Deriving Green's function for d'Alembert operator

indutny

(Dated: April 27, 2020)

Spin-$\frac{1}{2}$ particles are governed by Dirac equation and are described by 4-component objects named spinors. Given the influence of Special Relativity on the derivation of Dirac equation it is inevitable that the physical processes involving spin-$\frac{1}{2}$ particles have to be independent of observer and their inertial frame. Measurements made in one frame should agree with the measurements made in another, and thus the components of spinor have to transform between frames. In this paper we introduce the generators of Lorentz transformations that form a Lie algebra, find the decomposition of Lorentz algebra into a sum of two sub-algebras $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, build left- and right- handed representations of spin-$\frac{1}{2}$ particles and combine them to form a full Dirac spinor.

I. OVERVIEW

A. Conventions

The metric is:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

B. Green's function

Our goal is to solve:

$$\Box\Psi(x-y) = \eta^{\mu\nu}\partial_\mu\partial_\nu\Psi(x-y) = \delta^{(4)}(x-y) \tag{2}$$

for $\Psi(x-y)$. The resulting function can be used to compute the solutions to more general differential equation:

$$\Box h(x) = T(x). \tag{3}$$

Given the $\Psi(x-y)$ the reader can check that the solution to (3) is:

$$h(x) = \int d^4y\ \Psi(x-y)T(y). \tag{4}$$

II. SOLUTION