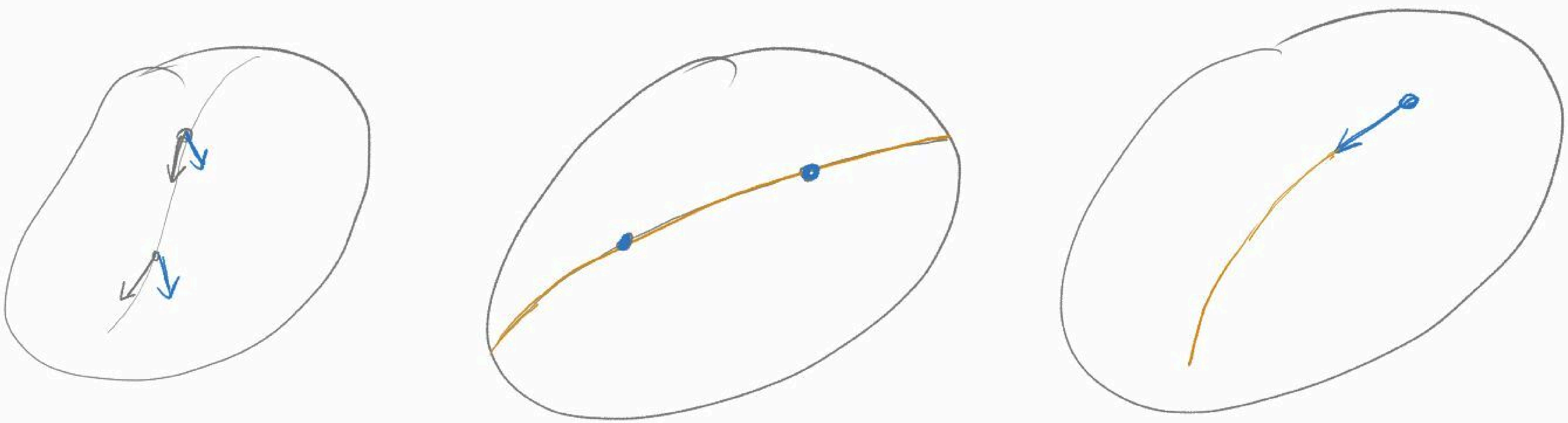
EleutherAI Math Reading Group

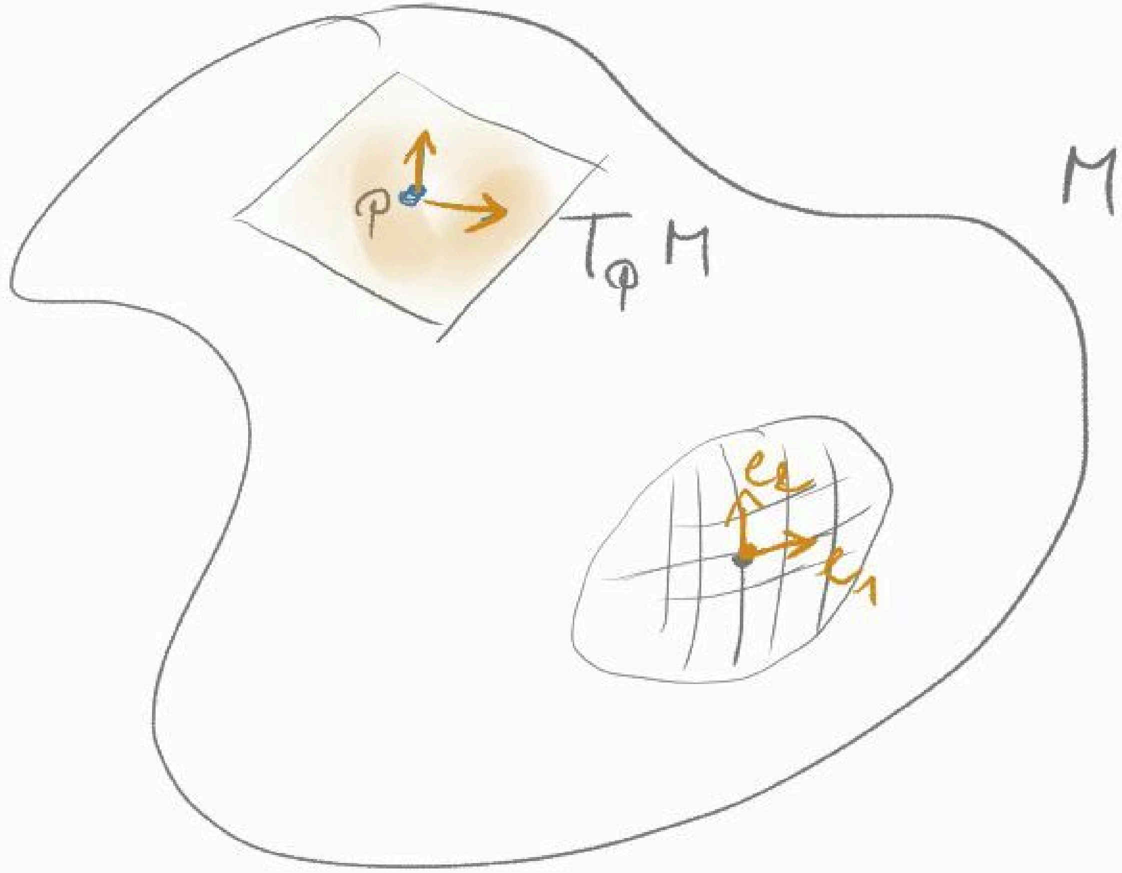
Differential Geometry II:

Covariant Derivative, Parallel Transport

Geodesics, (Curvature)



Notation Recap



Local coordinate system $x = x^i$

Induced Tangent Space basis

$e_i = \frac{\partial}{\partial x_i} = \partial_i \rightarrow$ Vector field $X = X^i e_i \in \mathcal{X}(M, TM)$

Local inner product in T_pM

$$\langle u, v \rangle_p = g_{ij} u^i v^j$$

$$g = g_{ij} \underbrace{dx^i dx^j}_{\text{induced basis}}$$

induced basis on T_p^*M $(dx^i e_j = \delta^i_j)$

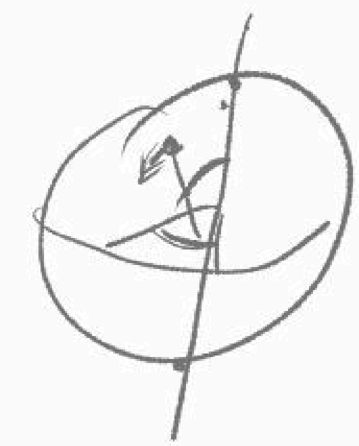
Einstein Summation:

$$X^i e_i \equiv \sum_{i=1}^{n} X^i e_i$$

$$g_{ij} u^i v^j \equiv \sum_{i=1}^{n} g_{ij} u^i v^j$$

Example (Metric on a sphere) $x^2+y^2+z^2=R^2$

spherical coordinates $(\theta, \phi) \longmapsto \begin{cases} x = R\cos\theta\sin\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\phi \end{cases}$



$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= \left(\frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi\right)^2 + \left(\frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi\right)^2 + \left(\frac{\partial z}{\partial\theta}d\theta + \frac{\partial z}{\partial\phi}d\phi\right)^2$$

$$= \left(-R\sin\theta\sin\phi\, d\theta + R\cos\theta\cos\phi\, d\phi\right)^2$$
$$+ \left(R\cos\theta\sin\phi\, d\theta + R\sin\theta\cos\phi\, d\phi\right)^2 + \left(-R\sin\phi\, d\phi\right)^2$$

$$= R^2\sin^2\theta\sin^2\phi\, d\theta^2 - 2R^2\sin\theta\sin\phi\cos\theta\cos\phi\, d\theta\, d\phi + R^2\cos^2\theta\cos^2\phi\, d\phi^2$$
$$+ R^2\cos^2\theta\sin^2\phi\, d\theta^2 + 2R^2\sin\theta\sin\phi\cos\theta\cos\phi\, d\theta\, d\phi + R^2\sin^2\theta\cos^2\phi\, d\phi^2$$
$$+ R^2\sin^2\phi\, d\phi^2$$

$$= R^2\sin^2\phi\, d\theta^2 + R^2 d\phi^2$$

① Directional derivative of a scalar function

Let $f : M \to \mathbb{R}$ be differentiable around $\varphi_0 \in M$

let $v \in T_\varphi M$

$$D_v f = \frac{d}{dt} f(\underbrace{\varphi(t)}) \Big|_{t=0} = \lim_{t \to 0} \frac{1}{t} \left(f(\varphi(t)) - f(\varphi_0) \right)$$

differentiable curve

$\varphi : [a, b] \to M$

$\varphi(0) = \varphi_0 \,, \; \dot{\varphi}(0) = v$

Can we generalize this for other types of functions?

Directional derivative of a vector field?

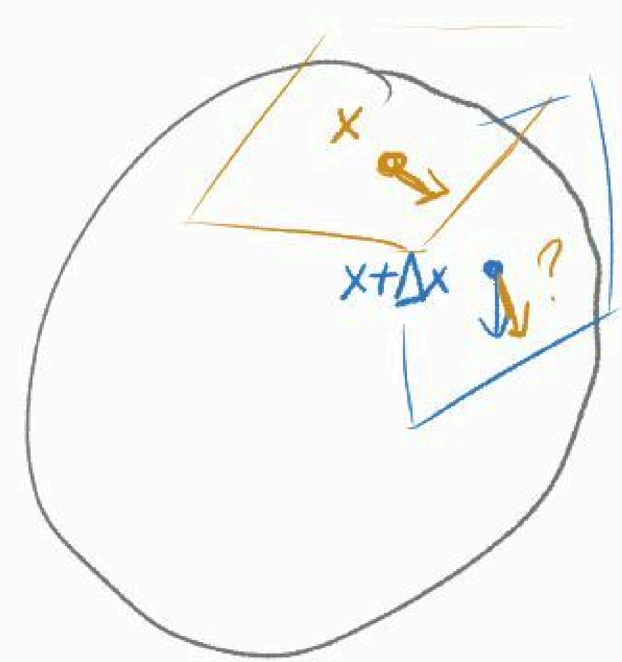
Let $X = X^i e_i \in \mathcal{X}(M, TM)$

$$\frac{\partial X^i}{\partial x^j} = \lim_{\Delta x^j \to 0} \frac{\underbrace{X^i(\ldots, x^j + \Delta x^j, \ldots) e_i}_{\in T_{x+\Delta x} M} - \underbrace{X^i(\ldots, x^j, \ldots) e_i}_{\in T_x M}}{\Delta x^j}$$

Type Error!

We need somehow to "translate" $X^i(x)$ over to $T_{x+\Delta x} M$

$\hookrightarrow$ "parallel transport"



let $\tilde{X}|_{x+\Delta x}$ be the parallel transport of $X|_x$

We will ask $\tilde{X}$ to satisfy some constraints:

$$\tilde{X}^i(x+\Delta x) - X^i(x) \sim \Delta x$$

$$\widetilde{(X^i+Y^i)}(x+\Delta x) = \tilde{X}^i(x+\Delta x) + \tilde{Y}^i(x+\Delta x)$$

Which implies

$$\underbrace{\tilde{X}^i(x+\Delta x)}_{\substack{\text{transported}\\ \text{coordinates}}} = \underbrace{X^i(x)}_{\substack{\text{old}\\ \text{coords}}} - \underbrace{X^k(x)\,\Gamma^i_{jk}(x)\,\Delta x^j}_{\text{Bilinear correction}}$$

$$\rightsquigarrow \nabla_j X = \lim_{\Delta x^j \to 0} \frac{X^i(x+\Delta x) - \tilde{X}^i(x+\Delta x)}{\Delta x^j}\frac{\partial}{\partial x^i} = \left(\frac{\partial X^i}{\partial x^j} + X^k \Gamma^i_{jk}\right)\frac{\partial}{\partial x^i}$$

Affine connection

$$\nabla : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X} \; ; \; X, Y \longmapsto \nabla_X Y$$

Such that

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$$

$$\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z$$

$$\nabla_{(fX)} Y = f \nabla_X Z$$

$$\nabla_X (f Y) = D_X f \; Y + f \nabla_X Y \qquad \text{(product rule)}$$

where f is a scalar field

Connection coefficients

$$\nabla_j e_i \equiv \nabla_{e_j} e_i = \Gamma^k_{ji} e_k$$

What happens to e_i, when transported in direction e_j.

Let $X = X^i e_i$, $Y = Y^j e_j$, then

$$\nabla_X Y = X^i \nabla_i (Y^j e_j) = X^i (\partial_i Y^j e_j + Y^j \nabla_i e_j)$$

$$= X^i \left(\frac{\partial Y^j}{\partial x^i} + \Gamma^j_{ik} Y^k \right) e_j$$

Covariant derivative

The operator ∇_X is called the covariant derivative along X

It acts on tensor fields:

- $\nabla_X f = \partial_X f$ (scalar fields)
- $\nabla_X Y = X^i \left(\frac{\partial Y^j}{\partial x^i} + \Gamma^j_{ik} Y^k \right) e_j$ (vector fields)
- $\nabla_X (T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2)$ (tensors)
- $\partial_X \langle \omega, Y \rangle = \nabla_X (\langle \omega, Y \rangle) = \langle \nabla_X \omega, Y \rangle + \langle \omega, \nabla_X Y \rangle$ (covector field $\equiv$ 1-form)

$$\Rightarrow \quad (\nabla_X \omega)_j = X^i \partial_i \omega_j - X^i \Gamma^k_{ij} \omega_k$$

Γ_{ij}^{k} is not a tensor!

$\boxed{\text{Proof}}$ let $x = x^i \to e_i = \frac{\partial}{\partial x^i} \to \Gamma_{ij}^{k} \rightsquigarrow \nabla_{e_j} e_i = \Gamma_{ji}^{k} e_k$

$y = y^j \to f_j = \frac{\partial}{\partial y^j} \to \tilde{\Gamma}_{ij}^{k}$ $\boxed{\nabla_{f_n} f_m = \tilde{\Gamma}_{mn}^{l} f_l}$

We have $f_j = \left(\frac{\partial x^i}{\partial y^j}\right) e_i$, hence

$$\nabla_{f_n} f_m = \nabla_{f_n}\left(\frac{\partial x^i}{\partial y^m} e_i\right) = \frac{\partial^2 x^i}{\partial y^n \partial y^m} e_i + \frac{\partial x^k}{\partial y^n}\frac{\partial x^i}{\partial y^m} \nabla_k e_i$$

$$= \left(\frac{\partial^2 x^j}{\partial y^n \partial y^m} + \frac{\partial x^k}{\partial y^n}\frac{\partial x^i}{\partial y^m} \Gamma_{ki}^{j}\right) e_j$$

and $\tilde{\Gamma}_{nm}^{l} f_l = \tilde{\Gamma}_{nm}^{l} \left(\frac{\partial x^j}{\partial y^l}\right) e_j$

$$\Rightarrow \tilde{\Gamma}_{nm}^{l} = \underbrace{\frac{\partial x^k}{\partial y^n}\frac{\partial x^i}{\partial y^m}\frac{\partial y^l}{\partial x^j} \Gamma_{ki}^{j}}_{\text{tensor change of coords}} + \underbrace{\frac{\partial^2 x^j}{\partial y^n \partial y^m} \cdot \frac{\partial y^l}{\partial x^j}}_{\text{extra term}}$$

Parallel Transport

Let $\varphi :]a,b[\to M$ be a differentiable curve,

$X = X^i e_i$ a vector field defined along $\varphi(t)$

If $\quad \nabla_V X = 0 \quad \forall t \in]a,b[$

$\hookrightarrow V = \frac{dx^k}{dt}(\varphi(t)) e_i \quad$ [tangent to $\varphi(t)$]

We say that X is parallel transported along $\varphi(t)$

$$\rightsquigarrow \quad \frac{dX^i}{dt} + \Gamma^i_{jk} \frac{dx^j(\varphi(t))}{dt} X^k = 0$$

"The coordinates of X are constant relative to $\dot{\varphi}(t)$"

Geodesic curve

A curve $\varphi :]a,b[\longrightarrow M$ s.t. its tangent vector $\dot{\varphi}(t) = \frac{dx^i}{dt}(\varphi(t))e_i$

$\overset{\shortparallel}{V}$

is parallel transported along $\varphi(t)$, i.e.

$$\nabla_V V = 0 \quad \Leftrightarrow \quad \frac{d^2x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

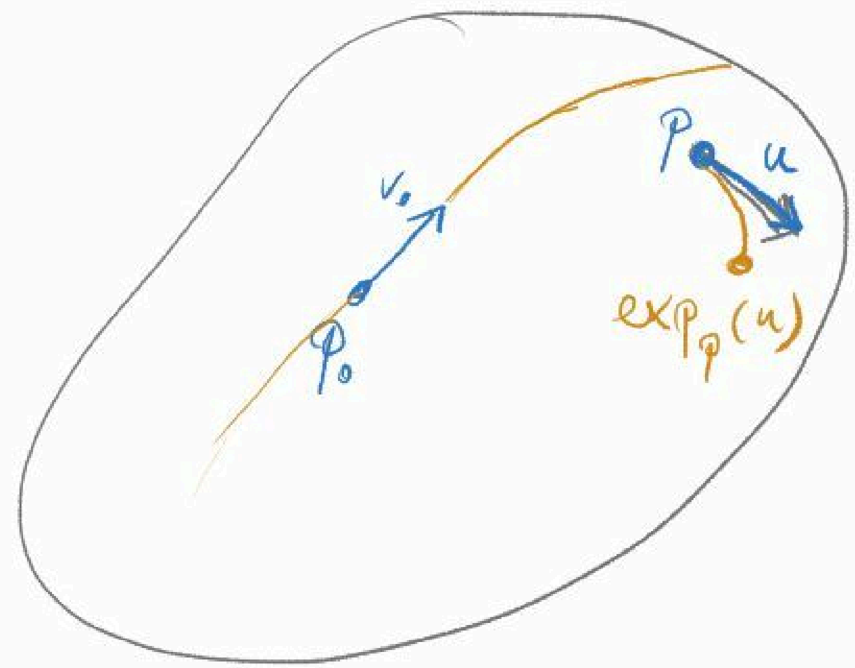
is called a <u>geodesic</u>

$\boxed{\text{Intuitively, geodesics just keep following the "same" direction (no acceleration)}}$

Exponential Map

Given a point $\varphi_0 \in M$ and a tangent vector $V_0 \in T_\varphi M$,
there is a unique geodesic $\varphi : [a, b] \to M$ such that

$\varphi(0) = \varphi_0$ and $\dot{\varphi}(0) = V_0$



NB: We can use this to perform Gradient Descent on a Manifold.

Given a scalar field $f : M \to \mathbb{R}$

its gradient $(\nabla f)^i = g^{ij} \partial_j f \in$ "ambient space"

$$\theta^{(t+1)} = \exp_{\theta(t)}(-\eta_t v) \quad \text{where } v = \operatorname{proj}_{T_{\theta(t)}M} \nabla f$$

But where did the metric go?

So far the connection coefficients Γ^k_{ji} are arbitrary

↳ many possible choices

For a Riemannian Manifold (M, g)

We can add constraints, e.g. that g is covariantly constant

→ The inner product of 2 vectors X, Y is invariant under parallel transport

↳ norm, angles

$$\rightsquigarrow 0 = \nabla_V [g(X,Y)] = V^k \Big[(\nabla_k g)(X,Y) + g(\underset{\substack{\shortparallel\\0}}{\nabla_k X}, Y) + g(X, \underset{\substack{\shortparallel\\0}}{\nabla_k Y}) \Big]$$

$$= V^k X^i Y^j (\nabla_k g)_{ij} \Rightarrow (\nabla_k g)_{ij} = \partial_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{li} = 0$$

A metric connection / compatible with the metric is a connection that satisfies

$$(\nabla_k g)_{ij} = 0$$

The components Γ^k_{ij} of a metric connection can be shown to be equal to

$$\Gamma^k_{ij} = \left\{ {k \atop ij} \right\} + K^k_{ij}$$

$$\left\{ {k \atop ij} \right\} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$

[Christoffel Symbols]

$$K^k_{ij} = \frac{1}{2} \left(T^k{}_{ij} + T_i{}^k{}_j + T_j{}^k{}_i \right)$$

[Contorsion Tensor]

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$$

Levi-Civita connection

We can ask the torsion tensor $T_{ij}^{\ \ k}$ to be zero everywhere (torsion-free)

$$\Rightarrow \quad \Gamma_{ij}^{k} = \left\{ {k \atop ij} \right\} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right)$$

$$= \Gamma_{ji}^{k}$$

This produces a symmetric, torsion-free, metric connection called the Levi-Civita connection.

It always exists and is unique.

Shortest paths on Riemannian Manifolds

$$\dot{x}^i = \frac{dx^i}{dt}$$

$$d(P,Q) = \min_x I[x] = \min_x \int_P^Q \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\, dt = \int_P^Q L[F(x,\dot{x})]\, dt$$

$$F = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j$$

Stationnary points of I are characterized by the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) - \frac{\partial L}{\partial x^k} = 0$$

Since $F = L^2/2$, we can rewrite

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right) - \frac{\partial F}{\partial x^k} = L\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) - \frac{\partial L}{\partial x^k}\right] + \frac{\partial L}{\partial \dot{x}^k}\cdot\frac{dL}{dt}$$

$$= \frac{\partial L}{\partial \dot{x}^k}\cdot\frac{dL}{dt}$$

$\rightarrow$ We can make this zero by making the curve arc-length parameterized

$$= 0$$

We now have

$$0 = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right) - \frac{\partial F}{\partial x^k} = \frac{d}{dt}\left(g_{ki}\dot{x}^i\right) - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j$$

$$= \frac{d g_{ki}}{dt}\dot{x}^i + g_{ki}\frac{d^2 x^i}{dt^2} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j$$

$$= \frac{\partial g_{ki}}{\partial x^j}\dot{x}^i\dot{x}^j + g_{ki}\frac{d^2 x^i}{dt^2} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j$$

$$= g_{ki}\frac{d^2 x^i}{dt^2} + \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)\frac{dx^i}{dt}\frac{dx^j}{dt}$$

$$= g_{ki}\left(\frac{d^2 x^i}{dt^2} + \underbrace{\frac{1}{2}g^{ki}\left(\partial_j g_{ki} + \partial_i g_{kj} - \partial_k g_{ij}\right)}_{\Gamma^k_{ij}}\frac{dx^i}{dt}\frac{dx^j}{dt}\right)$$

$$\Rightarrow \quad \frac{d^2 x^i}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \qquad \text{[Geodesic equation]}$$

Example (Geodesics on a sphere) $(R=1)$

$$ds^2 = \sin^2\phi \, d\theta^2 + d\phi^2$$

$$\begin{cases} \Gamma^{\phi}_{\theta\theta} = -\cos\phi\sin\phi \\ \Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} = \dfrac{\cos\phi}{\sin\phi} = \cot\phi \end{cases}$$

$$\rightarrow \begin{cases} \dfrac{d^2\phi}{dt^2} - \sin\phi\cos\phi\left(\dfrac{d\theta}{dt}\right)^2 = 0 \\ \dfrac{d^2\theta}{dt^2} + 2\cot\phi\left(\dfrac{d\phi}{dt}\right)\left(\dfrac{d\theta}{dt}\right) = 0 \end{cases}$$

solution

$$\rightsquigarrow \boxed{A\cos\theta\sin\phi + B\sin\theta\sin\phi - \cos\phi = 0}$$

$\Rightarrow$ Great circles