Morse theory

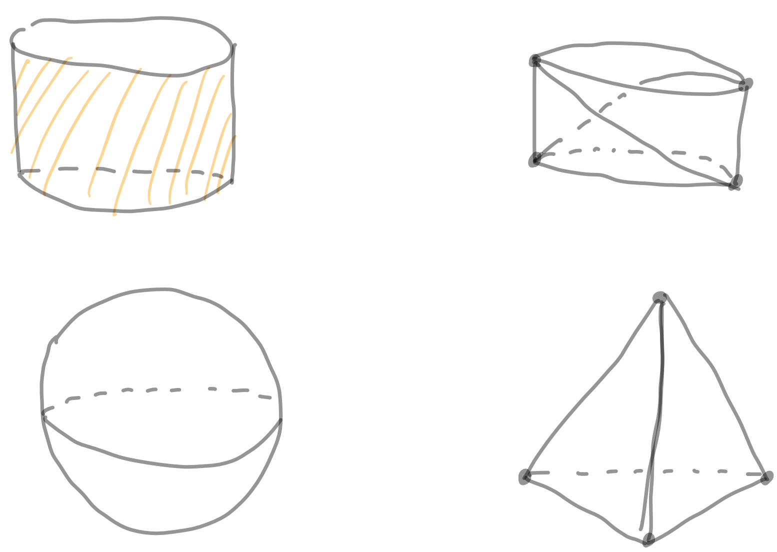
- Hand-wavy view of
  - How the game of differential topology feels like.

- More careful look at
  - What happens near gradient $= 0$ (Morse lemma)
  - Cartoons and toy examples

The game of cut and glue

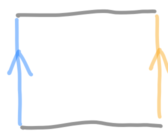
A large part of classical topology concerns how to iteratively construct a good topology space using simple ingredients.

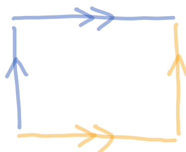
- Triangulations



- "cut - and - paste"

(Anybody played PORTAL ?)

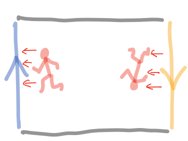
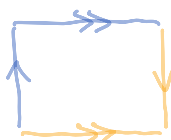
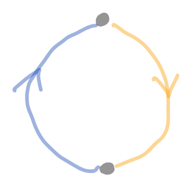
Cylinder :  

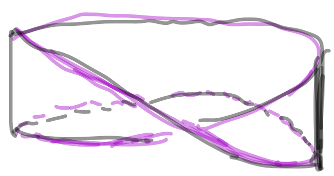
Torus :  ($\overset{\text{homeo}}{\simeq}$  , can you see ?)

Exercise :

What are these ?

$\mathbb{RP}^2$

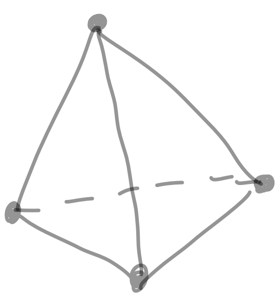


Euler characteristics

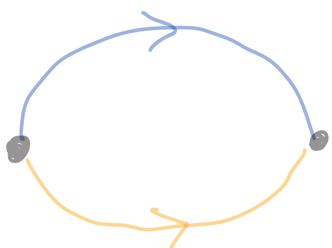
Invariance

$$\chi = \underset{\#\{\text{vertices}\}}{\overset{\#\{\text{vertices}\}}{}}\!\!\! \overset{\overset{\#\{\text{vertices}\}}{\|}}{V} - \overset{\overset{\#\{\text{edges}\}}{\|}}{E} + \overset{\overset{\#\{\text{faces}\}}{\|}}{F}$$

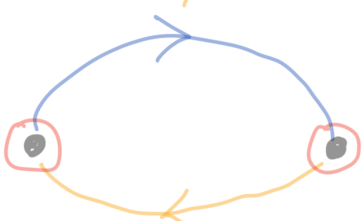
only depends on the topology of the space



$\chi = 4 - 6 + 4 = 2$



$\chi = 2 - 1 + 1 = 2$



the two are glued to be the same vertex

$\chi = 1 - 1 + 1 = 1$

Try others

Take-away:

- Topologists decompose manifolds into unit discs:

  <u>vertex</u> : D^0 0-dim "disc"

  <u>edge</u> : D^1 1-dim "disc"

  <u>face</u> : D^2 2-dim disc

  (you can go to higher dim !)

- Euler characteristic : alternating sum of them $\Rightarrow$ invariant under homeomorphism

Gradient flow and Morse lemma

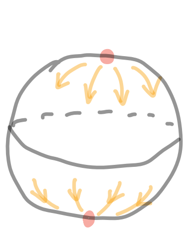
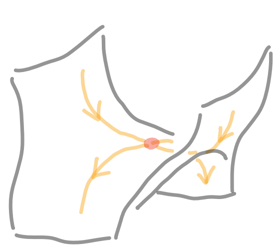
Given a differentiable submanifold $X \subset \mathbb{R}^N$, a differentiable function $f: X \longrightarrow \mathbb{R}$

It induces a gradient flow

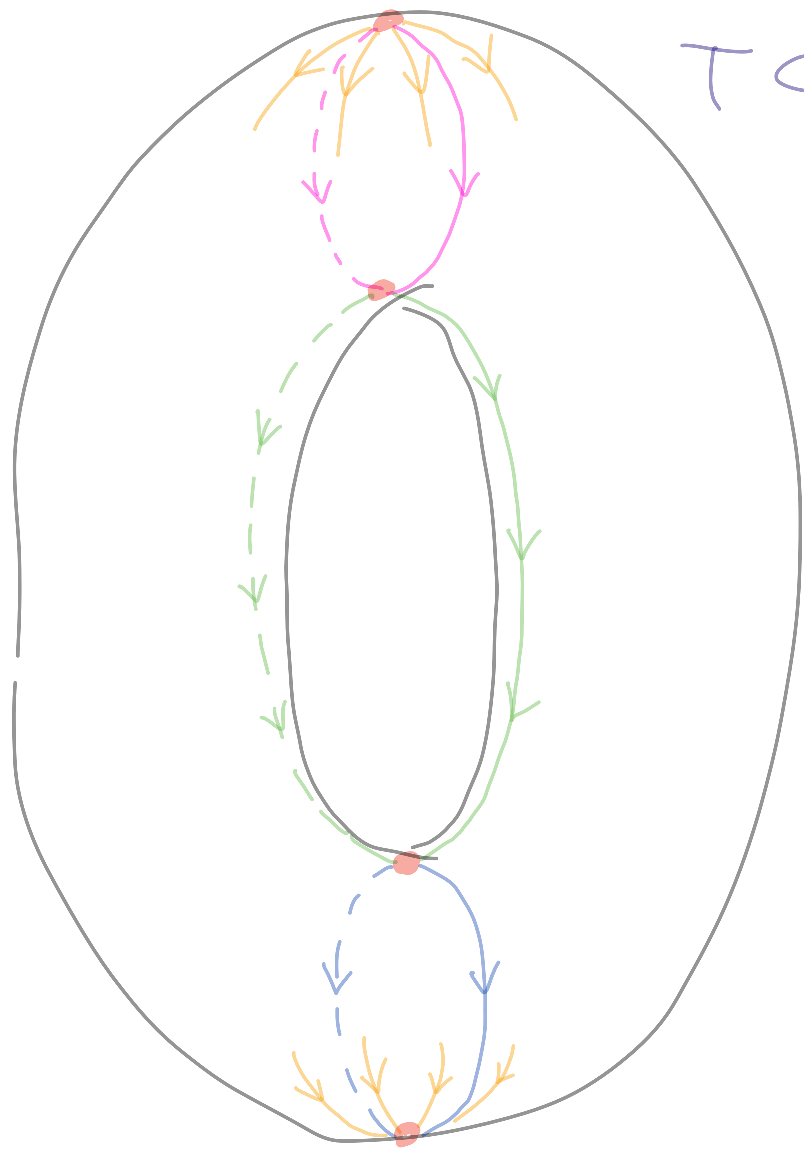
$\forall\, x \in X$, $v \in T_x X$,

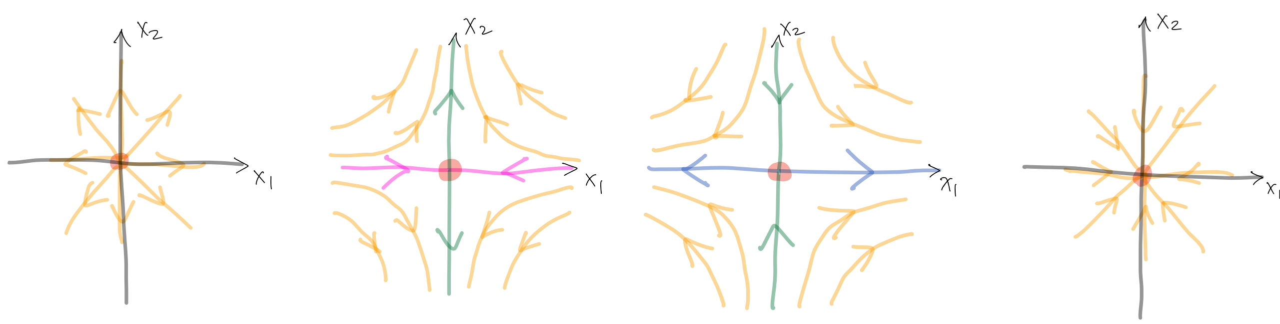
$$d_x f(v) = \langle \nabla_x f, v \rangle$$

Critical points : $x \in X$, $\nabla_x f = 0$

$T \subset \mathbb{R}$





e.g.

$f(x_1, x_2) = -x_1^2 - x_2^2$ $\quad$ $f(x_1, x_2) = x_1^2 - x_2^2$ $\quad$ $f(x_1, x_2) = -x_1^2 + x_2^2$ $\quad$ $f(x_1, x_2) = x_1^2 + x_2^2$

Hessian:

(Given a local coordinate)

$f: \mathbb{R}^M \to \mathbb{R}$

$$H_f(x) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{1 \le i, j \le M}$$

It depends on the local coordinates,

but if $(x_i = 0)$ is a critical pt,

Jacobian matrix acts on it as

$$J H_f(x) J^T = H_f(y)$$

"Invariant symmetric bilinear form"

"Tensor" (the real meaning haha)

When there is a change of variable $y_i = y_i(x_1, \cdots, x_M)$, $1 \leq i \leq M$

(adopting Einstein summation)

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial\left(\frac{\partial f}{\partial y_{i'}} \cdot \frac{\partial y_{i'}}{\partial x_i}\right)}{\partial y_{j'}} \frac{\partial y_{j'}}{\partial x_j}$$

$$= \frac{\partial^2 f}{\partial y_{i'} y_{j'}} \cdot \frac{\partial y_{i'}}{\partial x_i} \frac{\partial y_{j'}}{\partial x_j}$$

$$+ \frac{\partial f}{\partial y_{i'}} \cdot \frac{\partial^2 y_{i'}}{\partial x_i \partial y_{j'}} \frac{\partial y_{j'}}{\partial x_j}$$

$= 0$

Given a matrix M,
what are unchanged in JMJ^T
for nondegenerate J?

- rank
- the index

$$M \sim \begin{pmatrix} 0 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 0 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & -1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & -1 \end{pmatrix}$$

index

rank

Morse function:

$f: X \to \mathbb{R}$ such that the Hessian at any critical point is nondegenerate.

Morse lemma:

Given a Morse function $f: X \to \mathbb{R}$, a critical pt $p \in X$ there exists a coordinate $x_1, \cdots, x_M$

$$f = -x_1^2 - x_2^2 - \cdots - x_{\mathrm{ind}(p)}^2 + x_{\mathrm{ind}(p)}^2 + \cdots + x_M^2$$

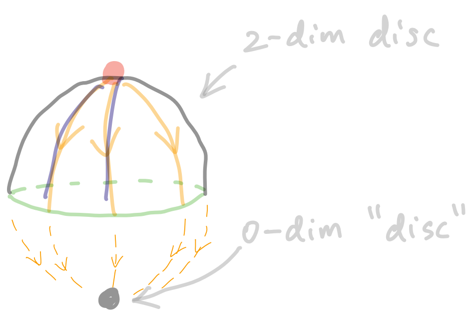
where $\mathrm{ind}(p) :=$ index of $H_f(p)$

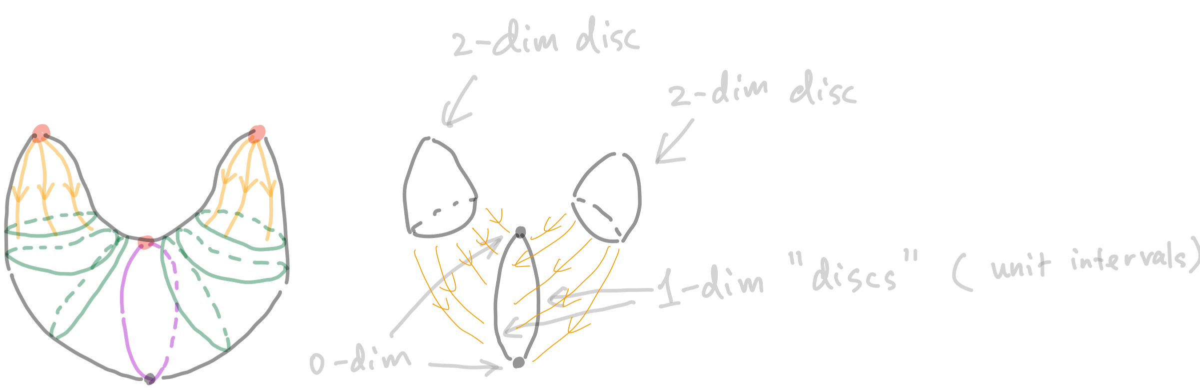
Take-away: X and f can be easily described near crit. pt, fully determined by index.

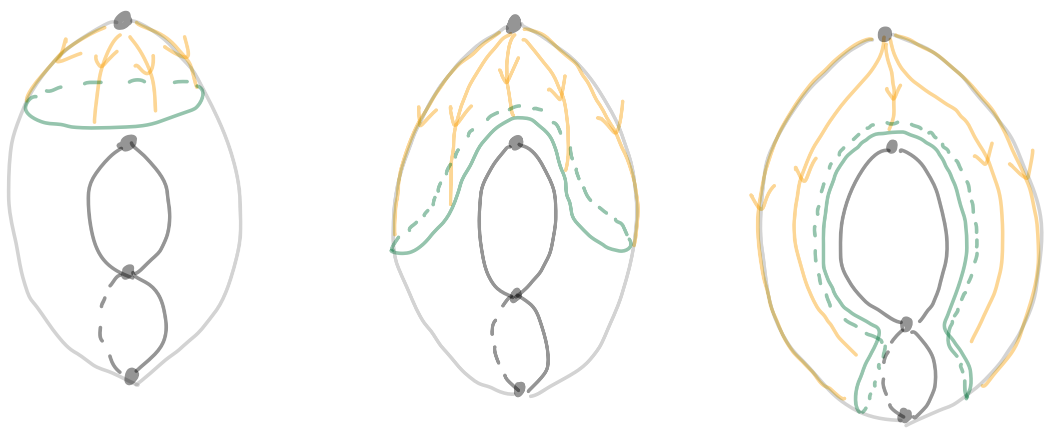
What's the point here?

- Any submfd $X \subset \mathbb{R}^N$ admits a Morse function.

- Gradient flow of Morse function induces a decomposition of X which unveils its topology.

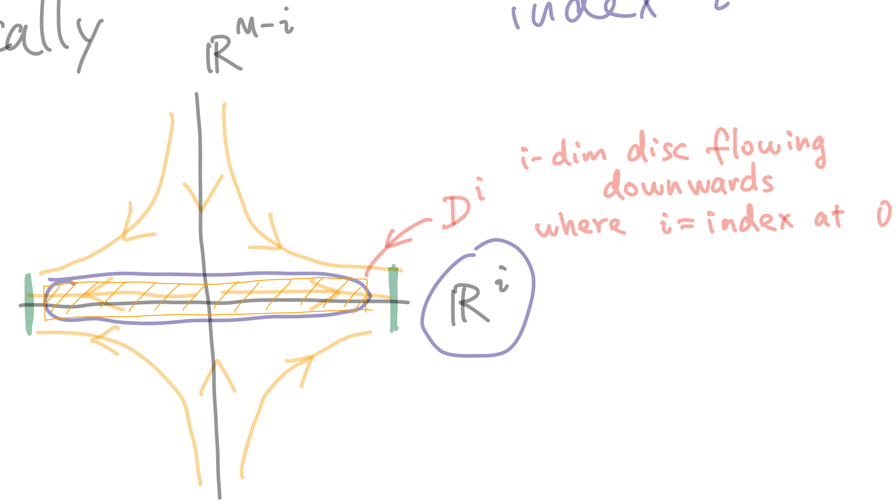






locally

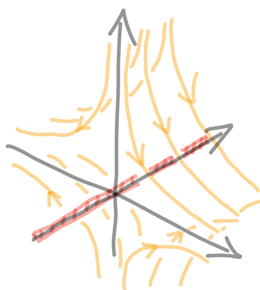
index i



Take-away:

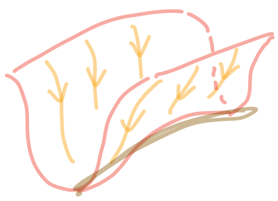
- Index i at p $\longleftrightarrow$ Attaching disc D^i at p
- Euler characteristic = alternating sum of the counts of indices

What happens if Hessian degenerates?



will form critical submanifold!

In other words, "loss valleys"



Some final referencing keywords:

	SLT	differential topology
nondeg. Hessian	regular model	Morse theory
deg. Hessian	singular model	Morse–Bott theory

(There is an infinite dim version, where function $\Leftrightarrow$ functional, "Floer theory")