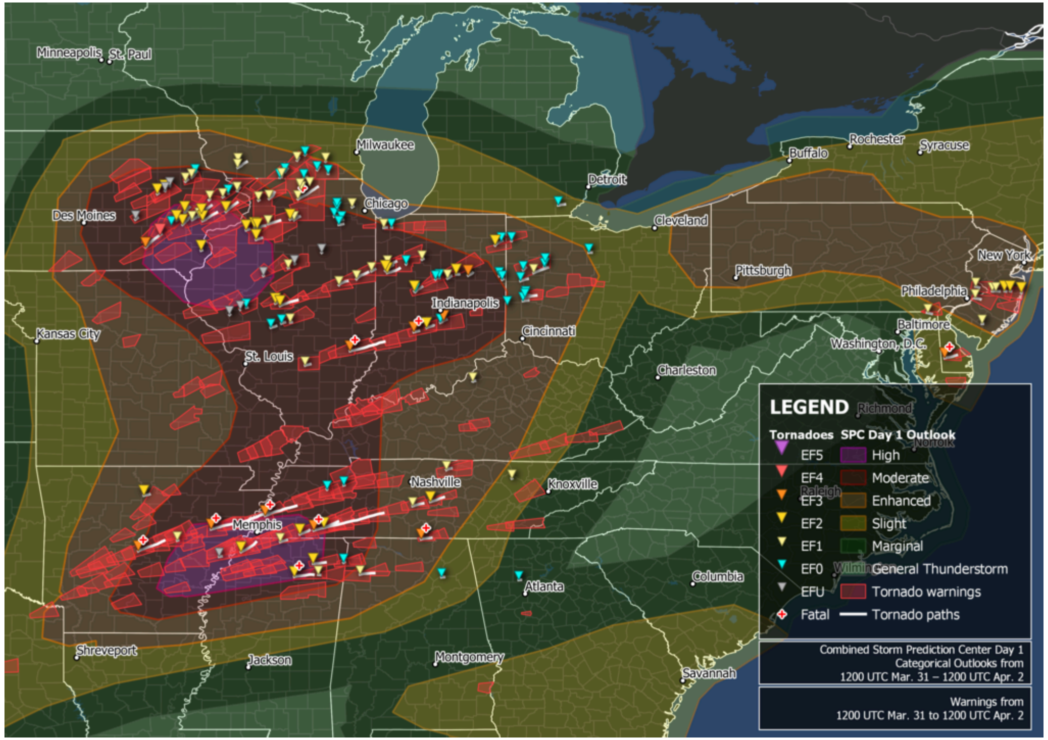
Chaos Theory

EAI Math Reading Group

31/03/2024

Today's notable anniversary



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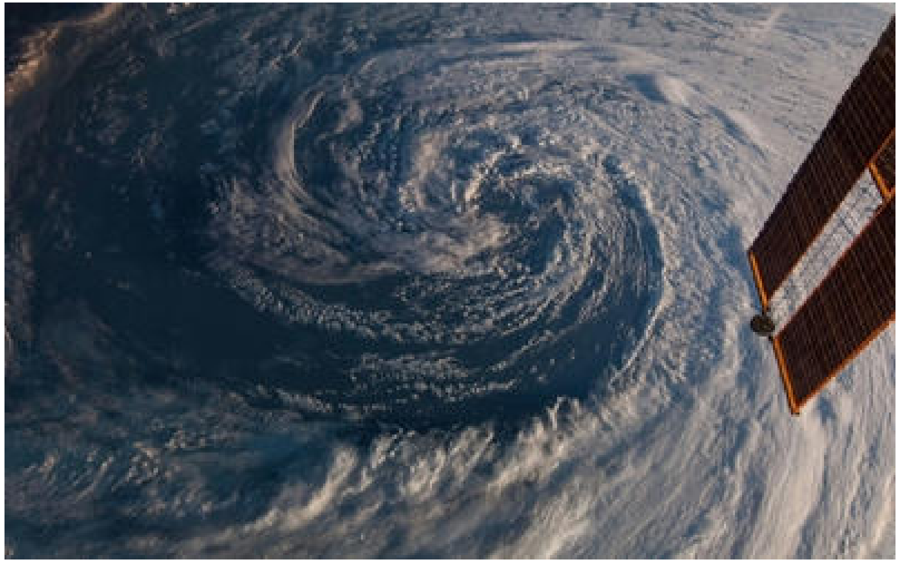
Motivation: Dynamical Systems are everywhere



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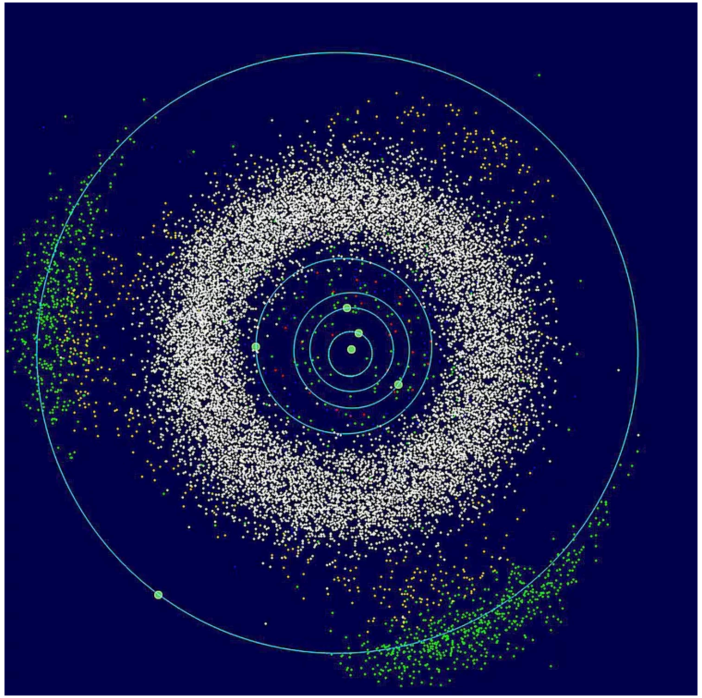
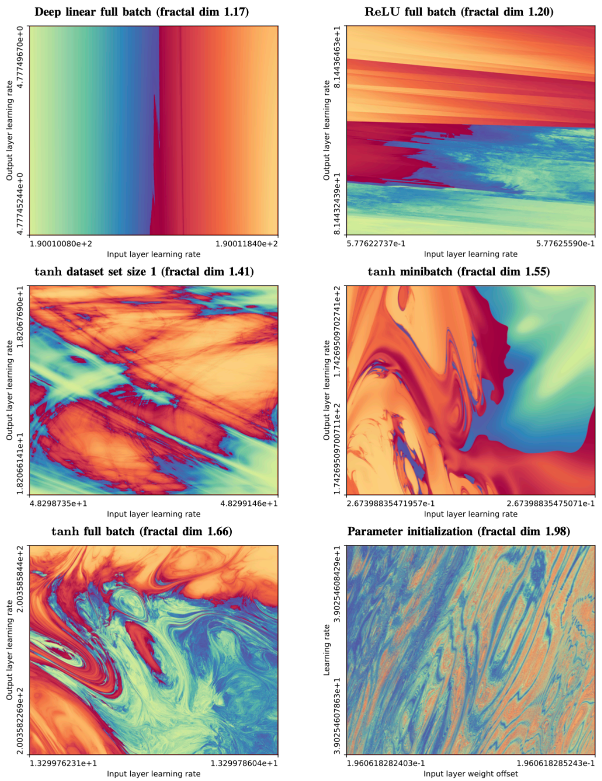


https://www.thoughtco.com/asteroids-and-the-asteroid-belt-3073446

Including in ML!

- SGD is a discrete-time dynamical system
  - can be chaotic
- Neural ODEs, SSMs, …
- Reservoir Computing
  - Echo-State Networks
  - Liquid-state machines
  - …



https://sohl-dickstein.github.io/2024/02/12/fractal.html

Outline

1. Discrete-time systems
2. Chaos in continuous-time systems
3. Lyapunov exponents
4. Fractals and strange attactors

Discrete-time systems

$$x_{k+1} = f(x_k), \qquad k \in \mathbb{N}$$

with $x_k \in X$, some set and $f : X \to X$

- X is called the *state-space*
- For $x_0 \in X$, $\{x_0, f(x_0), f(f(x_0)), ...\}$ is called the ***orbit*** of x_0
- The pair (X, f) is sometimes called a dynamical system

Example: $X = \mathbb{R}_+$, $f_r(x) = \frac{x^2-r}{2x}$, for some $r \in \mathbb{R}_+$

Fixed points

For a dynamical system (X, f), a ***fixed point*** is an element $x^* \in X$ s.t.

$$f(x^*) = x^*$$

Example: $\sqrt{r}$ is a fixed point of $f_r(x) = \frac{x^2 - r}{2x}$

Cyclic points x^* is n-cyclic if it is a fixed point of f^n

Stability of fixed points

A fixed point x^* of $f : \mathbb{R} \to \mathbb{R}$ is

- attracting (a sink) if $x_k \to x^*$ for all x_0 within some ball $B_\varepsilon(x^*)$ around x^*
- repelling (a source) if for any $x_0 \in B_\varepsilon(x^*)$, the orbit $(f^n(x_0))_{n\in\mathbb{N}}$ eventually leaves $B_\varepsilon(x^*)$.

Theorem [Strogatz]

Let f be a smooth map on $\mathbb{R}$ and x^* a fixed point of f, then

1. If $|f'(x^*)| < 1$, then x^* is a sink
2. If $|f'(x^*)| > 1$, then x^* is a source

What about $|f'(x^*)| = 1$?

Proof

1. Let $|f'(x^*)| < a < 1$. Since

$$|f'(x^*)| = \lim_{x \to x^*} \frac{|f(x) - f(x^*)|}{|x - x^*|},$$

there must be some ball $B_\varepsilon(x^*)$ such that

$$\frac{|f(x) - f(x^*)|}{|x - x^*|} < a,$$

for all $x \in B_\varepsilon(x^*), x \neq x^*$. This implies

$$|f(x) - x^*| < a \; |x - x^*| < |x - x^*| < \varepsilon.$$

So $f(x) \in B_\varepsilon(x^*)$, and moreover

$$|f^k(x) - x^*| < a^k \; |x - x^*| \to 0$$

So $f^k(x)$ converges to x^*. The proof for 2. is similar.

1D linear system

Let $a \in \mathbb{R}$ and consider

$$x_{k+1} = a x_k$$

$x^* = 0$ is always a fixed point, and orbits can be explicitly written as

$$x_k = a^k x_0$$

- $|a| < 1$: All orbits converge to 0
- $|a| > 1$: All orbits diverge to infinity
- $a = 1$: Every $x \in \mathbb{R}$ is a fixed point
- $a = -1$: Every $x \in \mathbb{R}$ is in a 2-cycle

Linear systems

Let $A \in \mathbb{R}^{n \times n}$ be some matrix, and consider the dynamical system

$$x_{k+1} = A x_k$$

Suppose that A can be diagonalized as $V \Lambda V^{-1}$, so that for any $x \in \mathbb{R}^n$, noting $y = V^{-1} x$

$$Ax = V \Lambda V^{-1} x = V \Lambda V^{-1} V y = V \Lambda y,$$

so the original system is equivalent to the diagonal system

$$y_{k+1} = \Lambda y_k,$$

or equivalently, for $i \in \{1, ..., n\}$

$$y_{k+1}^{(i)} = \lambda_i y_k^{(i)}$$

we now have just n decoupled 1D systems, and 0 is a sink iff $|\lambda_i| < 1, \forall i$.

Nonlinear Example: Logistic map

$$x_{k+1} = rx_k(1 - x_k)$$

Fixed points: $x = rx(1 - x) \Leftrightarrow x(r - 1 - rx) = 0$

$\Rightarrow 0$ and $\frac{r-1}{r}$ are fixed points

Stability: $[rx(1 - x)]' = r - 2rx$

$\Rightarrow 0$ is stable for $0 \leq r < 1$; $\frac{r-1}{r}$ stable for $1 < r < 3$

What happens after 3?

Recap

Extremely simple discrete-time systems can have *extremely* complicated long-term behavior

Temporary definition of Chaos:

- bounded (orbits do not diverge)
- non-periodic (no stable fixed points/cycles)
- sensitivity to initial conditions
- (dense periodic orbits)

Quasiperiodicity

Consider the map $f_q : [0, 1] \to [0, 1]$

$$f_q(x) = (x + q) \bmod 1$$

If viewed as a map on the circle, f_q corresponds to rotating by a constant angle q.

- If $q = \frac{p}{r} \in \mathbb{Q}$, then any point x returns to itself after pr iterations, so the system is pr-periodic
- If $q \notin \mathbb{Q}$, there are no periodic orbits, and any orbit gets arbitrarily close to any point

The second case is *not* chaotic, and is called *quasiperiodic*.

Chaos in continuous-time systems

Consider continuous-time systems of the form

$$\dot{x} = f(x),$$

with $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$

Can we still get chaotic behavior? Yes (obviously), ***but***

- only in 3 dimensions or more (Poincaré-Bendixson theorem),
- linear systems in finite dimensions are never chaotic[1]

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[1]They can be *quasiperiodic*, however

Lyapunov exponents

Given a smooth map $f : \mathbb{R} \to \mathbb{R}$, the *Lyapunov* number $L(x_0)$ of the orbit $\{x_0, x_1, ...\}$ is

$$L(x_0) = \lim_{n\to\infty} \left(|f'(x_0)| \; ... \; |f'(x_n)|\right)^{\frac{1}{n}}$$

(if the limit exists), and the corresponding *Lyapunov exponent* is

$$h(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln|f'(x_k)|$$

Morally: Lyapunov exponents measure how orbits nearby behave ($\rightsquigarrow$ sensitivity to initial conditions)

Lyapunov exponents and Chaos

For a map $f : \mathbb{R} \to \mathbb{R}$, a bounded orbit $\{x_0, x_1, ...\}$ is called *chaotic* if it is

1. not asymptotically periodic (i.e. doesn't converge to a fixed point/cycle)
2. The Lyapunov exponent $h(x_0)$ is positive

Generalizations

- multiple dimensions: singular values of Df^n (Jacobian of f^n)
- continuous-time: "Discretize" the system by only sampling at each $t \in \mathbb{N}$. This produces a discrete-time system.

Fractals

A *fractal* is a set (usually in a metric space) exhibiting certain properties:

- complicated structure at multiple scales
- repetition of structures (“self similarity”)
- non-integer “fractal dimension”

Fractal dimension

For a bounded set S in a metric space, its *box-counting dimension* is $d \in \mathbb{R}$ if the minimal number of *boxes*[2] of size ε required to cover S scales as

$$N(\varepsilon) = C\left(\frac{1}{\varepsilon}\right)^d,$$

where C is some constant.

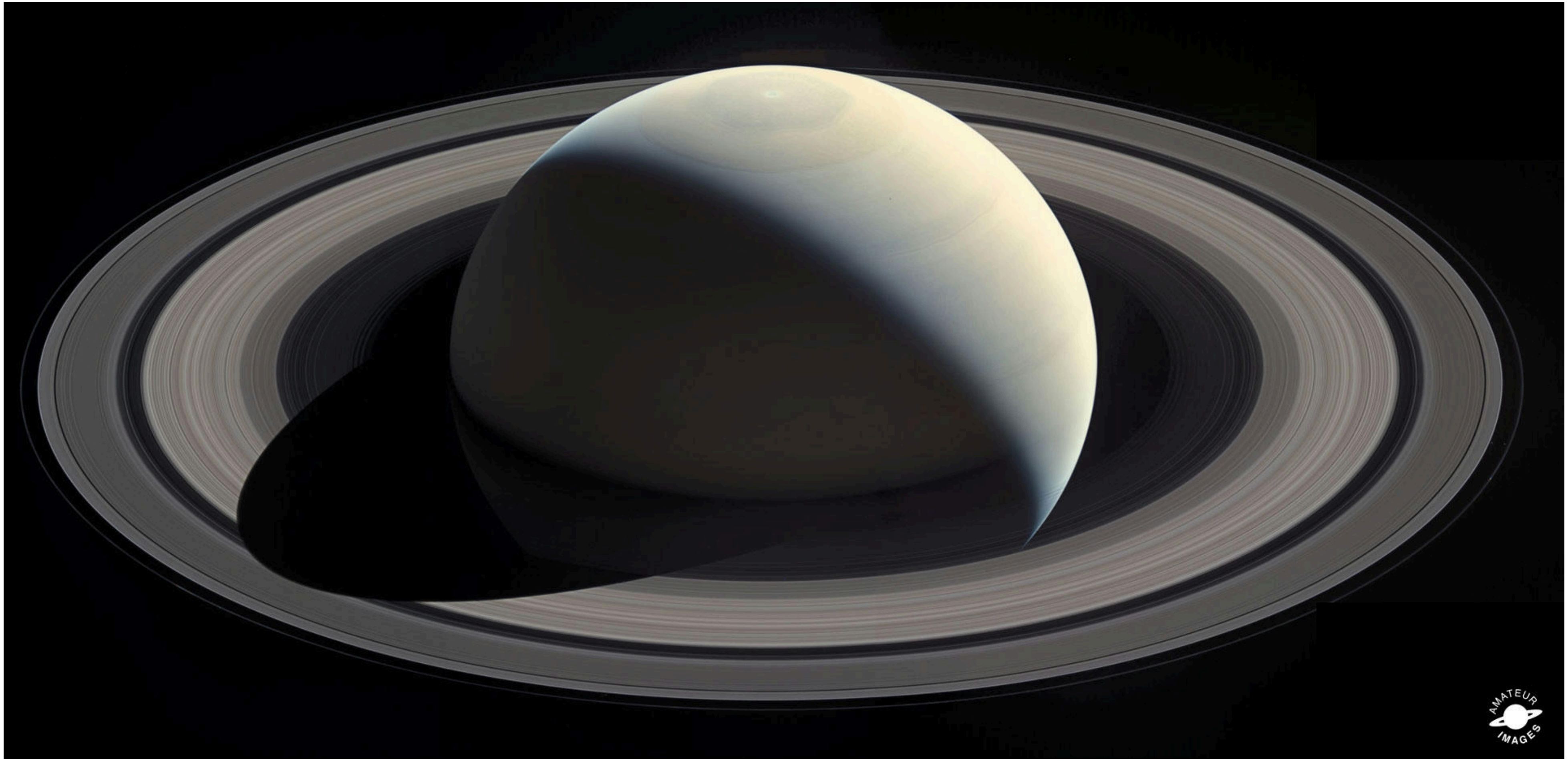
- smooth curves: $d = 1$
- smooth surfaces: $d = 2$
- …

A set S is called *fractal* if its box-counting dimension not an integer

[2] or balls

Closing thoughts

- Chaos can arise in *very* simple systems
- Need at least an *uncountable* state space
- No analytical solutions $\Rightarrow$ must simulate numerically, but
- Sensitivity to initial conditions implies chaotic systems can never be perfectly simulated/computed



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