3.3 in Baumgarte

indutny

(Dated: May 23, 2020)

I. RICCI TENSOR FOR CONFORMAL METRIC

A. Problem statement

Compute conformal Ricci tensor and scalar Ricci curvature using non-conformal formula:

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{ij}\Gamma^l_{kl} - \Gamma^k_{il}\Gamma^l_{jk}, \tag{1}$$

$$R = \gamma^{ij} R_{ij}, \tag{2}$$

for the metric

$$\overline{\gamma}_{ij} = \psi^{-4}\gamma_{ij},\ \overline{\gamma}^{ij} = \psi^4 \gamma^{ij}, \tag{3}$$

and Christoffel Symbol:

$$\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}\left(\gamma_{lj,k} + \gamma_{lk,j} - \gamma_{jk,l}\right), \tag{4}$$

where ψ is an arbitrary function of spatial slice.

B. Christoffel Symbol

$$\Gamma^i_{jk} = \frac{1}{2}\overline{\gamma}^{il}\left(\overline{\gamma}_{lj,k} + \overline{\gamma}_{lk,j} - \overline{\gamma}_{jk,l} + 4\overline{\gamma}_{lj}\partial_k \ln\psi + 4\overline{\gamma}_{lk}\partial_j \ln\psi - 4\overline{\gamma}_{jk}\partial_l \ln\psi\right) \tag{5}$$

$$= \overline{\Gamma}^i_{jk} + 2\left(\delta^i{}_j \partial_k \ln\psi + \delta^i{}_k \partial_j \ln\psi - \overline{\gamma}^{il}\overline{\gamma}_{jk}\partial_l \ln\psi\right) \tag{6}$$

$$= \overline{\Gamma}^i_{jk} + 2\left(\delta^i{}_j \overline{D}_k \ln\psi + \delta^i{}_k \overline{D}_j \ln\psi - \overline{\gamma}_{jk}\overline{\gamma}^{il}\overline{D}_l \ln\psi\right). \tag{7}$$

For future convenience (and to save electronic ink) we define $\alpha \equiv \ln\psi$, and thus (7) becomes:

$$\Gamma^i_{jk} = \overline{\Gamma}^i_{jk} + 2\left(\delta^i{}_j \overline{D}_k \alpha + \delta^i{}_k \overline{D}_j \alpha - \overline{\gamma}_{jk}\overline{\gamma}^{il}\overline{D}_l \alpha\right). \tag{8}$$

C. Ricci Tensor

Let's introduce one more symbol to simplify future (formidable) calculations:

$$A^i_{jk} \equiv \delta^i{}_j \overline{D}_k \alpha + \delta^i{}_k \overline{D}_j \alpha - \overline{\gamma}_{jk}\overline{\gamma}^{il}\overline{D}_l \alpha, \tag{9}$$

$$\Gamma^i_{jk} = \overline{\Gamma}^i_{jk} + 2A^i_{jk}. \tag{10}$$

With these definitions Ricci tensor becomes:

$$\begin{aligned}R_{ij} &= \partial_k\left(\overline{\Gamma}^k_{ij}+2A^k_{ij}\right)-\partial_j\left(\overline{\Gamma}^k_{ik}+2A^k_{ik}\right)\\ &\quad+\left(\overline{\Gamma}^k_{ij}+2A^k_{ij}\right)\left(\overline{\Gamma}^l_{kl}+2A^l_{kl}\right)-\left(\overline{\Gamma}^k_{il}+2A^k_{il}\right)\left(\overline{\Gamma}^l_{jk}+2A^l_{jk}\right) \qquad (11)\\ &= \overline{R}_{ij}+2\partial_k A^k_{ij}-2\partial_j A^k_{ik}\\ &\quad+2\overline{\Gamma}^k_{ij}A^l_{kl}+2\overline{\Gamma}^l_{kl}A^k_{ij}+4A^k_{ij}A^l_{kl}-2\overline{\Gamma}^k_{il}A^l_{jk}-2\overline{\Gamma}^l_{jk}A^k_{il}-4A^k_{il}A^l_{jk}. \qquad (12)\end{aligned}$$

D. $O(A)$ terms

Terms with two A would involve two α and thus do not mix with terms with single A. We can therefore start by considering the terms with just a single A:

$$2\partial_k A^k_{ij}-2\partial_j A^k_{ik}+2\overline{\Gamma}^k_{ij}A^l_{kl}+2\overline{\Gamma}^l_{kl}A^k_{ij}-2\overline{\Gamma}^k_{il}A^l_{jk}-2\overline{\Gamma}^l_{jk}A^k_{il}. \qquad (13)$$

The expression has a resemblance of a sum of covariant derivatives (A^i_{jk} is trivially a tensor, symmetric under interchange of lower indices j and k):

$$\begin{aligned}2\overline{D}_k A^k_{ij}-2\overline{D}_j A^k_{ik} &= 2\partial_k A^k_{ij}+2\overline{\Gamma}^k_{kl}A^l_{ij}-2\overline{\Gamma}^l_{ki}A^k_{lj}-\cancel{2\overline{\Gamma}^l_{kj}A^k_{il}}\\ &\quad-2\partial_j A^k_{ik}-2\overline{\Gamma}^k_{jl}A^l_{ik}+2\overline{\Gamma}^l_{ji}A^k_{lk}+\cancel{2\overline{\Gamma}^l_{jk}A^k_{il}} \qquad (14)\\ &= 2\partial_k A^k_{ij}-2\partial_j A^k_{ik}+2\overline{\Gamma}^k_{kl}A^l_{ij}-2\overline{\Gamma}^l_{ki}A^k_{lj}-2\overline{\Gamma}^l_{kj}A^k_{il}+2\overline{\Gamma}^l_{ji}A^k_{lk}, \qquad (15)\end{aligned}$$

and thus:

$$\begin{aligned}(13) &= 2\overline{D}_k A^k_{ij}-2\overline{D}_j A^k_{ik} \qquad (16)\\ &= 2\overline{D}_k\left(\delta^k_i\overline{D}_j\alpha+\delta^k_j\overline{D}_i\alpha-\overline{\gamma}_{ij}\overline{\gamma}^{kl}\overline{D}_l\alpha\right)-2\overline{D}_j\left(\delta^k_i\overline{D}_k\alpha+\delta^k_k\overline{D}_i\alpha-\overline{\gamma}_{ik}\overline{\gamma}^{kl}\overline{D}_l\alpha\right) \qquad (17)\\ &= 2\overline{D}_i\overline{D}_j\alpha+2\overline{D}_j\overline{D}_i\alpha-2\overline{\gamma}_{ij}\overline{D}^2\alpha-2\overline{D}_j\overline{D}_i\alpha-6\overline{D}_j\overline{D}_i\alpha+2\overline{D}_j\overline{D}_i\alpha \qquad (18)\\ &= 2\overline{D}_i\overline{D}_j\alpha-4\overline{D}_j\overline{D}_i\alpha-2\overline{\gamma}_{ij}\overline{D}^2\alpha. \qquad (19)\end{aligned}$$

Expression above requires some further massaging:

$$\overline{D}_j\overline{D}_i\alpha=\overline{D}_j\partial_i\alpha=\partial_j\partial_i\alpha-\overline{\Gamma}^l_{ji}\partial_l\alpha=\overline{D}_i\overline{D}_j\alpha, \qquad (20)$$

which we can substitute back into (19) to get a nice expression:

$$(13)=-2\left(\overline{D}_i\overline{D}_j\alpha+\overline{\gamma}_{ij}\overline{D}^2\alpha\right). \qquad (21)$$

E. $O(A^2)$ terms

We still have two terms with double A in them left:

$$4\left(A^k_{ij}A^l_{kl} - A^k_{il}A^l_{jk}\right) \tag{22}$$

The first term expands to

$$\begin{aligned}
A^k_{ij}A^l_{kl} = &\left(\delta^k_i\overline{D}_j\alpha + \delta^k_j\overline{D}_i\alpha - \overline{\gamma}_{ij}\overline{\gamma}^{km}\overline{D}_m\alpha\right)\left(\delta^l{}_k\overline{D}_l\alpha + \delta^l{}_l\overline{D}_k\alpha - \overline{\gamma}_{kl}\overline{\gamma}^{lp}\overline{D}_p\alpha\right) && (23)\\
&= \delta^k_i\delta^l{}_k\overline{D}_j\alpha\overline{D}_l\alpha + \delta^k_j\delta^l{}_k\overline{D}_i\alpha\overline{D}_k\alpha - \delta^l{}_k\overline{\gamma}_{ij}\overline{\gamma}^{km}\overline{D}_m\alpha\overline{D}_l\alpha\\
&\quad + 3\delta^k_i\overline{D}_j\alpha\overline{D}_k\alpha + 3\delta^k_j\overline{D}_i\alpha\overline{D}_k\alpha - 3\overline{\gamma}_{ij}\overline{\gamma}^{km}\overline{D}_m\alpha\overline{D}_k\alpha\\
&\quad - \delta^k_i\overline{\gamma}_{kl}\overline{\gamma}^{lp}\overline{D}_j\alpha\overline{D}_p\alpha - \delta^k_j\overline{\gamma}_{kl}\overline{\gamma}^{lp}\overline{D}_i\alpha\overline{D}_p\alpha + \overline{\gamma}_{ij}\overline{\gamma}^{km}\overline{\gamma}_{kl}\overline{\gamma}^{lp}\overline{D}_m\alpha\overline{D}_p\alpha && (24)\\
&= \overline{D}_i\alpha\overline{D}_j\alpha + \overline{D}_i\alpha\overline{D}_j\alpha - \overline{\gamma}_{ij}\overline{\gamma}^{kl}\overline{D}_k\alpha\overline{D}_l\alpha\\
&\quad + 3\overline{D}_i\alpha\overline{D}_j\alpha + 3\overline{D}_i\alpha\overline{D}_j\alpha - 3\overline{\gamma}_{ij}\overline{\gamma}^{kl}\overline{D}_k\alpha\overline{D}_l\alpha\\
&\quad - \overline{D}_i\alpha\overline{D}_j\alpha - \overline{D}_i\alpha\overline{D}_j\alpha + \overline{\gamma}_{ij}\overline{\gamma}^{jk}\overline{D}_k\alpha\overline{D}_l\alpha && (25)\\
&= 6\overline{D}_i\alpha\overline{D}_j\alpha - 3\overline{\gamma}_{ij}\overline{\gamma}^{kl}\overline{D}_k\alpha\overline{D}_l\alpha. && (26)
\end{aligned}$$

The second term expands to

$$\begin{aligned}
A^k_{il}A^l_{jk} = &\left(\delta^k_i\overline{D}_l\alpha + \delta^k_l\overline{D}_i\alpha - \overline{\gamma}_{il}\overline{\gamma}^{km}\overline{D}_m\alpha\right)\left(\delta^l{}_j\overline{D}_k\alpha + \delta^l{}_k\overline{D}_j\alpha - \overline{\gamma}_{jk}\overline{\gamma}^{lp}\overline{D}_p\alpha\right) && (27)\\
&= \delta^k_i\delta^l{}_j\overline{D}_l\alpha\overline{D}_k\alpha + \delta^k_l\delta^l{}_j\overline{D}_i\alpha\overline{D}_k\alpha - \delta^l{}_j\overline{\gamma}_{il}\overline{\gamma}^{km}\overline{D}_m\alpha\overline{D}_k\alpha\\
&\quad + \delta^k_i\delta^l{}_k\overline{D}_l\alpha\overline{D}_j\alpha + \delta^k_l\delta^l{}_k\overline{D}_i\alpha\overline{D}_j\alpha - \delta^l{}_k\overline{\gamma}_{il}\overline{\gamma}^{km}\overline{D}_j\alpha\overline{D}_m\alpha\\
&\quad - \delta^k_i\overline{\gamma}_{jk}\overline{\gamma}^{lp}\overline{D}_l\alpha\overline{D}_p\alpha - \delta^k_l\overline{\gamma}_{jk}\overline{\gamma}^{lp}\overline{D}_i\alpha\overline{D}_p\alpha + \overline{\gamma}_{il}\overline{\gamma}^{km}\overline{\gamma}_{jk}\overline{\gamma}^{lp}\overline{D}_m\alpha\overline{D}_p\alpha && (28)\\
&= \overline{D}_i\alpha\overline{D}_j\alpha + \overline{D}_i\alpha\overline{D}_j\alpha - \overline{\gamma}_{ij}\overline{\gamma}^{kl}\overline{D}_k\alpha\overline{D}_l\alpha\\
&\quad + \overline{D}_i\alpha\overline{D}_j\alpha + 3\overline{D}_i\alpha\overline{D}_j\alpha - \overline{D}_i\alpha\overline{D}_j\alpha\\
&\quad - \overline{\gamma}_{ij}\overline{\gamma}^{kl}\overline{D}_k\alpha\overline{D}_l\alpha - \overline{D}_i\alpha\overline{D}_j\alpha + \overline{D}_i\alpha\overline{D}_j\alpha && (29)\\
&= 5\overline{D}_i\alpha\overline{D}_j\alpha - 2\overline{\gamma}_{ij}\overline{\gamma}^{kl}\overline{D}_k\alpha\overline{D}_l\alpha. && (30)
\end{aligned}$$

Putting (26) and (30) together:

$$4\left(A^k_{ij}A^l_{kl} - A^k_{il}A^l_{jk}\right) = 4\left(\overline{D}_i\alpha\overline{D}_j\alpha - \overline{\gamma}_{ij}\overline{\gamma}^{kl}\overline{D}_k\alpha\overline{D}_l\alpha\right). \tag{31}$$

F. Final Ricci Tensor

Inserting (21) and (31) back into (11) we get:

$$R_{ij} = \overline{R}_{ij} - 2\left(\overline{D}_i\overline{D}_j\alpha + \overline{\gamma}_{ij}\overline{D}^2\alpha\right) + 4\left(\overline{D}_i\alpha\overline{D}_j\alpha - \overline{\gamma}_{ij}\overline{\gamma}^{kl}\overline{D}_k\alpha\overline{D}_l\alpha\right) \tag{32}$$

$$\begin{aligned} &= \overline{R}_{ij} - 2\left(\overline{D}_i\overline{D}_j\ln\psi + \overline{\gamma}_{ij}\overline{D}^2\ln\psi\right) \\ &\quad + 4\left(\left(\overline{D}_i\ln\psi\right)\left(\overline{D}_j\ln\psi\right) - \overline{\gamma}_{ij}\overline{\gamma}^{kl}\left(\overline{D}_k\ln\psi\right)\left(\overline{D}_l\ln\psi\right)\right). \end{aligned} \tag{33}$$

G. Ricci Scalar

Ricci Scalar is just a contraction of Ricci tensor:

$$R = \gamma^{ij}R_{ij} = \psi^{-4}\overline{\gamma}^{ij}R_{ij} \tag{34}$$

$$= \psi^{-4}\overline{\gamma}^{ij}\left(\overline{R}_{ij} - 2\left(\overline{D}_i\overline{D}_j\alpha + \overline{\gamma}_{ij}\overline{D}^2\alpha\right) + 4\left(\overline{D}_i\alpha\overline{D}_j\alpha - \overline{\gamma}_{ij}\overline{\gamma}^{kl}\overline{D}_k\alpha\overline{D}_l\alpha\right)\right) \tag{35}$$

$$= \psi^{-4}\left(\overline{R} - 2\left(1 + \overline{\gamma}^{ij}\overline{\gamma}_{ij}\right)\overline{D}^2\alpha + 4\left(1 - \overline{\gamma}^{ij}\overline{\gamma}_{ij}\right)\overline{\gamma}^{kl}\overline{D}_k\alpha\overline{D}_l\alpha\right) \tag{36}$$

$$= \psi^{-4}\left(\overline{R} - 8\overline{D}^2\alpha - 8\overline{\gamma}^{kl}\overline{D}_k\alpha\overline{D}_l\alpha\right) \tag{37}$$

$$= \psi^{-4}\left(\overline{R} - 8\overline{\gamma}^{kl}\overline{D}_k\overline{D}_l\ln\psi - 8\overline{\gamma}^{kl}\overline{D}_k\left(\ln\psi\right)\overline{D}_l\left(\ln\psi\right)\right) \tag{38}$$

It will be useful to work on the second term in isolation

$$\overline{\gamma}^{kl}\overline{D}_k\overline{D}_l\ln\psi = \overline{\gamma}^{kl}\overline{D}_k\left(\frac{\overline{D}_l\psi}{\psi}\right) = \overline{\gamma}^{kl}\left(-\psi^{-2}\overline{D}_k\psi\overline{D}_l\psi + \psi^{-1}\overline{D}_k\overline{D}_l\psi\right) \tag{39}$$

putting it back into (34)

$$R = \psi^{-4}\left(\overline{R} - 8\psi^{-1}\overline{D}^2\psi + \cancel{8\psi^{-2}\overline{\gamma}^{kl}\overline{D}_k\psi\overline{D}_l\psi} - \cancel{8\overline{\gamma}^{kl}\overline{D}_k\left(\ln\psi\right)\overline{D}_l\left(\ln\psi\right)}\right) \tag{40}$$

$$= \psi^{-4}\overline{R} - 8\psi^{-5}\overline{D}^2\psi. \tag{41}$$