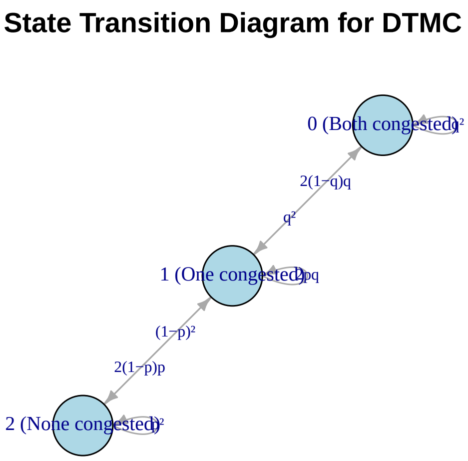
CSE 3504: Project 2

Isaac Piegat

12/2/2024

Problem 1:

1a.



1b.

$$P = \begin{bmatrix} q^2 & 2(1-q)q & (1-q)^2 \\ q^2 & 2pq & (1-p)^2 \\ p^2 & 2(1-p)p & (1-p)^2 \end{bmatrix}$$

1c.

$$\pi P = \pi, \quad \text{where } \sum_i \pi_i = 1$$

$p = 0.5$, $q = 0.5$

$$P = \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.5 & 0.25 \end{bmatrix}$$

R code:

```
P <- matrix(c(0.25, 0.5, 0.25,
              0.25, 0.5, 0.25,
              0.25, 0.5, 0.25),
            nrow = 3, byrow = TRUE)

n <- nrow(P)
A <- t(P) - diag(n)
A[n, ] <- 1
b <- c(rep(0, n-1), 1)
steady_state <- solve(A, b)

steady_state
```

Results:

$$\pi_0 = 0.25, \quad \pi_1 = 0.5, \quad \pi_2 = 0.25$$

Thus:
- π_0: Probability that no route is congested is 0.25.
- π_1: Probability that one route is congested is 0.5.
- π_2: Probability that both routes are congested is 0.25.

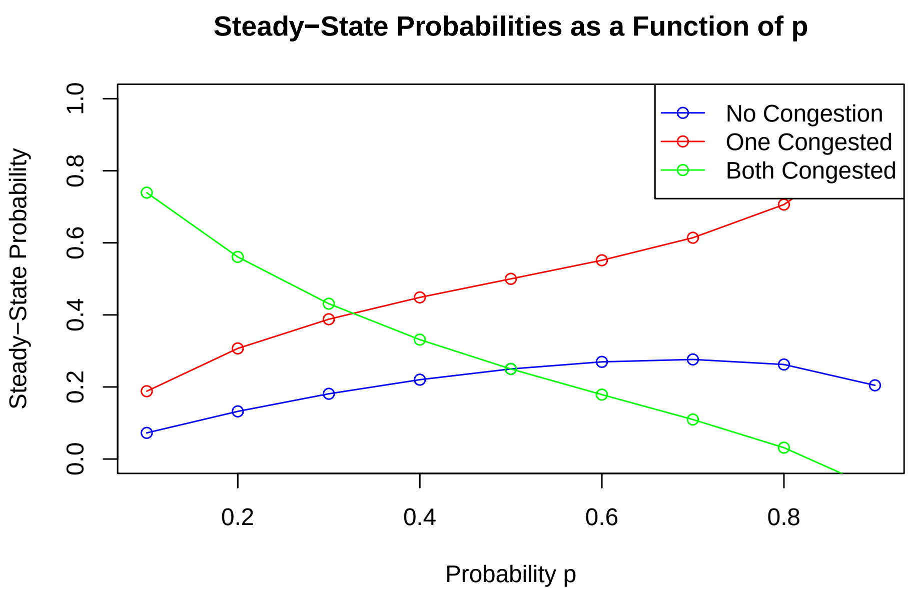
1d.

```
steady_state_probabilities <- function(p, q) {
  P <- matrix(c(q^2, 2*(1-q)*q, (1-q)^2,
                q^2, 2*p*q, (1-p)^2,
                p^2, 2*(1-p)*p, (1-p)^2),
              nrow = 3, byrow = TRUE)
  n <- nrow(P)
  A <- t(P) - diag(n)
  A[n, ] <- 1
  b <- c(rep(0, n-1), 1)
  steady_state <- solve(A, b)
  return(steady_state)
}

p_values <- seq(0.1, 0.9, by = 0.1)
q <- 0.5
results <- sapply(p_values, function(p) steady_state_probabilities(p, q))

plot(p_values, results[,1], type = "o", col = "blue", ylim = c(0, 1),
     xlab = "Probability p", ylab = "Steady-State Probability",
     main = "Steady-State Probabilities as a Function of p")
lines(p_values, results[,2], type = "o", col = "red")
lines(p_values, results[,3], type = "o", col = "green")
legend("topright", legend = c("No Congestion", "One Congested", "Both Congested"),
       col = c("blue", "red", "green"), lty = 1, pch = 1)
```

Resulting plot:



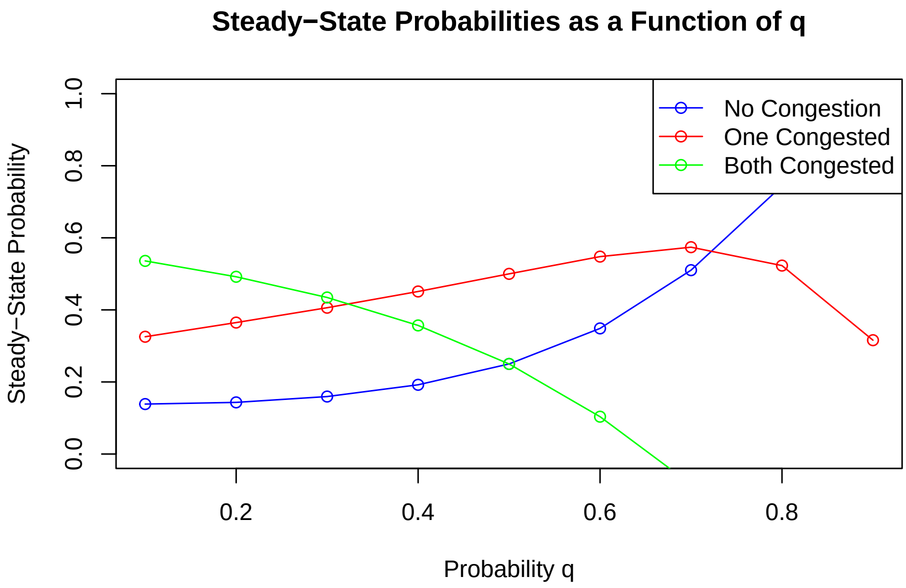
1e.

```
steady_state_probabilities <- function(p, q) {
  P <- matrix(c(q^2, 2*(1-q)*q, (1-q)^2,
                q^2, 2*p*q, (1-p)^2,
                p^2, 2*(1-p)*p, (1-p)^2),
              nrow = 3, byrow = TRUE)
  n <- nrow(P)
  A <- t(P) - diag(n)
  A[n, ] <- 1
  b <- c(rep(0, n-1), 1)
  steady_state <- solve(A, b)
  return(steady_state)
}

# Vary q between 0.1 and 0.9, p is fixed at 0.5
q_values <- seq(0.1, 0.9, by = 0.1)
p <- 0.5
results <- sapply(q_values, function(q) steady_state_probabilities(p, q))

# Plot the probabilities
plot(q_values, results[,1], type = "o", col = "blue", ylim = c(0, 1),
     xlab = "Probability q", ylab = "Steady-State Probability",
     main = "Steady-State Probabilities as a Function of q")
lines(q_values, results[,2], type = "o", col = "red")
lines(q_values, results[,3], type = "o", col = "green")
legend("topright", legend = c("No Congestion", "One Congested", "Both Congested"),
       col = c("blue", "red", "green"), lty = 1, pch = 1)
```

Resulting plot:

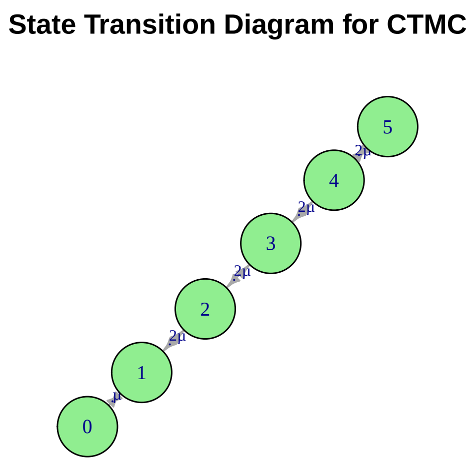


1f.

As p increases, the probability of congestion increases. When it decreases, both one and two congested routes decrease. This means that the higher p there is, there is an overall reduction in congestion. However, as q increases, the probability of both routes being congested increases, while the probabilities of no congestion and one congested route decreases. A higher q then means more ocngested states.

Problem 2:

2a.



2b.

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & 0 & 0 \\ 0 & 0 & 2\mu & -(\lambda+2\mu) & \lambda & 0 \\ 0 & 0 & 0 & 2\mu & -(\lambda+2\mu) & \lambda \\ 0 & 0 & 0 & 0 & 2\mu & -2\mu \end{bmatrix}$$

2c.

R code:

```
lambda <- 6  # Arrival rate (per hour)
mu <- 4      # Service rate per server (per hour)

Q <- matrix(c(
  -lambda, lambda, 0, 0, 0, 0,
  mu, -(lambda + mu), lambda, 0, 0, 0,
  0, 2*mu, -(lambda + 2*mu), lambda, 0, 0,
  0, 0, 2*mu, -(lambda + 2*mu), lambda, 0,
  0, 0, 0, 2*mu, -(lambda + 2*mu), lambda,
  0, 0, 0, 0, 2*mu, -2*mu
), nrow = 6, byrow = TRUE)

n <- nrow(Q)
A <- t(Q)
A[n, ] <- 1  # Replace last row for normalization
b <- c(rep(0, n-1), 1)  # Right-hand side
steady_state <- solve(A, b)

p_both_busy <- sum(steady_state[3:6])
p_turned_away <- steady_state[6]
```

The results are:

$$P(\text{both servers busy}) = 0.55, \quad P(\text{turned away}) = 0.085$$