

3.3 in Baumgarte

indutny

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I. RICCI TENSOR FOR CONFORMAL METRIC

A. Problem statement

Compute conformal Ricci tensor using non-conformal formula:

$$R_{ij} = \partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ik}^k + \Gamma_{ij}^k \Gamma_{kl}^l - \Gamma_{il}^k \Gamma_{jk}^l \quad (1)$$

for the metric

$$\bar{\gamma}_{ij} = \psi^{-4} \gamma_{ij}, \quad \bar{\gamma}^{ij} = \psi^4 \gamma^{ij}, \quad (2)$$

and Christoffel Symbol:

$$\Gamma_{jk}^i = \frac{1}{2} \gamma^{il} (\gamma_{lj,k} + \gamma_{lk,j} - \gamma_{jk,l}), \quad (3)$$

where ψ is an arbitrary function of spatial slice.

B. Christoffel Symbol

$$\Gamma_{jk}^i = \frac{1}{2} \bar{\gamma}^{il} (\bar{\gamma}_{lj,k} + \bar{\gamma}_{lk,j} - \bar{\gamma}_{jk,l} + 4\bar{\gamma}_{lj} \partial_k \ln \psi + 4\bar{\gamma}_{lk} \partial_j \ln \psi - 4\bar{\gamma}_{jk} \partial_l \ln \psi) \quad (4)$$

$$= \bar{\Gamma}_{jk}^i + 2 (\delta_j^i \partial_k \ln \psi + \delta_k^i \partial_j \ln \psi - \bar{\gamma}^{il} \bar{\gamma}_{jk} \partial_l \ln \psi) \quad (5)$$

$$= \bar{\Gamma}_{jk}^i + 2 (\delta_j^i \bar{D}_k \ln \psi + \delta_k^i \bar{D}_j \ln \psi - \bar{\gamma}_{jk} \bar{\gamma}^{il} \bar{D}_l \ln \psi). \quad (6)$$

For future convenience (and to save electronic ink) we define $\alpha \equiv \ln \psi$, and thus (6) becomes:

$$\Gamma_{jk}^i = \bar{\Gamma}_{jk}^i + 2 (\delta_j^i \bar{D}_k \alpha + \delta_k^i \bar{D}_j \alpha - \bar{\gamma}_{jk} \bar{\gamma}^{il} \bar{D}_l \alpha). \quad (7)$$

C. Ricci Tensor

Let's introduce one more symbol to simplify later (formidable) calculations:

$$A_{jk}^i \equiv \delta_j^i \bar{D}_k \alpha + \delta_k^i \bar{D}_j \alpha - \bar{\gamma}_{jk} \bar{\gamma}^{il} \bar{D}_l \alpha, \quad (8)$$

$$\Gamma_{jk}^i = \bar{\Gamma}_{jk}^i + 2A_{jk}^i. \quad (9)$$

With these definitions Ricci tensor becomes:

$$R_{ij} = \partial_k \left(\bar{\Gamma}_{ij}^k + 2A_{ij}^k \right) - \partial_j \left(\bar{\Gamma}_{ik}^k + 2A_{ik}^k \right) + \left(\bar{\Gamma}_{ij}^k + 2A_{ij}^k \right) \left(\bar{\Gamma}_{kl}^l + 2A_{kl}^l \right) - \left(\bar{\Gamma}_{il}^k + 2A_{il}^k \right) \left(\bar{\Gamma}_{jk}^l + 2A_{jk}^l \right) \quad (10)$$

$$= \bar{R}_{ij} + 2\partial_k A_{ij}^k - 2\partial_j A_{ik}^k + 2\bar{\Gamma}_{ij}^k A_{kl}^l + 2\bar{\Gamma}_{kl}^l A_{ij}^k + 4A_{ij}^k A_{kl}^l - 2\bar{\Gamma}_{il}^k A_{jk}^l - 2\bar{\Gamma}_{jk}^l A_{il}^k - 4A_{il}^k A_{jk}^l. \quad (11)$$

D. $O(A)$ terms

Terms with two A would involve two α and thus do not mix with terms with single A . We can therefore start by considering the terms with just a single A :

$$2\partial_k A_{ij}^k - 2\partial_j A_{ik}^k + 2\bar{\Gamma}_{ij}^k A_{kl}^l + 2\bar{\Gamma}_{kl}^l A_{ij}^k - 2\bar{\Gamma}_{il}^k A_{jk}^l - 2\bar{\Gamma}_{jk}^l A_{il}^k. \quad (12)$$

The expression has a resemblance of a sum of covariant derivatives (A_{jk}^i is trivially a tensor, symmetric under interchange of lower indices j and k):

$$2\bar{D}_k A_{ij}^k - 2\bar{D}_j A_{ik}^k = 2\partial_k A_{ij}^k + 2\bar{\Gamma}_{kl}^k A_{ij}^l - 2\bar{\Gamma}_{ki}^l A_{lj}^k - \cancel{2\bar{\Gamma}_{kj}^l A_{il}^k} - 2\partial_j A_{ik}^k - 2\bar{\Gamma}_{jl}^k A_{ik}^l + 2\bar{\Gamma}_{ji}^l A_{lk}^k + \cancel{2\bar{\Gamma}_{jk}^l A_{il}^k} \quad (13)$$

$$= 2\partial_k A_{ij}^k - 2\partial_j A_{ik}^k + 2\bar{\Gamma}_{kl}^k A_{ij}^l - 2\bar{\Gamma}_{ki}^l A_{lj}^k - 2\bar{\Gamma}_{kj}^l A_{il}^k + 2\bar{\Gamma}_{ji}^l A_{lk}^k, \quad (14)$$

and thus:

$$(12) = 2\bar{D}_k A_{ij}^k - 2\bar{D}_j A_{ik}^k \quad (15)$$

$$= 2\bar{D}_k \left(\delta_i^k \bar{D}_j \alpha + \delta_j^k \bar{D}_i \alpha - \bar{\gamma}_{ij} \bar{\gamma}^{kl} \bar{D}_l \alpha \right) - 2\bar{D}_j \left(\delta_i^k \bar{D}_k \alpha + \delta_k^k \bar{D}_i \alpha - \bar{\gamma}_{ik} \bar{\gamma}^{kl} \bar{D}_l \alpha \right) \quad (16)$$

$$= 2\bar{D}_i \bar{D}_j \alpha + 2\bar{D}_j \bar{D}_i \alpha - 2\bar{\gamma}_{ij} \bar{D}^2 \alpha - 2\bar{D}_j \bar{D}_i \alpha - 6\bar{D}_j \bar{D}_i \alpha + 2\bar{D}_j \bar{D}_i \alpha \quad (17)$$

$$= 2\bar{D}_i \bar{D}_j \alpha - 4\bar{D}_j \bar{D}_i \alpha - 2\bar{\gamma}_{ij} \bar{D}^2 \alpha. \quad (18)$$

Expression above requires some further massaging:

$$\bar{D}_j \bar{D}_i \alpha = \bar{D}_j \partial_i \alpha = \partial_j \partial_i \alpha - \bar{\Gamma}_{ji}^l \partial_l \alpha = \bar{D}_i \bar{D}_j \alpha, \quad (19)$$

which we can substitute back into (18) to get a nice expression:

$$(12) = -2 \left(\bar{D}_i \bar{D}_j \alpha + \bar{\gamma}_{ij} \bar{D}^2 \alpha \right). \quad (20)$$

E. $O(A^2)$ terms

We still have two terms with double A in them left:

$$4 \left(A_{ij}^k A_{kl}^l - A_{il}^k A_{jk}^l \right) \quad (21)$$

The first term expands to

$$A_{ij}^k A_{kl}^l = \left(\delta_i^k \bar{D}_j \alpha + \delta_j^k \bar{D}_i \alpha - \bar{\gamma}_{ij} \bar{\gamma}^{km} \bar{D}_m \alpha \right) \left(\delta_k^l \bar{D}_l \alpha + \delta_l^k \bar{D}_k \alpha - \bar{\gamma}_{kl} \bar{\gamma}^{lp} \bar{D}_p \alpha \right) \quad (22)$$

$$\begin{aligned} &= \delta_i^k \delta_k^l \bar{D}_j \alpha \bar{D}_l \alpha + \delta_j^k \delta_k^l \bar{D}_i \alpha \bar{D}_k \alpha - \delta_k^l \bar{\gamma}_{ij} \bar{\gamma}^{km} \bar{D}_m \alpha \bar{D}_l \alpha \\ &\quad + 3 \delta_i^k \bar{D}_j \alpha \bar{D}_k \alpha + 3 \delta_j^k \bar{D}_i \alpha \bar{D}_k \alpha - 3 \bar{\gamma}_{ij} \bar{\gamma}^{km} \bar{D}_m \alpha \bar{D}_k \alpha \\ &\quad - \delta_i^k \bar{\gamma}_{kl} \bar{\gamma}^{lp} \bar{D}_j \alpha \bar{D}_p \alpha - \delta_j^k \bar{\gamma}_{kl} \bar{\gamma}^{lp} \bar{D}_i \alpha \bar{D}_p \alpha + \bar{\gamma}_{ij} \bar{\gamma}^{km} \bar{\gamma}_{kl} \bar{\gamma}^{lp} \bar{D}_m \alpha \bar{D}_p \alpha \end{aligned} \quad (23)$$

$$\begin{aligned} &= \bar{D}_i \alpha \bar{D}_j \alpha + \bar{D}_i \alpha \bar{D}_j \alpha - \bar{\gamma}_{ij} \bar{\gamma}^{kl} \bar{D}_k \alpha \bar{D}_l \alpha \\ &\quad + 3 \bar{D}_i \alpha \bar{D}_j \alpha + 3 \bar{D}_i \alpha \bar{D}_j \alpha - 3 \bar{\gamma}_{ij} \bar{\gamma}^{kl} \bar{D}_k \alpha \bar{D}_l \alpha \\ &\quad - \bar{D}_i \alpha \bar{D}_j \alpha - \bar{D}_i \alpha \bar{D}_j \alpha + \bar{\gamma}_{ij} \bar{\gamma}^{jk} \bar{D}_k \alpha \bar{D}_l \alpha \end{aligned} \quad (24)$$

$$= 6 \bar{D}_i \alpha \bar{D}_j \alpha - 3 \bar{\gamma}_{ij} \bar{\gamma}^{kl} \bar{D}_k \alpha \bar{D}_l \alpha. \quad (25)$$

The second term expands to

$$A_{il}^k A_{jk}^l \quad (26)$$