4.9 in Baumgarte

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(Dated: July 10, 2020)

I. 4.9

A. Useful Identities

$$\sqrt{-g} = \alpha\sqrt{\gamma}, \tag{1}$$

$$\partial_t \ln\sqrt{\gamma} = -\alpha K + D_i\beta^i, \tag{2}$$

$$\partial_t K = -D^2\alpha + \alpha\left(K_{ij}K^{ij} + 4\pi(\rho + S)\right) + \beta^i D_i K, \tag{3}$$

$$g^{ab} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^i \\ -\alpha^{-2}\beta^j & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix}. \tag{4}$$

B. Problem statement

Using

$$^{(4)}\Gamma^a = -\frac{1}{\sqrt{-g}}\partial_b\left(\sqrt{-g}g^{ab}\right) = 0 \tag{5}$$

derive following equalities:

$$(\partial_t - \beta^j\partial_j)\alpha = -\alpha^2 K, \tag{6}$$

$$(\partial_t - \beta^j\partial_j)\beta^i = -\alpha^2\left(\gamma^{ij}\partial_j \ln\alpha + \gamma^{jk}\Gamma^i_{jk}\right). \tag{7}$$

C. Solution

It'd be useful to rewrite (5) for a 3-dimensional metric γ:

$$\Gamma^i = \gamma^{jk}\Gamma^i_{jk} = -\frac{1}{\sqrt{\gamma}}\partial_j\left(\sqrt{\gamma}\gamma^{ij}\right) = -\gamma^{ij}\partial_j \ln\sqrt{\gamma} - \partial_j\gamma^{ij}. \tag{8}$$

First let's rewrite the time component of (5) using (4) and (1):

$$\frac{1}{\alpha\sqrt{\gamma}}\partial_b(\alpha\sqrt{\gamma}g^{0b}) = \frac{1}{\alpha\sqrt{\gamma}}\partial_t(-\sqrt{\gamma}\alpha^{-1}) + \frac{1}{\alpha\sqrt{\gamma}}\partial_i(\sqrt{\gamma}\alpha^{-1}\beta^i) \tag{9}$$

$$= -\left(\alpha^{-2}\partial_t - \alpha^{-2}\beta^i\partial_i\right)\ln\sqrt{\gamma} + \left(\alpha^{-3}\partial_t - \alpha^{-3}\beta^i\partial_i\right)\alpha + \alpha^{-2}\partial_i\beta^i = 0 \tag{10}$$

$$\Longrightarrow -\alpha\left(\partial_t - \beta^i\partial_i\right)\ln\sqrt{\gamma} + \left(\partial_t - \beta^i\partial_i\right)\alpha + \alpha\partial_i\beta^i = 0 \tag{11}$$

$$\left(\partial_t - \beta^i\partial_i\right)\alpha = \alpha\left(\partial_t - \beta^i\partial_i\right)\ln\sqrt{\gamma} - \alpha\partial_i\beta^i. \tag{12}$$

Plugging (2) into (12):

$$
\begin{aligned}
\left(\partial_t - \beta^i \partial_i\right)\alpha &= -\alpha^2 K + \alpha\left(D_i \beta^i - \beta^i \partial_i \ln\sqrt{\gamma} - \partial_i \beta^i\right) && (13)\\
&= -\alpha^2 K + \alpha\gamma^{ij}\left(\partial_i \beta^i + \Gamma^i_{ij}\beta^j - \beta^i \partial_i \ln\sqrt{\gamma} - \partial_i \beta^i\right) && (14)\\
&= -\alpha^2 K + \alpha\gamma^{ij}\left(\Gamma^i_{ij}\beta^j - \frac{1}{\sqrt{\gamma}}\beta^i \partial_i \sqrt{\gamma}\right) && (15)\\
&= -\alpha^2 K + \alpha\gamma^{ij}\left(\Gamma^i_{ij}\beta^j - \frac{1}{\sqrt{\gamma}}\beta^i \frac{1}{2\sqrt{\gamma}}\gamma\gamma^{jk}\partial_i \gamma_{jk}\right) && (16)\\
&= -\alpha^2 K + \alpha\gamma^{ij}\beta^i\left(\Gamma^j_{ji} - \frac{1}{2}\gamma^{jk}\partial_i \gamma_{jk}\right) && (17)\\
&= -\alpha^2 K + \alpha\gamma^{ij}\gamma^{jk}\beta^i \frac{1}{2}\left(\cancel{\gamma_{ki,j}} + \cancel{\gamma_{kj,i}} - \cancel{\gamma_{ij,k}} - \cancel{\gamma_{jk,i}}\right) && (18)\\
&= -\alpha^2 K. && (19)
\end{aligned}
$$

which is exactly (6).

For the spatial components of (5):

$$
\begin{aligned}
\frac{1}{\alpha\sqrt{\gamma}}\partial_b(\alpha\sqrt{\gamma}g^{ib}) &= \frac{1}{\alpha\sqrt{\gamma}}\partial_t(\alpha\sqrt{\gamma}g^{i0}) + \frac{1}{\alpha\sqrt{\gamma}}\partial_j(\alpha\sqrt{\gamma}g^{ij}) && (20)\\
&= \frac{1}{\alpha\sqrt{\gamma}}\partial_t(\alpha^{-1}\sqrt{\gamma}\beta^i) + \frac{1}{\alpha\sqrt{\gamma}}\partial_j\left(\alpha\sqrt{\gamma}\left(\gamma^{ij} - \alpha^{-2}\beta^i\beta^j\right)\right) && (21)\\
&= \frac{1}{\alpha\sqrt{\gamma}}\partial_t(\alpha^{-1}\sqrt{\gamma}\beta^i) + \frac{1}{\alpha\sqrt{\gamma}}\partial_j\left(\alpha\sqrt{\gamma}\gamma^{ij}\right) - \frac{1}{\alpha\sqrt{\gamma}}\partial_j\left(\alpha^{-1}\sqrt{\gamma}\beta^i\beta^j\right). && (22)
\end{aligned}
$$

Let's work it term by term. First:

$$
\begin{aligned}
\frac{1}{\alpha\sqrt{\gamma}}\partial_t(\alpha^{-1}\sqrt{\gamma}\beta^i) &= -\alpha^{-2}\beta^i\partial_t \ln\alpha + \alpha^{-2}\beta^i\partial_t \ln\sqrt{\gamma} + \alpha^{-2}\partial_t\beta^i\\
&= \alpha^{-2}\left(-\beta^i\partial_t \ln\alpha + \beta^i\partial_t \ln\sqrt{\gamma} + \partial_t\beta^i\right), && (23)
\end{aligned}
$$

the second:

$$
\frac{1}{\alpha\sqrt{\gamma}}\partial_j\left(\alpha\sqrt{\gamma}\gamma^{ij}\right) = \gamma^{ij}\partial_j \ln\alpha + \gamma^{ij}\partial_j \ln\sqrt{\gamma} + \partial_j\gamma^{ij}, \tag{24}
$$

and the third:

$$
\begin{aligned}
\frac{1}{\alpha\sqrt{\gamma}}\partial_j\left(\alpha^{-1}\sqrt{\gamma}\beta^i\beta^j\right) &= -\alpha^{-2}\beta^i\beta^j\partial_j \ln\alpha + \alpha^{-2}\beta^i\beta^j\partial_j \ln\sqrt{\gamma} + \alpha^{-2}\beta^j\partial_j\beta^i + \alpha^{-2}\beta^i\partial_j\beta^j\\
&= \alpha^{-2}\left(-\beta^i\beta^j\partial_j \ln\alpha + \beta^i\beta^j\partial_j \ln\sqrt{\gamma} + \beta^j\partial_j\beta^i + \beta^i\partial_j\beta^j\right). && (25)
\end{aligned}
$$

Combining (23), (24), (25):

$$
\begin{aligned}
(22) = \alpha^{-2}&\left(-\beta^i\left(\partial_t - \beta^j\partial_j\right)\ln\alpha + \beta^i\left(\partial_t - \beta^j\partial_j\right)\ln\sqrt{\gamma} + \left(\partial_t - \beta^j\partial_j\right)\beta^i - \beta^i\partial_j\beta^j\right)\\
&+ \gamma^{ij}\partial_j \ln\alpha + \gamma^{ij}\partial_j \ln\sqrt{\gamma} + \partial_j\gamma^{ij}. && (26)
\end{aligned}
$$

We can use (11) to simplify further:

$$\alpha^2(26) = \left(\partial_t - \beta^j\partial_j\right)\beta^i + \alpha^2\left(\gamma^{ij}\partial_j \ln\alpha + \gamma^{ij}\partial_j \ln\sqrt{\gamma} + \partial_j\gamma^{ij}\right) = 0 \tag{27}$$

and using (8):

$$\left(\partial_t - \beta^j\partial_j\right)\beta^i = -\alpha^2\left(\gamma^{ij}\partial_j \ln\alpha - \Gamma^i\right) \tag{28}$$

which is (7) up to a pesky sign.