Solving problem in Carroll's Spacetime

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I. GENERAL RELATIVITY PROBLEM

A. Problem statement

Prove that:

$$t_{\mu\nu} = \langle (\partial_\mu h_{\rho\sigma}) (\partial_\nu h^{\rho\sigma}) - \frac{1}{2} (\partial_\mu h) (\partial_\nu h) - (\partial_\rho h^{\rho\sigma}) (\partial_\mu h_{\nu\sigma}) - (\partial_\rho h^{\rho\sigma}) (\partial_\nu h_{\mu\sigma}) \rangle \tag{1}$$

is invariant under gauge transformation:

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \tag{2}$$

The "$\langle$", "$\rangle$" brackets mean time averaging and allow moving partial derivatives around:

$$\langle A \partial_\mu B \rangle = -\langle (\partial_\mu A) B \rangle. \tag{3}$$

B. First Attempt

The variation of h is:

$$\delta h = \partial^{\mu\nu} \delta h_{\mu\nu} = 2 \partial^\lambda \xi_\lambda. \tag{4}$$

Let's simplify it term by term:

$$\begin{aligned}
&(\partial_\mu h_{\rho\sigma}) (\partial_\nu h^{\rho\sigma}) && (5)\\
=&(\partial_\mu \partial_\rho \xi_\sigma + \partial_\mu \partial_\sigma \xi_\rho) (\partial_\nu h^{\rho\sigma}) + (\partial_\mu h_{\rho\sigma}) (\partial_\nu \partial^\rho \xi^\sigma + \partial_\nu \partial^\sigma \xi^\rho) && (6)\\
=&- 2h^{\rho\sigma} (\partial_\mu \partial_\nu \partial_\rho \xi_\sigma + \partial_\mu \partial_\nu \partial_\sigma \xi_\rho), && (7)
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{2} (\partial_\mu h) (\partial_\nu h) &= - \left(\partial_\mu \partial^\lambda \xi_\lambda\right) (\partial_\nu h) - (\partial_\mu h) \left(\partial_\nu \partial^\lambda \xi_\lambda\right) && (8)\\
&= 2h^{\rho\sigma} \left(\eta_{\rho\sigma} \partial_\mu \partial_\nu \partial^\lambda \xi_\lambda\right), && (9)
\end{aligned}$$

$$\begin{aligned}
- (\partial_\rho h^{\rho\sigma}) (\partial_\mu h_{\nu\sigma}) &= - (\partial_\rho \partial^\rho \xi^\sigma + \partial_\rho \partial^\sigma \xi^\rho) (\partial_\mu h_{\nu\sigma}) - (\partial_\rho h^{\rho\sigma}) (\partial_\mu \partial_\nu \xi_\sigma + \partial_\mu \partial_\sigma \xi_\nu) && (10)\\
&= h^{\rho\sigma} \left[\eta_{\rho\nu} \left(\partial_\mu \Box \xi_\sigma + \partial_\mu \partial_\sigma \partial^\lambda \xi_\lambda\right) + \partial_\rho \partial_\mu \partial_\nu \xi_\sigma + \partial_\rho \partial_\mu \partial_\sigma \xi_\nu\right], && (11)
\end{aligned}$$

and similarly:

$$- (\partial_\rho h^{\rho\sigma}) (\partial_\nu h_{\mu\sigma}) = h^{\rho\sigma} \left[\eta_{\rho\mu} \left(\partial_\nu \Box \xi_\sigma + \partial_\nu \partial_\sigma \partial^\lambda \xi_\lambda\right) + \partial_\rho \partial_\mu \partial_\nu \xi_\sigma + \partial_\rho \partial_\nu \partial_\sigma \xi_\mu\right]. \tag{12}$$

Combining (7), (9), (11), and (12) together we gett:

$$
\begin{aligned}
&h^{\rho\sigma}\Big[-2\left(\partial_\mu\partial_\nu\partial_\rho\xi_\sigma+\partial_\mu\partial_\nu\partial_\sigma\xi_\rho\right)+2\eta_{\rho\sigma}\partial_\mu\partial_\nu\partial^\lambda\xi_\lambda \\
&+\eta_{\rho\nu}\left(\partial_\mu\Box\xi_\sigma+\partial_\mu\partial_\sigma\partial^\lambda\xi_\lambda\right)+\partial_\rho\partial_\mu\partial_\nu\xi_\sigma+\partial_\rho\partial_\mu\partial_\sigma\xi_\nu \\
&+\eta_{\rho\mu}\left(\partial_\nu\Box\xi_\sigma+\partial_\nu\partial_\sigma\partial^\lambda\xi_\lambda\right)+\partial_\rho\partial_\mu\partial_\nu\xi_\sigma+\partial_\rho\partial_\nu\partial_\sigma\xi_\mu\Big]
\end{aligned} \tag{13}
$$