Notes on Category Theory — Pieces

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Contents

1 Basic vocabulary

1.1 Categories

It is quite easy to make examples motivating the definition of categories and the evolution that follows through these pages.

Example 1.1 (Set Theory)**.** Here, we have *sets* and *functions*. Whereas the concepts of set ad membership are primitive, functions are formalised as follows: for A and B sets, a function from A to B is any $f \subseteq A \times B$ such that for every $x \in A$ there exists one and only one $y \in B$ such that $(x, y) \in f$. We write

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

to say 'f is a function from A to B' and, for $x \in A$, we write $f(x)$ the element of B bound to x by f. Consecutive functions can be combined in a quite natural way: for A, B and C sets and functions

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

the *composite* of g and f is the function

$$g \circ f : A \to C\,,\ g \circ f(x) := g\big(f(x)\big).$$

Informally speaking: f takes one input and gives one output; it is then passed to g, which then provides one result. Such operation is called *composition* and has some nice basic properties

1. Every set A has associated an *identity*

$$1_A : A \to A\,,\ 1_A(x) := x$$

is such that for every set B and function $g : B \to A$ we have

$$1_A \circ g = g$$

and for every set C and function $h : A \to C$ we have

$$h \circ 1_A = h.$$

2. $\circ$ is *associative*, that is for A, B, C and D sets and

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

functions, we have the identity

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Example 1.2 (Topology)**.** A *topological space* is a set where some of its subsets have the status of 'open' sets. Being sets at their core, we have functions between topological spaces, but some of them are more interesting than others. Namely, *continuous functions* are functions that care about the label of open: for if X and Y are topological spaces, a function $f : X \to Y$ is said *continuous* whenever for every open set U of Y the set $f^{-1}U$ is an open set of X. Being function, consecutive continuous functions can be composed: is the resulting function continuous as well? Yes: if X, Y and Z are topological spaces and f and g continuous, for if U is open, then so is $(g \circ f)^{-1}U$.[1] We can state the following basic properties for the composition of continuous functions:

1. Every topological space A has associated the continuous function

$$1_A : A \to A\,,\ 1_A(x) := x$$

is such that for every topological space B and continuous function $g : B \to A$ we have

$$1_A \circ g = g$$

and for every topological space C and continuous function $h : A \to C$ we have

$$h \circ 1_A = h.$$

2. $\circ$ is *associative*, that is for A, B, C and D topological spaces and

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

continuous functions, we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

That is: take the properties listed in the previous example and replace 'set' with 'topological space' and 'function' with 'continuous function'.

1. It is a fact of Set Theory that for X, Y and Z sets and $f : X \to Y$ and $g : Y \to Z$ functions, we have $(g \circ f)^{-1}U = f^{-1}\left(g^{-1}U\right)$ for every $U \subseteq Z$.

Exercise 1.3. In Measure Theory, we have *σ-algebras*, that is sets where some of its subsets are said to be *measurable*. We can define *measurable functions* too, that is functions that care about the property od being measurable as continuous functions do of the property of being open.[2] Of course, you can found other example of categories in Algebra, Linear Algebra, Geometry and Analysis. Go and catch as many as you can within you mathematical knowledge. And yes, it may be boring sometimes, and you are right, but as we progress there are remarkable differences from one category to another one.

It should be clear a this point what the pattern is:

Definition 1.4 (Categories)**.** A *category* amounts at assigning some things called *objects* and, for each couple of objects a and b, other things named *morphisms* from a to b. We write $f : a \to b$ to say that f is a morphism from a to b, where a is the *domain* of f and b the *codomain*. Besides, for a, b and c objects and $f : a \to b$ and $g : b \to c$ morphisms, there is associated the *composite morphism*

$$gf : a \to c.$$

All those things are regulated by the following axioms:

1. for every object x there is a morphism, 1_x, from x to x such that for every object y and morphism $g : y \to x$ we have

$$1_x g = g$$

and for every object z and morphism $h : x \to z$ we have

$$h 1_x = h;$$

2. for a, b, c and d objects and morphisms

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

we have the identity

$$(hg)f = h(gf).$$

Sometimes, instead of 'morphism' you may find written 'map' or 'arrow'. The former is quite used outside Category Theory, whereas the latter refers to the fact that the symbol $\to$ is employed.

We have started with sets and functions, afterwards we have made an example based on the previous one; if you have accepted the invite of the exercise above, you have likely found categories where objects are sets at their core and morphisms are functions with extra property. We have given an abstract definition of categories because out there are many other categories that deserve attention.

Example 1.5 (Monoids are categories)**.** Consider a category $\mathcal{G}$ with a single object, that we indicate with a bare $\bullet$. All of its morphisms have $\bullet$ as domain and codomain: then any two morphisms are composable and the composite

2. Perhaps, you are taught that measurable functions are functions $f : \Omega \to \mathbb{R}$ from a measurable space Ω such that $f^{-1}(-\infty, a]$ is a measurable for every $a \in \mathbb{R}$. Anyway, $\mathbb{R}$ has the Borel σ-algebra, which is defined as the smallest of the σ-algebras containing the open subsets of $\mathbb{R}$ under the Euclidean topology. It can be easily shown that $f : \Omega \to \mathbb{R}$ is measurable if and only if $f^{-1}B$ is measurable for every Borel subset B of $\mathbb{R}$.

of two morphisms $\bullet \to \bullet$ is a morphism $\bullet \to \bullet$. This motivates us to proceed as follows: let G be the collection of the morphisms of $\mathcal{G}$ and consider the operation of composing morphism

$$G \times G \to G, \ (x, y) \to xy.$$

Being $\mathcal{G}$ a category implies this function is associative and $\mathcal{G}$ has the identity of $\bullet$, that is G has one element we call 1 and such that $f1 = 1f = f$ for every $f \in G$. In other words, we are saying G is a monoid. We say the single object category $\mathcal{G}$ *is* a monoid.
Conversely, take a monoid G and a symbol $\bullet$: make such thing acquire the status of object and the elements of G that of morphisms; in that case, the operation of G has the right to be called composition because the axioms of monoid say so. Here, $\bullet$ is something we care of just because by definition morphisms require objects and it has no role other than this.

In Mathematics, a lot of things are monoids, so this is nice. In particular, a *group* is a single object category where for every morphism f there is a morphism g such that gf and fg are the identity of the the unique object. We will deal with isomorphism later in this chapter.

Example 1.6 (Preordered sets are categories)**.** A *preordered set* (sometimes contracted as *proset*) consists of a set A and a relation $\leq$ on A such that:

1. $x \leq x$ for every $x \in A$;
2. for every $x, y, z \in A$ we have that if $x \leq y$ and $y \leq z$ then $x \leq z$.

Now we do this: for $x, y \in A$, whenever $x \leq y$ take $(a, b) \in A \times A$. We operate with these couples as follows:

$$(y, z)(x, y) := (x, z), \tag{1.1}$$

where $x, y, z \in A$. This definition is perfectly motivated by (2): in fact, if $x \leq y$ and $y \leq z$ then $x \leq z$, and so there is (x, z). By (1), for every $x \in A$ we have the couple (x, x), which has the following property: for every $y \in A$

$$\begin{aligned} (x, y)(x, x) &= (x, y) \quad \text{for every } y \in A \\ (x, x)(z, x) &= (z, x) \quad \text{for every } z \in A. \end{aligned} \tag{1.2}$$

Another remarkable feature is that for every $x_1, x_2, x_3, x_4 \in A$

$$\big((x_3, x_4)(x_2, x_3)\big)(x_1, x_2) = (x_3, x_4)\big((x_2, x_3)(x_1, x_2)\big) \tag{1.3}$$

We have a category indeed: its objects are the elements of A, the morphisms are the couples (x, y) such that $x \leq y$ and (1.1) gives the notion of composition; (1.2) says what are identities while (1.3) tells the compositions are associative.

Several things are prosets, so this is nice. Namely, *partially ordered sets*, or *posets*, are prosets where every time there are morphisms going opposite directions

$$a \rightleftarrows b$$

then $a = b$. Later in this chapter, we will meet *skeletal* categories.

Example 1.7 (Matrices)**.** We need to clarify some terms and notations before. Fixed some field k, for m and n positive integers, a *matrix* of type $m \times n$ is a table of elements of k arranged in m rows and n columns:

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,n} \end{pmatrix}$$

If A is the name of a matrix, then $A_{i,j}$ is the element on the intersection of the ith row and the jth column. Matrices can be multiplied: if A and B are matrices of type $m \times n$ and $n \times r$ respectively, then AB is the matrix of type $m \times r$ where

$$(AB)_{i,j} := \sum_{p=1}^{n} A_{i,p} B_{p,j}.$$

Our experiment is this: consider the positive integers in the role of objects and, for m and n integers, the matrices of type $m \times n$ as morphisms from n to m; now, take AB as the composition of A and B. Let us investigate whether categorial axioms hold.

- For n positive integer, we have the *identity matrix* I_n, the one of type $n \times n$ defined by

  $$(I_n)_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

  One, in fact, can verify that such matrix is an 'identity' in categorial sense: for every positive integer m, an object, and for every matrix A of type $m \times n$, a morphism from n to m, we have

  $$AI_n = A,$$

  that is composing A with I_n returns A; similarly, for every positive integer r and for every matrix B of type $r \times n$ we have

  $$I_n B = B.$$

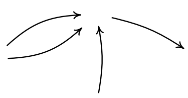
- For A, B and C matrices of type $m \times n$, $n \times r$ and $r \times s$ respectively, we have

  $$(AB)C = A(BC).$$

  Again, this identity can be regarded under a categorial light.

The category of matrices over a field k just depicted is written $\mathbf{Mat}_k$. This example may seem quite useless, but it really does matter when you know there is the category of finite vector spaces $\mathbf{FDVect}_k$: just wait until we talk about equivalence of categories. [We may leave something to think about in the meantime, right?]

A diagram is a drawing made of 'nodes', that is empty slots, and 'arrows', that part from some nodes and head to other ones. Here is an example:



(1.4)

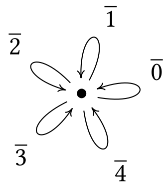


Figure 1. The group $\mathbb{Z}_5$ in a diagrammatic vest.

Nodes are the places where to put objects' names and arrows are to be labelled with morphisms' names. The next step is putting labels indeed, something like this:

$$\begin{tikzcd} & b \arrow[dr, "q"] & \\ a \arrow[ur, bend left, "f"] \arrow[ur, "g"'] & & d \\ & c \arrow[uu, "h"'] & \end{tikzcd} \tag{1.5}$$

The idea we want to capture is: having a scheme of nodes and arrows, as in (1.4), and then assigning labels, as in (1.5). Since diagrams serve to graphically show some categorial structure, there should exist the possibility to 'compose' arrows: two consecutive arrows

$$\begin{tikzcd} \cdot \arrow[r, bend left] & \cdot \arrow[r, bend right] & \cdot \end{tikzcd} \tag{1.6}$$

naturally yields that one that goes from the first node and heads to the last one; if in (1.6) we label the arrows with f and g, respectively, then the composite arrow is to be labelled with the composite morphism gf. That operation shall be associative and there should exist identity arrows too, that is arrows that represent and behave exactly as identity morphisms. In other words, our drawings shall care of the categorial structure.

If we want to formalize the idea just outlined, the definition of diagram sounds something like this:

Definition 1.8 (Diagrams)**.** A *diagram* in a category $\mathcal{C}$ is having:

- a scheme of nodes and arrows, that is a category $\mathcal{I}$;
- labels for nodes, that is for every object i of $\mathcal{I}$ one object x_i of $\mathcal{C}$;
- labels for arrows, that is for every pair of objects i and j of $\mathcal{I}$ and morphism $\alpha : i \to j$ of $\mathcal{I}$, one morphism $f_\alpha : x_i \to x_j$ of $\mathcal{C}$

with all this complying the following rules:

1. $f_{1_i} = 1_{x_i}$ for every i object of $\mathcal{I}$;
2. $f_\beta f_\alpha = f_{\beta\alpha}$, for α and β two consecutive morphisms of $\mathcal{I}$.

Rather than thinking diagrams abstractly — like in the form stated in the definition —, one usually draws them. In general, it is not a good idea to draw all the compositions. For example, consider four nodes and three arcs displayed as

$$\begin{tikzcd} a \arrow[dr, "f"'] & & c \arrow[dr, "h"] & \\ & b \arrow[ur, "g"] & & d \end{tikzcd}$$

and draw all the compositions: you will convince yourself it may be a huge mess even for small diagrams. In fact, why waste an arrow to represent the composite gf in

$$\begin{tikzcd} a \arrow[dr, "f"] & & c \\ & b \arrow[ur, "g"'] & \end{tikzcd}$$

when gf is walking along f before and g then? Neither identities need to be drawn: we know every object has one and only one identity and thus the presence of an object automatically carries the presence of its identity.

[A finer formalisation of commutativity?] Consecutive arrows form a 'path'; in that case, we refer to the domain of its first arrow as the domain of the path and to the codomain of the last one as the codomain of the path. Two paths are said *parallel* when they share both domain and codomain. A diagram is said to be *commutative* whenever any pair of parallel paths yields the same composite morphism.

Let us express the categorial axioms in a diagrammatic vest. Let $\mathcal{C}$ be a category and x an object of $\mathcal{C}$. The fact that 1_x the identity of x can be translated as follows: the diagrams

$$\begin{tikzcd} & x \arrow[dd, "1_x"] \\ a \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & x \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dd, "1_x"'] \arrow[dr, "g"] & \\ & b \\ x \arrow[ur, "g"'] & \end{tikzcd} \tag{1.7}$$

commute for every a and b objects and f and g morphisms of $\mathcal{C}$. Associativity can be rephrased by saying:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "gf"'] & b \arrow[dr, "hg"] & \\ & c \arrow[r, "h"'] & d \end{tikzcd}$$

commutes for every a, b, c and d objects and f, g and h morphisms in $\mathcal{C}$.

Example 1.9 (Semigroup axioms)**.** A *semigroup* is a set X together with a function $\mu : X \times X \to X$ which is associative, that is

$$\mu(\mu(a,b),c) = \mu(a,\mu(b,c)) \text{ for every } a,b,c \in X.$$

The aim of this example is to see how can we put in diagrams all this. We have a triple of elements, to start with, $(a,b,c) \in X \times X \times X$. On the left side of the equality above, a and b are multiplied first, and the result is multiplied with c:

$$\begin{array}{ccccc} X \times X \times X & \longrightarrow & X \times X & \longrightarrow & X \\ (a,b,c) & \longrightarrow & (\mu(a,b),c) & \longrightarrow & \mu(\mu(a,b),c) \end{array}$$

It is best we some effort in naming these functions. While it is clear that $X \times X \to X$ is our μ, how do we write $X \times X \times X \to X$? There is notation for it: $\mu \times 1_X$.[3] Instead, on the other side, b and c are multiplied first, and then a is multiplied to their product:

$$\begin{array}{ccccc} X \times X \times X & \longrightarrow & X \times X & \longrightarrow & X \\ (a,b,c) & \longrightarrow & (a,\mu(b,c)) & \longrightarrow & \mu(a,\mu(b,c)) \end{array}$$

3. In general, if you have two functions $f : A_1 \to A_2$ and $g : B_1 \to B_2$, the function $f \times g : A_1 \times B_1 \to A_2 \times B_2$ is the one defined by $f \times g(a,b) = (f(a), g(b))$.

The first function is $1_X \times \mu$ and the second one is μ. Thus the equality of the definition of semigroup is equivalent to the fact that the diagram

$$\begin{tikzcd} X \times X \times X \arrow[r, "\mu \times 1_X"] \arrow[d, "1_X \times \mu"'] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

commutes.

Exercise 1.10. Recall that a monoid is a semigroup (X, μ) with $e \in X$ such that $\mu(x, e) = \mu(e, x) = x$ for every $x \in X$. We usually write a monoid as a triple (X, μ, e). A *group* is a monoid (G, μ, e) such that for every $x \in G$ there exists $y \in G$ such that $\mu(x, y) = \mu(y, x) = e$. Rewrite these structures using commutative diagrams. (The work about associativity is already done, so you should focus how to express the property of the identity in a monoid and the property of inversion for groups.) Also recall that a *monoid homomorphism*, then, from a monoid (X, μ, e_X) to a monoid (Y, λ, e_Y) is any function $f : X \to Y$ such that $f(\mu(a, b)) = \lambda(f(a), f(b))$ for every $a, b \in X$ and $f(e_X) = e_Y$. Use commutative diagrams. Observe that group homomorphisms are defined by requiring to preserve multiplication, whereas the the preservation of identities can be deduced.

[There is some remark on this.]

1.2 Foundations

Let us return at the beginning, namely the definition of category. Why not formulate it in terms of sets? That is, why don't muster the objects into a set, for any pair of objects, the morphisms into a set and writing compositions as functions?

Let us analyse what happens if we do that. A basic and quite popular fact that fatally crushes our hopes is:

there is no set of all sets.[4]

The first aftermath is that the existence of **Set** would not be legal, because otherwise a set would gather all sets.

Another example comes from both Algebra and Set Theory. In general, it's not a so profound result, but it is interesting for our discourse:

every pointed set $(X, 1)$ has an operation that makes it a group.[5]

Viz there exists no set of all groups, and then neither **Grp** would be supported.

As if the previous examples were not enough, Topology provides another irreducible case. Any set has the corresponding powerset, thus any set gives rise to at least one topological space. Our efforts are doomed, again: there is no set of all topological spaces, and so also **Top** would not be allowed!

It seems that using Set Theory requires the sacrifice of nice categories; and we do not want that, of course. From the few examples above one could

4. If we want a set X to be the set of all sets, then it has all its subsets as elements, which is an absurd. In fact, Cantor's Theorem states that for every set X there is no surjective function $f : X \to 2^X$.

5. Actually, this fact is equivalent to the Axiom of Choice.

surmise it is a matter of *size*: sets sometimes are not appropriate for collecting all the stuff that makes a category. Luckily, there is not a unique Set Theory and, above all, there is one that could help us.

The *von Neumann-Bernays-Gödel approach*, usually shortened as NBG, was born to solve size problems, and may be a good ground for our purposes. In NBG we have *classes*, the most general concept of 'collection'. But not all classes are at the same level: some, the *proper classes*, cannot be element of any class, whilst the others are the *sets*. Here is how the definition of category would look like.

Definition 1.11 (Categories)**.** A category $\mathcal{C}$ consists of:

- a class of objects, denoted $|\mathcal{C}|$;
- for every $a, b \in |\mathcal{C}|$, a class of morphisms from a to b, written as $\mathcal{C}(a,b)$;
- for every $a, b, c \in |\mathcal{C}|$, a composition, viz a function

$$\mathcal{C}(b,c) \times \mathcal{C}(a,b) \to \mathcal{C}(a,c)\,,\ (g,f) \to gf$$

with the following axioms:

1. for every $x \in |\mathcal{C}|$ there exists a $1_x \in \mathcal{C}(x,x)$ such that for every $y \in |\mathcal{C}|$ and $g \in \mathcal{C}(y,x)$ we have
$$1_x g = g$$
and for every $z \in |\mathcal{C}|$ and $h \in \mathcal{C}(x,z)$ we have
$$h 1_x = h;$$
2. for $a, b, c, d \in |\mathcal{C}|$ and $f \in \mathcal{C}(a,b)$, $g \in \mathcal{C}(b,c)$ and $h \in \mathcal{C}(c,d)$ we have the identity
$$(hg)f = h(gf).$$

How does this double ontology of NBG actually apply at our discourse? For example, in NBG the class of all sets is a legit object: it is a proper class, because it cannot be an actual set. Thus, **Set** exists on NBG, and so exists **Grp**, **Top** and other big categories. Which is nice.

Hence, it is sensible to introduce some terms that distinguish categories by the size of their class of objects. [...]

[What can go wrong if $\mathcal{C}(a,b)$ are proper classes?]

1.3 Isomorphisms

[This sections requires a heavy rewriting.]

Let us step back to the origins. The categorial axioms state identities that deals with morphisms, since equality between morphisms is involved. For that reason, we shall regard these axioms as ones about morphisms, since objects barely appear as start/end point of morphisms.

Thus categories have a notion of sameness between morphisms, the equality, but nothing is said about objects. Of course, there is equality for objects too, but we can craft a better notion of sameness of objects. Not because equality is bad, but we shall look for something that can be stated solely in categorial terms. As usual, simple examples help us to isolate the right notion.

Cantor, the father of Set Theory, conducted its enquiry on cardinalities and not on equality of sets.

Example 1.12 (Isomorphisms of sets)**.** For A and B sets, there exists a bijective function $A \to B$ if and only if there exist two functions

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

such that $gf = 1_A$ and $fg = 1_B$. In Set Theory, the adjective 'bijective' is defined by referring of the fact that sets are things that have elements:

for every $y \in B$ there is one and only one $x \in A$ such that $f(x) = y$.

In contrast,

there exist two functions $A \underset{g}{\overset{f}{\rightleftarrows}} B$ such that $gf = 1_A$ and $fg = 1_B$

is written in terms of functions and compositions of functions, that is it is written in a categorial language.

Example 1.13 (Isomorphisms in **Grp**)**.** [Yet to be TEXed...]

Example 1.14 (Isomorphisms in **Top**)**.** In Topology, things work a little differently. There are bijective continuous functions that that are not homeomorphisms. For instance,

$$f : [0, 2\pi) \to \mathbb{S}^1\,,\ f(x) := (\cos x, \sin x)$$

is continuous and bijective, but fails to be a homeomorphism because $\mathbb{S}^1$ is compact while $[0, 1)$ is not. 'Fortunately', in Topology there are two basic facts:

- bijective continuous functions that are also closed are homeomorphisms
- continuous functions from compact spaces to Hausdorff spaces are closed

As a consequence, **Top** has a subcategory in which bijections are homeomorphisms: the sub*category of compact Hausdorff spaces* **CHaus**.

Fine, there is some idea that we can formulate in categorial language.

Definition 1.15 (Isomorphic objects)**.** In a category $\mathcal{C}$, let a and b two objects and $f : a \to b$ a morphism. A morphism $g : b \to a$ of the same category is said *inverse* of f whenever $gf = 1_a$ and $fg = 1_b$. In that case

- $f : a \to b$ of $\mathcal{C}$ is an *isomorphism* when it has an inverse.
- a is said *isomorphic* to b when there is an isomorphism $a \to b$ in $\mathcal{C}$, and write $a \cong b$.

Lemma 1.16. Every morphism has at most one inverse.

That is it may not exist, but if it does it is unique. We write the inverse of f as f^{-1}.

Proof. Fixed a certain category $\mathcal{C}$ and given a morphism $f : a \to b$ with inverses $g_1, g_2 : b \to a$, we have $g_1 = g_1 1_b = g_1(fg_2) = (g_1 f)g_2 = 1_a g_2 = g_2$. □

Example 1.17 (Isomorphisms in $\mathbf{Mat}_k$)**.** In this category, morphisms are matrices with entries in some field k and isomorphisms are exactly invertible matrices. Recall that a square matrix A is said invertible whenever the is some matrix B of the same order such that $AB = BA = I$. From Linear Algebra, we know that a matrix A is invertible if and only if (for example) $\det A \neq 0$. One thing a careful reader may ask is: why restrict to only square matrices? We can easily prove that

if a matrix of type $m \times n$ has an inverse, then $m = n$.

This means for $\mathbf{Mat}_k$ that two different objects cannot be isomorphic, or equivalently isomorphic objects are equal.

Categories like this one have a dedicated name.

Definition 1.18 (Skeletal categories)**.** A category is said *skeletal* whenever its isomorphic objects are equal.

Exercise 1.19. Write **FinSet** for the category of finite sets and functions between sets. Find one skeleton.

Exercise 1.20. Find one skeleton of $\mathbf{FDVect}_k$.

1.4 Mono- and Epimorphisms

[This section has to be rewritten.]

Definition 1.21 (Monomorphisms and epimorphisms)**.** A morphism $f : a \to b$ of a category $\mathcal{C}$ is said to be:

- a *monomorphism* whenever if

$$\begin{tikzcd} c \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & a \arrow[r, "f"] & b \end{tikzcd}$$

  commutes for every object c and morphisms $g_1, g_2 : c \to a$ of $\mathcal{C}$, then $g_1 = g_2$;
- an *epimorphism* whenever if

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, bend left, "h_1"] \arrow[r, bend right, "h_2"'] & c \end{tikzcd}$$

  commutes for every object d and morphisms $h_1, h_2 : c \to a$ of $\mathcal{C}$, then $h_1 = h_2$;

[...]

Another way to express the things of the previous definition is this: $f : a \to b$ is a monomorphism whenever for every $c \in |\mathcal{C}|$ the function

$$\mathcal{C}(c, a) \to \mathcal{C}(c, b)\,,\ g \to fg \tag{1.8}$$

is injective. Similarly, $f : a \to b$ is an epimorphism when for every $d \in |\mathcal{C}|$ the function

$$\mathcal{C}(a, d) \to \mathcal{C}(b, d)\,,\ h \to hf \tag{1.9}$$

is injective. Category theorists call the functions (1.8) *precompositions* with f and (1.9) *postcompositions* with f.

1.5 Basic constructions

In this section, we will present the first and most basic constructions involving categories.

For $\mathcal{C}$ a category, its *dual* (or *opposite*) category is denoted $\mathcal{C}^{\mathrm{op}}$ and is described as follows. Here, the objects are the same of $\mathcal{C}$ and 'being a morphism $a \to b$' exactly means 'being a morphism $b \to a$ in $\mathcal{C}$'. In other words, passing from a category to its dual leaves the objects unchanged, whereas the morphisms have their verses reversed. To dispel any ambiguity, by 'reversing' the morphisms we mean that morphisms $f : a \to b$ of $\mathcal{C}$ can be found among the morphisms $b \to a$ of $\mathcal{C}^{\mathrm{op}}$ and, vice versa, morphisms $a \to b$ of $\mathcal{C}^{\mathrm{op}}$ among the morphisms $b \to a$ of $\mathcal{C}$. Nothing is actually constructed out of the blue. Some authors suggest to write f^{op} to indicate that one f once it has domain and codomain interchanged, but we do not do that here, because they really are the same thing but in different places. So, if f is the name of a morphism of $\mathcal{C}$, the name f is kept to indicate that morphism as a morphism of $\mathcal{C}^{\mathrm{op}}$; obviously, the same convention applies in the opposite direction. It may seem we are going to nowhere, but it makes sense when it comes to define the compositions in $\mathcal{C}^{\mathrm{op}}$: for $f : a \to b$ and $g : b \to c$ morphsisms of $\mathcal{C}^{\mathrm{op}}$ the composite arrow is so defined

$$gf := fg.$$

This is not a commutative property, though. Such definition is to be read as follows. At the left side, f and g are to be intended as morphisms of $\mathcal{C}^{\mathrm{op}}$ that are to be composed therein. Then the composite gf is calculated as follows:

1. look at f and g as morphisms of $\mathcal{C}$ and compose them as such: so $f : b \to a$ and $g : c \to b$ and $fg : c \to a$ according to $\mathcal{C}$;
2. now regard fg as a morphism of $\mathcal{C}^{\mathrm{op}}$: this is the value gf is bound to.

Let us see now whether the categorial axioms are respected. For x object of $\mathcal{C}^{\mathrm{op}}$ there is 1_x, which is a morphism $x \to x$ in either of $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$. For every object y and morphism $f : y \to x$ of $\mathcal{C}^{\mathrm{op}}$ we have

$$1_x f = f 1_x = f.$$

Similarly, we have that

$$g 1_x = g$$

for every object z and morphism $g : x \to z$ of $\mathcal{C}^{\mathrm{op}}$. Hence, 1_x is an identity morphism in $\mathcal{C}^{\mathrm{op}}$ too. Consider now four objects and morphisms of $\mathcal{C}^{\mathrm{op}}$

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

and let us parse the composition

$$h(gf).$$

In $h(gf)$ regard both h and gf as morphisms of $\mathcal{C}$. In that case, $h(gf)$ is exactly $(gf)h$, where gf is fg once f and g are taken as morphisms of $\mathcal{C}$ and composed there. So $h(gf) = (fg)h$, where at left hand side compositions are performed in $\mathcal{C}$: being the composition is associative, $h(gf) = (fg)h = f(gh)$. We go back to $\mathcal{C}^{\mathrm{op}}$, namely $f(gh)$ becomes $(gh)f$ and gh becomes hg, so that we eventually get the associativity

$$h(gf) = (hg)f.$$

It may seem hard to believe, but duality is one of the biggest conquest of Category Theory. [Talk more about duality here...] This construction may seem a useless sophistication for now, but later we will discover how this serves the scope to make functors encompass a broader class of constructions. However, as for now, let us see all this under a new light: what does duality mean for diagrammatic reasoning? Commuting triangles

$$\begin{tikzcd} & b \arrow[d, "g"] \\ a \arrow[ur, "f"] \arrow[r, "h"'] & c \end{tikzcd}$$

of $\mathcal{C}$ are exactly commuting triangles

$$\begin{tikzcd} & b \arrow[dl, "f"'] \\ a & c \arrow[l, "h"] \arrow[u, "g"'] \end{tikzcd}$$

in $\mathcal{C}^{\mathrm{op}}$.

Exercise 1.22. In this way, it should be even more immediate to prove the two categorial axioms. Give it a try. Observe this approach is the mere translation of what we have conveyed with words above.

Example 1.23 (Dual prosets)**.** We already know how prosets are categories; let $(P, \leqslant)$ be one of them. Here the morphisms are exactly the pairs (b, a) such that $(a, b) \in \leqslant$. If we rephrase all this, we can introduce the dual relation $\geqslant$ defined by: $b \geqslant a$ if and only if $a \geqslant b$.

Exercise 1.24. Consider a single object category $\mathcal{G}$ — what is called moonid —. What is $\mathcal{G}^{\mathrm{op}}$? What is $\mathcal{G}^{\mathrm{op}}$ if $\mathcal{G}$ is a group?

The concept of duality for categories has one important consequence on statements written in a 'categorial language'. We do not need to be fully precise here: they are statements written in a sensible way using the usual logical connectives, names for objects, names for morphisms and quantifiers acting on such names.

Example 1.25. If we have a morphism $f : a \to b$ in some category $\mathcal{C}$, consider the statement

> For every object c of $\mathcal{C}$ and morphisms $g_1, g_2 : c \to a$ in $\mathcal{C}$, if $f g_1 = f g_2$ then $g_1 = g_2$.

If you remember, it is just said that f is a monomorphism. We operate a translation that doesn't modify the truth of the sentence: that is, if it is true, it remains so; it is false, it remains false.

> For every object c of $\mathcal{C}^{\mathrm{op}}$ and morphisms $g_1, g_2 : a \to c$ in $\mathcal{C}^{\mathrm{op}}$, if $g_1 f = g_2 f$ then $g_1 = g_2$.

If we regard f as a morphism $b \to a$ of $\mathcal{C}^{\mathrm{op}}$, then f is an epimorphism in $\mathcal{C}^{\mathrm{op}}$.

Let us try to settle this explicitly: if we have a categorial statement p, the dual of p — we may call p^{op} — is the statement obtained from p keeping the connectives and the quantifiers of p, whereas the other parts are replaced by their dual counterparts.

Example 1.26. [Anticipate products and co-products...]

Another useful construction is that of product of categories. Assuming we have two categories $\mathcal{C}_1$ and $\mathcal{C}_2$, the product $\mathcal{C}_1 \times \mathcal{C}_2$ is the category in which

- The objects are the pairs $(a, b) \in |\mathcal{C}_1| \times |\mathcal{C}_2|$.
- Being a morphism $(a_1, a_2) \to (b_1, b_2)$ means being a pair of two morphisms

$$\left(\begin{array}{c} a_1 \\ {\scriptstyle f_1}\downarrow \\ b_1 \end{array} , \begin{array}{c} a_2 \\ \downarrow{\scriptstyle f_2} \\ b_2 \end{array} \right)$$

  where f_1 is in $\mathcal{C}_1$ and f_2 in $\mathcal{C}_2$; we write such morphism as (f_1, f_2).
- The composition is defined component-wise

$$\left(\begin{array}{c} b_1 \\ {\scriptstyle g_1}\downarrow \\ c_1 \end{array} , \begin{array}{c} b_2 \\ \downarrow{\scriptstyle g_2} \\ c_2 \end{array} \right) \left(\begin{array}{c} a_1 \\ {\scriptstyle f_1}\downarrow \\ b_1 \end{array} , \begin{array}{c} a_2 \\ \downarrow{\scriptstyle f_2} \\ b_2 \end{array} \right) := (g_1 f_1, g_2 f_2).$$

Exercise 1.27. Verify categorial axioms hold for the product of two categories.

In future, we will need to consider product categories of the form $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$. For objects, there is nothing weird to say; about morphism, observe that a morphism

$$\begin{array}{c} (a, b) \\ \downarrow{\scriptstyle (f, g)} \\ (a', b') \end{array}$$

is precisely the pair

$$\left(\begin{array}{c} a \\ {\scriptstyle f}\uparrow \\ a' \end{array} , \begin{array}{c} b \\ \downarrow{\scriptstyle g} \\ b \end{array} \right)$$

whose first component comes from $\mathcal{C}$ while the second one from $\mathcal{D}$.

Let as now talk about comma categories.

Example 1.28. Words are labels humans attach to things to refer to them. Different groups of speakers have developed different names for the surrounding world, which resulted in different languages. We can use sets to gather the words present in any language. Now, if we are given a set Ω of things and a set L of the words of a chosen language, then a function $\lambda : \Omega \to L$ can be seen as the act of labelling things with names.[6] We will call such functions as

6. Of course, our discourse is rather simplified here: everything in Ω has one and only one dedicated word, which is not always the case. In fact, the existence of synonyms within a language undermines the requirement of uniqueness. Further, a language might not have words for everything: for instance, German has the word Schilderwald, which has not a corresponding single word in English — if you want to explain the meaning it bears, you can say it is 'a street that is so overcrowded and rammed with so many street signs that you are getting lost rather than finding your way.'

vocabularies for Ω, although this might not be the official name.
Languages do not live in isolation with others: if we know how to translate words, we can understand what speakers of other languages are saying. Imagine now you have two vocabularies

$$\begin{tikzcd} A & & B \\ & \Omega \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] \end{tikzcd}$$

To illustrate concept, think some $\omega \in \Omega$: if $\omega \in \Omega$ has the name $\alpha(\omega)$ under the vocabulary α and ω is called $\beta(\omega)$ according to β, then a translation would be a correspondence between $\alpha(\omega)$ and $\beta(\omega)$. We could say, a *translation* from α to β is a function $\tau : A \to B$ such that

$$\begin{tikzcd} A \arrow[rr, "\tau"] & & B \\ & \Omega \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] \end{tikzcd}$$

commutes. As you may expect, translations can be composed to obtain translations: if we have two translations η and θ as in the diagram

$$\begin{tikzcd} & B \arrow[dr, "\theta"] & \\ A \arrow[ur, "\eta"] & \Omega \arrow[l, "\alpha"] \arrow[u, "\beta"] \arrow[r, "\gamma"'] & C \end{tikzcd}$$

with the two triangles commuting, we have also the commuting

$$\begin{tikzcd} A \arrow[rr, "\theta\eta"] & & C \\ & \Omega \arrow[ul, "\alpha"] \arrow[ur, "\gamma"'] \end{tikzcd}$$

This is interesting if we look at things with categories: objects are functions that have a domain in common, and we have selected as morphisms the functions between the codomains that make certain triangles commute. Other examples in such spirit follow.

Example 1.29 (Covering spaces)**.** Probably you have heard of covering spaces when you had to calculate the first homotopy group of $\mathbb{S}^1$. More generally, a *covering space* of a topological space X is continuous function $p : \widetilde{X} \to X$ with the following property: there is a open cover $\{U_i \mid i \in I\}$ of X such that every $p^{-1}U_i$ is the disjoint union of a family $\left\{V_j^i \mid j \in J\right\}$ of open subsets of $\widetilde{X}$ and the restriction of p to V_j^i is a homeomorphism $V_j^i \to U_i$ for every $i \in I$ and $j \in J$. [To be continued...]

Example 1.30 (Field extensions)**.** [Yet to be TEX-ed...]

1.6 Functors

Definition 1.31 (Functors)**.** A functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is having the following functions, all indicated by F:

- one 'function on objects'

$$F : |\mathcal{C}| \to |\mathcal{D}| \ , \ x \to F(x)$$

- for every objects a and b, one 'function on morphisms'

$$F : \mathcal{C}(a,b) \to \mathcal{D}(F(a),F(b)) \ , \ f \to F(f)$$

such that

1. for every object x of $\mathcal{C}$ we have $F(1_x) = 1_{F(x)}$;
2. for every objects x, y, z and morphisms $f : x \to y$ and $g : y \to z$ of $\mathcal{C}$ we have $F(g)F(f) = F(gf)$.

To say that F is a functor from $\mathcal{C}$ to $\mathcal{D}$ we use $F : \mathcal{C} \to \mathcal{D}$, a symbolism that recalls that one of morphism in categories.

A first straightforward consequence of functoriality is contained in the following proposition.

Proposition 1.32. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If f is an isomorphism of $\mathcal{C}$, then so is $F(f)$.

Proof. **Exercise 1.33**. □

As often happens, let us start with simple exmaples: in this context, the simplest ones can be obtained by choosing very simple categories.

Example 1.34 (Functors from sets)**.** Classes can be regarded as categories with no morphisms apart identities: in any category, every object carries its own identity, but if these are the only morphisms, they become redundant information. We will restrict our attention to classes that are actual sets.
So, what is a functor $F : \mathcal{S} \to \mathcal{C}$ out of a set $\mathcal{S}$? As functors do by definition, it maps objects to objects and morphisms to morphisms; but the only morphisms of $\mathcal{S}$ are identities, which are taken to identities of $\mathcal{C}$. Since F involves only objects and identities, F is just a families of objects of $\mathcal{C}$.[7] In particular, functors from sets to sets are just functions!

Example 1.35 (Functors from prosets)**.** Consider a functor $F : (A, \leq) \to \mathcal{C}$ out of a proset. We know that $(A, \leq)$ regarded as a category has at most one morphism for each ordered couple in $A \times A$. For that reason, let us adopt this notation: for every $i, j \in A$ such that $i \leq j$ indicate by $F_{i,j}$ the image of the unique morphism $i \to j$ of $(A, \leq)$ via F. That being said, our functor F is just a collection $\{F_i \mid i \in A\}$ with the morphisms $F_{i,j}$'s for $i, j \in A$ with $i \leq j$.
As a particular instance of this, let us examine functors

$$H : (\mathbb{N}, \leq) \to \mathcal{C}$$

with $\leq$ being the usual ordering of $\mathbb{N}$. For $i, j \in \mathbb{N}$ with $i \leq j$, the morphism $i \to j$ can be factored into consecutive morphisms

$$i \to j = (j-1 \to j) \cdots (i+1 \to i+2)(i \to i+1).$$

For that reason, our H 'is' just a sequence

$$H_0 \xrightarrow{\partial_0} H_1 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{n+1}} H_n \xrightarrow{\partial_n} \cdots$$

of objects and morphisms in $\mathcal{C}$, where we have written ∂_j for $H_{j,j+1}$.

7. You probably are used to write $\{X_\alpha \mid \alpha \in I\}$ to indicate a family of sets. Actually, $\{X_\alpha \mid \alpha \in I\}$ is a function from the set of indexes I to some set the X_i's are picked from.

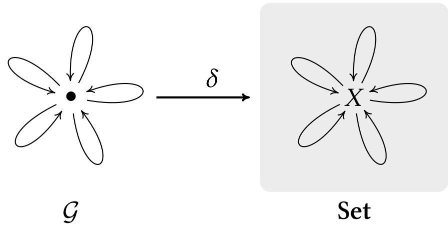


Figure 2. A group action as functor.

Exercise 1.36. Yes, diagrams are functors!

Example 1.37 (Monotonic functions)**.** We have met before, how a preordered set is a category; recall also the pure set-theoretic definition of this notion. For $(A, \leq_A)$ and $(B, \leq_B)$ preordered sets, a function $f : A \to B$ is said *monotonic* whenever for every $x, y \in A$ we have $f(x) \leq_B f(y)$ provided that $x \leq_A y$. In bare set-theoretic terms, this can be rewritten as follows: for every $x, y \in A$ such that $(x, y) \in \leq_A$, then $(f(x), f(y)) \in \leq_B$, where we make explicit the pairs, that are morphisms of the preordered sets seen as categories.

Example 1.38 (Monoid homomorphsisms)**.** We have previously seen that a monoid 'is' a single-object category. Consider now two such categories, say $\mathcal{G}$ and $\mathcal{H}$, and a functor $f : \mathcal{G} \to \mathcal{H}$ is. Denoting by $\bullet_{\mathcal{G}}$ and $\bullet_{\mathcal{H}}$ the object of $\mathcal{G}$ and $\mathcal{H}$ respectively, there is a unique possibility: mapping $\bullet_{\mathcal{G}}$ to $\bullet_{\mathcal{H}}$. The functorial axioms in that case are:

$$f(xy) = f(x)f(y)$$

for every morphisms x and y of $\mathcal{G}$ and

$$f(1_{\mathcal{G}}) = 1_{\mathcal{H}},$$

with $1_{\mathcal{G}}$ and $1_{\mathcal{H}}$ being the identities of $\mathcal{G}$ and $\mathcal{H}$ respectively. These two properties say that f is a monoid homomorphism; in this case there is also an equation that about objects but these two are a mere subtlety that adds nothing. It is easy to do the converse: a monoid homomorphism is a functor.

Let us remain to Algebra. What happens if the domain of a functor is a group and the codomain is **Set**? (Recall that 'group' is just a fancier abbreviation of 'single object groupoid'.) The answer is contained in the following example.

Example 1.39 (Group actions)**.** If $\mathcal{G}$ is a group, what is a functor $\delta : \mathcal{G} \to$ **Set**? Let us write G the set of its morphisms. The single object of $\mathcal{G}$ is mapped to one set X. Since all the morphisms of $\mathcal{G}$ are isomorphisms, then δ takes each of them to bijections from X to X. Thus we can say δ is the assignment of a certain set X and, for every g element of the group, of one isomorphism $\delta_g : X \to X$. Does this sound familiar? What we have described is a group *action* over the set X, that is a group isomorphism from G to the symmetric group of X.

Example 1.40 (The category **Eqv**)**.** A *setoid* [nlab uses this term...] is having a set and an equivalence relation defined on it. If X is a set and $\sim$ an equivalence relation over X, the setoid amounting of these data is written as $(X, \sim)$. Any set X has of course its own equality, that we denote by $=_X$.[8]
For if X and Y sets, a function $f : X \to Y$ respects this rule by definition:

8. In Set Theory, $=_X$ is the set $\{(a, a) \mid a \in X\}$.

for every $a, b \in X$, if $a =_X b$ then $f(a) =_Y b$.

We would like to replace the equalities above with equivalence relations: for if $(X, \sim_X)$ and $(Y, \sim_Y)$ are setoids, a *functoid* [ok, let me find/craft a nicer name...] from $(X, \sim_X)$ to $(Y, \sim_Y)$ is any function $f : X \to Y$ such that

for every $a, b \in X$, if $a \sim_X b$ then $f(a) \sim_Y f(b)$.

Functoids are certain type of functions, and composing two of them as such returns a funtoid. Categorial axioms hold almost for free, so we really have a *category of setoids and functoids*, **Eqv**.
Let us now involve functoriality. There is a nice theorem:

Let X and Y be two sets with $\sim_X$ and $\sim_Y$ equivalence relations on X and Y respectively. Then for every $f : X \to Y$ such that $f(a) \sim_Y f(b)$ for every $a, b \in X$ such that $a \sim_X b$, there exists one and only one $\phi : X/\sim_X \to Y/\sim_Y$ that makes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda a.[a]_X"'] & Y \arrow[d, "\lambda b.[b]_Y"] \\ X/\sim_X \arrow[r, "\phi"'] & Y/\sim_Y \end{tikzcd}$$

commute. (The vertical functions are the canonical projections.)

This underpins the functor

$$\pi : \mathbf{Eqv} \to \mathbf{Set}$$

that maps setoids $(X, \sim)$ to sets $X/\sim$ and functoids $f : (X, \sim_X) \to (Y, \sim_Y)$ to functions

$$\pi_f : X/\sim_X \to Y/\sim_Y$$
$$\pi_f([a]_X) := [f(a)]_Y,$$

whose existence and uniqueness is claimed by the just mentioned Proposition.

Example 1.41 (Free groups). Suppose given a *group alphabet* S, which is a set of things we decide to name 'letters'. Then a *group word* with system S is a string obtained by juxtaposition of a finite amount of 'x^1' and 'x^{-1}', where $x \in S$. The *empty word* is obtained by writing no letter, and we shall denote it by something, say e; instead, the other words appear as

$$x_1^{\phi_1} \cdots x_n^{\phi_n},$$

with $x_1, \dots, x_n \in S$ and $\phi_1, \dots, \phi_n \in \{-1, 1\}$. [9] [10]
The length of a word is the number of letters it is made of. We define equality only on words having the same length: we say $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is equal to $y_1^{\beta_1} \cdots y_n^{\beta_n}$ whenever $x_i = y_i$ and $\alpha_i = \beta_i$ for every $i \in \{1, \dots, n\}$.
A group word $x_1^{\phi_1} \cdots x_n^{\phi_n}$ is said *irreducible* whenever $x_i^{\phi_i} \neq x_{i+1}^{-\phi_{i+1}}$ for every $i \in \{1, \dots, n-1\}$; the empty word is irreducible by convention. Let us write $\langle S \rangle$ the set of all irreducible words written using the alphabet S. It is natural

9. Something that may irk you is that our words can be redundant, being consecutive repetitions of the same letter allowed. If you want, you can let exponents range over all the integers, but this needs you to modify what comes after.
10. Here, we can choose any pair of distinct symbols instead of -1 and 1. If we do so, we need a function that maps each of them into the other one. In this presentation we employ the function that takes one integer and returns its opposite.

to join two words by bare juxtaposition, but the resulting word may not be irreducible; this issue has to be fixed:

$$
\begin{aligned}
&\cdot : \langle S\rangle \times \langle S\rangle \to \langle S\rangle \\
&e\cdot w := w\,,\ \ w\cdot e := w \\
&(x_1^{\lambda_1}\cdots x_m^{\lambda_m})\cdot(y_1^{\mu_1}\cdots y_n^{\mu_n}) := \begin{cases} (x_1^{\lambda_1}\cdots x_{m-1}^{\lambda_{m-1}})\cdot(y_2^{\mu_2}\cdots y_n^{\mu_n}) & \text{if } x_m^{\lambda_m} = y_1^{-\mu_1} \\ x_1^{\lambda_1}\cdots x_m^{\lambda_m} y_1^{\mu_1}\cdots y_n^{\mu_n} & \text{otherwise.}\end{cases}
\end{aligned}
$$

Let us define a function that either reverses the order of the letters and changes each exponent to the other one:

$$i : \langle S\rangle \to \langle S\rangle\ ,\ i\left(x_1^{\xi_1}\cdots x_i^{\xi_i} x_{i+1}^{\xi_{i+1}}\cdots x_n^{\xi_n}\right) := x_n^{-\xi_n}\cdots x_{i+1}^{-\xi_{i+1}} x_i^{-\xi_i}\cdots x_1^{-\xi_1}.$$

It is immediate to show that $w\cdot i(w) = i(w)\cdot x = e$ for every $w \in \langle S\rangle$. Only the associativity of $\cdot$ is a a bit tricky to prove. At this point we have endowed $\langle S\rangle$ with a group structure.
Thus from a set S we are able to build a group $\langle S\rangle$, that is called *free group* with base S, or group generated by S. Now, if take two sets S and T and a function $f : S \to T$, we have the group homomorphism

$$\langle f\rangle : \langle S\rangle \to \langle T\rangle\ ,\ \langle f\rangle\,(x_1^{\delta_1}\cdots x_n^{\delta_n}) := \big(f(x_1)\big)^{\delta_i}\cdots\big(f(x_n)\big)^{\delta_n}.$$

It is immediate to demonstrate that we ended up with having a functor

$$\langle\ \ \rangle : \mathbf{Set} \to \mathbf{Grp}.$$

In future we will provide other examples involving free modules.

Example 1.42 (Free modules)**.** The explicit construction of the free *abelian* group given a set is simpler than that of free group in general. Since an abelian group is a $\mathbb{Z}$-module, let us show how to build a free module.
Let R be a ring and S be a set, as in the previous example. Intuitively, the module generated by S are linear combination of a finite amount of elements of S, that is expressions of the form

$$\sum_{i=1}^{n} \lambda_i x_i$$

for $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in R$ and $x_1, \dots, x_n \in S$. Observe, however that the 'sum' here is just a formal expression: there is no link to the an operation of sum yet. We can rethink this linear combination as something more manageable during computations:

$$\sum_{x\in S} \lambda_x x$$

where $\lambda : S \to R$ is non zero for a finite amount of elements of S. Observe that it is just formalism: S may be an infinite set, but the sum $\sum_{x\in S}\lambda_x x$ is not to be understood as a series in Analysis; consider also $\lambda_x \neq 0$ for finitely many x, so if S is infinite, the most of the terms are useless. [Instead of using the device of 'formal expressions', we can define the module words as functions $\lambda : S \to R$ that assume non-zero values for a finite amount of elements. Isn't that the same stuff of a formal sum?]
Thus, let us write the explicit definition:

$$\langle S\rangle := \left\{ \sum_{x\in S}\lambda_x x \,\middle|\, \lambda : S \to R\,,\ \lambda_x \neq 0 \text{ for finitely many times} \right\}.$$

This is only the first step to make a module with such set. We give a sum

$$\begin{aligned}+ : \langle S\rangle \times \langle S\rangle &\to \langle S\rangle \\ \left(\sum_{x\in S}\alpha_x x\right) + \left(\sum_{x\in S}\beta_x x\right) &:= \sum_{x\in S}(\alpha_x + \beta_x)x\end{aligned}$$

and an external product

$$\begin{aligned}\cdot : R \times \langle S\rangle &\to \langle S\rangle \\ \eta \cdot \left(\sum_{x\in S}\alpha_x x\right) &:= \sum_{x\in S}(\eta\alpha_x)x\end{aligned}$$

It is simple to verify that $\langle S\rangle$ is a R-module now.
So far, we only have an process that takes sets and emits R-modules: to make a functor, we also need to instruct how to construct a linear function from a simple function of sets. For $f : S \to T$, we give

$$\begin{aligned}\langle f\rangle : \langle S\rangle &\to \langle T\rangle \\ \langle f\rangle \left(\sum_{x\in S}\lambda_x x\right) &:= \sum_{x\in S}\lambda_x f(x).\end{aligned}$$

It is simple to verify we have a functor

$$\langle\ \ \rangle : \mathbf{Set} \to \mathbf{Mod}_R.$$

Exercise 1.43. There is a plenty of 'free stuff' around that can give arise to functors like the one above. Find and illustrate some of them.

Example 1.44 (The First Homotopy Group)**.** A *pointed topological space* is a topological space X with one point $x_0 \in X$; we write it as (X, x_0). We define a *pointed continuous function* $(X, x_0) \to (Y, y_0)$ to be a continuous functions $X \to Y$ taking x_0 to y_0. Furthermore, composing such functions yields a pointed continuous function. So, really we have the category of pointed topological spaces, we denote by $\mathbf{Top}_*$.

$$\Omega(X, x_0) := \{\text{continuous } \phi : [0,1] \to X \mid \phi(0) = \phi(1) = x_0\}$$

and call its elements *loops* of X based at x_0. Two loops can be joined, that is traversing one loop after another one: for if $\phi, \psi \in \Omega(X, x_0)$ we introduce the loop $\phi * \psi : [0,1] \to X$ with

$$(\phi * \psi)(t) := \begin{cases}\phi(2t) & \text{if } t \le \frac{1}{2} \\ \psi(2t-1) & \text{otherwise.}\end{cases}$$

This gives us the operation of *junction* of loops

$$* : \Omega(X, x_0) \times \Omega(X, x_0) \to \Omega(X, x_0).$$

Now it's time to find a suitable equivalence relation that is compatible with this operation. For if $\phi, \psi \in \Omega(X, x_0)$, we say ϕ is *homotopic* to ψ whenever there exists a *homotopy* from ϕ to ψ, viz a continuous function

$$H : [0,1] \times [0,1] \to X$$

such that $H(\ ,0) = \phi$, $H(\ ,1) = \psi$, $H(s,0) = H(s,1) = x_0$ for every $s \in [0,1]$. This relation is an equivalence one and it is compatible with $*$. It remains to verify some properties to define a group structure:

- $(\alpha * \beta) * \gamma$ is homotopic to $\alpha * (\beta * \gamma)$ for every $\alpha, \beta, \gamma \in \Omega(X, x_0)$.
- the paths $\alpha * c_{x_0}$, $c_{x_0} * \alpha$ and α are homotpic for every $\alpha \in \Omega(X, x_0)$; here, c_{x_0} is the loop defined by $c_{x_0}(t) = x_0$.
- $\alpha * \alpha^{-1}$, $\alpha^{-1} * \alpha$ and c_{x_0} are homotopic for every $\alpha \in \Omega(X, x_0)$; here, α^{-1} is the loop with $\alpha^{-1}(t) := \alpha(1-t)$.

We have now all the ingredients to introduce a group: define $\pi_1(X, x_0)$ to be the set obtained identifying homotopic elements of $\Omega(X, x_0)$; this set is a group once you consider the operation

$$\begin{aligned} \pi_1(X, x_0) \times \pi_1(X, x_0) &\to \pi_1(X, x_0) \\ ([\alpha], [\beta]) &\to [\alpha][\beta] := [\alpha * \beta]. \end{aligned}$$

Here, we have written $[\phi]$ for the set of loops homotopic to ϕ. Sometimes — especially if we are considering more topological spaces —, we need to specify the topological space we are taking loops, for example writing $[\phi]_X$. Now, it is the turn to define induced homomorphisms: for a pointed continuous function $f : (X, x_0) \to (Y, y_0)$ we have the group homomorphism

$$\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)\,,\ \pi_1(f)[\phi]_X := [f\phi]_Y.$$

(You may have a look at Example 1.40.) Instead of $\pi_1(f)$, you may have been get used to f_*. In conclusion, we have just defined one functor

$$\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}.$$

The *first fundamental group* is not just a group, and that the actual group is just a piece of larger picture.

Traditionally, functors of Definition 1.31 above are called 'covariant', because there are *contra*variant functors too. However, there is no sensible reason to maintain these two adjectives; at least, almost everyone agrees to not use the first adjective, whilst the second one still survives.

For if $\mathcal{C}$ and $\mathcal{D}$ are categories, a *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is just a functor $\mathcal{C}^{\text{op}} \to \mathcal{D}$. It is best that we say what functors $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$ do. They map objects to objects and morphisms $f : a \to b$ of $\mathcal{C}^{\text{op}}$ to morphisms $F(f) : F(a) \to F(b)$ of $\mathcal{D}$. But, remembering how dual categories are defined, what F actually does is this:

> it maps objects of $\mathcal{C}$ to objects of $\mathcal{D}$, and morphisms $f : b \to a$ of $\mathcal{C}$ to morphisms $F(f) : F(a) \to F(b)$ of $\mathcal{D}$ (mind that a and b have their roles flipped).

Now, what about functoriality axioms? Neither with identities F does something different and the composite gf of $\mathcal{C}^{\text{op}}$ is mapped to the composite $F(g)F(f)$ of $\mathcal{D}$. Again by definition of dual categories, this can be translated as follows:

> the composite fg of $\mathcal{C}$ is mapped to $F(g)F(f)$ (notice here how f and g have their places switched).

You can think of contravariant functors as a trick to do what we want.

Example 1.45. The set of natural numbers $\mathbb{N}$ has the order relation of divisibility, that we denote $|$: regard this poset as a category. From Group Theory, we know that for every $m, n \in \mathbb{N}$ such that $m \mid n$ there is a homomorphism

$$f_{m,n} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}\,,\ f_{m,n}(a + n\mathbb{Z}) := a + m\mathbb{Z}.$$

In fact, $\mathbb{Z}/m\mathbb{Z}$ is the kernel of the homomorphism

$$\pi_m : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}\,,\ \pi_m(x) := x + m\mathbb{Z}$$

and, because $m \mid n$, we have $n\mathbb{Z} \subseteq m\mathbb{Z}$. In that case, some Isomorphism Theorem[11] justifies the existence of $f_{m,n}$. This offers us a nice functor:

$$F : (\mathbb{N}, \mid)^{\mathrm{op}} \to \mathbf{Grp}$$

that maps naturals n to groups $\mathbb{Z}/n\mathbb{Z}$ and $m \mid n$ to the homomorphism $f_{m,n}$ defined above.

[Clearly, this section needs more work...]

Functors can be composed — and I think at this point it is not a secret. Take $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ categories and functors

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}.$$

The sensible way to define the composite functor $GF : \mathcal{C} \to \mathcal{E}$ is mapping the objects x of $\mathcal{C}$ to the objects $GF(x)$ of $\mathcal{E}$, and the morphisms $f : x \to y$ of $\mathcal{C}$ to the morphisms $GF(f) : GF(x) \to GF(y)$ of $\mathcal{E}$. That being set, the composition is associative and there is an identity functor too.

There are all the conditions, so what prevents us to consider a category — we can call **Cat** — that has categories as objects and functors as morphisms?

If we work upon NBG, we can think of any proper class as a category, for this statement have a closer look at Example 1.34. What happens now is that the class of objects of **Cat** has an element that is a proper class, which isn't legal in NBG.

Is a category of *locally small* categories and functors problematic? Take $\mathcal{C}$ such that $\mathcal{C}(a, b)$ is a proper class for some a and b objects: consider $\mathcal{C}/b$. In this case $|\mathcal{C}/b|$ is a proper class too, and here we go again.

Now what? If we stick to NBG, this is a limit we have to take into account. From now on, **Cat** is the category of *small* categories and functors between small categories.

1.7 The hom functor

In a locally small [have we defined somewhere that?] category $\mathcal{C}$, take two morphisms

$$\begin{array}{ccc} a_1 & & a_2 \\ {\scriptstyle g_1}\big\uparrow & \text{and} & \big\downarrow{\scriptstyle g_2} \\ b_1 & & b_2 \end{array}$$

For $f : a_1 \to a_2$ we have the composite $g_2 f g_1 : b_1 \to b_2$, that is a function

$$\mathcal{C}(a_1, a_2) \to \mathcal{C}(b_1, b_2).$$

11. How theorems are named sometimes varies, so for sake of clarity let us explicit the statement we are referring to: Let G and H be two groups, $f : G \to H$ an homomorphism and N some normal subgroup of G. Consider also the homomorphism $p_N : G \to G/N$, $p_N(x) := xN$. If $N \subseteq \ker f$ then there exists one and only one homomorphism $\overline{f} : G/N \to H$ such that $f = \overline{f} p_N$. (Moreover, $\overline{f}$ is surjective if and only if so is f.)

We will refer to this function using lambda calculus notation: $\lambda f.g_2 f g_1$. If we have

$$\begin{tikzcd} a_1 \arrow[r, "f"] & a_2 \arrow[d, "g_2"] \\ b_1 \arrow[u, "g_1"] & b_2 \arrow[d, "h_2"] \\ c_1 \arrow[u, "h_1"] & c_2 \end{tikzcd}$$

we can say that

$$\lambda f.(h_2 g_2) f (g_1 h_1) = (\lambda f'.h_2 f' h_1)(\lambda f.g_2 f g_1),$$

which can be derived by uniquely using the associativity of the composition. Another remarkable property can be obtained when $a_1 = b_1$, $g_1 = 1_{a_1}$, $a_2 = b_2$ and $g_2 = 1_{a_2}$:

$$\lambda f.1_{a_2} f 1_{a_1} = \lambda f.f = 1_{\mathcal{C}(a_1,b_1)}.$$

There is functoriality, to understand that, we need to package all this machinery in one functor. The functor we are looking for is

$$\hom_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$$

which takes every $(x,y) \in |\mathcal{C}^{\mathrm{op}} \times \mathcal{C}|$ to $\mathcal{C}(x,y)$ and

$$\hom_{\mathcal{C}}\left(\begin{array}{ccc} a_1 & & a_2 \\ {\scriptstyle g_1}\uparrow & , & \downarrow{\scriptstyle g_2} \\ b_1 & & b_2 \end{array} \right)\left(a_1 \xrightarrow{f} a_2 \right) := g_2 f g_1.$$

1.8 Constructions involving functors

[Yet to be TEXed...]

1.9 Natural transformations

For $\mathcal{C}$ and $\mathcal{D}$ categories and $F, G : \mathcal{C} \to \mathcal{D}$ functors, a *transformation* from F to G amounts at having for every $x \in |\mathcal{C}|$ one morphism $F(x) \to G(x)$ of $\mathcal{D}$. In other words, a transformation is aimed to measure the difference of two parallel functor by the unique means we have, viz morphisms.

In general, we stick to the following convention: if η is the name of a transformation from F to G, then η_x indicates the component $F(x) \to G(x)$ of the transformation.

We are not interested in all transformations, of course.

Definition 1.46 (Natural transformations)**.** A transformation η from a functor $F : \mathcal{C} \to \mathcal{D}$ to a functor $G : \mathcal{C} \to \mathcal{D}$ is said to be *natural* whenever for every $a, b \in |\mathcal{C}|$ and $f \in \mathcal{C}(a,b)$ the square

$$\begin{tikzcd} F(a) \arrow[r, "\eta_a"] \arrow[d, "F(f)"'] & G(a) \arrow[d, "G(f)"] \\ F(b) \arrow[r, "\eta_b"'] & G(b) \end{tikzcd}$$

commutes. This property is the 'naturality' of η.

There are some notations for referring to natural transformations: one may write $\eta : F \Rightarrow G$ or even

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd}$$

if they want to explicit also categories.

Natural transformations can be composed: taken two consecutive natural transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{description, name=G}] \arrow[r, bend right=60, "H"'{name=H}] & \mathcal{D} \arrow[Rightarrow, from=F, to=G, "\eta"] \arrow[Rightarrow, from=G, to=H, "\theta"] \end{tikzcd}$$

the transformation $\theta\eta$ that have the components $\theta_x\eta_x : F(x) \to H(x)$, for $x \in |\mathcal{C}|$ of $\mathcal{D}$ is natural. Such composition is associative. Moreover, for every functor $F : \mathcal{C} \to \mathcal{D}$ there is the natural transformation $1_F : F \Rightarrow F$ with components $1_{F(x)} : F(x) \to F(x)$, for $x \in |\mathcal{C}|$; they are identities in categorial sense:

$$\eta 1_F = \eta \text{ for every natural transformation } \eta : F \Rightarrow G$$
$$1_F \mu = \mu \text{ for every natural transformation } \mu : H \Rightarrow F.$$

All this suggests to, given two categories $\mathcal{C}$ and $\mathcal{D}$, form a category with functors $\mathcal{C} \to \mathcal{D}$ as objects and natural transformations as morphism, them being composable as explained above. [...]

[Consider `https://mathoverflow.net/q/39073`...]

1.10 Equivalent categories

Let us give a definition that will motivate our discourse.

Definition 1.47 (Full- and faithfulness)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is said *full*, respectively *faithful*, whenever for every $a, b \in |\mathcal{C}|$ the functions

$$F : \mathcal{C}(a, b) \to \mathcal{D}(F(a), F(b))$$

are surjective, respectively injective; we say that F is *fully faithful* [how lame, lol...] whenever it is both full and faithful.

What do we want 'two categories are the same' to mean? [Craft a nicer exposition... Let us try with categories being isomorphic first, and then with *essentially surjective* functors. Talk about *skeletons* of categories, and how can help to say whether two categories are equivalent.]

Example 1.48 (A functor $\mathbf{Mat}_k \to \mathbf{FDVect}_k$)**.** For k field, consider the functor

$$M : \mathbf{Mat}_k \to \mathbf{FDVect}_k$$

that maps $n \in |\mathbf{Mat}_k| = \mathbb{N}$ to $M(n) := k^n$ and $A \in \mathbf{Mat}_k(r, s)$ to the linear function

$$M_A : k^r \to k^s$$
$$M_A(x) = Ax.$$

(Here the elements of k^n are matrices of type $n \times 1$.) [...]

1.11 The Yoneda Lemma

[This section is to be moved elsewhere...][Maybe, I should stick to *small* categories...] [Use `cramped` for some `tikzcds`...]

We have the *evaluation functor*

$$\mathrm{ev}_{\mathcal{C}} : \mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$$

that on objects

$$\mathrm{ev}_{\mathcal{C}}(x, F) := F(x)$$

and on morphisms

$$\mathrm{ev}\left(\begin{array}{c} a \\ \downarrow f \\ b \end{array} , \begin{array}{c} F \\ \eta \Downarrow \\ G \end{array} \right) := \eta_b F(f) = G(f)\eta_a .$$

Lemma 1.49 (A lemma for the Yoneda Lemma)**.** Let $\mathcal{C}$ be a locally small category. Then for every $x \in |\mathcal{C}|$ and functor $F : \mathcal{C} \to \mathbf{Set}$,

$$[\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F) \cong F(x).$$

In particular, the classes $[\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F)$ are actual sets.

Proof. For x and F as in the hypothesis, take functions

$$\lambda_{x,F} : [\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F) \to F(x)\,,\; \lambda_{x,F}(\alpha) := \alpha_x(1_x).$$

Now, for every $a \in F(x)$ we have the transformation $\mu_{x,F}(a)$ from $\mathcal{C}(x, \bullet)$ to F which has the components

$$\mathcal{C}(x, c) \to F(c)\,,\; f \to \big(F(f)\big)(a);$$

it is immediate to show that it is natural. Thus we have functions

$$\mu_{x,F} : F(x) \to [\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F).$$

We prove

$$\begin{aligned} \lambda_{x,F}\mu_{x,F} &= 1_{F(x)} \\ \mu_{x,F}\lambda_{x,F} &= 1_{[\mathcal{C},\mathbf{Set}](\mathcal{C}(x,-),F)}. \end{aligned}$$

In fact, for $a \in F(x)$ we have $\lambda_{x,F}\big(\mu_{x,F}(a)\big)$ is the component $\mathcal{C}(x, x) \to F(x)$ of $\mu_{x,F}(a)$ evaluated at 1_x, viz $1_{F(x)}(a) = a$. Besides, for if $\alpha : \mathcal{C}(x, \bullet) \Rightarrow F$ natural transformation we have $\mu_{x,F}\big(\lambda_{x,F}(\alpha)\big) = \mu_{x,F}\big(\alpha_x(1_x)\big)$ is the natural transformation $\mathcal{C}(x, \bullet) \Rightarrow F$ with components

$$\mathcal{C}(x, c) \to F(c)\,,\; f \to \big(F(f)\big)(\alpha_x(1_x)) = \alpha_c(f)$$

for $c \in |\mathcal{C}|$; that is $\mu_{x,F}\lambda_{x,F}(\alpha) = \alpha$. The proof is complete now. □

Let $\mathcal{C}$ be a locally small category. We have the functor

$$\mathcal{Y}_{\mathcal{C}} : \mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$$

given on objects as follows

$$\mathcal{Y}_{\mathcal{C}}(x, F) := [\mathcal{C}, \mathbf{Set}]\big(\mathcal{C}(x, -), F\big)$$

and on morphisms

$$\left[\mathcal{Y}_{\mathcal{C}}\left(\begin{array}{c}a\\ f\big\downarrow\\ b\end{array},\ \begin{array}{c}F\\ \eta\Downarrow\\ G\end{array}\right)\right]\left(\begin{array}{c}\mathcal{C}(a,-)\\ \Downarrow\alpha\\ F\end{array}\right):=\left\{\mathcal{C}(b,c)\xrightarrow{\eta_c\alpha_c(\_f)}G(c)\,\middle|\,c\in|\mathcal{C}|\right\}.$$

Observe that Lemma 1.49 solves annoying size issues in the definition of $\mathcal{Y}_{\mathcal{C}}$ on objects. While the statement of this lemma is important for technical reasons, its proof guides us to the following completion.

Proposition 1.50 (Yoneda Lemma)**.** For $\mathcal{C}$ locally small category, $\mathcal{Y}_{\mathcal{C}} \cong \mathrm{ev}_{\mathcal{C}}$.

Proof. The transformation $\lambda : \mathcal{Y}_{\mathcal{C}} \Rightarrow \mathrm{ev}_{\mathcal{C}}$ having as components the functions $\lambda_{x,F}$ of the proof of Lemma 1.49 is natural, that is

$$\begin{tikzcd}
\mathcal{Y}_{\mathcal{C}}(a,F) \arrow[r, "\lambda_{a,F}"] \arrow[d, "\mathcal{Y}_{\mathcal{C}}(f,\eta)"'] & \mathrm{ev}_{\mathcal{C}}(a,F) \arrow[d, "\mathrm{ev}_{\mathcal{C}}(f,\eta)"] \\
\mathcal{Y}_{\mathcal{C}}(b,G) \arrow[r, "\lambda_{b,G}"'] & \mathrm{ev}_{\mathcal{C}}(b,G)
\end{tikzcd}$$

commutes for every $f \in \mathcal{C}(a,b)$ and $\eta \in [\mathcal{C},\mathbf{Set}](F,G)$. In fact, for every natural transformation $\eta \in \mathcal{Y}_{\mathcal{C}}(a,F)$ we have

$$\mathrm{ev}_{\mathcal{C}}(f,\eta)\big(\lambda_{a,F}(\alpha)\big) = \eta_b\alpha_b\mathcal{C}(a,f)(1_a) = \eta_b\alpha_b f;$$

besides,

$$\lambda_{b,G}\big(\mathcal{Y}_{\mathcal{C}}(f,\eta)(\alpha)\big) = \eta_b\alpha_b(\_f)(1_b) = \eta_b\alpha_b f.$$

We can conclude λ is an isomorphism, as the proof of Lemma 1.49 tells us its components are isomorphisms. $\square$