Why you should care about Topology

Applications in Machine learning

- Principled way to talk about <u>Convergence</u> for
  - functions, sets, graphs, computer programs, metric spaces, ...

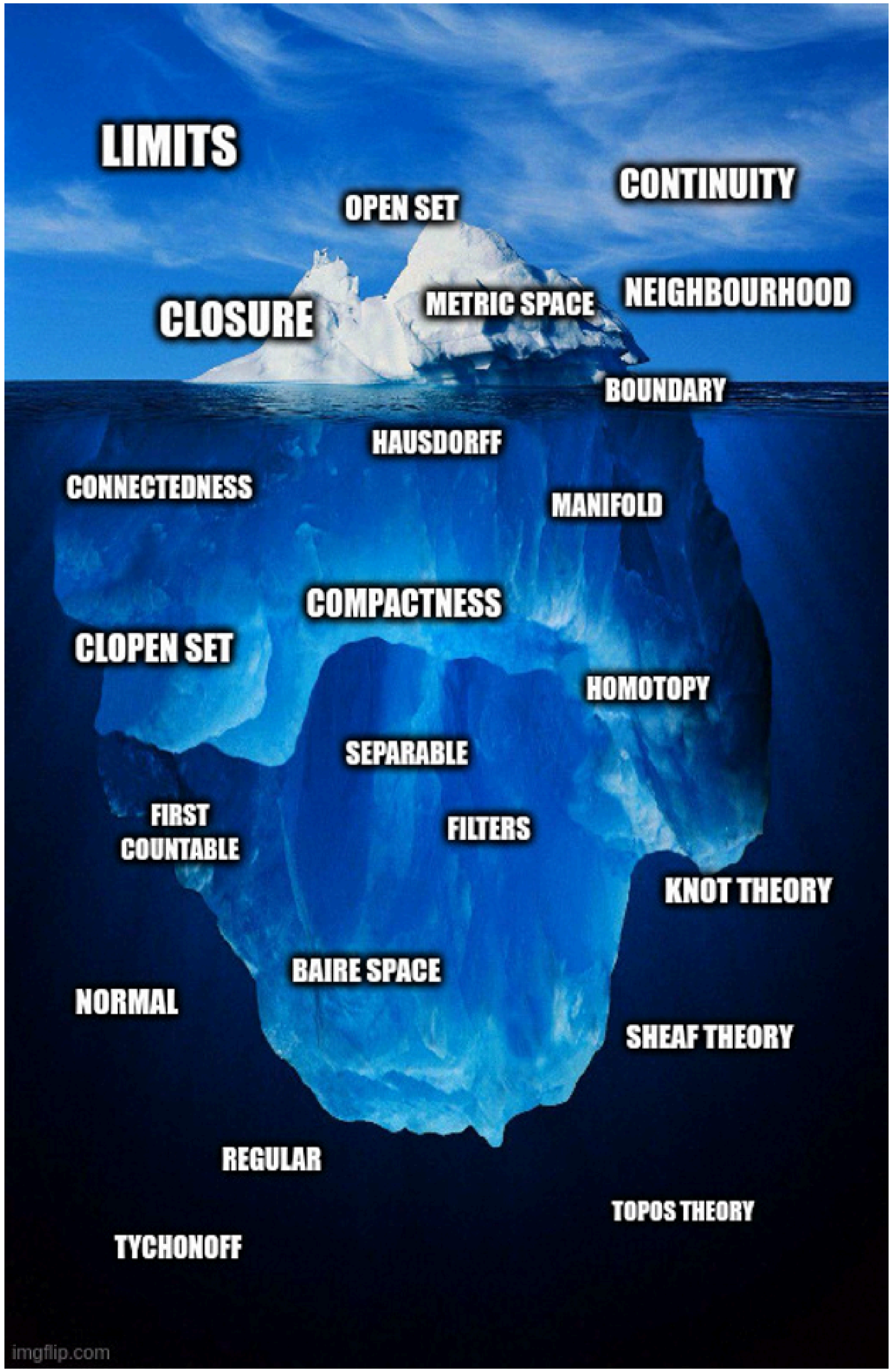
    ↳ approximation theorems

- Insights about Neural Network hyperparams (width)

  (see Chris Olah – Neural Networks, Manifolds and Topology)

  + embeddings (Manifold Hypothesis)

The Topology Iceberg



Separation axioms in topological spaces

Kolmogorov classification

T_0	(Kolmogorov)
T_1	(Fréchet)
T_2	(Hausdorff)
$T_{2½}$	(Urysohn)
completely T_2	(completely Hausdorff)
T_3	(regular Hausdorff)
$T_{3½}$	(Tychonoff)
T_4	(normal Hausdorff)
T_5	(completely normal Hausdorff)
T_6	(perfectly normal Hausdorff)

History

Why should you care about Topology?

17th Century: development of Calculus by Newton / Leibniz
⇝ Problem: No rigorous foundations, weird questions around infinitesimals

18th Century: Rigorous foundations by Bolzano, Cauchy, Weierstrass, ...
("$\varepsilon - \delta$" formulation)

⇝ Topology is about taming infinity
→ rigorous definitions of limits, continuity, ...

+ Graph Theory, Euler characteristic, Hairy Ball theorem, Homotopy, Homology, ... ⇝ "Geometry without numbers"
(lengths, angles)

General Topology ~~Crash Course~~ Speedrun

Topological Space (X, τ) where $X :: Set$

$\tau \subset \mathcal{P}(X)$ s.t.

- $\emptyset, X \in \tau$
- $U, V \in \tau \Rightarrow U \cap V \in \tau$ (finite intersections)
- $\{U_i\}_{i \in I} \Rightarrow \bigcup_{i \in I} U_i \in \tau$ (arbitrary unions)

- The elements of τ are called the open sets of the topology τ
- A subset F of X is closed if $X \setminus F \in \tau$

Characterization by closed sets

1 N

If (X, τ) is a topological space

- $\emptyset, X$ are closed sets
- F, G closed $\Rightarrow$ $F \cup G$ closed (finite unions)
- $\{F_i\}_{i \in I}$ family of closed subsets $\Rightarrow$ $\bigcap_{i \in I} F_i$ closed

(arbitrary intersections)

Note: sets can be both open and closed (clopen)
or neither (e.g $[a, b)$)

Intuition: A topology is the set of "pictures that can be drawn with a given set of "brushes"

Trivial topology: Fill Tool 

Discrete topology: Infinitely thin "point brush"

$\{x\}$

Euclidian topology: Brushes with non-zero thickness

Examples

① Discrete topology

$(X, \mathcal{P}(X))$ discrete topology (every subset open)

Trivial topology

$(X, \{\emptyset, X\})$

Usual topology on $\mathbb{R}$

let $\tau \triangleq \{U \subseteq \mathbb{R} \mid U$ is the union of open intervals$\}$

↳ $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

Sierpinsky space

$X = \{0,1\}$ $\tau = \{\emptyset, \{1\}, \{0,1\}\}$ (simplest non trivial topological space)

__Metric space__ (X, d) $\quad d: X \times X \to \mathbb{R}_+$

$$\begin{cases} d(x,x) = 0, \quad d(x,y) = 0 \Rightarrow x = y \\ d(x,y) = d(y,x) \\ d(x,y) \leq d(x,z) + d(z,y) \end{cases}$$

generic open ball $\mathcal{B}(x,\varepsilon) = \{ y \in X \mid d(x,y) < \varepsilon \}$

$\rightsquigarrow$ generate the __metric__ topology

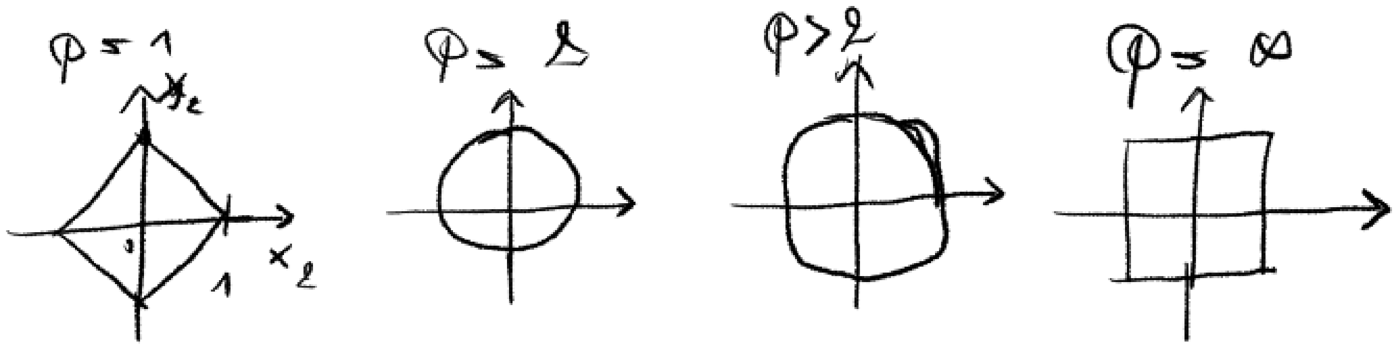
(X, τ) is __metrizable__ if there is a metric that generates τ

↳ ⚠ Can be multiple metrics!

e.g. $\quad d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$

l_p-norms all generate the euclidian topology on $\mathbb{R}^n$

(but not in infinite dimensions)



Euclidian topology

$X = \mathbb{R}^n$

open ball : $\mathcal{B}(x, \varepsilon) = \{ y \in \mathbb{R}^n \mid \|x - y\|_2 < \varepsilon \}$

euclidian topology $\tau = \{$ arbitrary unions of open balls $\}$

Finite Complement topology

X arbitrary set $\tau = \{\emptyset, X\} \cup \{U \mid X \setminus U \text{ is finite}\}$

Order Topology

partially ordered set $(X, \leq)$

$x \leq x$

$x \leq y$ and $y \leq x \Rightarrow x = y$

$x \leq y$ and $y \leq z \Rightarrow x \leq z$

order topology : $\tau = \{$arbitrary unions one sided intervals$\}$

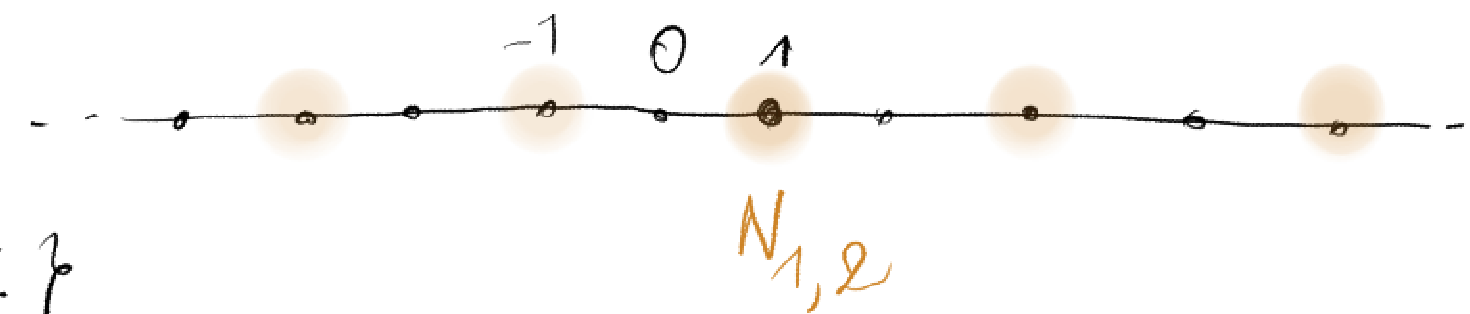
$\{b \in X \mid a < b\}$

$\{b \in X \mid b < a\}$

On $(\mathbb{R}, \leq)$ ~ equivalent to euclidian topology

Topological proof that there are infinitely many primes

let $a, b \in \mathbb{Z}$, $b \neq 0$ and let



$$N_{a,b} = \{ a + bn \mid n \in \mathbb{Z} \}$$

and generate a topology using unions of such sets, i.e. declare $U \subseteq \mathbb{Z}$ to be open if it is a union of such sets.

$\emptyset = N_{0,2} \cap N_{1,2}$ is open. $\mathbb{Z} = N_{0,1}$ is clearly open

. let U_1, U_2 be open sets and let $a \in U_1 \cap U_2$ s.t. $U_1 \cap U_2 \neq \emptyset$

$$\Rightarrow \begin{array}{l} \exists\, b_1 \text{ s.t } N_{a,b_1} \subset U_1 \\ \exists\, b_2 \text{ s.t } N_{a,b_2} \subset U_2 \end{array} \Rightarrow N_{a,b_1 b_2} \subseteq N_{a,b_1} \cap N_{a,b_2} \subseteq U_1 \cap U_2$$

. the $N_{a,b}$ sets are also closed: $N_{a,b} = \mathbb{Z} \setminus \bigcup_{i=1}^{b-1} N_{a+i,b}$

Since any integer $c \in \mathbb{Z} \setminus \{-1,1\}$ admits at least one prime divisor p

$\Rightarrow c \in N_{0,p} \quad \Rightarrow \quad \mathbb{Z} \setminus \{-1,1\} = \bigcup_{p \text{ prime}} N_{0,p}$

Suppose there are only finitely many primes in $\mathbb{Z}$, then $\mathbb{Z} \setminus \{-1,1\}$ is a closed set hence $\{-1,1\}$ is open which is a contradiction as it is finite

Moral of the story: Choose the topology for the specific problem you're studying

Why the inverse image?

$f : (X, \tau_X) \to (Y, \tau_Y)$ is an <u>open map</u> if

$$U \in \tau_X \Rightarrow f(U) \in \tau_Y$$

Are open maps continuous? No

Counter example: $f : \mathbb{R} \to \mathbb{R}$ (usual topology)

$f(x) = x^2$ $\quad f$ is continuous

but $f(]-1, 1[) = \underbrace{[0, 1[}_{\text{not an open set}}$ $\quad f$ is not open

Continuity

Metric spaces $f : (X, d_X) \to (Y, d_Y)$ is **continuous** at $x \in X$

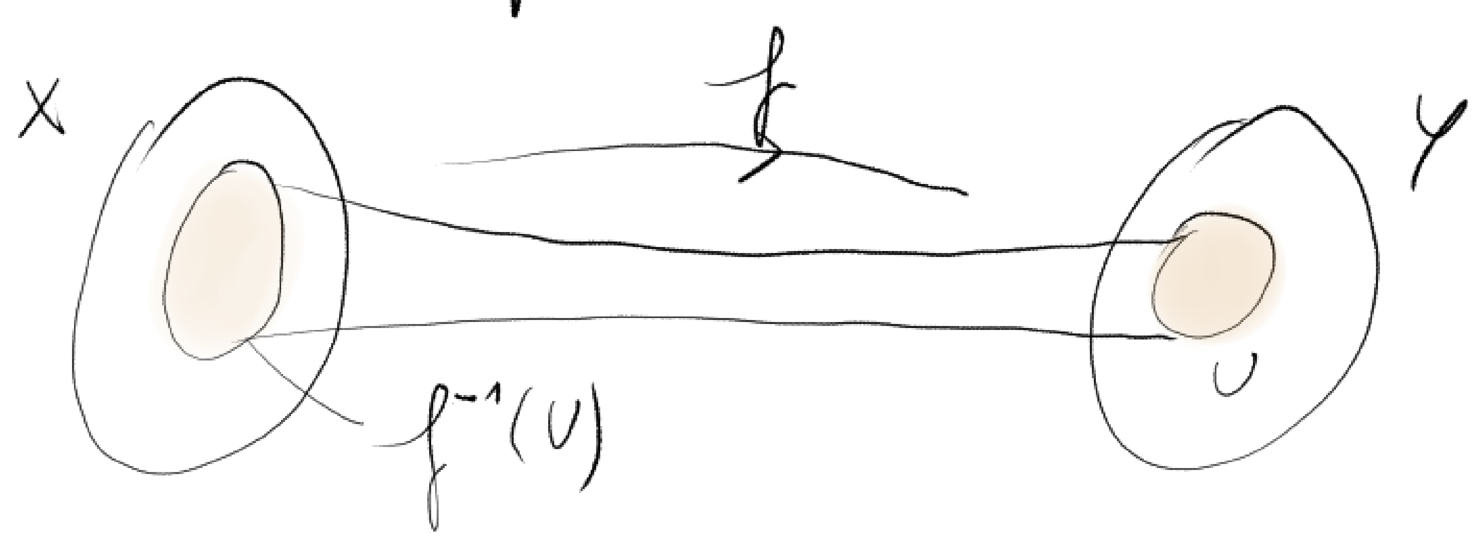
if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$$

$$\Leftrightarrow \quad f(B_X(x,\delta)) \subseteq B_Y(f(x), \varepsilon)$$

General continuity $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous

$$\Leftrightarrow \quad \forall U \in \tau_Y \quad f^{-1}(U) = \{x \in X \mid f(x) \in U\} \in \tau_X$$

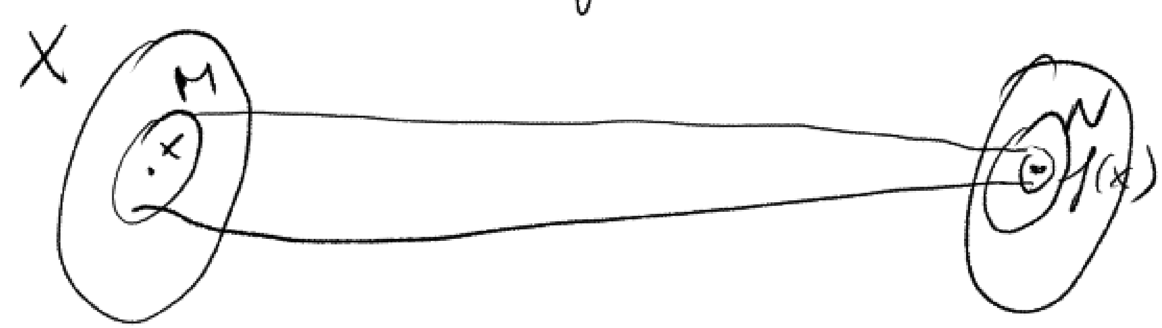


Examples

- constant functions are always continuous
- the identity function $id: (X, \tau) \to (X, \tau)$ is always continuous
- If $f: X \to Y$ and $g: Y \to Z$ are continuous functions then $g \circ f: X \to Z$ is continuous
- Any function from a discrete topological space is continuous
- Any function to a trivial topological space is continuous

Continuity at a point

$f: X \to Y$ is continuous at a point $x \in X$ if for every neighbourhood N of $f(x)$ in Y, there is a neighbourhood M of x in X s.t. $f(M) \subseteq N$



Note: Checking this condition on the elements of a neighbourhood basis of $f(x)$ is enough, i.e.

f continuous at x $\Leftrightarrow$ for any neighbourhood basis $\mathcal{B}_{f(x)}$ of $f(x)$, $\{f^{-1}(N) \mid N \in \mathcal{B}_{f(x)}\}$ is a neighbourhood basis of x

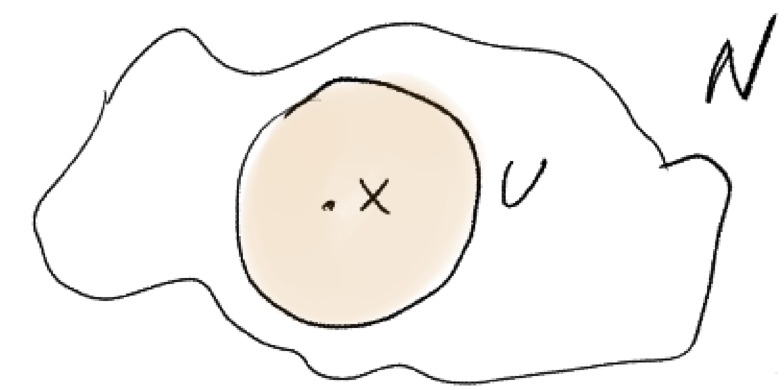
Theorem: f continuous $\Leftrightarrow$ f continuous at x $\forall x \in X$

Neighborhoods

Let (X, τ) be a top. space and $x \in X$.

$N \subseteq X$ is a neighborhood of x if there is $U \in \tau$ s.t.

$$x \in U \subseteq N$$



Neighborhood basis

$\mathcal{B}_x \subseteq \mathcal{P}(X)$ is a neighborhood basis of x if

- $\forall B \in \mathcal{B}_x$ B is a neighborhood of x
- every neighborhood of x contains an element of $\mathcal{B}_x$

Example: $\{B(x, \frac{1}{n}) \mid n \in \mathbb{N}_0\}$ (for any metric topology)

Homeomorphism

- $f : X \to Y$ is a homeomorphism

  $\Leftrightarrow$ f is bijective and f, f^{-1} are continuous

- X and Y are homeomorphic

  if there is an homeomorphism $X \to Y$

  $\rightsquigarrow$ This is an equivalence relation!

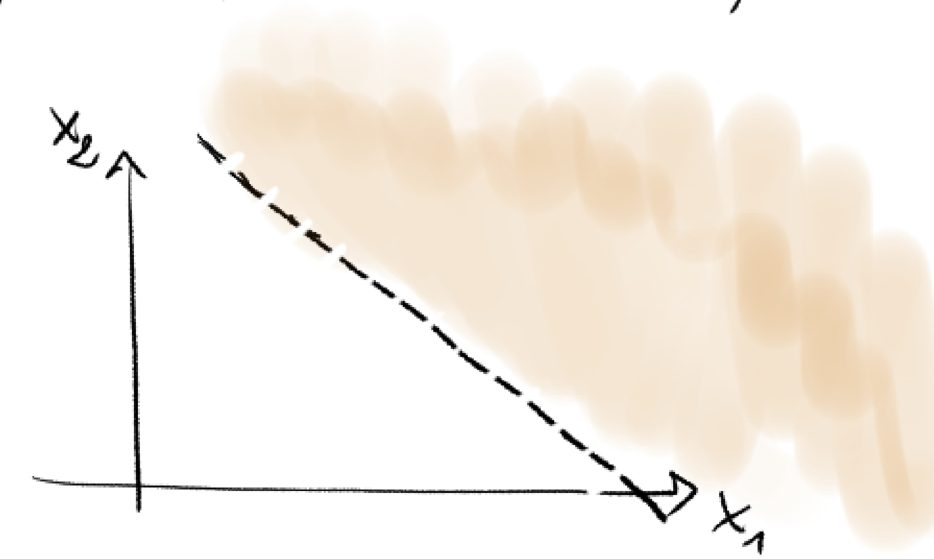
Homeomorphisms are one-to-one correspondances between collections of open sets

$\rightsquigarrow$ Can't distinguish two homeomorphic spaces based on their topological structure

Example : Half-space topology

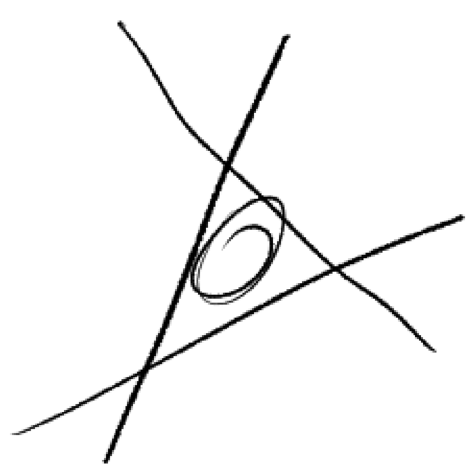
half space $\quad \Pi_{w,b} = \{ x \in \mathbb{R}^n \mid w^T x + b > 0 \}$ for $w \in \mathbb{R}^n$, $b \in \mathbb{R}$

- use these a a sub-basis to construct a topology τ_h



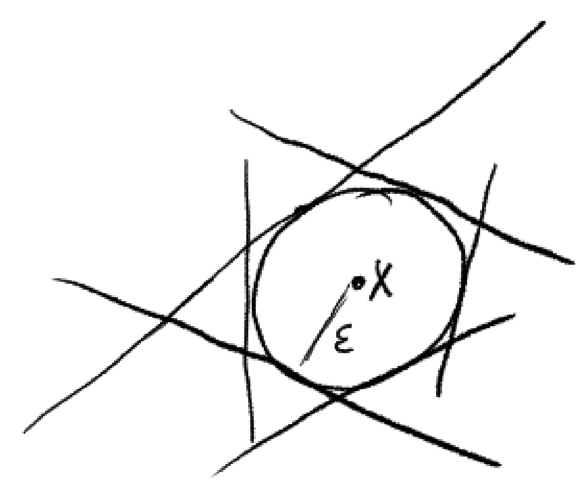
Claim : the "half-space topology" is equivalent to the usual topology τ_u

- $\Pi_{w,b} \in \tau_u$



$\Rightarrow \forall U \in \tau_h$, $U \in \tau_u$

$\Rightarrow \tau_h \subset \tau_u$

- 

$\overline{B}_{x,\varepsilon} = \bigcap_{\|w\|=1} \Pi^c_{w, -w^T x - \varepsilon}$

$w^T(y-x) \leq \varepsilon$

$w^T y - w^T x - \varepsilon \leq 0$

$\Rightarrow \tau_u \subset \tau_h$

Basis of a topology

- $\mathcal{B}$ is a basis of (X, τ) if

$$\tau = \{\text{arbitrary unions of elements of } \mathcal{B}\}$$

- S is a subbasis of (X, τ) if

$$\mathcal{B}(S) = \{\text{finite intersections of elements of } S\}$$

is a basis of (X, τ)

↳ Any subset S can be used to construct a topology ($\equiv$ the topology generated by S)

Example: open balls $B(x, \varepsilon)$ generate the metric topology

First countable (X, τ) is first countable if $\forall\, x \in X$ there is a countable neighborhood basis

Second countable : τ has a countable basis

Example: • Every metric space is first countable, as

$\{ B(x, \frac{1}{n}) \mid n \in \mathbb{N}_0 \}$ is a countable neighbourhood basis for x

• Euclidian spaces are second countable, as

$\{ B(q, \frac{1}{n}) \mid q \in \mathbb{Q}^n,\ n \in \mathbb{N}_0 \}$

Constructing Topologies

- Subspace Topology Given (X, τ) a top. space and $A \subset X$, the subspace topology on A is defined as $\tau_A = \{A \cap U \mid U \in \tau\}$

Other definition: Given the inclusion function $i : A \hookrightarrow X$
τ_A is the smallest topology which makes i continuous.

Let (X, τ) be a top. space and $A \subseteq X$

1) There is a largest open set $U \subseteq A$, called the interior

$$\text{int}_X(A) ; \ \text{int}(A) \ \text{or} \ \mathring{A} = \bigcup \{ U \in \tau \mid U \subseteq A \}$$

2) There is a smallest closed set $F \supseteq A$, called the closure

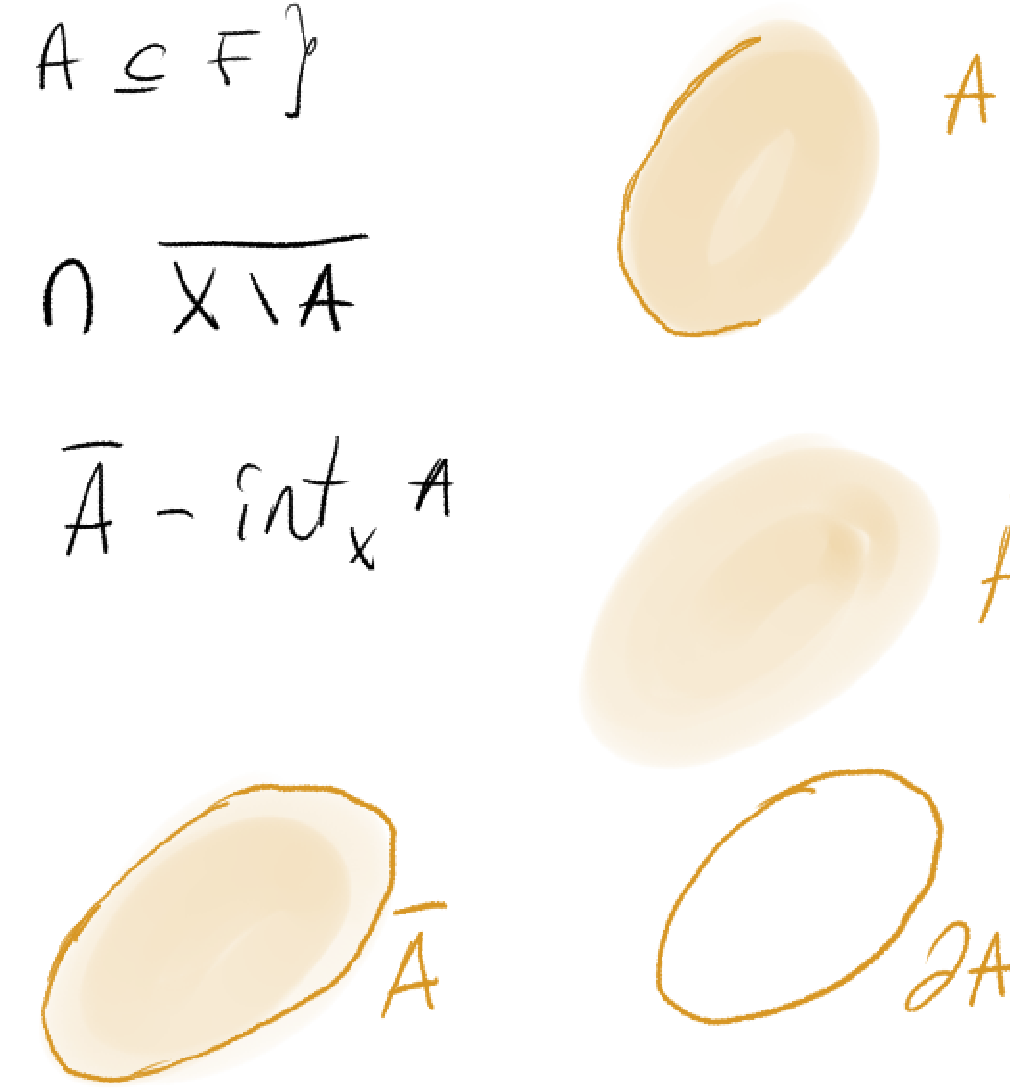
$$\overline{A}^X ; \ \overline{A} = \bigcap \{ F \text{ closed} \mid A \subseteq F \}$$

- The <u>boundary</u> of A : $\partial A = \overline{A} \cap \overline{X \setminus A}$

$$= \overline{A} - \text{int}_X A$$

- A subset is dense if $\overline{A} = X$

e.g. $\mathbb{Q}$ is dense in $\mathbb{R}$



Limit points

x is a point of accumulation of A if $x \in \overline{A \setminus \{x\}}$
(limit point)

isolated point : $\exists$ neighborhood U of x s.t. $A \cap U = \emptyset$

$\bar{A} = A \cup A'$
$\hookrightarrow$ limit points of A

$\Rightarrow$ A is closed $\Leftrightarrow$ A contains all its limit points

Convergence of a sequence : $(x_n) \subset X$ converges to x in X

$\Leftrightarrow$ for every neighbourhood B of x, there is $M \geq 0$ s.t.

$n \geq M \Rightarrow x_n \in B$

NB. limits of sequences are not necessarily unique!

Product Topology

Cartesian Product

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ (X, \tau_X) & & (Y, \tau_Y) \end{tikzcd}$$

We want to construct a topology on $X \times Y$

$\rightarrow$ projections π_X, π_Y should be continuous

$$\pi_X^{-1}(U) = U \times Y \qquad (U \times Y) \cap (X \times V) = U \times V$$

$$\pi_Y^{-1}(V) = X \times V$$

$U \times V$ must be an open set!

product topology: $\{$arbitrary union of sets $U_i \times V_i$, $i \in I\}$

Product of infinitely many topologies

$\{X_i\}_{i \in I}$ family of topological spaces

product topology is generated by $\times_{i \in I} U_i$

where $U_i \in X_i$ and $U_i = X_i$ for all but finitely many i's

(because we can only take finite intersections of open sets)

Uniqueness of the product topology

For every topological space Z with continuous maps $f_X: Z \to X$

$$f_Y: Z \to Y$$

$$\begin{tikzcd}
Z \arrow[dr, "f"] \arrow[drr, bend left, "f_Y"] \arrow[ddr, bend right, "f_X"'] & & \\
& X \times Y \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \\
& X &
\end{tikzcd}$$

We can always find $f: Z \to X \times Y$ s.t.

$$f_X = \pi_X \circ f$$

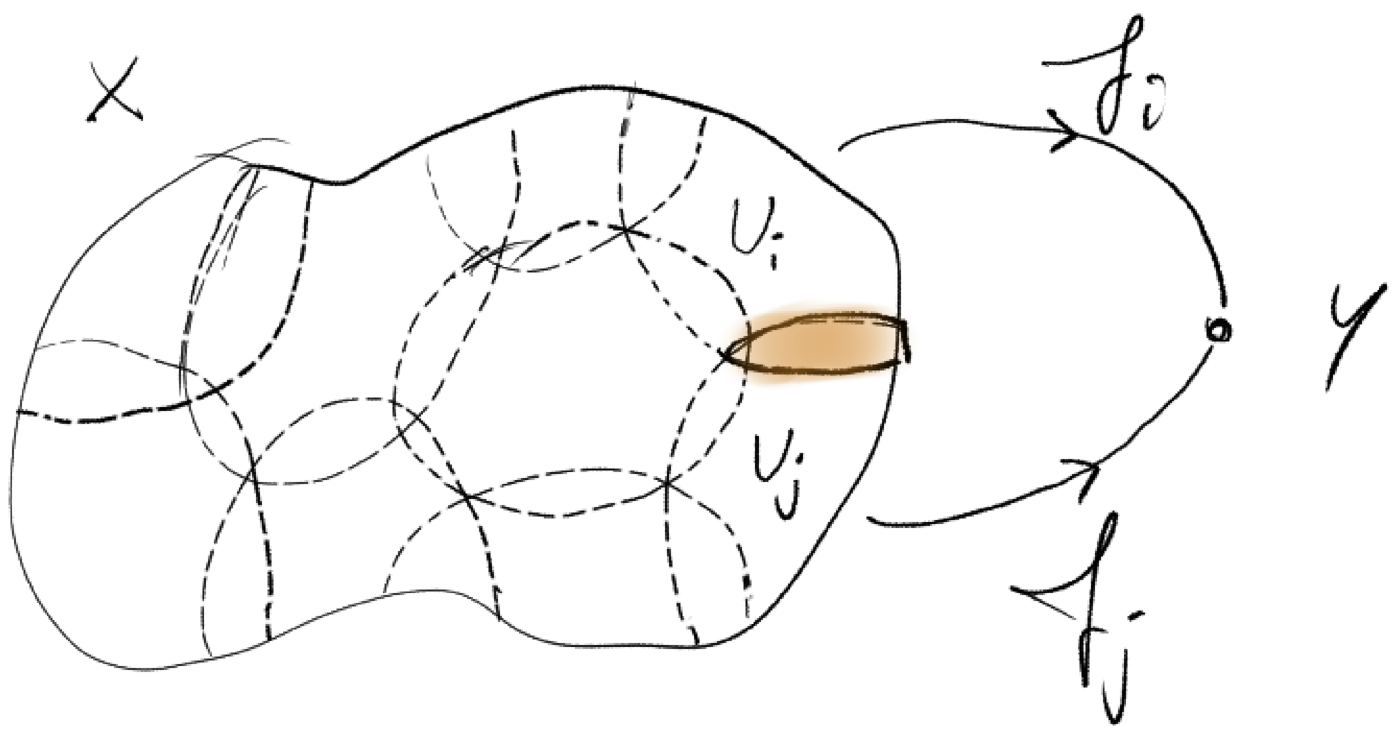
$$f_Y = \pi_Y \circ f$$

Gluing Topologies

Let X be a topological space, $\{U_i\}_{i\in I}$ an open cover of X

- U open in $X \Leftrightarrow U \cap U_i$ open in U_i $\forall i \in I$
- Let $f: X \to Y$ be continuous $\rightsquigarrow f_i = f|_{U_i}: U_i \to Y$

  s.t. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ (*)



- Conversely given $f_i: U_i \to Y$ $\forall i \in I$ s.t. (*), we can "glue" the f_i to form a map $f: X \to Y$ that is continuous

<u>Theorem</u> Let X be an arbitrary set, $\{X_i\}_{i \in I}$ a collection of subsets of X whose union is X. Suppose that for each $i \in I$ there is a topology τ_i on X_i s.t. $\forall i,j \in I$, $X_i \cap X_j$ is open in (X_i, τ_i) and (X_j, τ_j) and the induced topologies on $X_i \cap X_j$ coincide.

Then there is a unique topology τ on X that induce τ_i on each X_i

<u>Note</u>: • This is important for defining (differential) manifolds

• $\rightarrow$ Sheaf Theory, Topos Theory