2.21 in Baumgarte

indutny

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I. 2.21

A. Problem statement

Useful identities:

$$\Omega_a = \nabla_a t, \tag{1}$$

$$\nabla_{[a}\Omega_{b]} = 0, \tag{2}$$

$$-\frac{1}{\alpha^2} = g^{ab}\Omega_a\Omega_b, \tag{3}$$

$$\omega_a = \alpha\Omega_a, \tag{4}$$

$$n^a = -g^{ab}\omega_b, \tag{5}$$

$$n^a n_a = -1, \tag{6}$$

$$\gamma^a_{\ b} = \delta^a_{\ b} + n^a n_b, \tag{7}$$

$$n^a\nabla_a n_b \equiv a_b, \tag{8}$$

$$\gamma^a_{\ b}\frac{\nabla_a\alpha}{\alpha} = a_b. \tag{9}$$

Prove that:

$$\mathcal{L}_{\alpha n}\gamma^a_{\ b} = 0. \tag{10}$$

B. Attempted solution

$$\begin{aligned}
\mathcal{L}_{\alpha n}\gamma^a_{\ b} =& \alpha n^c\nabla_c\gamma^a_{\ b} + \nabla_b(\alpha n^c)\gamma^a_{\ c} - \nabla_c(\alpha n^a)\gamma^c_{\ b} && (11)\\
=& \alpha n^c\nabla_c(n^a n_b) + \underbrace{\gamma^a_{\ c}n^c}_{0}\nabla_b\alpha + \alpha\gamma^a_{\ c}\nabla_b n^c - n^a\underbrace{\gamma^c_{\ b}\nabla_c\alpha}_{\alpha a_b} - \alpha\gamma^c_{\ b}\nabla_c n^a && (12)\\
=& \alpha n_b\underbrace{n^c\nabla_c n^a}_{a^a} + \alpha n^a\underbrace{n^c\nabla_c n_b}_{a_b} + \alpha\nabla_b n^a + \alpha n^a n_c\nabla_b n^c - \alpha n^a a_b && (13)\\
&- \alpha\nabla_b n^a - \alpha n_b\underbrace{n^c\nabla_c n^a}_{a^a} && (14)\\
=& \alpha n^a n_c\nabla_b n^c = \alpha n^a\underbrace{\nabla_b(n_c n^c)}_{0} - \alpha n^a n^c\nabla_b n_c && (15)\\
=& -n^a n^c\underbrace{\nabla_b(\alpha n_c)}_{-\nabla_b\omega_c} + n^a\underbrace{n^c n_c}_{-1}\nabla_b\alpha && (16)
\end{aligned}$$