4.9 in Baumgarte

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(Dated: July 8, 2020)

I. 4.9

A. Useful Identities

$$\sqrt{-g} = \alpha\sqrt{\gamma}, \tag{1}$$

$$\partial_t \ln|\gamma|^{1/2} = -\alpha K + D_i\beta^i, \tag{2}$$

$$\partial_t K = -D^2\alpha + \alpha\left(K_{ij}K^{ij} + 4\pi(\rho + S)\right) + \beta^i D_i K, \tag{3}$$

$$g^{ab} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^i \\ -\alpha^{-2}\beta^j & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix}. \tag{4}$$

B. Problem statement

Using

$$^{(4)}\Gamma^a = -\frac{1}{\sqrt{-g}}\partial_b\left(\sqrt{-g}g^{ab}\right) = 0 \tag{5}$$

derive following equalities:

$$(\partial_t - \beta^j\partial_j)\alpha = -\alpha^2 K, \tag{6}$$

$$(\partial_t - \beta^j\partial_j)\beta^i = -\alpha^2\left(\gamma^{ij}\partial_j\ln\alpha + \gamma^{jk}\Gamma^i_{jk}\right). \tag{7}$$

C. Solution

Let's rewrite $^{(4)}\Gamma^0$ first:

$$-\frac{1}{\alpha\sqrt{\gamma}}\partial_b(\alpha\sqrt{\gamma}g^{0b}) = -\frac{1}{\alpha\sqrt{\gamma}}\partial_t(\sqrt{\gamma}\alpha^{-1}) - \frac{1}{\alpha\sqrt{\gamma}}\partial_i(\sqrt{\gamma}\alpha^{-1}\beta^i) \tag{8}$$

$$= -\left(\alpha^{-2}\partial_t + \alpha^{-2}\beta^i\partial_i\right)\ln\sqrt{\gamma} + \left(\alpha^{-2}\partial_t + \alpha^{-2}\beta^i\partial_i\right)\alpha - \alpha^{-2}\partial_i\beta^i = 0. \tag{9}$$