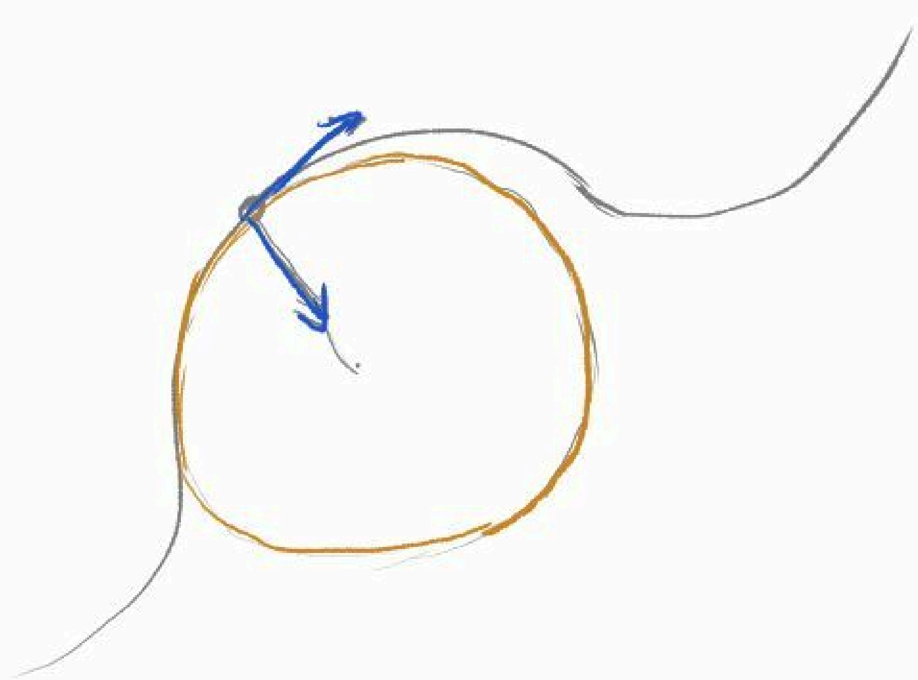
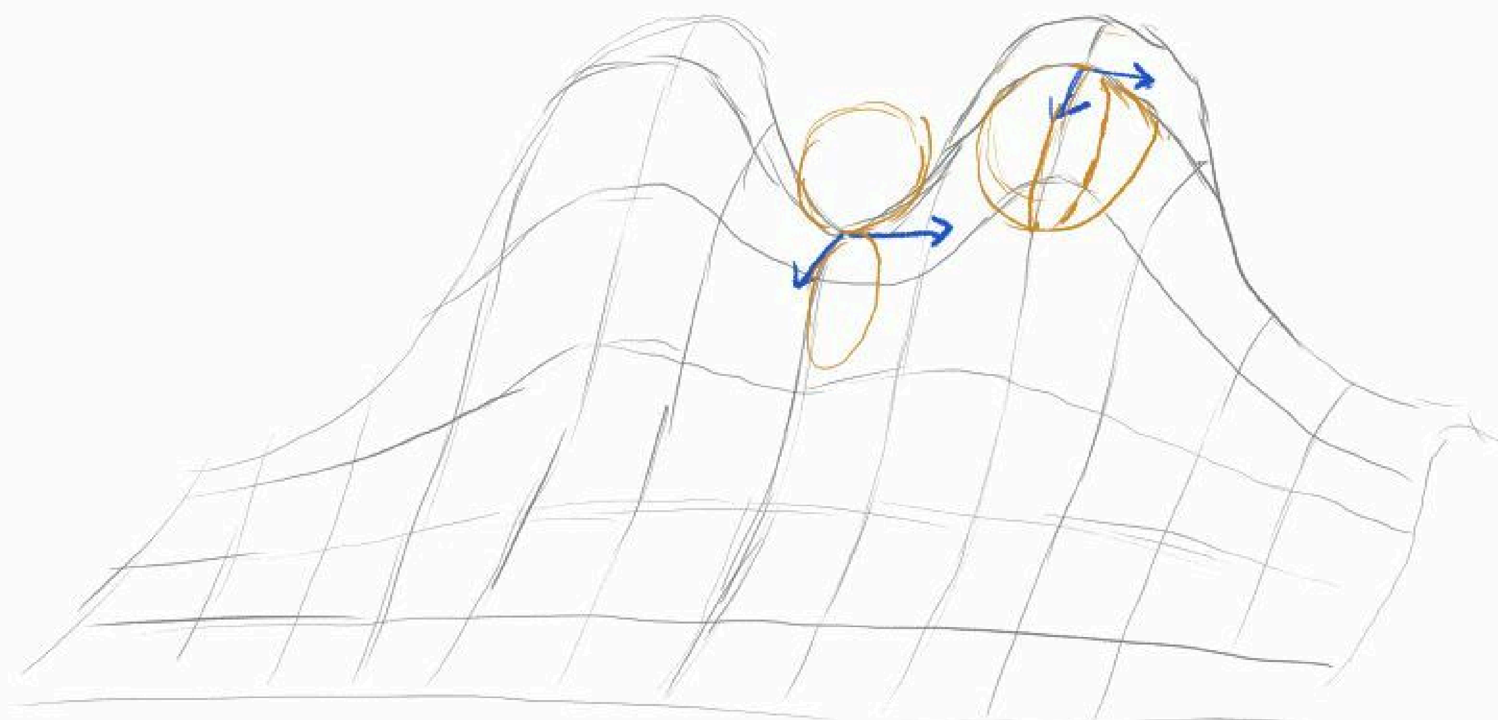
EAI Math Reading Group

① Differential Geometry III

Curvature and Torsion





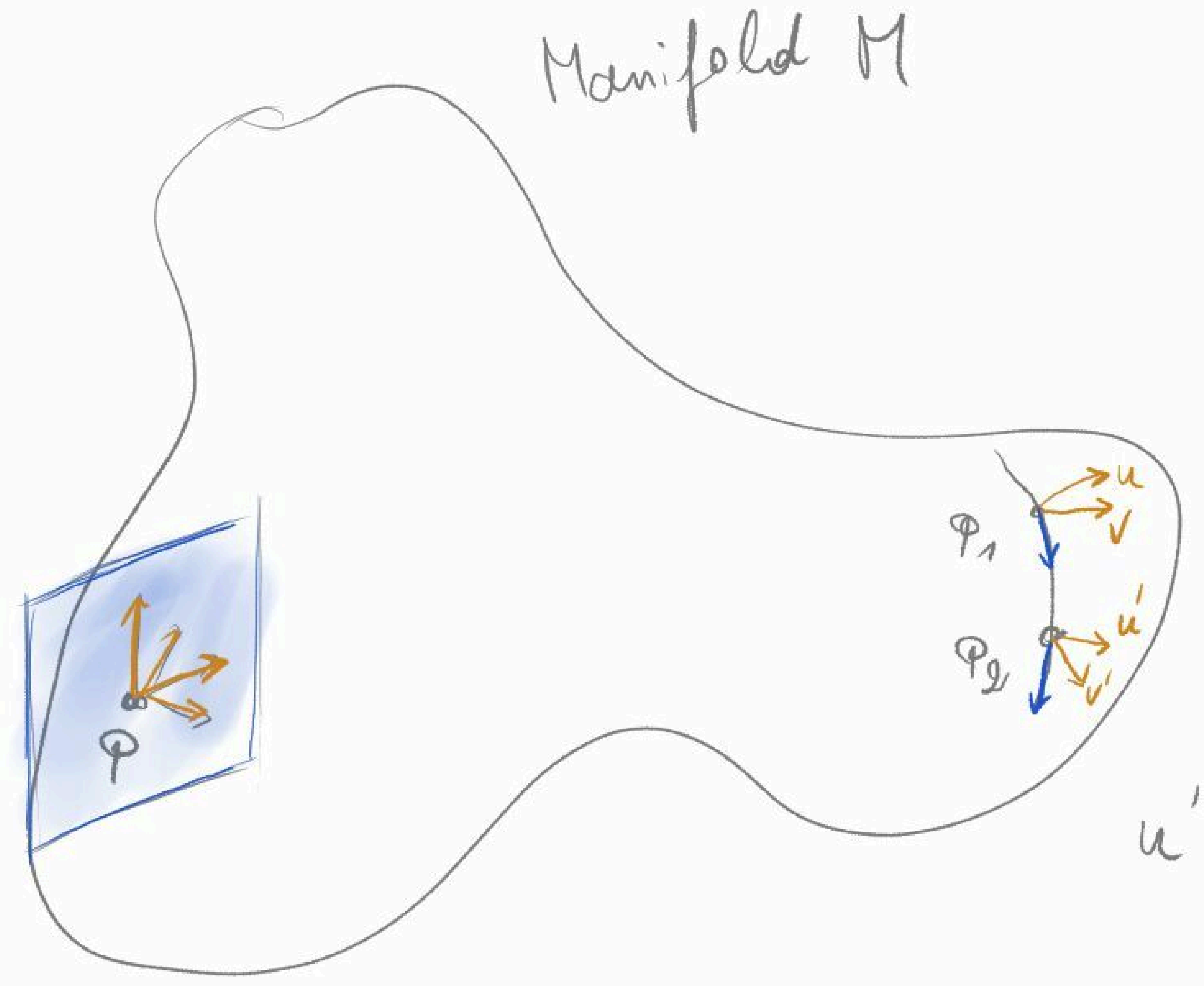
Outline

1) Curvature and torsion for curves in Euclidian space

2) Riemann curvature and torsion for manifolds

3) Gauss curvature

0) ① Differential Geometry Refresher



Tangent Space

$T_\varphi M$

vector space w/ inner product

$\langle u, v\rangle_\varphi = g_{ij} u^i v^j$ → Riemannian metric

∇ → Connection

$u' = \Pi u$

$= u - u^j dx^i \, \Gamma_{ij}^{k} \partial_k$

↓ Connection coefficients

$\nabla_j \partial_i = \Gamma_{ij}^{k} \partial_k$

1) Curves in $\mathbb{R}^3$ $\left(g_{ij} = \delta_{ij} \,,\ \Gamma^k_{ij} = 0 \right)$

Consider $\gamma : [a,b] \longrightarrow \mathbb{R}^3$

$$t \longmapsto \gamma(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

s.t. x_1, x_2, x_3 are $\mathcal{C}^1$ (C^k in general)

- tangent vector $v_i = \dfrac{dx_i}{dt}$

- length of a curve $s(t_0,t_1) = \displaystyle\int_{t_0}^{t_1} \| v(t) \| \, dt$

↳ length parameterized curve $\tilde{x}_i(s) = x_i(t(s))$

$\downarrow$

unique $t \geq t_0$ s.t.
$s(t_0,t) = s$

Frenet - Serret Frame

At each point of the curve γ, we can associate an orthonormal basis $\{\vec{t}, \vec{n}, \vec{b}\}$

- $\vec{t} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{dx}{d\alpha}\frac{d\alpha}{ds} = \frac{dx}{ds}$ tangent vector

- $\vec{n} = \frac{\frac{d\vec{t}}{d\alpha}}{\left\|\frac{d\vec{t}}{d\alpha}\right\|} = \frac{\frac{d\vec{t}}{ds}}{\left\|\frac{d\vec{t}}{ds}\right\|}$ normal vector

NB. $\langle \vec{n}, \vec{t} \rangle = 0$ because $0 = \frac{d}{d\alpha}\langle \vec{t}, \vec{t} \rangle = 2\langle \vec{t}, \underbrace{\frac{d\vec{t}}{d\alpha}}_{\parallel \vec{n}} \rangle$

- $\vec{b} = \vec{t} \times \vec{n}$ binormal vector

Curvature and torsion of a curve

By definition, we have

$$\frac{d\vec{t}}{ds} = \underbrace{K(s)}_{\text{curvature}}\,\vec{n} \quad \text{with} \quad \boxed{K(s) = \left\| \frac{d\vec{t}}{ds} \right\|}$$

and

- $$\frac{d\vec{b}}{ds} = \underbrace{\frac{d\vec{t}}{ds} \times \vec{n}}_{0} + \vec{t} \times \frac{d\vec{n}}{ds} \Rightarrow \frac{d\vec{b}}{ds} \perp \vec{t} \text{ and } \frac{d\vec{b}}{ds} \perp \vec{b}$$

$$\frac{d\vec{b}}{ds} = \underbrace{-\tau(s)}_{\text{torsion}}\,\vec{n}$$

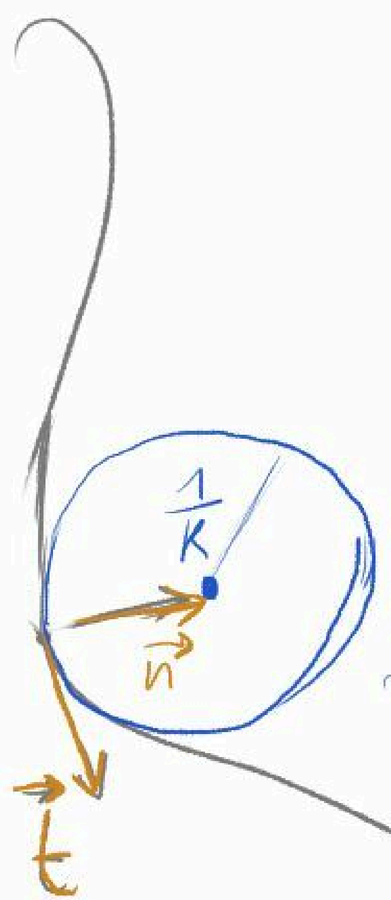
- $$\frac{d\vec{n}}{ds} = \frac{d\vec{b}}{ds} \times \vec{t} + \vec{b} \times \frac{d\vec{t}}{ds} = \tau(s)\vec{b} - K(s)\vec{t}$$

Osculating circle

Consider the Taylor approximation of the curve around $\gamma(s_0)$:

$$\gamma(s_0 + ds) \simeq \gamma(s_0) + \frac{d\vec{x}}{ds} ds + \frac{1}{2} \frac{d^2\vec{x}}{ds^2} ds^2 + \dots$$

$$= \underbrace{\gamma(s_0) + \vec{t}\, ds + \frac{1}{2} \kappa(s_0)\, \vec{n}\, ds^2}_{\text{at second order, } \gamma \text{ is a planar curve in } \underbrace{\gamma(s_0) + \operatorname{span}\{\vec{t}, \vec{n}\}}_{\text{osculating plane}}} + \dots$$



$$\gamma(s_0) + \frac{1}{\kappa}\vec{n} + \frac{\sin\alpha}{\kappa}\vec{t} - \frac{\cos\alpha}{\kappa}\vec{n}$$

osculating circle = circle tangent to γ at $\gamma(s_0)$ that fits best

What about torsion ?

- Curvature describes how much γ bends <u>in</u> the osculating plane

- $\frac{d\vec{b}}{ds} = 0 = \tau(s)\,\vec{n} \Rightarrow \gamma$ is planar !

- torsion describes how much γ bends <u>out</u> of the plane

2) Back to the Manifold

Let (M, ∇) be a manifold with connection ∇, and consider a "2D slice" of M $(\sigma, \tau) \mapsto \varphi(\sigma, \tau)$

$U = \frac{\partial \varphi}{\partial \sigma}$, $V = \frac{\partial \varphi}{\partial \tau}$ are tangent vector fields to φ

$$T(U, V) = \nabla_\tau \frac{\partial \varphi}{\partial \sigma} - \nabla_\sigma \frac{\partial \varphi}{\partial \tau}$$

$$\boxed{T(X, Y) = \nabla_X Y - \nabla_Y X - \underline{[X, Y]}}$$

$$= T^k_{ij} U^i V^j \frac{\partial}{\partial x^k}$$

where $\boxed{T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}}$ is the <u>torsion tensor</u>

Morally, torsion measures how "commutative" the covariant derivative is
(curvature)

let W be a vector field along φ

" $W(\sigma, \tau)$

$$R(U,V)W = \nabla_\tau \nabla_\sigma W - \nabla_\sigma \nabla_\tau W$$

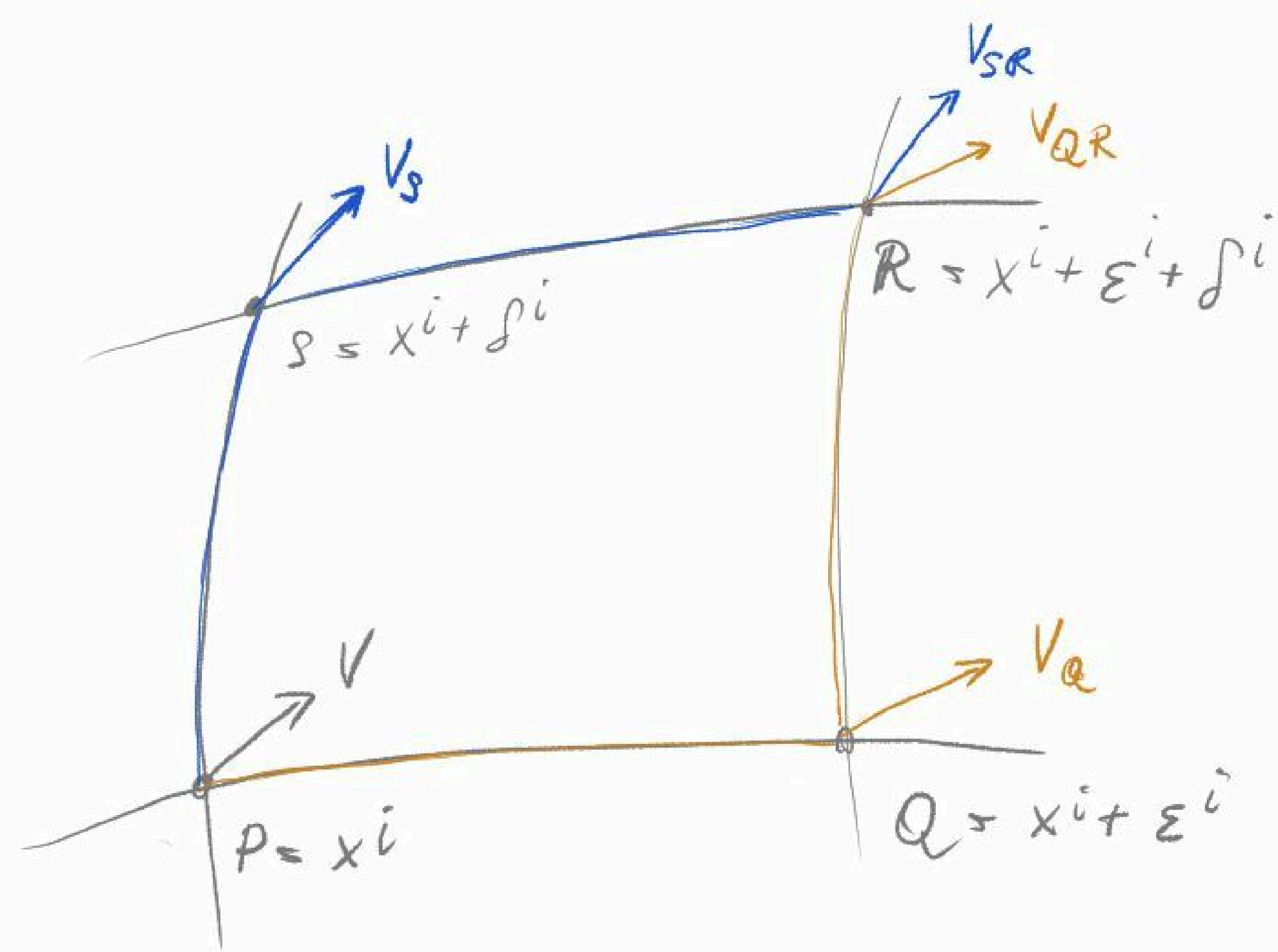
$$= R^i_{mkl} U^k V^l W^m \frac{\partial}{\partial x^i}$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$\boxed{R^i_{mkl} = \left(\partial_l \Gamma^i_{mk} - \partial_k \Gamma^i_{ml} + \Gamma^p_{mk} \Gamma^i_{pl} - \Gamma^p_{ml} \Gamma^i_{pk} \right)}$$

↳ curvature tensor

Interpretation



Consider the vector $V \in T_p M$ being parallel-transported along two different paths to the same point

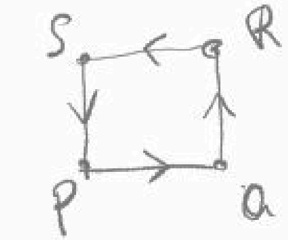
At second order we have

$$V^i_{QR} \approx V^i - V^k \Gamma^i_{jk} \varepsilon^j - V^k \Gamma^i_{jk} \delta^j - V^k \left(\partial_l \Gamma^i_{jk} - \Gamma^m_{lk} \Gamma^i_{jm} \right) \delta^j \varepsilon^l$$

$$V^i_{SR} \approx V^i - V^k \Gamma^i_{jk} \delta^j - V^k \Gamma^i_{jk} \varepsilon^j - V^k \left(\partial_j \Gamma^i_{lk} - \Gamma^m_{jk} \Gamma^i_{lm} \right) \delta^j \varepsilon^l$$

$$\Rightarrow V^i_{QR} - V^i_{SR} \approx V^k R^i_{klj} \delta^j \varepsilon^l$$

Interpretation of torsion



Consider the effect of $[\nabla_i, \nabla_j] = \nabla_i \nabla_j - \nabla_j \nabla_i$

on a vector field X^m

$$[\nabla_i, \nabla_j] X^m = \left(\partial_i \Gamma^m_{jn} - \partial_j \Gamma^m_{in} + \Gamma^m_{il} \Gamma^l_{jn} - \Gamma^m_{jl} \Gamma^l_{in} \right) X^n$$

$$- \left(\Gamma^l_{ij} - \Gamma^l_{ji} \right) \nabla_l X^m$$

$$= \underbrace{R^m{}_{nij}}\, X^n - \underbrace{T^l_{ij}\, \nabla_l}\, X^m$$

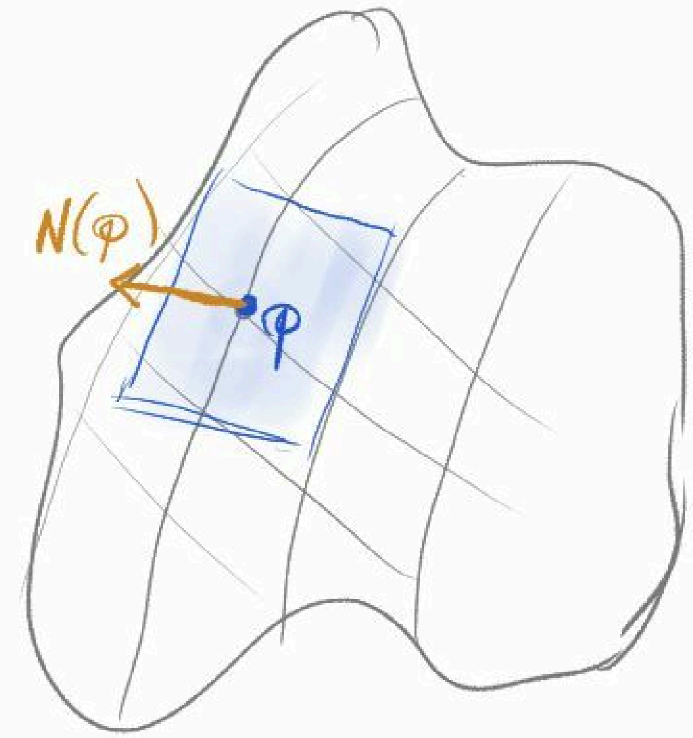
curvature measures the part of parallel transport proportional to X

on the loop

torsion measures the part of parallel transport proportional to the covariant derivative

Gauss curvature

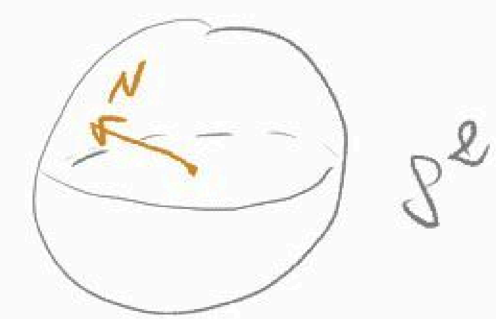
Consider a surface S embedded in $\mathbb{R}^3$



- Gauss map $\quad p \longmapsto N(p) =$ a $\overset{\text{unit}}{\text{normal}}$ vector to $T_p S$

$$N : S \longrightarrow S^2$$

$\hookrightarrow$ unit sphere



- taking the differential dN yields a self-adjoint linear map $T_p S \longrightarrow T_N S^2 = T_p S$

i.e. $\langle dN\, U, V \rangle = \langle U, dN\, V \rangle$

Induced connection

We have an affine connection ∇ on S defined by the euclidean connection $\nabla^{E} = D$ on $\mathbb{R}^3$:

$$\nabla_Y X = \mathrm{Proj}_{T_\varphi S} D_Y X$$

$$= D_Y X - \langle D_Y X, N \rangle N$$

$\Rightarrow$

- $$T\left(\frac{\partial \varphi}{\partial \sigma}, \frac{\partial \varphi}{\partial \tau}\right) = \mathrm{Proj}_{T_\varphi S}\Big(\underbrace{\frac{\partial}{\partial \tau}\frac{\partial \varphi}{\partial \sigma} - \frac{\partial}{\partial \sigma}\frac{\partial \varphi}{\partial \tau}}_{=0}\Big)$$

- $$R\left(\frac{\partial \varphi}{\partial \sigma}, \frac{\partial \varphi}{\partial \tau}\right) W = \dots = \left\langle W, \frac{\partial N}{\partial \sigma}\right\rangle \frac{\partial N}{\partial \tau} - \left\langle W, \frac{\partial N}{\partial \tau}\right\rangle \frac{\partial N}{\partial \sigma}$$

$$R(U,V)W = \langle W, dN U\rangle dN V - \langle W, dN V\rangle dN U$$

Taking one more inner product:

$$\langle R(U,V)W, Z\rangle = \langle W, dNU\rangle\langle dNV, Z\rangle - \langle W, dNV\rangle\langle dNU, Z\rangle$$

$$= \det \begin{bmatrix} \langle W, dNU\rangle & \langle W, dNV\rangle \\ \langle Z, dNU\rangle & \langle Z, dNV\rangle \end{bmatrix}$$

$$= \ldots$$

$$= \underbrace{\det dN}_{K} \det \begin{bmatrix} \langle W, U\rangle & \langle W, V\rangle \\ \langle Z, U\rangle & \langle Z, V\rangle \end{bmatrix}$$

(gaussian curvature)

Examples

1) $S = \{ \varphi \in \mathbb{R}^3 \,/\, \|\varphi\|^2 = r^2 \} \rightsquigarrow N = r^{-1}\varphi$

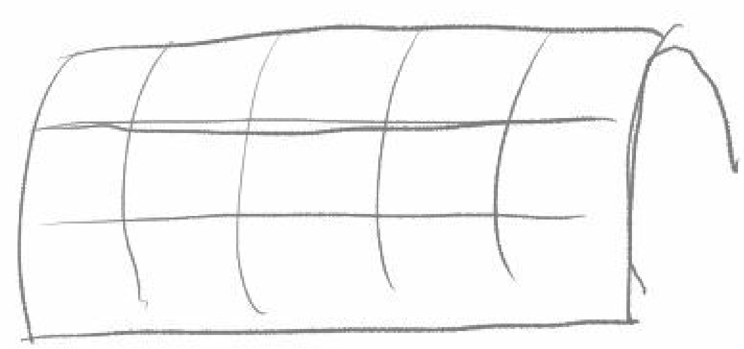
$$\Rightarrow K = \frac{\det\left(\frac{1}{r}\,\varphi_i\,\varphi_j\right)}{\det\left(\varphi_i\,\varphi_j\right)} = \frac{1}{r^2}$$

2) $S \equiv ax^2 + by^2 - z = 0$

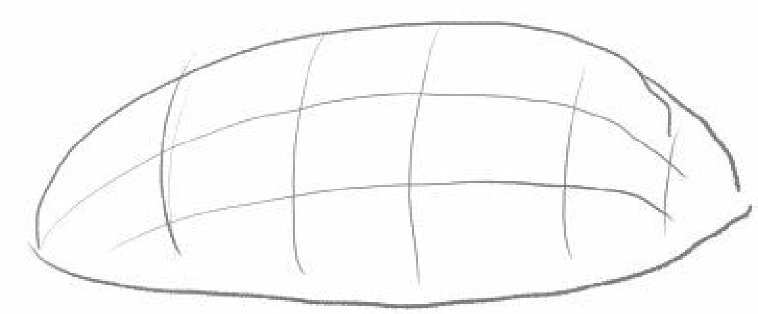
$$K = \frac{4ab}{(4a^2x^2 + 4b^2y^2 + 1)^2}$$

$K < 0$

$K = 0$



$K > 0$



<u>Other interpretation</u>

dN is a symmetric map on T_pS

↳ it admits real eigenvalues k_1, k_2
with orthogonal eigenvectors u_1, u_2