Solving problem in Carroll's Spacetime

indutny

(Dated: April 30, 2020)

I. GENERAL RELATIVITY PROBLEM

A. Problem statement

Prove that:

$$t_{\mu\nu} = \langle (\partial_\mu h_{\rho\sigma}) (\partial_\nu h^{\rho\sigma}) - \frac{1}{2} (\partial_\mu h) (\partial_\nu h) - (\partial_\rho h^{\rho\sigma}) (\partial_\mu h_{\nu\sigma}) - (\partial_\rho h^{\rho\sigma}) (\partial_\nu h_{\mu\sigma}) \rangle \tag{1}$$

is invariant under gauge transformation:

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu . \tag{2}$$

The "$\langle$", "$\rangle$" brackets mean time averaging and allow moving partial derivatives around:

$$\langle A \partial_\mu B \rangle = -\langle (\partial_\mu A) B \rangle . \tag{3}$$

B. First Attempt

The variation of h is:

$$\delta h = \partial^{\mu\nu} \delta h_{\mu\nu} = 2\partial^\lambda \xi_\lambda . \tag{4}$$

Let's simplify it term by term:

$$\begin{aligned}
& (\partial_\mu h_{\rho\sigma}) (\partial_\nu h^{\rho\sigma}) && (5)\\
={} & (\partial_\mu \partial_\rho \xi_\sigma + \partial_\mu \partial_\sigma \xi_\rho) (\partial_\nu h^{\rho\sigma}) + (\partial_\mu h_{\rho\sigma}) (\partial_\nu \partial^\rho \xi^\sigma + \partial_\nu \partial^\sigma \xi^\rho) && (6)\\
={} & -2h^{\rho\sigma} (\partial_\mu \partial_\nu \partial_\rho \xi_\sigma + \partial_\mu \partial_\nu \partial_\sigma \xi_\rho) , && (7)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2} (\partial_\mu h) (\partial_\nu h) &= \left(\partial_\mu \partial^\lambda \xi_\lambda\right) (\partial_\nu h) + (\partial_\mu h) \left(\partial_\nu \partial^\lambda \xi_\lambda\right) && (8)\\
&= -2h^{\rho\sigma} \left(\eta_{\rho\sigma} \partial_\mu \partial_\nu \partial^\lambda \xi_\lambda\right) , && (9)
\end{aligned}$$

$$\begin{aligned}
(\partial_\rho h^{\rho\sigma}) (\partial_\mu h_{\nu\sigma}) &= (\partial_\rho \partial^\rho \xi^\sigma + \partial_\rho \partial^\sigma \xi^\rho) (\partial_\mu h_{\nu\sigma}) + (\partial_\rho h^{\rho\sigma}) (\partial_\mu \partial_\nu \xi_\sigma + \partial_\mu \partial_\sigma \xi_\nu) && (10)\\
&= -h^{\rho\sigma} \left[\eta_{\rho\nu} \left(\partial_\mu \Box \xi_\sigma + \partial_\mu \partial_\sigma \partial^\lambda \xi_\lambda\right) + \partial_\rho \partial_\mu \partial_\nu \xi_\sigma + \partial_\rho \partial_\mu \partial_\sigma \xi_\nu\right] , && (11)
\end{aligned}$$

and similarly:

$$(\partial_\rho h^{\rho\sigma}) (\partial_\nu h_{\mu\sigma}) = -h^{\rho\sigma} \left[\eta_{\rho\mu} \left(\partial_\nu \Box \xi_\sigma + \partial_\nu \partial_\sigma \partial^\lambda \xi_\lambda\right) + \partial_\rho \partial_\mu \partial_\nu \xi_\sigma + \partial_\rho \partial_\nu \partial_\sigma \xi_\mu\right] . \tag{12}$$

Combining (7), (9), (11), and (12) together and leaving $-h^{\rho\sigma}$ implicit we get:

$$\begin{aligned}&2\partial_\mu\partial_\nu\partial_\rho\xi_\sigma + 2\partial_\mu\partial_\nu\partial_\sigma\xi_\rho + 2\eta_{\rho\sigma}\partial_\mu\partial_\nu\partial^\lambda\xi_\lambda \\ &+ \eta_{\rho\nu}\left(\partial_\mu\Box\xi_\sigma + \partial_\mu\partial_\sigma\partial^\lambda\xi_\lambda\right) + \partial_\rho\partial_\mu\partial_\nu\xi_\sigma + \partial_\rho\partial_\mu\partial_\sigma\xi_\nu \\ &+ \eta_{\rho\mu}\left(\partial_\nu\Box\xi_\sigma + \partial_\nu\partial_\sigma\partial^\lambda\xi_\lambda\right) + \partial_\rho\partial_\mu\partial_\nu\xi_\sigma + \partial_\rho\partial_\nu\partial_\sigma\xi_\mu \neq 0? \end{aligned} \tag{13}$$

C. Second Attempt

We could as well move all partial derivatives from ξ to h, and so the terms are:

$$(\partial_\mu h_{\rho\sigma})(\partial_\nu h^{\rho\sigma}) = -2h^{\rho\sigma}\left(\partial_\mu\partial_\nu\partial_\rho\xi_\sigma + \partial_\mu\partial_\nu\partial_\sigma\xi_\rho\right) \tag{14}$$

$$= 2\xi_\lambda\left(\partial_\mu\partial_\nu\partial_\rho h^{\lambda\rho} + \partial_\mu\partial_\nu\partial_\sigma h^{\rho\sigma}\right), \tag{15}$$

$$\frac{1}{2}(\partial_\mu h)(\partial_\nu h) = -2h^{\rho\sigma}\left(\eta_{\rho\sigma}\partial_\mu\partial_\nu\partial^\lambda\xi_\lambda\right) = 2\xi_\lambda\left(\eta_{\rho\sigma}\partial_\mu\partial_\nu\partial^\lambda h^{\rho\sigma}\right), \tag{16}$$

$$(\partial_\rho h^{\rho\sigma})(\partial_\mu h_{\nu\sigma}) = -h^{\rho\sigma}\left[\eta_{\rho\nu}\left(\partial_\mu\Box\xi_\sigma + \partial_\mu\partial_\sigma\partial^\lambda\xi_\lambda\right) + \partial_\rho\partial_\mu\partial_\nu\xi_\sigma + \partial_\rho\partial_\mu\partial_\sigma\xi_\nu\right] \tag{17}$$

$$= \xi_\lambda\left[\left(\partial_\mu\Box h_\nu^{\ \lambda} + \partial_\mu\partial_\sigma\partial^\lambda h_\nu^{\ \sigma}\right) + \partial_\rho\partial_\mu\partial_\nu h^{\rho\lambda} + \eta_{\lambda\nu}\partial_\rho\partial_\mu\partial_\sigma h^{\rho\sigma}\right], \tag{18}$$

and finally:

$$(\partial_\rho h^{\rho\sigma})(\partial_\nu h_{\mu\sigma}) \tag{19}$$

$$= \xi_\lambda\left[\left(\partial_\nu\Box h_\mu^{\ \lambda} + \partial_\nu\partial_\sigma\partial^\lambda h_\mu^{\ \sigma}\right) + \partial_\rho\partial_\mu\partial_\nu h^{\rho\lambda} + \eta_{\lambda\mu}\partial_\rho\partial_\nu\partial_\sigma h^{\rho\sigma}\right]. \tag{20}$$

Combining (15), (16), (18), and (20) and leaving ξ_λ implicit:

$$\begin{aligned}&2\partial_\mu\partial_\nu\partial_\rho h^{\lambda\rho} + 2\partial_\mu\partial_\nu\partial_\sigma h^{\rho\sigma} + 2\eta_{\rho\sigma}\partial_\mu\partial_\nu\partial^\lambda h^{\rho\sigma} \\ &+ \left(\partial_\mu\Box h_\nu^{\ \lambda} + \partial_\mu\partial_\sigma\partial^\lambda h_\nu^{\ \sigma}\right) + \partial_\rho\partial_\mu\partial_\nu h^{\rho\lambda} + \eta_{\lambda\nu}\partial_\rho\partial_\mu\partial_\sigma h^{\rho\sigma} \\ &+ \left(\partial_\nu\Box h_\mu^{\ \lambda} + \partial_\nu\partial_\sigma\partial^\lambda h_\mu^{\ \sigma}\right) + \partial_\rho\partial_\mu\partial_\nu h^{\rho\lambda} + \eta_{\lambda\mu}\partial_\rho\partial_\nu\partial_\sigma h^{\rho\sigma}. \end{aligned} \tag{21}$$