EAI Math Reading Group VI

Wishart Matrices

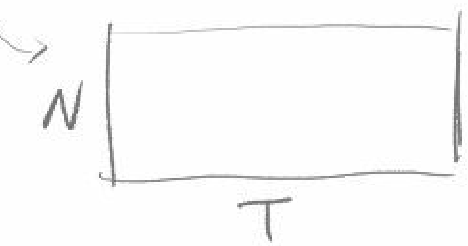
&

The Marčenko-Pastur distribution

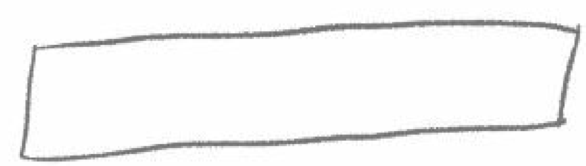
Setup: Sample Covariance Matrices

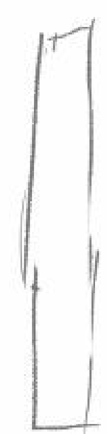
data: T observations of N variables $\{x_i^t\}_{i,t}$ $\quad i \in \{1, \dots, N\}$, $t \in \{1, \dots, T\}$

↳ e.g. time series from stocks, neurons, ...



- Main parameter: $q = \frac{N}{T}$

$q < 1$:  more samples than variables

$q > 1$:  more variables than samples (e.g. genetic studies)

Problem: Estimate the true covariance of the data

Assuming zero-mean x_i^t (otherwise remove the mean)

sample covariance matrix

$$E = \frac{1}{T} H H^T \longrightarrow$$ symmetric, positive definite

$$E_{ij} = \frac{1}{T} \sum_{t=1}^{T} x_i^t x_j^t$$ (covariance of the variables)

$$F = \frac{1}{N} H^T H$$ (covariance of the observations)

- TE and NF have the same eigenvalues
- $q < 1$: F has N non-zero eigenvalues $\frac{1}{q} \lambda_k^E$, $T-N$ zero eigenvals

$$\Rightarrow \quad g_T^F(z) = \frac{1}{T} \sum_{k=1}^{T} \frac{1}{1-\lambda_k^F} = q^{2} g_N^E(z) + \frac{1-q}{z}$$

First & Second Moments of a Wishart Matrix

Suppose the columns of H are drawn iid $\sim N(0, C)$ ↳ true covariance

$$\Rightarrow \mathbb{E}\left[H_{it} H_{js}\right] = C_{ij} \delta_{ts}$$

- $E = \frac{1}{T} H H^T$ (Wishart Matrix)

$$\mathbb{E}\left[E_{ij}\right] = \frac{1}{T} \mathbb{E}\left[\sum_{t=1}^{T} H_{it} H_{jt}\right] = \frac{1}{T} \sum_{t=1}^{T} C_{ij} = C_{ij}$$

$\Rightarrow \mathbb{E}[E] = C$ ⤳ the sample covariance matrix is an unbiased estimator of the true covariance

- $\underline{\tau(E^2)} = \frac{1}{NT^2} \mathbb{E}\left[\operatorname{tr}(HH^T HH^T)\right] = \frac{1}{NT^2} \sum_{i,j,t,s} \mathbb{E}\left[H_{it} H_{jt} H_{js} H_{is}\right]$

$$= \underset{\uparrow \text{ Wick's theorem}}{\cdots} = \tau(C^2) + \frac{N}{T} \tau(C)^2 + \frac{1}{T} \tau(C^2) \rightarrow \tau(C^2) + q\, \tau(C)^2$$

Law of Wishart Matrices

$$P(\{H_{ij}\}) = \frac{1}{\sqrt{(2\pi)^N \det C}} \exp\left[-\frac{1}{2} H_{it} (C^{-1})_{ij} H_{jt}\right]$$

$$P[H] = \frac{1}{\sqrt{(2\pi)^N} \det C^{T/2}} \exp\left[-\frac{1}{2} \underbrace{\mathrm{tr}(H^T C^{-1} H)}_{= \mathrm{tr}(E C^{-1})}\right] \qquad H \rightarrow E$$

$\vdots$

$$P[E] = \underbrace{\frac{(T/2)^{NT/2}}{\Gamma_N(T/2)}}_{} \frac{(\det E)^{\frac{T-N-1}{2}}}{(\det C)^{\frac{T}{2}}} \exp\left[-\frac{T}{2} \mathrm{tr}(E C^{-1})\right]$$

$\hookrightarrow$ multivariate gamma distribution $\rightsquigarrow$

$N=1$: $P(E) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$

$b = \frac{T}{2C} \quad a = \frac{T}{2}$

$\rightarrow$ The Wishart Ensemble generalizes the Gamma distribution to Gaussian matrices

White Wishart Matrices

$$E = O \Lambda O^T$$
$$\log E = O \underbrace{\log \Lambda} O^T$$

By the identity $\det E = \exp(\mathrm{tr} \log E)$, we get

$$P[E] = \frac{(T/2)^{\frac{NT}{2}}}{\Gamma_N(T/2)} \frac{1}{(\det C)^{\frac{T}{2}}} \exp\left[-\frac{T}{2} \mathrm{tr}(E C^{-1}) + \frac{T-N-1}{2} \mathrm{tr} \log E \right]$$

- Let $C = I$ (white case), then

$$\underbrace{P[W] \propto \exp\left[-\frac{N}{2} \mathrm{tr}\, V(W) \right]} \quad \text{where } V(W) = \left(1 - \frac{1}{q}\right) \log W + \frac{1}{q} W$$

↳ rotationally invariant

The Marčenko-Pastur distribution (White case $C = I$)

Let $H \in \mathbb{R}^{N \times T} \sim$ iid $\mathcal{N}(0,1)$, $W = \frac{1}{T} H H^T$

Let $z \in \mathbb{C} \setminus \sigma(W)$, $M = zI - W$, then by Schur's complement

$$\frac{1}{(G(z))_{11}} = M_{11} - M_{12} (M_{22})^{-1} M_{21} \qquad \text{where} \quad M = \begin{array}{cc|c} & & \\ \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] & \begin{array}{c} 1 \\ N-1 \end{array} \\ \begin{array}{cc} 1 & N-1 \end{array} & \end{array}$$

$$G(z) = (zI - W)^{-1}$$

$$= z - W_{11} - \frac{1}{T^2} \sum_{t,s=1}^{T} \sum_{j,k=2}^{N} H_{1t} \underbrace{H_{jt} (M_{22}^{-1})_{jk} H_{ks}} H_{1s}$$

$$= z - \underbrace{W_{11}}_{= 1 + O(T^{-\frac{1}{2}})} - \frac{1}{T} \sum_{j,k=2}^{N} \underbrace{\frac{\sum_t H_{kt} H_{jt}}{T}}_{W_{kj}} (M_{22}^{-1})_{jk} + \underbrace{O(T^{-\frac{1}{2}})}$$

$$\frac{1}{(G(z))_{11}} = z - 1 - \frac{1}{T} \underbrace{\operatorname{tr} W_2 G_2(z)} + O(T^{-\frac{1}{2}})$$

$$= \operatorname{tr}\left(W_2 (zI - W_2)^{-1} \right)$$

$$= -\operatorname{tr} I + z \operatorname{tr}\left((zI - W_2)^{-1} \right)$$

$$= -\operatorname{tr} I + z \operatorname{tr} G_2(z)$$

$$\Rightarrow \quad \frac{1}{(G(z))_{11}} = \underbrace{z - 1 + q - qz\, g(z)} + \underbrace{O(N^{-\frac{1}{2}})}$$

$$\Big\downarrow \; N, T \to \infty \quad \frac{N}{T} = q$$

$$\boxed{\frac{1}{g(z)} = z - 1 + q - qz\, g(z)}$$

Solution of the Stieltjes transform

$$g(z) = \frac{z + q - 1 \pm \sqrt{(z+q-1)^2 - 4qz}}{2qz}$$

$$\lambda_\pm = (1 \pm \sqrt{q})^2$$

Correct branch:

$$g(z) = \frac{z - (1-q) - \sqrt{z-\lambda_+}\sqrt{z-\lambda_-}}{2qz}$$

Eigenvalue density

$$\rho(x) = \frac{1}{\pi} \lim_{\eta \to 0} \operatorname{Im} g(x - i\eta) = \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{2\pi q x} + \underbrace{\frac{q-1}{q}\delta(x)}_{\text{zero eigenvalues when } N>T}$$

General Wishart Matrices

$$E_C = \frac{1}{T} H_C H_C^T \qquad H_C \sim N(0, C)$$

let $C = O \Lambda O^T$ (diagonalized form) $\quad \Lambda = \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_N^2 \end{pmatrix}$

$$C^{\frac{1}{2}} = O \Lambda^{\frac{1}{2}} O^T \qquad \Lambda^{\frac{1}{2}} = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_N \end{pmatrix}$$

. To generate $y \sim N(0, C)$, generate $x \sim N(0, I) \rightsquigarrow y = C^{\frac{1}{2}} x$

$$\Rightarrow \mathbb{E}[y y^T] = \mathbb{E}\left[C^{\frac{1}{2}} x x^T C^{\frac{1}{2}}\right] = C$$

$$E_C = \frac{1}{T} H_C H_C^T = \frac{1}{T} \underbrace{C^{\frac{1}{2}} H H^T C^{\frac{1}{2}}}_{\text{"Free Product"}} = C^{\frac{1}{2}} \underbrace{W_q}_{\text{White Wishart w/ } q = \frac{N}{T}} C^{\frac{1}{2}}$$

↳ "Free Probability"