19.1 in MTW

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I. DERIVATION OF METRIC FAR OUTSIDE A WEAKLY GRAVITATING BODY

A. tt - component

The trace-reversed stress-energy tensor is defined as:

$$\bar{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho}T_{\sigma\rho}, \tag{1}$$

where

$$\eta^{\sigma\rho}T_{\sigma\rho} = -T_{00} + T_{ii}. \tag{2}$$

The tt component is:

$$\bar{T}_{00} = T_{00} - \frac{1}{2}T_{00} + \frac{1}{2}T_{ii} = \frac{1}{2}\left(T_{00} + T_{ii}\right) = \frac{1}{2}\left(T^{00} + T^{ii}\right). \tag{3}$$

$$h_{00} = \int d^3x' \frac{4\bar{T}_{00}(t - |x - x'|, x')}{|x - x'|} = \tag{4}$$

$$= 4\int d^3x' \left(\bar{T}_{00}(t - r, x') + \bar{T}_{00,0}(t - r, x')\left(r - |x - x'|\right) + \cdots\right)\frac{1}{|x - x'|} \tag{5}$$

$$= 4\int d^3x' \left(\bar{T}'_{00} + \bar{T}'_{00,0}\left(x^j\frac{x'^j}{r} + \frac{1}{2}\frac{x^jx^k}{r}\frac{x'^jx'^k - r'^2\delta_{jk}}{r^2} + O\left(\frac{1}{r^2}\right)\right) + \cdots\right) \times$$
$$\left(\frac{1}{r} + \frac{x^lx'^l}{r^3} + O\left(\frac{1}{r^3}\right)\right). \tag{6}$$

The term with $\bar{T}'_{00}$ (*no time derivatives*) is:

$$\frac{2}{r}\int d^3x' \left(T^{00} + T^{ii}\right) + \frac{2x^l}{r^3}\int d^3x' \left(T^{00} + T^{ii}\right)x'^l + O\left(\frac{1}{r^3}\right). \tag{7}$$

Using (19.4b), (19.6a), (19.7a), and (19.7b) from the textbook it becomes:

$$(7) = \frac{2M}{r} + \frac{1}{r}\int d^3x'\, T^{00}_{\ ,00}r'^2 + \frac{2x^l}{r^3}\int d^3x' \left(T^{0i}x'^ix'^l - \frac{1}{2}T^{0l}r'^2\right)_{,0} + O\left(\frac{1}{r^3}\right). \tag{8}$$

The term with $\bar{T}'_{00,0}$ (*first-order time derivative*) is:

$$\frac{2}{r^2}\int d^3x' \left(T^{00} + T^{ii}\right)_{,0}\left(x^jx'^j + \frac{x^kx^kx'^jx'^k}{r^2} + \frac{1}{2}x^jx^k\frac{x'^jx'^k - r'^2\delta_{jk}}{r^2}\right) + O\left(\frac{1}{r^3}\right) \tag{9}$$

$$= 2\int d^3x' \left(T^{00} + T^{ii}\right)_{,0}\left(\frac{x^jx'^j}{r^2} + \frac{\left(3x'^jx'^k - r'^2\delta_{jk}\right)x^jx^k}{2r^4}\right) + O\left(\frac{1}{r^3}\right). \tag{10}$$

Every term in (8) and (10) except for $\frac{2M}{r}$ is precisely canceled by gauge transformation: $\tilde{h}_{00} = h_{00} - 2\partial_0\xi_0$.

B. ti - components

Similarly, we can compute the $n = 0$ term of the h_{0i}:

$$\frac{4}{r}\int d^3x' T_{0i} + \frac{4x^k}{r^3}\int d^3x'\, T_{0i}x'^k + O\left(\frac{1}{r^3}\right). \tag{11}$$

Applying zero momentum (19.4a) condition and using (19.7c) from the textbook:

$$(11) = -\frac{4x^k}{r^3}\int d^3x'\, T^{0i}x'^k + O\left(\frac{1}{r^3}\right) \tag{12}$$

$$= -\frac{2x^k}{r^3}\int d^3x' \left(T^{0k}x'^l - T^{0k}x'^i\right) \tag{13}$$