Notes on Category Theory — Pieces

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1 Introduction to Topoi

1.1 Subobject classifiers

Throughout the current section, we assume $\mathcal{E}$ is a category with initial object 1. That being the setting, we can give the following definition.

Definition 1. A *subobject classifier* for $\mathcal{E}$ is any morphism $t : 1 \to \Omega$ such that: for every monomorphism $f : a \to b$ of $\mathcal{E}$ there is one and only one morphism $\chi_f : b \to \Omega$ in $\mathcal{E}$ for which there is a pullback square

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "!"'] & b \arrow[d, "\chi_f"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd} \tag{1.1}$$

That is we can assign to every monomorphism $f : a \to b$ the morphism $\chi_f : b \to \Omega$ satisfying the property of the definition. Let us introduce then some symbolism: for $b \in |\mathcal{E}|$ we write $\mathrm{Sub}_{\mathcal{E}}\, b$ for the class of all the monomorphisms of $\mathcal{E}$ with codomain b. Hence we can introduce the function

$$\chi : \mathrm{Sub}_{\mathcal{E}}\, b \to \mathcal{E}(b, \Omega)$$

with χ_f defined to be that morphism $b \to \Omega$ for which there is a pullback square as the diagram (1.1).

It is worth to observe $\mathrm{Sub}_{\mathcal{E}}(b)$ has a natural structure of preorder: for

$$\begin{tikzcd} a_1 \arrow[dr, "f_1"'] & & a_2 \arrow[dl, "f_2"] \\ & b & \end{tikzcd}$$

monomorphisms of $\mathcal{E}$, write $f_1 \le f_2$ to say there is some $h : a_1 \to a_2$ in $\mathcal{E}$ for which

$$\begin{tikzcd} a_1 \arrow[rr, "h"] \arrow[dr, "f_1"'] & & a_2 \arrow[dl, "f_2"] \\ & b & \end{tikzcd}$$

commutes. Note that, being here f_1 and f_2 monomorphisms, there is at most one h as such and it is a monomorphism as well.

We show now the relation $\simeq$ on $\mathrm{Sub}_{\mathcal{E}}\, b$ defined by

$$f_1 \simeq f_2 \text{ if and only if } f_1 \leq f_2 \text{ and } f_2 \leq f_1$$

for $f_1, f_2 \in \mathrm{Sub}_{\mathcal{E}}\, b$ is an equivalence relation. [...]

Yes, $\mathrm{Sub}_{\mathcal{E}}\, b$ is the full subcategory of $\mathcal{E} \downarrow b$ whose objects are all the monomorphisms of $\mathcal{E}$ with codomain b, and whose isomorphism relation is $\simeq$.

Proposition 2. Let

$$\begin{tikzcd} a_1 \arrow[dr, "f_1"'] & & a_2 \arrow[dl, "f_2"] \\ & b & \end{tikzcd}$$

be monomorphisms. $\chi_{f_1} = \chi_{f_2}$ if and only if $f_1 \simeq f_2$.

Proof. Assume $\chi_{f_1} = \chi_{f_2}$. By definition of subobject classifiers, χ_{f_1} is the morphism for which

$$\begin{tikzcd} a_1 \arrow[r, "f_1"] \arrow[d, "!"'] & b \arrow[d, "\chi_{f_1}"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} a_2 \arrow[r, "f_2"] \arrow[d, "!"'] & b \arrow[d, "\chi_{f_1}"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$

are pullback squares. Consequently, we must infer that there is one isomorphism $h : a_1 \to a_2$ such that $f_1 = f_2 h$. Hence $f_1 \leq f_2$, and $f_2 \leq f_1$ too, because $f_1 h^{-1} = f_2$.

For the remaining part of the proof, let us write $!_1$ the unique morphism $a_1 \to 1$ and $!_2$ the unique morphism $a_2 \to 1$. Also remember that triangles

$$\begin{tikzcd} a_1 \arrow[rr] \arrow[dr, "!_1"'] & & a_2 \arrow[dl, "!_2"] \\ & 1 & \end{tikzcd} \qquad \begin{tikzcd} a_1 \arrow[dr, "!_1"'] & & a_2 \arrow[ll] \arrow[dl, "!_2"] \\ & 1 & \end{tikzcd}$$

always commute.

Now we suppose $f_1 \simeq f_2$. The plan for the proof is: if we show that

$$\begin{tikzcd} a_1 \arrow[r, "f_1"] \arrow[d, "!_1"'] & b \arrow[d, "\chi_{f_2}"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$

is a pullback square, then, being $\chi_{f_1} : b \to \Omega$ the one for which there is a pullback square like this, we can conclude $\chi_{f_1} = \chi_{f_2}$. First of all such square commutes: if we call h the morphism $a_1 \to a_2$ such that $f_1 = f_2 h$, then

$$\chi_{f_2} f_1 = \chi_{f_2} f_2 h = t!_2 h = t!_1.$$

Consider

$$\begin{tikzcd} \bullet \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\ & a_1 \arrow[r, "f_1"] \arrow[d, "!_1"'] & b \arrow[d, "\chi_{f_2}"] \\ & 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$

where $\chi_{f_2} u = tv$. Being

$$\begin{tikzcd} a_2 \arrow[r, "f_2"] \arrow[d, "!_2"'] & b \arrow[d, "\chi_{f_2}"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$

a pullback square we have one $z : \bullet \to a_2$ for which $f_2 z = u$ and $!_2 z = v$. From the assumption $f_1 \simeq f_2$, we have $f_2 \leq f_1$, that is $f_2 = f_1 q$ for some $q : a_2 \to a_1$. Then $u = f_1 qz$ and $v =!_1 qz$. Let us see if qz is what we are looking for.

$$\begin{tikzcd} \bullet \arrow[dr, shift left, "qz"] \arrow[dr, shift right, "r"'] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\ & a_1 \arrow[r, "f_1"] \arrow[d, "!_1"'] & b \arrow[d, "\chi_{f_2}"] \\ & 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$

where we suppose $!_1 r = v$ and $f_1 r = u$. Being f_1 a monomorphism, the sole second identity is enough to conclude $r = qz$. $\square$