 1.

Proposition 1.31 (Finite products, reduction from left)**.** Let $\mathcal{C}$ be a category a finite set $\{x_1, \dots, x_n\}$, with $n \geq 2$, of objects of $\mathcal{C}$. Let $x_1 \xleftarrow{l_2} p_2 \xrightarrow{r_2} x_2$ be one of the products of $\{x_1, x_2\}$ and let $p_i \xleftarrow{l_{i+1}} p_{i+1} \xrightarrow{r_{n+1}} x_{i+1}$ be one of the products of $\{p_i, x_{i+1}\}$. Then the morphisms

$$\begin{aligned} l_2 \cdots l_n &: p_n \to x_1 \\ r_j l_{j+1} \cdots l_n &: p_n \to x_j \quad \text{for } j \in \{2, \dots, n-1\} \\ r_n &: p_n \to x_n \end{aligned}$$

of $\mathcal{C}$ do form a product of $\{x_1, \dots, x_n\}$.

reduction from left (Proposition 1.31)

$$\begin{tikzcd}
 & & & & & p_n \arrow[dl, "l_n"'] \arrow[dr, "r_n"] & \\
 & & & & p_{n-1} \arrow[dl, "l_{n-1}"'] \arrow[dr, "r_{n-1}"] & & x_n \\
 & & & \cdots \arrow[dl, "l_4"'] \arrow[dr, "\cdots"] & & x_{n-1} & \\
 & & p_3 \arrow[dl, "l_3"'] \arrow[dr, "r_3"] & & \cdots & & \\
 & p_2 \arrow[dl, "l_2"'] \arrow[dr, "r_2"] & & x_3 & & & \\
x_1 & & x_2 & & & &
\end{tikzcd}$$

reduction from right (Proposition 1.32)

$$\begin{tikzcd}
 & p_n \arrow[dl, "l_n"'] \arrow[dr, "r_n"] & & & & & \\
x_1 & & p_{n-1} \arrow[dl, "l_{n-1}"'] \arrow[dr, "r_{n-1}"] & & & & \\
 & x_2 & & \cdots \arrow[dl, "\cdots"'] \arrow[dr, "r_4"] & & & \\
 & & \cdots & & p_3 \arrow[dl, "l_3"'] \arrow[dr, "r_3"] & & \\
 & & & x_{n-2} & & p_2 \arrow[dl, "l_2"'] \arrow[dr, "r_2"] & \\
 & & & & x_{n-1} & & x_n
\end{tikzcd}$$

Figure 1. Finite products recursively constructed.

Proof. The proof is conducted by induction on $n \geq 2$. The case $n = 2$ is the base case of our recursive definition. To proceed with the inductive step, let us picture the situation:

$$\begin{tikzcd}
 & & & & p_{n+1} \arrow[dll, "l_{n+1}"'] \arrow[drr, "r_{n+1}"] & & \\
 & & p_n \arrow[dll, "l_2\cdots l_n"'] \arrow[d, "r_j l_{j+1}\cdots l_n"] \arrow[drr, "r_n"] & & & & x_{n+1} \\
x_1 & & x_j & & x_n & & \\
 & & & & & a \arrow[ulllll, "f_1"] \arrow[ulll, "f_j"] \arrow[ul, "f_n"] \arrow[uur, "f_{n+1}"'] \arrow[uulll, dotted, "h"] \arrow[uuul, dotted, "k"'] &
\end{tikzcd}$$

where $j \in \{2, \dots, n-1\}$, a is an arbitrary object with morphisms $f_1, \dots, f_n, f_{n+1}$. By the universal property of product, we have

$$\begin{aligned}
f_1 &= l_1 \cdots l_n h \\
f_j &= r_j l_{j+1} \cdots l_n h \\
f_n &= r_n h
\end{aligned}$$

for one and only one $h : a \to p_n$. Again by the universal property of product.

$$\begin{aligned}
h &= l_{n+1} k \\
f_{n+1} &= r_{n+1} k
\end{aligned}$$

for a unique $k : a \to p_{n+1}$. Thus

$$\begin{aligned}
f_1 &= l_1 \cdots l_n l_{n+1} k \\
f_j &= r_j l_{j+1} \cdots l_n l_{n+1} k \\
f_n &= r_n l_{n+1} k \\
f_{n+1} &= r_{n+1} k
\end{aligned}$$

and we have concluded. $\square$

Proposition 1.32 (Finite products, reduction from right)**.** Let $\mathcal{C}$ be a category a finite set $\{x_1, \dots, x_n\}$, with $n \geq 2$, of objects of $\mathcal{C}$. Let $x_{n-1} \xleftarrow{l_2} p_2 \xrightarrow{r_2} x_n$ be one of the products of $\{x_{n-1}, x_n\}$ and let $x_{n-i} \xleftarrow{l_{i+1}} p_{i+1} \xrightarrow{r_{n+1}} p_i$ be one of the products of $\{x_{n-i}, p_i\}$. Then the morphisms

$$\begin{aligned}
r_2 \cdots r_n &: p_n \to x_n \\
l_j r_{j+1} \cdots r_n &: p_n \to x_{n-j+1} \quad \text{for } j \in \{2, \dots, n-1\} \\
r_n &: p_n \to x_1
\end{aligned}$$

of $\mathcal{C}$ do form a product of $\{x_1, \dots, x_n\}$.

Proof. This is **exercise 1.33**. You should expect some work like in the proof of Proposition 1.31. $\square$

Corollary 1.34 (Associativity of product)**.** In a category $\mathcal{C}$, let

$$x_1 \xleftarrow{p_1} x_1 \times x_2 \xrightarrow{p_2} x_2$$

a product of $\{x_1, x_2\}$,

$$x_1 \times x_2 \xleftarrow{p_{1,2}} (x_1 \times x_2) \times x_3 \xrightarrow{p_3} x_3$$

a product of $\{x_1 \times x_2, x_3\}$,

$$x_2 \xleftarrow{q_2} x_2 \times x_3 \xrightarrow{q_3} x_3$$

a product of $\{x_2, x_3\}$,

$$x_1 \xleftarrow{q_1} x_1 \times (x_2 \times x_3) \xrightarrow{q_{2,3}} x_2 \times x_3$$

a product of $\{x_1, x_2 \times x_3\}$.

$$\begin{tikzcd}
 & & (x_1 \times x_2) \times x_3 \arrow[dl, "p_{1,2}"'] \arrow[ddrr, "p_3"] & & \\
 & x_1 \times x_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & & \\
x_1 & & x_2 & & x_3 \\
 & & & x_2 \times x_3 \arrow[ul, "q_2"] \arrow[ur, "q_3"'] & \\
 & & x_1 \times (x_2 \times x_3) \arrow[uull, "q_1"] \arrow[ur, "q_{2,3}"'] & &
\end{tikzcd}$$

Then

$$(x_1 \times x_2) \times x_3 \cong x_1 \times (x_2 \times x_3).$$

Proof. It follows from Proposition 1.31 and Proposition 1.32. □

Corollary 1.35. A category has all finite products if and only if has a terminal object and all binary products.

Proof. One implication is easy. For the opposite one: terminal objects are empty products; an object with identity is a product of itself; if you are given at least two objects, either of Proposition 1.31 and Proposition 1.32 tell you finite product are consecutive binary products. □

Exercise 1.36. In a category with terminal object 1, we have $a \times 1 \cong 1 \times a \cong a$.

1.4 Pullbacks and pushouts

Let $\mathcal{C}$ be a category. The limits of the functors

$$\left(\begin{tikzcd} \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd}\right) \to \mathcal{C}$$

are called *pullbacks.* Dually, the colimits of the functors

$$\left(\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \end{tikzcd}\right) \to \mathcal{C}$$

are said *pushouts.* More explicitly:

Definition 1.37 (Pullbacks & pushouts, explicit)**.** Let $\mathcal{C}$ a category and a pair of morphisms with the same codomain

$$\begin{tikzcd} a \arrow[dr, "f"'] & & b \arrow[dl, "g"] \\ & c & \end{tikzcd} \tag{1.1}$$

in $\mathcal{C}$. A *pullback* of (1.1) is any pair of morphisms with a common domain

$$\begin{tikzcd} & p \arrow[dl, "h"'] \arrow[dr, "k"] & \\ a & & b \end{tikzcd}$$

in $\mathcal{C}$ such that:

- the square

  $$\begin{tikzcd} & p \arrow[dl, "h"'] \arrow[dr, "k"] & \\ a \arrow[dr, "f"'] & & b \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

  commutes
- for every

  $$\begin{tikzcd} & p' \arrow[dl, "h'"'] \arrow[dr, "k'"] & \\ a & & b \end{tikzcd}$$

  in $\mathcal{C}$ making

  $$\begin{tikzcd} & p \arrow[dl, "h'"'] \arrow[dr, "k'"] & \\ a \arrow[dr, "f"'] & & b \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

  commute there exists one and only one $r : p' \to p$ in $\mathcal{C}$ such that

  $$\begin{tikzcd} & p' \arrow[dl, "h'"'] \arrow[d, dashed, "r"] \arrow[dr, "k'"] & \\ a & p \arrow[l, "h"] \arrow[r, "k"'] & b \end{tikzcd}$$

  commutes.

Assuming now we have a pair of morphisms with the same domain

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ a & & b \end{tikzcd} \tag{1.2}$$

in $\mathcal{C}$, a *pushout* of (1.2) is any pair of morphisms in $\mathcal{C}$ with a common codomain

$$\begin{tikzcd} a \arrow[dr, "h"'] & & b \arrow[dl, "k"] \\ & q & \end{tikzcd}$$

such that:

- the square

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ a \arrow[dr, "h"'] & & b \arrow[dl, "k"] \\ & q & \end{tikzcd}$$

  commutes
- for every

$$\begin{tikzcd} a \arrow[dr, "h'"'] & & b \arrow[dl, "k'"] \\ & q' & \end{tikzcd}$$

  in $\mathcal{C}$ making

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ a \arrow[dr, "h'"'] & & b \arrow[dl, "k'"] \\ & q & \end{tikzcd}$$

  commute there exists one and only one $s : q \to q'$ in $\mathcal{C}$ such that

$$\begin{tikzcd} a \arrow[r, "h"] \arrow[dr, "h'"'] & q \arrow[d, dashed, "s"] & b \arrow[l, "k"'] \arrow[dl, "k'"] \\ & q' & \end{tikzcd}$$

  commutes.

The definitions above become can be more concise though: pullbacks (pushouts) are products (coproducts) in certain categories.

Proposition 1.38. A pullback of

$$\begin{tikzcd} a \arrow[dr, "f"'] & & b \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

in $\mathcal{C}$ is any of the products that pair of morphisms in $\mathcal{C}{\downarrow}c$. Dually, a pushout of

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ a & & b \end{tikzcd}$$

in $\mathcal{C}$ is any of the coproducts such pair of morphisms in $c{\downarrow}\mathcal{C}$.

Proof. This is **exercise 1.39**. □

Exercise 1.40. Let $\mathcal{C}$ be a category with initial object 0 and terminal object 1. What are pullbacks of a pair of morphisms with codomain 1? What are pushouts of a pair of morphisms with domain 0?

Example 1.41 (Pullbacks in **Set**). Now we consider sets and functions as in

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

with the aim to find a pullback of it. From the set

$$D := \{(a,b) \in A \times B \mid f(a) = g(b)\}$$

we can make the functions

$$p : D \to A\,,\ p(a,b) := a$$
$$q : D \to B\,,\ q(a,b) := b$$

Hence we can draw at least the commuting square

$$\begin{tikzcd} D \arrow[r, "p"] \arrow[d, "q"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

Now consider

$$\begin{tikzcd} D' \arrow[drr, bend left, "p'"] \arrow[ddr, bend right, "q'"'] & & \\ & D \arrow[r, "p"] \arrow[d, "q"'] & A \arrow[d, "f"] \\ & B \arrow[r, "g"'] & C \end{tikzcd}$$

with $fp' = gq'$. This hypothesis implies that $(p'(x), q'(x)) \in D$ for every $x \in D'$, and allows us to introduce the function

$$r : D' \to D\,,\ r(x) := (p'(x), q'(x))$$

which is such that $pr = p'$ and $qr = q'$. Finally, r is the unique one to do so, which fact is immediate for how r is defined.

Let us remain in **Set**. Consider a function $f : A \to B$ and the diagram

$$\begin{tikzcd} & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where we have duplicated f. The example above tells us we have the pullback square

$$\begin{tikzcd} R_f \arrow[r, "p"] \arrow[d, "q"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

with $R_f := \{(a,b) \in A \times A \mid f(a) = f(b)\}$. This subset of $A \times A$ is a certain equivalence relation over A, namely the *kernel relation* of f. [Did we mention kernel relations in the intro?]

There is nothing special of **Set** that prevents us to generalize it to any category $\mathcal{C}$: we define the *kernel relation* of a $f : a \to b$ in $\mathcal{C}$ to be any of the pullbacks of

$$\begin{tikzcd} & a \arrow[d, "f"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$$

As soon as we meet coequalizers, we will have the tool to express the quotient X/R_f in a categorial fashion, and thus to motivate the general concept of *quotient object.*

Example 1.42 (Pushouts in **Set**). Recall what we have done in Example 1.25, but change a bit the notation. Take a family of two sets A_1 and A_2: write $A_1 + A_1$ instead of using the $\sum$ or $\coprod$ notation, write `left` and `right` in place of in_1 and in_2, respectively.
To get started, let us consider sets and functions

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

Let us draw a diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "\texttt{left}"] \\ B \arrow[r, "\texttt{right}"'] & A + B \end{tikzcd} \tag{1.3}$$

By definition of $A + B$, it can be $\texttt{left}f(x) \neq \texttt{right}g(x)$ for some $x \in C$. However, we can make them ‘equal’ under an adequate equivalence relation $\sim$: the smallest in which, for $x \in C$, the elements $\texttt{left}f(x)$ and $\texttt{right}g(x)$ are identified; that is we define $\sim$ to be the smallest equivalence relation containing

$$R := \{(\texttt{left}f(x), \texttt{right}g(x)) \mid x \in C\}.$$

In this case, let us write p the projection $A + B \to \frac{A+B}{\sim}$. The new square is

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "h"] \\ B \arrow[r, "k"'] & \frac{A+B}{\sim} \end{tikzcd} \tag{1.4}$$

with $h := p\texttt{left}$ and $k := p\texttt{right}$ and it is commutative. Now, pick

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "h"] \arrow[ddr, bend left, "h'"] & \\ B \arrow[r, "k"'] \arrow[drr, bend right, "k'"'] & \frac{A+B}{\sim} & \\ & & X \end{tikzcd}$$

such that $h'f = k'g$. By the universal property of coproduct, we have the function

$$h' + k' : A + B \to X$$

such that $(h' + k')\,\texttt{left} = h'$ and $(h' + k')\,\texttt{right} = k'$. Taken, for any $x \in C$, one $(\texttt{left}f(x), \texttt{right}g(x)) \in R$, we have

$$(h' + g')(\texttt{right}f(x)) = h'f(x)$$
$$(h' + g')(\texttt{left}g(x)) = k'g(x)$$

which are equal for every $x \in C$, by assumption. [Talk about generated equivalence relations and what follows.] Thus the triangle

$$\begin{tikzcd} A+B \arrow[rr, "h'+k'"] \arrow[dr, "p"'] & & X \\ & \frac{A+B}{\sim} \arrow[ur, dashed] & \end{tikzcd}$$

commutes for exactly one dashed function. This function is the one we are looking for. [Complete...]

Exercise 1.43. In the previous example, what is $\frac{A+B}{\sim}$ if $C = A \cap B$ and f and g are just the inclusions of C in A and B respectively? [There is other material to put here...]

Exercise 1.44. Go back to Example 1.42. What if we started our discourse from

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, hook] \\ B \arrow[r, hook] & A \cup B \end{tikzcd}$$

instead?

Exercise 1.45 (Gluing topological spaces)**.** If you are given two spaces X and A, a subspace $E \subseteq A$ and a continuous function $f : E \to X$, then $X \sqcup_f A$ denotes the disjoint union $X \sqcup A$ where every $x \in E$ is identified to $f(x)$, that is

$$X \sqcup_f A := \frac{X \sqcup A}{x \sim f(x) \text{ for } x \in E}.$$

Find a pushout square

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "f"'] & A \arrow[d] \\ X \arrow[r] & X \sqcup_f A \end{tikzcd}$$

in **Top**. The exercise requires you to work about the topologies involved and about continuity.

Example 1.46 (CW complexes)**.** In Topology, several spaces often employed are homotopic — or even homeomorphic — to other spaces glued together. Although you can glue everything to everything, very simple spaces to attach are disks $\mathbb{D}^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ along their boundaries $\mathbb{S}^{n-1} := \partial\mathbb{D}^n$. (Pay attention to superscripts.) For any topological space X, we can perform the following recursive construction:

- Let X_0 be the space X but with the discrete topology.
- For $n \in \mathbb{N}$, from topological space X_n we prescribe the construction of another space X_{n+1}. If we are given a family $\{D_\alpha \mid \alpha \in \Lambda\}$ of copies of $\mathbb{D}^{n+1}$ and collection of continuous functions

  $$\{f_\alpha : \partial D_\alpha \to X_n | \alpha \in \Lambda\}$$

  then we can consider the following topological space

  $$X_{n+1} := \frac{X_n \sqcup \coprod_{\alpha \in \Lambda} D_\alpha}{x \sim f_\alpha(x) \text{ for } \alpha \in \Lambda \text{ and } x \in \partial D_\alpha}.$$

In other words, X_{n+1} is X_n with $(n+1)$-dimensional disks attached to it along their boundaries.

As always, we are striving to find some universal property worth of consideration. A square comes easily if you consider the inclusions running in parallel $\mathbb{S}^n \hookrightarrow \mathbb{D}^{n+1}$ and $X_n \hookrightarrow X_{n+1}$ together with $\mathbb{D}^{n+1} \hookrightarrow X_{n+1}$ of the construction above. The other pieces are the attaching maps: indeed we have a commuting square

$$\begin{tikzcd} \coprod_{\alpha\in\Lambda} \partial D_\alpha \arrow[r, hook] \arrow[d, "\coprod_{\alpha\in\Lambda} f_\alpha"'] & \coprod_{\alpha\in\Lambda} D_\alpha \arrow[d, hook] \\ X_n \arrow[r, hook] & X_{n+1} \end{tikzcd}$$

This square is a pushout one in **Top**, which is easy to prove.

Exercise 1.47 (Spheres are CW complexes)**.** Consider the case in which X_0 is a single point space and so are the spaces X_i for $1 \le i \le n-1$. Such situation can be achieved by attaching no disk for a while; afterwards, attach one disk $\mathbb{D}^n$ along $\mathbb{S}^{n-1}$ to X_{n-1}. Hence we have a homeomorphism $X_n \cong \mathbb{D}^n/\mathbb{S}^{n-1}$, but you will show something more, that is $X_n \cong \mathbb{S}^n$.
In your Topology course, you might have managed to show this as follows:

1. You have constructed a surjective continuous function $f : \mathbb{D}^n \to \mathbb{S}^n$ and considered the quotient space $\mathbb{D}^n/\sim_f$ where $\sim_f$ is the kernel relation [talk about kernel relations!] of f. This relation is not a random relation: for every $x, y \in \mathbb{D}^n$, we have $x \sim_f y$ if and only if $x = y$ or $x, y \in \mathbb{S}^{n-1}$. As consequence, $\mathbb{D}^n/\sim_f = \mathbb{D}^n/\mathbb{S}^{n-1}$.
2. Thanks to the universal property of quotients, the function $\phi : \mathbb{D}^n/\mathbb{S}^{n-1} \to \mathbb{S}^n$ such that $f = \phi p_n$, with the p_n the canonical projection, is continuous and bijective. Now, recalling that continuous functions from compact spaces to Hausdorff spaces are closed, conclude that indeed ϕ is a homeomorphism.

The aim of this exercise it that you can arrive to the same result in a different manner. If you can recollect your memories or retrieve your notes, see if you can recycle the f above and write a pushout square

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, hook] \arrow[d, "!"'] & \mathbb{D}^n \arrow[d, "f"] \\ X_{n-1} \arrow[r] & \mathbb{S}^n \end{tikzcd} \tag{1.5}$$

If you do not know how $\mathbb{D}^n/\mathbb{S}^{n-1} \cong \mathbb{S}^n$, it does not matter since you will force yourself to search for a pushout square like (1.5).

Exercise 1.48 ($\mathbb{RP}^n$ is a CW complex)**.** In Topology, the n-th *real projective space* is

$$\mathbb{RP}^n := \frac{\mathbb{R}^{n+1} \setminus \{0\}}{x \sim \lambda x \text{ for } x \in \mathbb{R}^{n+1}, \lambda \in \mathbb{R}}$$

which is known to be homeomorphic to the sphere $\mathbb{S}^n$ which has the antipodal points identified:

$$\frac{\mathbb{S}^n}{x \sim -x \text{ for } x \in \mathbb{S}^n}.$$

This observation is the key for the coming arguments. In fact, $\mathbb{S}^n$ is the boundary of $\mathbb{D}^{n+1}$ and we already have a continuous function $p_n : \mathbb{S}^n \to \mathbb{RP}^n$

that attaches the the disc to the projective space along the boundary. Find a pushout square of the form

$$\begin{tikzcd} \mathbb{S}^n \arrow[r, hook] \arrow[d, "p_n"'] & \mathbb{D}^{n+1} \arrow[d, "?"] \\ \mathbb{RP}^n \arrow[r, hook] & \mathbb{RP}^{n+1} \end{tikzcd} .$$

Topology, again, but combined with Group Theory.

Example 1.49 (Seifert-van Kampen Theorem)**.** Suppose given a topological space X, two open subsets $A, B \subseteq X$ such that $A \cup B = X$ and one point x_0 of $A \cap B$. Let us denote by i_A, i_B, j_A and j_B the group morphisms induced by the inclusions $A \cap B \hookrightarrow A$, $A \cap B \hookrightarrow B$, $A \hookrightarrow X$ and $B \hookrightarrow X$, respectively. If A, B and $A \cap B$ are path-connected then,

$$\begin{tikzcd} & \pi_1(A, x_0) \arrow[dr, "j_A"] & \\ \pi_1(A \cap B, x_0) \arrow[ur, "i_A"] \arrow[dr, "i_B"'] & & \pi_1(X, x_0) \\ & \pi_1(B, x_0) \arrow[ur, "j_B"'] & \end{tikzcd}$$

is a pushout square of **Grp**.

[The Pullback Lemma is dropped here without a precise plan to embed it nicely with examples and further development. It's an issue that must be fixed.]

Proposition 1.50 (The Pullback Lemma)**.** In a category $\mathcal{C}$ consider a diagram

$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, "d"'] \arrow[dr, phantom, "Q_1"] & \bullet \arrow[r, "b"] \arrow[d, "g"] \arrow[dr, phantom, "Q_2"] & \bullet \arrow[d, "c"] \\ \bullet \arrow[r, "e"'] & \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

where the perimetric rectangle commutes and the square on the right is a pullback one. Then that on the left is a pullback square is and only if so is the outer rectangle.

Proof. Let us assume Q_1 is a pullback square first. Consider any choice of h and k such that $ch = f(ek)$:

$$\begin{tikzcd} \bullet \arrow[drrr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & & \\ & \bullet \arrow[r, "a"] \arrow[d, "d"] \arrow[dr, phantom, "Q_1"] & \bullet \arrow[r, "b"] \arrow[d, "g"] \arrow[dr, phantom, "Q_2"] & \bullet \arrow[d, "c"] \\ & \bullet \arrow[r, "e"'] & \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

Being Q_2 a pullback square, there exists one and only one l such that $h = bl$ and $gl = ek$.

$$\begin{tikzcd} \bullet \arrow[drrr, bend left, "h"] \arrow[ddr, bend right, "k"'] \arrow[drr, "l"] & & & \\ & \bullet \arrow[r, "a"] \arrow[d, "d"] \arrow[dr, phantom, "Q_1"] & \bullet \arrow[r, "b"] \arrow[d, "g"] \arrow[dr, phantom, "Q_2"] & \bullet \arrow[d, "c"] \\ & \bullet \arrow[r, "e"'] & \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

We have just said that this square in red commutes:

$$\begin{tikzcd}
\bullet \arrow[drr, bend left, "l"] \arrow[drrr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & & \\
& \bullet \arrow[r, "a"'] \arrow[d, "d"] \arrow[dr, phantom, "Q_1"] & \bullet \arrow[r, "b"'] \arrow[d, "g"] \arrow[dr, phantom, "Q_2"] & \bullet \arrow[d, "c"] \\
& \bullet \arrow[r, "e"'] & \bullet \arrow[r, "f"'] & \bullet
\end{tikzcd}$$

Now, being Q_1 a pullback square, we have one m such that$l = am$ and $k = dm$:

$$\begin{tikzcd}
\bullet \arrow[dr, "m"] \arrow[drr, bend left, "l"] \arrow[drrr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & & \\
& \bullet \arrow[r, "a"'] \arrow[d, "d"] \arrow[dr, phantom, "Q_1"] & \bullet \arrow[r, "b"'] \arrow[d, "g"] \arrow[dr, phantom, "Q_2"] & \bullet \arrow[d, "c"] \\
& \bullet \arrow[r, "e"'] & \bullet \arrow[r, "f"'] & \bullet
\end{tikzcd}$$

At this point, we have $bam = bl = h$ and $dm = k$. To conclude the first half of the theorem, you have to pick any m' making commute the triangles in green:

$$\begin{tikzcd}
\bullet \arrow[dr, "m'"] \arrow[drr, bend left, "am"] \arrow[drrr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & & \\
& \bullet \arrow[r, "a"'] \arrow[d, "d"] \arrow[dr, phantom, "Q_1"] & \bullet \arrow[r, "b"'] \arrow[d, "g"] \arrow[dr, phantom, "Q_2"] & \bullet \arrow[d, "c"] \\
& \bullet \arrow[r, "e"'] & \bullet \arrow[r, "f"'] & \bullet
\end{tikzcd}$$

Being Q_2 a pullback square, we have $am' = am$. In conclusion, being Q_1 a pullback square too, we have $m = m'$. [Finer explanation here...] □

Exercise 1.51. Prove the remaining part of the theorem above.

Example 1.52 (Character functions)**.** Consider a subset A of some larger set X. You sure know a simple function called *character function* with just says if an element of X is a member of A:

$$\chi_A : X \to \{\mathtt{true}, \mathtt{false}\} \ , \ \chi_A(x) := \begin{cases} \mathtt{true} & \text{if } x \in A \\ \mathtt{false} & \text{otherwise} \end{cases}$$

From now on, let us write Ω to mean $\{\mathtt{true}, \mathtt{false}\}$. As always, let us draw what we have:

$$\begin{tikzcd}
A \arrow[r, hook] & X \arrow[d, "\chi_A"] \\
& \Omega
\end{tikzcd}$$

with $A \hookrightarrow X$ being the usual inclusion. The composition of such functions is function constant to `true`. We know that constant functions are such because they can be factored through some function $A \to 1$ [write about this explicitly somewhere], which results in a commuting square

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "!"'] & X \arrow[d, "\chi_A"] \\
1 \arrow[r, "\lambda x.\mathtt{true}"'] & \Omega
\end{tikzcd}$$

Well, this square is a pullback square. Of course, that is not all we have to say. [To be continued.]

1.5 Equalizers and coequalizers

For $\mathcal{C}$ category, the limits of functors

$$\left(\bullet \rightrightarrows \bullet \right) \to \mathcal{C}$$

are called *equalizers*. The colimits are called *coequalizers* instead.

Definition 1.53 (Equalizers & coequalizers, explicit)**.** Let $\mathcal{C}$ be a category and

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd} \tag{1.6}$$

a pair of morphisms in $\mathcal{C}$. An *equalizer* of this pair is any morphism $i : c \to a$ such that:

- the diagram

$$\begin{tikzcd} c \arrow[r, "i"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$$

  commutes
- for every $i' : c' \to a$ of $\mathcal{C}$ making

$$\begin{tikzcd} c' \arrow[r, "i'"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$$

  commute, there is one and only one $k : c' \to c$ in $\mathcal{C}$ such that

$$\begin{tikzcd} c' \arrow[d, "k"'] \arrow[dr, "i'"] & \\ c \arrow[r, "i"'] & a \end{tikzcd}$$

  commutes.

Dually, a *coequalizer* of the pair (1.6) is any morphism $j : b \to d$ such that

- the diagram

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "j"] & d \end{tikzcd}$$

  commutes;
- for every $j' : b \to d'$ of $\mathcal{C}$ making

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "j'"] & d' \end{tikzcd}$$

  commute, there exists one and only one $h : d \to d'$ such that

$$\begin{tikzcd} & d \arrow[d, "h"] \\ b \arrow[ur, "j"] \arrow[r, "j'"'] & d' \end{tikzcd}$$

  commutes.

Before we analyse some example, the following lemma, may be quite useful to guide us.

Lemma 1.54. Equalizers are monomorphisms. Dually, coequalizers are epimorphisms.

Proof. Consider

$$
\begin{tikzcd}
e' \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & e \arrow[r, "i"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b
\end{tikzcd}
$$

with i equalizer of f and g and $is = it$. We can redraw this diagram as follows:

$$
\begin{tikzcd}
e \arrow[dr, "i"] & & \\
& a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\
e' \arrow[uu, bend left, "s"] \arrow[uu, bend right, "t"'] \arrow[ur, "is=it"'] & &
\end{tikzcd}
$$

In this case, we have $f(is) = f(it) = g(is) = g(it)$. Thus, by definition of equalizer, it must be $s = t$. □

How could this be of aid? For example, in **Set** this means we have to look for inclusions in the domain of the domain of the parallel arrows. That is the case, indeed.

Example 1.55 (Equalizers in **Set**)**.** In **Set**, consider two functions

$$
\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd} .
$$

The subset

$$E := \{x \in X \mid f(x) = g(x)\}$$

has the inclusion in X, we call it $i : E \hookrightarrow X$. Of course, we have $fi = gi$, so one part of the work is done. Now, let us take a commuting diagram

$$
\begin{tikzcd}
E' \arrow[r, "i'"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd} .
$$

It follows that $i'(x) \in E$ for every $x \in E'$. Hence, we shall consider the function

$$h : E' \to E,\ h(x) := i'(x).$$

It is immediate now that $i : E \hookrightarrow X$ is an equalizer of f and g.

If we want one example of coequalizer in **Set**, we have to look for some surjection out of the codomain of the given parallel arrows.

Example 1.56 (Coequalizers in **Set**)**.** In **Set**, consider two functions

$$
\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd} .
$$

Having a commuting diagram like

$$
\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "j"] & A
\end{tikzcd}
$$

means that we must look for some $j : Y \to A$ such that, for $x \in X$, the elements $f(x)$ and $g(x)$ are brought to the same element of A. The thing works if take A to be the quotient

$$\frac{Y}{f(x) \sim g(x) \text{ for } x \in X}$$

and define j as quotient map $Y \to Y/\!\sim$. It only remains to verify the universal property of coequalizer. Take any $j' : Y \to Z$ such that $j'f = j'g$.

$$\begin{tikzcd} & & Y/\!\sim \arrow[d, dashed] \\ X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[ur, "j"] \arrow[r, "j'"'] & Z \end{tikzcd}$$

The existence of the unique function $Y/\!\sim \to Z$ easily follows from Corollary 1.5.

Exercise 1.57. Retrieve Example 1.42 and prepare to combine it with the example we have just made. The square (1.3) doesn't even commute, but $A + B$ with the two injections is a coproduct, not a random thing out there. You can rearrange that diagram too

$$\begin{tikzcd} C \arrow[r, shift left, "\texttt{left}f"] \arrow[r, shift right, "\texttt{right}g"'] & A + B \end{tikzcd}$$

and summon the canonical projection $A + B \to \frac{A+B}{\sim}$ which is a coequalizer. At this point, we have the commutative square (1.4), which we proved to be a pushout square. Luckily, this works in general, and it is up to you to realize why and make the dual of the result too.
In a category $\mathcal{C}$, suppose you have a coproduct

$$a \xleftarrow{\texttt{left}} a + b \xrightarrow{\texttt{right}} b$$

a square

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "g"'] & a \arrow[d, "\texttt{left}"] \\ b \arrow[r, "\texttt{right}"'] & a + b \end{tikzcd}$$

and a coequalizer $p : a + b \to d$ of $\texttt{left}f$ and $\texttt{right}g$. Prove that

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "g"'] & a \arrow[d, "p\texttt{left}"] \\ b \arrow[r, "p\texttt{right}"'] & d \end{tikzcd}$$

is a pushout square in $\mathcal{C}$.

Exercise 1.58. In a category $\mathcal{C}$, consider two parallel morphisms

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$$

and a product

$$a \xleftarrow{p_a} a \times b \xrightarrow{p_b} b$$

The universal property of products gives the morphisms $\overline{f}, \overline{g} : a \to a \times b$ such that $p_a\overline{f} = 1_a$, $p_b\overline{f} = f$, $p_a\overline{g} = 1_a$ and $p_b\overline{g} = g$:

$$\begin{tikzcd} & a \arrow[dl, "1_a"'] \arrow[d, shift right, "\overline{f}"'] \arrow[d, shift left, "\overline{g}"] \arrow[dr, shift left, "g"] \arrow[dr, shift right, "f"'] & \\ a & a \times b \arrow[l, "p_a"] \arrow[r, "p_b"'] & b \end{tikzcd}$$

Consider the pullback square

$$\begin{tikzcd} c \arrow[r, "m"] \arrow[d, "n"'] & a \arrow[d, "\overline{f}"] \\ a \arrow[r, "\overline{g}"'] & a \times b \end{tikzcd}$$

and prove that:

1. $m = n$.
2. m is equalizer of f and g.

Proposition 1.59. Categories with binary products/coproducts have equalizers/coequalizers if and only if have pullbacks/pushouts. That is: in a category with products/coproducts, pullbacks/pushouts can be obtained by equalizers/coequalizers and vice versa.

Ok, let us step back to Lemma 1.54, for there is something curious. Equalizers are monomorphisms, but what are the consequences if an equalizer is in addition epic?

Proposition 1.60. An epic equalizer is an isomorphism. Dually, a monic epimonomorphism is an isomorphism.

Proof. Assume $i : e \to a$ is an epic equalizer of

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$$

From $fi = gi$, we can derive $f = g$, being i epic. Hence

$$\begin{tikzcd} a \arrow[r, "1_a"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$$

commutes: by the universal property of equalizers, $1_a = ik$ for a unique $k : a \to e$. Moreover, by simple computation $iki = i = i1_e$, which implies that $ki = 1_e$ since i is monic. In conclusion, i is invertible. □

How could this become even interesting to us? If a category is such that every monomorphism is an equalizer of some pair of parallel arrows, then there monic and epic morphisms are isomorphisms, which we know it does not occur in every category.

Example 1.61. In **Set** that phenomenon does occur, let us look in it more closely. The problem we have can be stated as follows: take and injective function $f : A \to B$ and find two parallel functions parting from B to which f is an equalizer. We know, how to construct equalizers in **Set** and, even though that is not what we want, that example may guide our exploration. The problem we want to solve requires to find a certain set C and two certain functions $h, k : B \to C$. [Yet to be TEXed...]

1.6 (Co)Completeness

[Some of the parts here are to be rewritten...] [Better notation needed here...] Consider a functor $F : \mathbf{I} \to \mathcal{C}$. Let I be the *underlying discrete category* of $\mathbf{I}$ [say something about that elsewhere...] and $X : I \to \mathcal{C}$ the functor introduced as $X_\lambda := F_\lambda$ for $\lambda \in I$ [the *underlying discrete functor*...]. In other words, X is just F without no morphism $F_f : X_\alpha \to X_\beta$, where $f : \alpha \to \beta$ in $\mathbf{I}$, since I itself has not any morphism apart the identities. For the same reason, any cone $c : k_v \Rightarrow X$ is a just cone $k_x \Rightarrow F$ that cannot care about the morphisms F_f; precisely, $c : k_v \Rightarrow X$ is just a family $\{v \to X_\lambda \mid \lambda \in I\}$, while $k_v \Rightarrow F$ is the same family but also satisfying the naturality condition:

$$\begin{tikzcd} & X_\alpha \arrow[dd, "F_f"] \\ v \arrow[ur, "c_\alpha"] \arrow[dr, "c_\beta"'] & \\ & X_\beta \end{tikzcd}$$

commutes for every α, β and $f : \alpha \to \beta$ in $\mathbf{I}$. So you should expect that in general categories having products cannot guarantee the existence of limits.

Let us indicate by $\{p_\lambda : P \to X_\lambda \mid \lambda \in I\}$ one of the products of X: we have the morphisms

$$\begin{tikzcd} & X_\alpha \arrow[dr, "F_f"] & \\ P \arrow[ur, "p_\alpha"] \arrow[rr, "p_\beta"'] & & X_\beta \end{tikzcd}$$

that run in parallel for $\alpha, \beta \in I$ and $f : \alpha \to \beta$ in $\mathbf{I}$. Observe that for every $\beta \in I$ there is one morphism $P \to X_\beta$, namely p_β. On the other hand, for $\alpha \in I$ there may be one $f : \alpha \to \beta$ or more or even none of such; this means in general we do not have only one F_f present in the last diagram. To be safe, we will consider may copies of X_β as needed, so that there is one $p_\alpha F_f$ going towards its own copy of X_β. [Use a better notation here.] For that scope, consider the set

$$J := \bigcup_{\alpha,\beta \in I} \mathbf{I}(\alpha, \beta)$$

and the functor $\widetilde{X} : J \to \mathcal{C}$ with $\widetilde{X}_f$ defined to be X_β where β is the codomain of f. If $\{q_f : Q \to \widetilde{X}_f \mid f \in J\}$ is one of the products of $\widetilde{X}$, then we have one morphism $r : P \to Q$ such that $q_f r = F_f p_\alpha$ for every $\alpha \in I$ and morphism $f \in J$ with domain α, and one morphism $s : P \to Q$ such that $q_f s = p_\beta$ for every $\beta \in I$ and morphism $f \in J$ with codomain β.

We have thus constructed two parallel morphisms

$$\begin{tikzcd} P \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & Q \end{tikzcd}$$

Assume there is an equalizer $i : L \to P$ of the pair r and s. We are going to show that:

the morphisms $p_\alpha i : L \to P_\alpha$ for $\alpha \in I$ do form a limit for F.

First of all, we verify that they form a natural transformation $k_L \Rightarrow X$. In fact,

$$F_f p_\alpha i = \underbrace{q_f r i = q_f s i}_{i \text{ equalizer of } r \text{ and } s} = p_\beta i$$

for every $\alpha, \beta \in I$ and $(f : a \to b) \in J$. We consider now any natural transformation $j : k_{L'} \Rightarrow F$ and show the existence and the uniqueness of a morphism $h : L' \to L$ such that

$$\begin{tikzcd}
L' \arrow[d, "h"'] \arrow[dr, "j_\alpha"] & \\
L \arrow[r, "p_\alpha i"'] & X_\alpha
\end{tikzcd}$$

commutes for every $\alpha \in I$. Forming the morphisms p_α for $\alpha \in I$ a product in $\mathcal{C}$, let $g : L' \to P$ in $\mathcal{C}$ be the morphism such that $p_\alpha g = j_\alpha$ for every $\alpha \in I$. We can arrange a picture like this:

$$\begin{tikzcd}
L \arrow[r, "i"] & P \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] \arrow[d, "p_\alpha"] & Q \arrow[d, "q_f"] \\
L' \arrow[ur, "g"] \arrow[r, "j_\alpha"'] & X_\alpha \arrow[r, "F_f"'] & X_\beta
\end{tikzcd}$$

Here, we have

$$q_f r g = F_f p_\alpha g = \underbrace{F_f j_\alpha = j_\beta}_{\substack{j:k_{L'} \Rightarrow F \text{is a} \\ \text{natural transformation}}} = p_\beta g = q_f s g.$$

Being the family of the morphisms $q_f : Q \to \widetilde{X}_f$ a product, we must have $st = rt$. And being $i : L \to P$ an equalizer of r and s, it must be $g = ih$ for a unique $h : L' \to L$. Hence, $j_\alpha = p_\alpha g = (p_\alpha i)h$, that is h works fine for our scope. To conclude, let $h' : L' \to L$ such that $(p_\alpha i)h = (p_\alpha i)h'$ for every $\alpha \in I$: by the universal property of products, $ih = ih'$; but, being equalizers monomorphisms, we can conclude $h = h'$.

Definition 1.62. A category $\mathcal{C}$ is said *(co)complete* whenever any functor $\mathbf{I} \to \mathcal{C}$ has a (co)limit. $\mathcal{C}$ is said *finitely (co)complete* when every functor $\mathbf{I} \to \mathcal{C}$ with $\mathbf{I}$ finite admits a (co)limit.

[No concerns about the size of $\mathbf{I}$? It is important.] In general, it may be difficult to demonstrate that a certain category is complete. We have just proved a criterion that may be of aid:

Proposition 1.63 (Completeness Theorem)**.** Categories that have products and equalizers are complete.

A special place is for finite (co)limits.

Proposition 1.64 (Finite Completeness Theorem I)**.** Categories having terminal objects, binary products and equalizers are finitely complete. [Write a definition for 'finite completeness'.]

Proof. Use Corollary 1.35 and the argument to prove the Completeness Theorem. □

Actually, we have another finite completeness theorem, which requires some preliminary work.

Proposition 1.65 (Finite Completeness Theorem II)**.** Categories that have terminal objects and pullbacks are finitely complete.

Proof. Use the previous Lemma and the Finite Completeness Theorem I. □

Let us sum all up in one corollary:

Corollary 1.66 (Finite Completeness Theorem)**.** For any category, the following facts are equivalent:

1. it is finitely complete
2. it has a terminal, binary products and equalizers
3. it has a terminal object and pullbacks

1.7 Other exercises

Here is a collection of exercises that do have a precise placement in the the previous sections, for they deal with the interplay between different kinds of limits and with the previous chapters in a manner that would burden the flow of the discourse.

Exercise 1.67. Consider a category with binary products, and for $a, b \in |\mathcal{C}|$ write the corresponding products as

$$a \xleftarrow{p_{a,b}} a \times b \xrightarrow{q_{a,b}} b$$

1. Construct the functor $(c\times) : \mathcal{C} \to \mathcal{C}$. Without much other effort, you can define functors $(\times c) : \mathcal{C} \to \mathcal{C}$ as well.
2. Can you construct natural transformations $(a\times) \Rightarrow (b\times)$ and $(\times a) \Rightarrow (\times b)$?

Dually, if in a category $\mathcal{C}$ with binary coproducts you have coproducts

$$a \xrightarrow{l_{a,b}} a + b \xleftarrow{r_{a,b}} b$$

you can:

1. Construct functors $(c+)$ and $(+c)$ from $\mathcal{C}$ to $\mathcal{C}$.
2. Construct natural transformations $(a+) \Rightarrow (b+)$ and $(+a) \Rightarrow (+b)$.

Exercise 1.68 (Elements of objects). [This part needs a heavy rewrite.] Let us take advantage of a basic fact about sets:

$$X \cong \mathbf{Set}(1, X) \text{ for every set } X.$$

In general, the isomorphism relation above is not made possible by a unique bijection, but there is one really meaningful for us: the function that takes $x \in X$ to the function $\widehat{x} : 1 \to X$ mapping the unique element of 1 into x. The great deal here is that functions $1 \to X$ inspect X and this isomorphism just outlined identifies every x to $\widehat{x}$.
Let us step back for a moment and turn our attention to the act of defining functions. To define a function $f : X \to Y$, one writes an expression like

$$f(x) := \Gamma \tag{1.7}$$

with Γ being a formula that may contain the symbol x or not. By writing something like (1.7), you are prescribing the images of each element of the domain. This deeply relies on these two facts:

1. Sets are things you can look inside.
2. We have a principle that guarantees the function we are defining in such manner is uniquely determined:

   Given two functions $f_1, f_2 : X \to Y$, if we have $f_1(a) = f_2(a)$ for every $a \in X$, then it must be $f_1 = f_2$.

   This is crucial, since once you have assigned a function as in (1.7), it cannot behave any different from what prescribed.

How this can be interesting to us at this point? First of all, 1 is a terminal object. We have showed earlier how elements of X can be thought as functions $1 \to X$. In this case, the application of f to x is the mere composition

$$fx : 1 \xrightarrow{x} X \xrightarrow{f} Y.$$

The principle aforementioned can be restated as:

If the diagram

$$\begin{tikzcd} 1 \arrow[r, "x"] & X \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & Y \end{tikzcd}$$

commutes for every $x : 1 \to X$, then $f_1 = f_2$.

[To be continued...]

Exercise 1.69 (Natural number objects)**.** Assume you have a category $\mathcal{C}$ with terminal object 1 and with *natural number object*

$$1 \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$$

[I should spend more words for natural number objects.] It is not necessarily the $\mathbb{N}$ in **Set** you are used too. Prove the *primitive recursion theorem*:

If

$$\mathbb{N} \xleftarrow{p_\mathbb{N}} \mathbb{N} \times X \xrightarrow{p_X} X$$

is a product of $\mathcal{C}$, then for every $x_0 : 1 \to X$ and $f : \mathbb{N} \times X \to X$ in $\mathcal{C}$ there is one and only one $x : \mathbb{N} \to X$ such that

$$\begin{tikzcd} & \mathbb{N} \arrow[r, "s"] \arrow[d, "\langle 1_\mathbb{N}, x \rangle"] & \mathbb{N} \arrow[d, "x"] \\ 1 \arrow[ur, "0"] \arrow[r, "\langle 0, x_0 \rangle"'] & \mathbb{N} \times X \arrow[r, "f"'] & X \end{tikzcd}$$

[We just used some notation we have never used before here.]

Indeed, in a category with a terminal object and binary products, *recursion theorem* is equivalent to *primitive recursion theorem*. [Prepare a hint.]

Exercise 1.70 (Epic-monic factorization)**.** Consider in category $\mathcal{C}$ one pullback square

$$\begin{tikzcd} r \arrow[r, "s"] \arrow[d, "t"'] & a \arrow[d, "f"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$$

and let $p : a \to a/r$ be a coequalizer of s and t.

1. Thanks to the universal property of coequalizers, there is exactly one $\overline{f} : a/r \to b$ in $\mathcal{C}$ that satisfies $f = \overline{f}p$. Show that $\overline{f}$ is monic.
2. The factorization $\overline{f}p$ is an *epic-monic factorization* of f. Consider now $q : a \to c$ epic and $g : c \to b$ monic in $\mathcal{C}$ such that $f = gq$, that is another epic-monic factorization of f. Show that there exists one and only one $h : a/r \to c$ such that

$$\begin{tikzcd}
 & a/r \arrow[dr, "\overline{f}"] \arrow[dd, "h"] & \\
a \arrow[ur, "p"] \arrow[dr, "q"'] & & b \\
 & c \arrow[ur, "g"'] &
\end{tikzcd}$$

   commutes. Show that moreover h is an isomorphism.
3. Explore by yourself: f is monic if and only if ....
4. Apply all that above to some concrete example. Does this sound familiar now?

Consider in category $\mathcal{C}$ one pushout square

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "f"'] & b \arrow[d, "u"] \\
b \arrow[r, "v"'] & z
\end{tikzcd}$$

and let $\operatorname{im} f : f(a) \to b$ be an equalizer of u and v. Make the dual of above. (Spoiler: again about epic-monic factorizations.)

Exercise 1.71 (Equivalence relations)**.** [Yet to be TeXed...]

Exercise 1.72 (Subobject classifiers and some properties)**.** [Yet to be TeXed...]