CSE3504 Homework

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Problem 1:

(a) The expected waiting time for $n = 3$ arrivals is:

$$\mathbb{E}[T_3] = \frac{n}{\lambda} = \frac{3}{0.1} = 30\,\text{minutes}$$

Thus, the expected waiting time is 30 minutes.

(b) The probability that fewer than 3 patients arrive in the first 60 minutes is:

$$P(N(60) < 3) = P(N(60) = 0) + P(N(60) = 1) + P(N(60) = 2)$$

Using the Poisson probability formula:

$$P(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

with $\lambda = 0.1$ and $t = 60$, we compute:

$$P(N(60) = 0) = e^{-6}, \quad P(N(60) = 1) = 6e^{-6}, \quad P(N(60) = 2) = 18e^{-6}$$

Adding these:

$$P(N(60) < 3) = e^{-6} + 6e^{-6} + 18e^{-6} = 25e^{-6}$$

Evaluating numerically in R:

```
lambda <- 0.1
t <- 60
prob <- exp(-lambda * t) * (1 + lambda * t + (lambda * t)^2 / 2)
prob
```

The result is approximately:

$$P(N(60) < 3) \approx 0.002478$$

Thus, the probability is approximately 0.25%.

Problem 2:

(a)

$$P = \begin{bmatrix} 0 & 0.75 & 0.2 & 0.05 \\ 0.05 & 0.2 & 0 & 0.45 \\ 0 & 0.4 & 0.3 & 0.2 \\ 0 & 0.15 & 0.3 & 0.55 \end{bmatrix}$$

(b)

$$P(\text{Average} \to \text{Poor}) = 0.00$$

Thus, the probability is 0.00.

(c)

$$P(\text{Rich} \to \text{Poor in 3 years}) = (P^3)_{1,3}$$

Using R to calculate P^3:

```
P <- matrix(c(0, 0.75, 0.2, 0.05,
              0.05, 0.2, 0, 0.45,
              0, 0.4, 0.3, 0.2,
              0, 0.15, 0.3, 0.55), nrow=4, byrow=TRUE)

P3 <- P %*% P %*% P
prob <- P3[1, 3]
prob
```

The result is:

$$P(\text{Rich} \to \text{Poor in 3 years}) \approx 0.174$$

Thus, the probability is 0.174.

(d) $\pi = [\pi_{\text{Rich}}, \pi_{\text{Average}}, \pi_{\text{Poor}}, \pi_{\text{In Debt}}]$ are:

$$\pi \cdot P = \pi$$

Expanded, these equations are:

$$\begin{aligned} \pi_{\text{Rich}} &= 0.75\pi_{\text{Rich}} + 0.05\pi_{\text{Average}} \\ \pi_{\text{Average}} &= 0.2\pi_{\text{Rich}} + 0.2\pi_{\text{Average}} + 0.4\pi_{\text{Poor}} + 0.15\pi_{\text{In Debt}} \\ \pi_{\text{Poor}} &= 0.2\pi_{\text{Rich}} + 0.3\pi_{\text{Poor}} + 0.3\pi_{\text{In Debt}} \\ \pi_{\text{In Debt}} &= 0.05\pi_{\text{Rich}} + 0.45\pi_{\text{Average}} + 0.2\pi_{\text{Poor}} + 0.55\pi_{\text{In Debt}} \end{aligned}$$

The normalization condition is:

$$\pi_{\text{Rich}} + \pi_{\text{Average}} + \pi_{\text{Poor}} + \pi_{\text{In Debt}} = 1$$

(e) Using R:

```
P <- matrix(c(0, 0.75, 0.2, 0.05,
              0.05, 0.2, 0, 0.45,
              0, 0.4, 0.3, 0.2,
              0, 0.15, 0.3, 0.55), nrow=4, byrow=TRUE)

P_steady <- P %^% 100
steady_state <- P_steady[1, ]
steady_state
```

The result is:

$$\pi = [\pi_{\text{Rich}}, \pi_{\text{Average}}, \pi_{\text{Poor}}, \pi_{\text{In Debt}}] \approx [0.052, 0.313, 0.375, 0.260]$$

Thus, the steady-state probabilities are:

$$\pi_{\text{Rich}} \approx 0.052, \quad \pi_{\text{Average}} \approx 0.313, \quad \pi_{\text{Poor}} \approx 0.375, \quad \pi_{\text{In Debt}} \approx 0.260$$

(f) **R Code:**

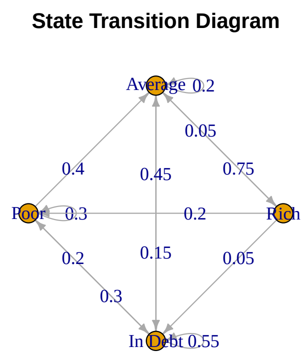
```
# Install and load igraph
install.packages("igraph")
library(igraph)

nodes <- c("Rich", "Average", "Poor", "In Debt")
edges <- c(
  "Rich", "Average", "Rich", "Poor", "Rich", "In Debt",
  "Average", "Rich", "Average", "Average", "Average", "In Debt",
  "Poor", "Average", "Poor", "Poor", "Poor", "In Debt",
  "In Debt", "Average", "In Debt", "Poor", "In Debt", "In Debt"
)
weights <- c(0.75, 0.2, 0.05, 0.05, 0.2, 0.45, 0.4, 0.3, 0.2, 0.15, 0.3, 0.55)

g <- graph(edges = edges, directed = TRUE)
E(g)$weight <- weights  # Assign weights to edges

plot(
  g,
  vertex.label = nodes,
  edge.label = round(E(g)$weight, 2),
  layout = layout.circle,
  edge.arrow.size = 0.5,
  main = "State Transition Diagram"
)
```

Result below:



Problem 3:

(a)

$$X \sim \text{Binomial}(n = 5, p = 0.85)$$

The probability of acceptable attendance is:

$$P(X \geq 4) = P(X = 4) + P(X = 5)$$

Using the binomial probability formula:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Compute each term:

$$P(X = 4) = \binom{5}{4} (0.85)^4 (0.15)^1$$

$$P(X = 5) = \binom{5}{5} (0.85)^5 (0.15)^0$$

Add the probabilities:

$$P(X \geq 4) = P(X = 4) + P(X = 5)$$

Using R:

```
# R Code
n <- 5
p <- 0.85
prob <- dbinom(4, size = n, prob = p) + dbinom(5, size = n, prob = p)
prob
```

Thus, the probability that a student has acceptable attendance is approximately:

$$P(X \geq 4) \approx 0.754$$

(b)

$$Y \sim \text{Binomial}(n = 200, p = 0.754)$$

The mean and variance of Y are:

$$\mu = np = 200 \cdot 0.754 = 150.8, \quad \sigma^2 = np(1-p) = 200 \cdot 0.754 \cdot (1-0.754) = 36.9688$$

Approximating Y as a Normal distribution:

$$Y \approx \mathcal{N}(\mu = 150.8, \sigma = \sqrt{36.9688} \approx 6.08)$$

We compute:

$$P(Y \geq 170) = P\left(Z \geq \frac{170 - 150.8}{6.08}\right)$$

The z-score is:

$$Z = \frac{170 - 150.8}{6.08} \approx 3.15$$

Using the standard normal distribution table:

$$P(Z \geq 3.15) \approx 0.0008$$

In R:

```
# R Code
n <- 200
p <- 0.754
mu <- n * p
sigma <- sqrt(n * p * (1 - p))
prob <- 1
```

(c)

$$\mathbb{E}[Y] = n \cdot p$$

Substituting the values $n = 200$ and $p = 0.754$:

$$\mathbb{E}[Y] = 200 \cdot 0.754 = 150.8$$

Thus, the expected number of students with acceptable attendance is:

$$\mathbb{E}[Y] = 150.8$$

Problem 4:

(a)

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{24}{25} & \frac{1}{25} & 0 & 0 & 0 \\ 0 & \frac{47}{50} & \frac{3}{50} & 0 & 0 \\ 0 & 0 & \frac{47}{50} & \frac{3}{50} & 0 \\ 0 & 0 & 0 & \frac{24}{25} & \frac{1}{25} \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{24}{25} & \frac{1}{25} & 0 & 0 \\ 0 & \frac{47}{50} & \frac{3}{50} & 0 \\ 0 & 0 & \frac{47}{50} & \frac{3}{50} \end{bmatrix}$$

The fundamental matrix N is:

$$N = (I - Q)^{-1}$$

Sum rows of N for result in R:

```
# R Code
library(MASS)

Q <- matrix(c(24/25, 1/25, 0, 0,
              0, 47/50, 3/50, 0,
              0, 0, 47/50, 3/50), nrow=3, byrow=TRUE)

I <- diag(nrow(Q))
N <- solve(I - Q)

expected_time <- rowSums(N)
expected_time
```

The result is:

$$\text{Expected time to absorption} \approx [1.04, 1.98, 3.93]$$

Thus, the expected number of interactions before absorption for each transient state is approximately 1.04, 1.98, and 3.93.

(b) **R Code:**

```
# Install and load igraph
install.packages("igraph")
library(igraph)

nodes <- c("1 Heard (Absorbing)", "2 Heard", "3 Heard", "4 Heard", "5 Heard (Absorbing)
```

```
edges <- c(
  2, 1,   # 2 Heard -> 1 Heard (Absorbing)
  2, 3,   # 2 Heard -> 3 Heard
  3, 2,   # 3 Heard -> 2 Heard
  3, 4,   # 3 Heard -> 4 Heard
  4, 3,   # 4 Heard -> 3 Heard
  4, 5    # 4 Heard -> 5 Heard (Absorbing)
)
weights <- c(1/25, 24/25, 3/50, 47/50, 3/50, 47/50)

g <- make_graph(edges = edges, directed = TRUE)
E(g)$weight <- weights  # Assign weights to edges

V(g)$name <- nodes

plot(
  g,
  vertex.label = V(g)$name,
  edge.label = round(E(g)$weight, 2),
  layout = layout.circle,
  edge.arrow.size = 0.5,
  main = "State Transition Diagram for Rumor Spread"
)
```

Diagram: