3.3 in Baumgarte

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(Dated: May 23, 2020)

I. RICCI TENSOR FOR CONFORMAL METRIC

A. Problem statement

Compute conformal Ricci tensor using non-conformal formula:

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{ij}\Gamma^l_{kl} - \Gamma^k_{il}\Gamma^l_{jk} \tag{1}$$

for the metric

$$\overline{\gamma}_{ij} = \psi^{-4}\gamma_{ij},\ \overline{\gamma}^{ij} = \psi^4\gamma^{ij}, \tag{2}$$

and Christoffel Symbol:

$$\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}\left(\gamma_{lj,k} + \gamma_{lk,j} - \gamma_{jk,l}\right), \tag{3}$$

where ψ is an arbitrary function of spatial slice.

B. Christoffel Symbol

$$\begin{aligned}
\Gamma^i_{jk} &= \frac{1}{2}\overline{\gamma}^{il}\left(\overline{\gamma}_{lj,k} + \overline{\gamma}_{lk,j} - \overline{\gamma}_{jk,l} + 4\overline{\gamma}_{lj}\partial_k \ln\psi + 4\overline{\gamma}_{lk}\partial_j \ln\psi - 4\overline{\gamma}_{jk}\partial_l \ln\psi\right) && (4)\\
&= \overline{\Gamma}^i_{jk} + 2\left(\delta^i{}_j\partial_k \ln\psi + \delta^i{}_k\partial_j \ln\psi - \overline{\gamma}^{il}\overline{\gamma}_{jk}\partial_l \ln\psi\right) && (5)\\
&= \overline{\Gamma}^i_{jk} + 2\left(\delta^i{}_j\overline{D}_k \ln\psi + \delta^i{}_k\overline{D}_j \ln\psi - \overline{\gamma}_{jk}\overline{\gamma}^{il}\overline{D}_l \ln\psi\right). && (6)
\end{aligned}$$

For future convenience (and to save electronic ink) we define $\alpha \equiv \ln\psi$, and thus (6) becomes:

$$\Gamma^i_{jk} = \overline{\Gamma}^i_{jk} + 2\left(\delta^i{}_j\overline{D}_k\alpha + \delta^i{}_k\overline{D}_j\alpha - \overline{\gamma}_{jk}\overline{\gamma}^{il}\overline{D}_l\alpha\right). \tag{7}$$

C. Ricci Tensor

Let's introduce one more symbol to simplify later (formidable) calculations:

$$\begin{aligned}
A^i_{jk} &\equiv \delta^i{}_j\overline{D}_k\alpha + \delta^i{}_k\overline{D}_j\alpha - \overline{\gamma}_{jk}\overline{\gamma}^{il}\overline{D}_l\alpha, && (8)\\
\Gamma^i_{jk} &= \overline{\Gamma}^i_{jk} + 2A^i_{jk}. && (9)
\end{aligned}$$

With these definitions Ricci tensor becomes:

$$\begin{aligned} R_{ij} &= \partial_k \left(\overline{\Gamma}^k_{ij} + 2A^k_{ij} \right) - \partial_j \left(\overline{\Gamma}^k_{ik} + 2A^k_{ik} \right) \\ &\quad + \left(\overline{\Gamma}^k_{ij} + 2A^k_{ij} \right) \left(\overline{\Gamma}^l_{kl} + 2A^l_{kl} \right) - \left(\overline{\Gamma}^k_{il} + 2A^k_{il} \right) \left(\overline{\Gamma}^l_{jk} + 2A^l_{jk} \right) \qquad (10) \\ &= \overline{R}_{ij} + 2\partial_k A^k_{ij} - 2\partial_j A^k_{ik} \\ &\quad + 2\overline{\Gamma}^k_{ij} A^l_{kl} + 2\overline{\Gamma}^l_{kl} A^k_{ij} + 4A^k_{ij} A^l_{kl} - 2\overline{\Gamma}^k_{il} A^l_{jk} - 2\overline{\Gamma}^l_{jk} A^k_{il} - 4A^k_{il} A^l_{jk}. \qquad (11) \end{aligned}$$

D. $O(A)$ terms

Terms with two A would involve two α and thus do not mix with terms with single A. We can therefore start by considering the terms with just a single A:

$$2\partial_k A^k_{ij} - 2\partial_j A^k_{ik} + 2\overline{\Gamma}^k_{ij} A^l_{kl} + 2\overline{\Gamma}^l_{kl} A^k_{ij} - 2\overline{\Gamma}^k_{il} A^l_{jk} - 2\overline{\Gamma}^l_{jk} A^k_{il}. \qquad (12)$$

The expression has a resemblance of a sum of covariant derivatives (A^i_{jk} is trivially a tensor, symmetric under interchange of lower indices j and k):

$$\begin{aligned} 2\overline{D}_k A^k_{ij} - 2\overline{D}_j A^k_{ik} &= 2\partial_k A^k_{ij} + 2\overline{\Gamma}^k_{kl} A^l_{ij} - 2\overline{\Gamma}^l_{ki} A^k_{lj} - \cancel{2\overline{\Gamma}^l_{kj} A^k_{il}} \\ &\quad - 2\partial_j A^k_{ik} - 2\overline{\Gamma}^k_{jl} A^l_{ik} + 2\overline{\Gamma}^l_{ji} A^k_{lk} + \cancel{2\overline{\Gamma}^l_{jk} A^k_{il}} \qquad (13) \\ &= 2\partial_k A^k_{ij} - 2\partial_j A^k_{ik} + 2\overline{\Gamma}^k_{kl} A^l_{ij} - 2\overline{\Gamma}^l_{ki} A^k_{lj} - 2\overline{\Gamma}^l_{kj} A^k_{il} + 2\overline{\Gamma}^l_{ji} A^k_{lk}, \qquad (14) \end{aligned}$$

and thus:

$$\begin{aligned} (12) &= 2\overline{D}_k A^k_{ij} - 2\overline{D}_j A^k_{ik} \qquad (15) \\ &= 2\overline{D}_k \left(\delta^k_i \overline{D}_j \alpha + \delta^k_j \overline{D}_i \alpha - \overline{\gamma}_{ij} \overline{\gamma}^{kl} \overline{D}_l \alpha \right) - 2\overline{D}_j \left(\delta^k_i \overline{D}_k \alpha + \delta^k_k \overline{D}_i \alpha - \overline{\gamma}_{ik} \overline{\gamma}^{kl} \overline{D}_l \alpha \right) \qquad (16) \\ &= 2\overline{D}_i \overline{D}_j \alpha + 2\overline{D}_j \overline{D}_i \alpha - 2\overline{\gamma}_{ij} \overline{D}^2 \alpha - 2\overline{D}_j \overline{D}_i \alpha - 6\overline{D}_j \overline{D}_i \alpha + 2\overline{D}_j \overline{D}_i \alpha \qquad (17) \\ &= 2\overline{D}_i \overline{D}_j \alpha - 4\overline{D}_j \overline{D}_i \alpha - 2\overline{\gamma}_{ij} \overline{D}^2 \alpha. \qquad (18) \end{aligned}$$

Expression above requires some further massaging:

$$\overline{D}_j \overline{D}_i \alpha = \overline{D}_j \partial_i \alpha = \partial_j \partial_i \alpha - \overline{\Gamma}^l_{ji} \partial_l \alpha = \overline{D}_i \overline{D}_j \alpha, \qquad (19)$$

which we can substitute back into (18) to get a nice expression:

$$(12) = -2 \left(\overline{D}_i \overline{D}_j \alpha + \overline{\gamma}_{ij} \overline{D}^2 \alpha \right). \qquad (20)$$

E. $O(A^2)$ terms

We still have two terms with double A in them left:

$$4A^k_{ij}A^k_{kl} - 4A^k_{il}A^l_{jk} \tag{21}$$