# Notes on Category Theory: Topoi

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1 Subobject classifiers

## Subobject classifiers

Throughout the current section, we assume  $\mathcal{E}$  is a category with initial object 1. That being the setting, we can give the following definition.

**Definition 1.1.** A *subobject classifier* for  $\mathcal E$  is any morphism  $t:1\to \Omega$  such that: for every monomorphism  $f:a\to b$  of  $\mathcal E$  there is one and only one morphism  $\chi_f:b\to\Omega$  in  $\mathcal E$  for which there is a pullback square

$$\begin{array}{ccc}
a & \xrightarrow{f} & b \\
\downarrow \downarrow & & \downarrow \chi_f \\
1 & \xrightarrow{t} & \Omega
\end{array}$$
(1.1)

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That is we can assign to every monomorphism  $f:a\to b$  the morphism  $\chi_f:b\to\Omega$  satisfying the property of the definition. Let us introduce then some symbolism: for  $b\in|\mathcal{E}|$  we write  $\mathrm{Sub}_{\mathcal{E}}\,b$  for the class of all the monomorphisms of  $\mathcal{E}$  with codomain b. Hence we can introduce the function

$$\chi : \operatorname{Sub}_{\mathcal{E}} b \to \mathcal{E}(b, \Omega)$$

with  $\chi_f$  defined to be that morphism  $b \to \Omega$  for which there is a pullback square as the diagram (1.1).

It is worth to observe  $Sub_{\mathcal{E}}(b)$  has a natural structure of preorder: for



monomorphisms of  $\mathcal{E}$ , write  $f_1 \leq f_2$  to say there is some  $h: a_1 \to a_2$  in  $\mathcal{E}$  for which



commutes. Note that, being here  $f_1$  and  $f_2$  monomorphisms, there is at most one h as such and it is a monomorphism as well.

We show now the relation  $\simeq$  on  $Sub_{\mathcal{E}} b$  defined by

$$f_1 \simeq f_2$$
 if and only if  $f_1 \leq f_2$  and  $f_2 \leq f_1$ 

for  $f_1, f_2 \in \operatorname{Sub}_{\mathcal{E}} b$  is an equivalence relation. [...]

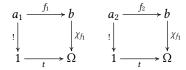
Yes,  $\operatorname{Sub}_{\mathcal{E}} b$  is the full subcategory of  $\mathcal{E} \downarrow b$  whose objects are all the monomorphisms of  $\mathcal{E}$  with codomain b, and whose isomorphism relation is  $\simeq$ .

#### **Proposition 1.2.** Let



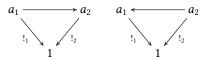
be monomorphisms.  $\chi_{f_1} = \chi_{f_2}$  if and only if  $f_1 \simeq f_2$ .

*Proof.* Assume  $\chi_{f_1} = \chi_{f_2}$ . By definition of subobject classifiers,  $\chi_{f_1}$  is the morphism for which



are pullback squares. Consequently, we must infer that there is one isomorphism  $h: a_1 \to a_2$  such that  $f_1 = f_2 h$ . Hence  $f_1 \le f_2$ , and  $f_2 \le f_1$  too, because  $f_1 h^{-1} = f_2$ .

For the remaining part of the proof, let us write  $!_1$  the unique morphism  $a_1 \to 1$  and  $!_2$  the unique morphism  $a_2 \to 1$ . Also remember that triangles



always commute.

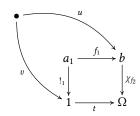
Now we suppose  $f_1 \simeq f_2$ . The plan for the proof is: if we show that



is a pullback square, then, being  $\chi_{f_1}: b \to \Omega$  the one for which there is a pullback square like this, we can conclude  $\chi_{f_1} = \chi_{f_2}$ . First of all such square commutes: if we call h the morphism  $a_1 \to a_2$  such that  $f_1 = f_2 h$ , then

$$\chi_{f_2}f_1 = \chi_{f_2}f_2h = t!_2h = t!_1.$$

Consider

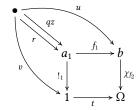


### 1. Subobject classifiers

where  $\chi_{f_2}u = tv$ . Being



a pullback square we have one  $z: \bullet \to a_2$  for which  $f_2z=u$  and  $!_2z=v$ . From the assumption  $f_1\simeq f_2$ , we have  $f_2\leq f_1$ , that is  $f_2=f_1q$  for some  $q:a_2\to a_1$ . Then  $u=f_1qz$  and  $v=!_1qz$ . Let us see if qz is what we are looking for.



where we suppose  $!_1r = v$  and  $f_1r = u$ . Being  $f_1$  a monomorphism, the sole second identity is enough to conclude r = qz.