2.32 in Baumgarte

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I. RICCI TENSOR FOR CONFORMAL METRIC

A. Problem statement

Compute Ricci tensor:

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{ij}\Gamma^l_{kl} - \Gamma^k_{il}\Gamma^l_{jk} \tag{1}$$

for the metric

$$\gamma_{ij} = \psi^4(r)\mathrm{diag}\left(1, r^2, r^2\sin^2\theta\right), \tag{2}$$

where Christoffel Symbol is defined as:

$$\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}\left(\gamma_{lj,k} + \gamma_{lk,j} - \gamma_{jk,l}\right) \tag{3}$$

and ψ is an arbitrary function of the radial coordinate r.

B. Preparations

The inverse of the diagonal metric is obviously:

$$\gamma^{ij} = \frac{1}{\psi^4(r)}\mathrm{diag}\left(1, \frac{1}{r^2}, \frac{1}{r^2\sin^2\theta}\right). \tag{4}$$

C. Christoffel Symbols

Let's start by meticulously computing all Christoffel symbols. With upper index "r" they are:

$$\Gamma^r_{rr} = \frac{1}{2}\gamma^{rr}\left(\gamma_{rr,r} + \gamma_{rr,r} - \gamma_{rr,r}\right) = \frac{1}{2}\gamma^{rr}\gamma_{rr,r} = \frac{1}{2\psi^4}\partial_r\left(\psi^4\right) = \frac{2\partial_r\psi}{\psi}, \tag{5}$$

$$\Gamma^r_{r\theta} = \frac{1}{2}\gamma^{rr}\left(\gamma_{rr,\theta} + \gamma_{r\theta,r} - \gamma_{r\theta,r}\right) = 0, \tag{6}$$

$$\Gamma^r_{r\phi} = \frac{1}{2}\gamma^{rr}\left(\gamma_{rr,\phi} + \gamma_{r\phi,r} - \gamma_{r\phi,r}\right) = 0, \tag{7}$$

$$\Gamma^r_{\theta\theta} = \frac{1}{2}\gamma^{rr}\left(\gamma_{r\theta,\theta} + \gamma_{r\theta,\theta} - \gamma_{\theta\theta,r}\right) = -\frac{1}{2\psi^4}\partial_r\left(\psi^4 r^2\right) = -\frac{2r^2\partial_r\psi}{\psi} - r, \tag{8}$$

$$\Gamma^r_{\theta\phi} = \frac{1}{2}\gamma^{rr}\left(\gamma_{r\theta,\phi} + \gamma_{r\phi,\theta} - \gamma_{\theta\phi,r}\right) = 0, \tag{9}$$

$$\Gamma^r_{\phi\phi} = \frac{1}{2}\gamma^{rr}\left(\gamma_{r\phi,\phi} + \gamma_{r\phi,\phi} - \gamma_{\phi\phi,r}\right) = -\frac{1}{2\psi^4}\partial_r\left(\psi^4 r^2\sin^2\theta\right) = \sin^2\theta\,\Gamma^r_{\theta\theta}. \tag{10}$$

With upper index "θ":

$$\Gamma^{\theta}_{rr} = \frac{1}{2}\gamma^{\theta\theta}\left(\gamma_{\theta r,r} + \gamma_{\theta r,r} - \gamma_{rr,\theta}\right) = 0, \tag{11}$$

$$\Gamma^{\theta}_{r\theta} = \frac{1}{2}\gamma^{\theta\theta}\left(\gamma_{\theta r,\theta} + \gamma_{\theta\theta,r} - \gamma_{r\theta,\theta}\right) = \frac{1}{2}\gamma^{\theta\theta}\gamma_{\theta\theta,r} = \frac{1}{2\psi^4 r^2}\partial_r\left(\psi^4 r^2\right) = -\frac{1}{r^2}\Gamma^{r}_{\theta\theta}, \tag{12}$$

$$\Gamma^{\theta}_{r\phi} = \frac{1}{2}\gamma^{\theta\theta}\left(\gamma_{\theta r,\phi} + \gamma_{\theta\phi,r} - \gamma_{r\phi,\theta}\right) = 0, \tag{13}$$

$$\Gamma^{\theta}_{\theta\theta} = \frac{1}{2}\gamma^{\theta\theta}\left(\gamma_{\theta\theta,\theta} + \gamma_{\theta\theta,\theta} - \gamma_{\theta\theta,\theta}\right) = -\frac{1}{2r^2\psi^4}\partial_\theta\left(\psi^4 r^2\right) = 0, \tag{14}$$

$$\Gamma^{\theta}_{\theta\phi} = \frac{1}{2}\gamma^{\theta\theta}\left(\gamma_{\theta\theta,\phi} + \gamma_{\theta\phi,\theta} - \gamma_{\theta\phi,\theta}\right) = 0, \tag{15}$$

$$\Gamma^{\theta}_{\phi\phi} = \frac{1}{2}\gamma^{\theta\theta}\left(\gamma_{\theta\phi,\phi} + \gamma_{\theta\phi,\phi} - \gamma_{\phi\phi,\theta}\right) = -\frac{1}{2r^2\psi^4}\partial_\theta\left(\psi^4 r^2\sin^2\theta\right) = -\sin\theta\cos\theta. \tag{16}$$

With upper index "ϕ":

$$\Gamma^{\phi}_{rr} = \frac{1}{2}\gamma^{\phi\phi}\left(\gamma_{\phi r,r} + \gamma_{\phi r,r} - \gamma_{rr,\phi}\right) = 0, \tag{17}$$

$$\Gamma^{\phi}_{r\theta} = \frac{1}{2}\gamma^{\phi\phi}\left(\gamma_{\phi r,\theta} + \gamma_{\phi\theta,r} - \gamma_{r\theta,\phi}\right) = 0, \tag{18}$$

$$\Gamma^{\phi}_{r\phi} = \frac{1}{2}\gamma^{\phi\phi}\left(\gamma_{\phi r,\phi} + \gamma_{\phi\phi,r} - \gamma_{r\phi,\phi}\right) = \frac{1}{2\psi^4 r^2\sin^2\theta}\partial_r\left(\psi^4 r^2\sin^2\theta\right) = \Gamma^{\theta}_{r\theta}, \tag{19}$$

$$\Gamma^{\phi}_{\theta\theta} = \frac{1}{2}\gamma^{\phi\phi}\left(\gamma_{\phi\theta,\theta} + \gamma_{\phi\theta,\theta} - \gamma_{\theta\theta,\phi}\right) = 0, \tag{20}$$

$$\Gamma^{\phi}_{\theta\phi} = \frac{1}{2}\gamma^{\phi\phi}\left(\gamma_{\phi\theta,\phi} + \gamma_{\phi\phi,\theta} - \gamma_{\theta\phi,\phi}\right) = \frac{1}{2\psi^4 r^2\sin^2\theta}\partial_\theta\left(\psi^4 r^2\sin^2\theta\right) = \frac{\cos\theta}{\sin\theta}, \tag{21}$$

$$\Gamma^{\phi}_{\phi\phi} = \frac{1}{2}\gamma^{\phi\phi}\left(\gamma_{\phi\phi,\phi} + \gamma_{\phi\phi,\phi} - \gamma_{\phi\phi,\phi}\right) = 0. \tag{22}$$

Let's summarize the above computation by writing down non-zero components of Γ^{i}_{jk}:

$$\Gamma^{r}_{rr} = \frac{2\partial_r\psi}{\psi},\ \Gamma^{r}_{\theta\theta} = -\frac{2r^2\partial_r\psi}{\psi} - r,\ \Gamma^{r}_{\phi\phi} = \sin^2\theta\ \Gamma^{r}_{\theta\theta}, \tag{23}$$

$$\Gamma^{\theta}_{r\theta} = -\frac{1}{r^2}\Gamma^{r}_{\theta\theta},\ \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \tag{24}$$

$$\Gamma^{\phi}_{r\phi} = \Gamma^{\theta}_{r\theta},\ \Gamma^{\phi}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}. \tag{25}$$

D. Ricci Tensor