Topology Speed Run II: Connectedness, Compactness and Hausdorff spaces

Why you should care:

- Important properties for approximation

Hausdorff $\Rightarrow$ "can approximate any particular point to arbitrary precision"

Compactness $\Rightarrow$ "can approximate to arbitrary precision with finitely many objects"

Example: (Universal approximation Theorem)

Let $\sigma \in C(\mathbb{R}, \mathbb{R})$, then σ not polynomial

$\Leftrightarrow \forall n \in \mathbb{N}, K \subseteq \mathbb{R}^n$ compact, $f \in C(K, \mathbb{R}^m)$, $\varepsilon > 0$, $\exists\, k \in \mathbb{N}$, $A \in \mathbb{R}^{k \times n}$, $b \in \mathbb{R}^k$, $C \in \mathbb{R}^{m \times k}$

$\sup_{x \in K} \| f(x) - g(x) \| < \varepsilon$

$\hookrightarrow \underbrace{C \cdot (\sigma \cdot (Wx + b))}$ NN w/ 1 hidden layer

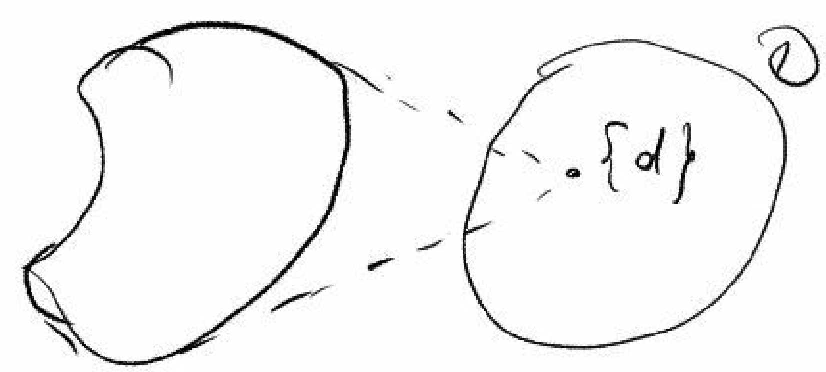
Connectedness: (X, τ) is connected $\Leftrightarrow$ X cannot be written as $U_1 \cup U_2$ where $U_1, U_2 \in \tau \setminus \{\emptyset\}$, $U_1 \cap U_2 = \emptyset$

- Clopen set: U is clopen $\Leftrightarrow$ $U \in \tau$ and $U^c \in \tau$
  $\Rightarrow$ U clopen $\Rightarrow$ U^c clopen

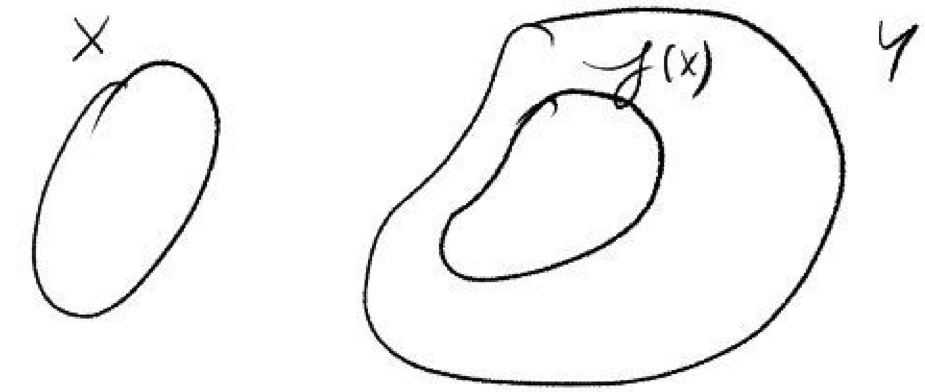
  } X connected $\Leftrightarrow$ only $\emptyset$ and X are clopen

- Discrete-valued map $f : X \to \mathcal{D}$ ↳ discrete topological space

  } X connected $\Leftrightarrow$ every continuous discrete-valued map on X is constant



Proposition: if $f: X \to Y$ is continuous and X connected, then $f(X)$ is connected



Proof

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "d \circ f"'] & f(X) \arrow[d, "d"] \\ & \mathcal{D} \end{tikzcd}$$

d ← discrete-valued map

↳ discrete-valued

⇒ constant

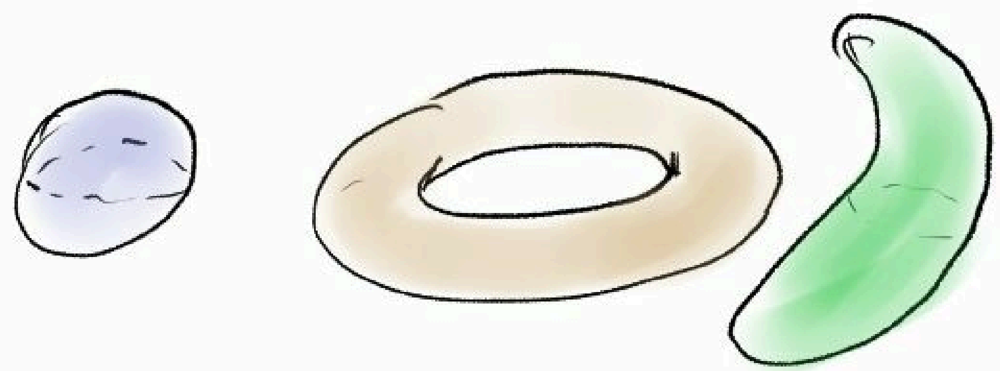
Proposition If $\{Y_i\}_{i \in I}$ is a collection of connected subsets in X and $Y_i \cap Y_j \neq \phi \ \forall i, j \in I$, then $\bigcup_{i \in I} Y_i$ is connected

Connected Components

$p \sim q \equiv p$ and q belong to a connected subset of X

$\rightsquigarrow$ equivalence relation

$\rightarrow$ The equivalence classes $X/\sim$ are the <u>connected components</u>



Connected components are

- connected and closed
- contain all connected subsets
- disjoint
- cover X

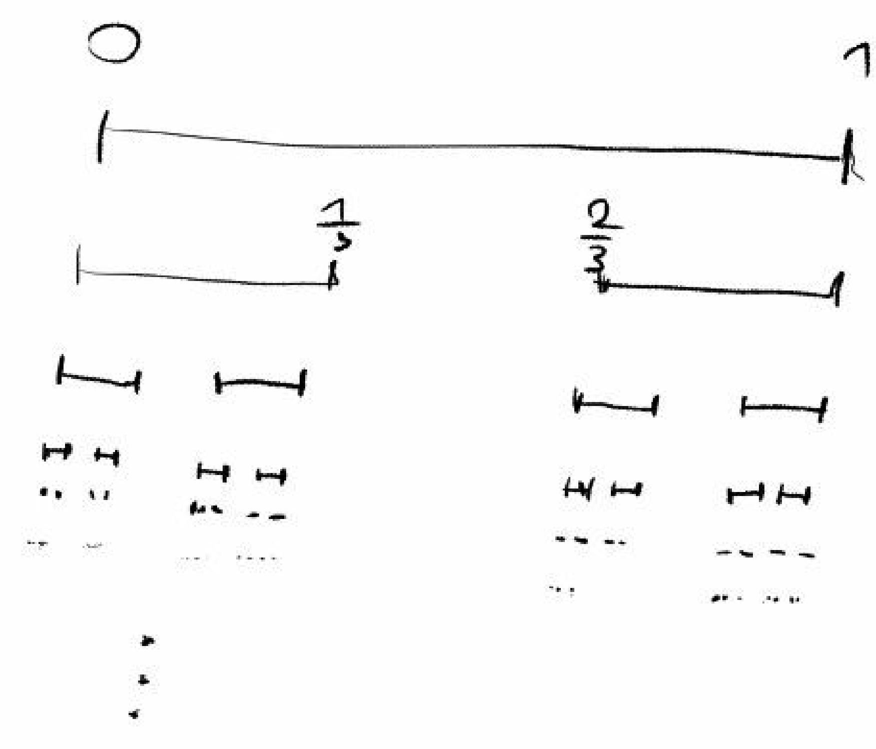
- Connected components are not always open:

  $\mathbb{Q} \subset \mathbb{R}$ only admits singletons $\{q\}$ $q \in \mathbb{Q}$ as connected subsets

  ↳ closed, but not open

- (X, τ) is totally disconnected if the only connected subsets of X are singletons

Example: Cantor's set



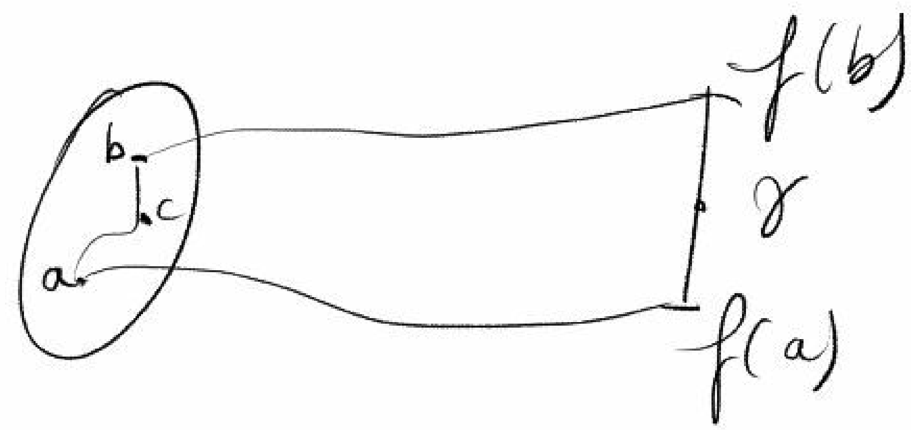
uncountable, totally disconnected, ...

Theorem: (Intermediate value theorem)

Let X be a connected topological space, $f : X \to \mathbb{R}$ continuous

If $a, b \in X$, $f(a) < \gamma < f(b)$ for $\gamma \in \mathbb{R}$, then

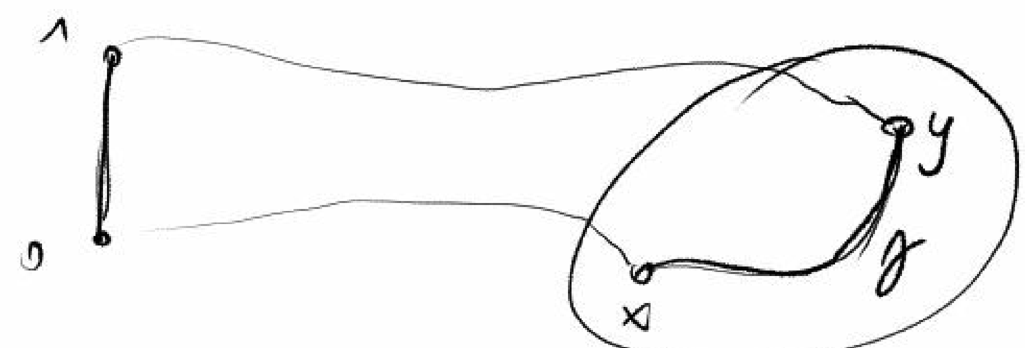
there exists $c \in X$ s.t. $f(c) = \gamma$



Path-connected: (X, τ) is path-connected if for all $x, y \in X$

there is a path $\gamma : [0,1] \to X$ $\gamma(0) = x$, $\gamma(1) = y$

↳ cont.



N.B. path-connected

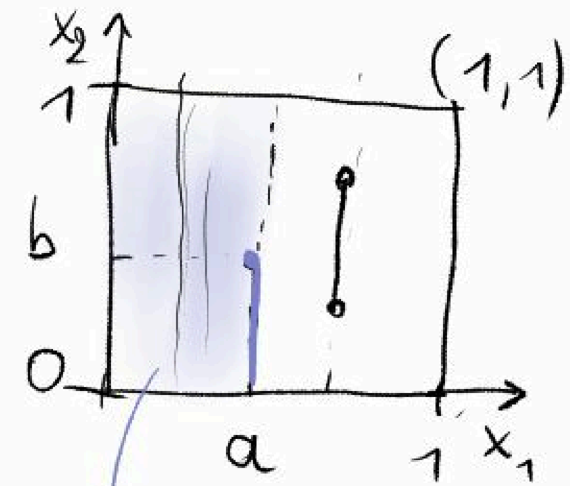
$\Rightarrow$ connected

Example $X = [0,1] \times [0,1]$ with lexicographic order $(a,b) \leq (c,d)$

$\Leftrightarrow$ $a < c$ or

$(a = c$ and $b \leq d)$

X with the order topology is connected but not path-connected



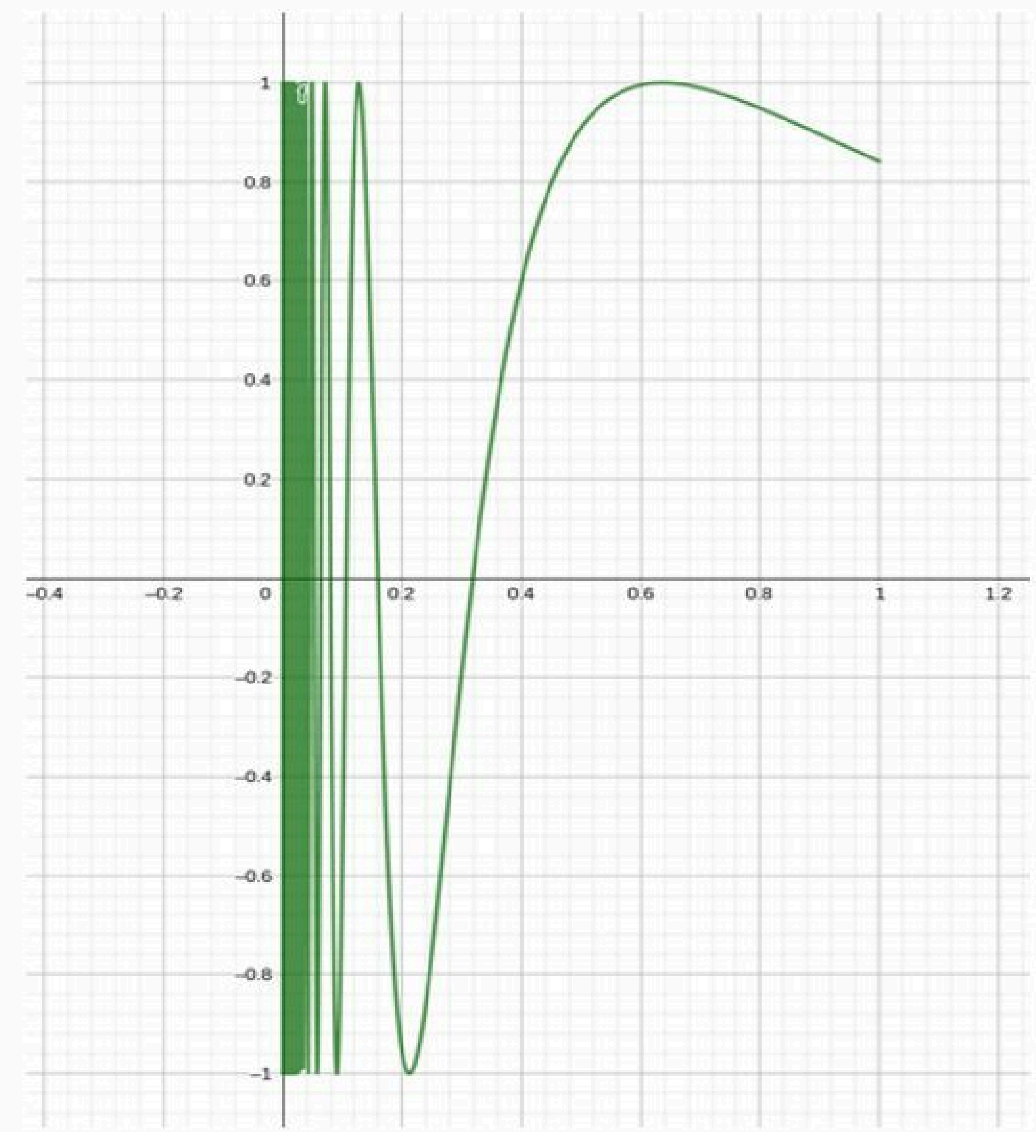
$\{x \in X \mid x \leq (a,b)\}$

Example: Topologist's sine curve

$$S = \left\{ \left(x, \sin\tfrac{1}{x}\right) \,\middle|\, 0 < x \leq 1 \right\} \subseteq \mathbb{R}^2$$

$$\bar{S} = S \cup \left(\{0\} \times [-1,1]\right)$$

is connected, but not path-connected

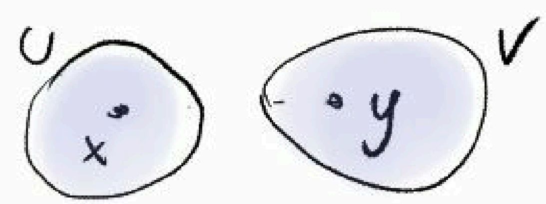


Separation Axioms

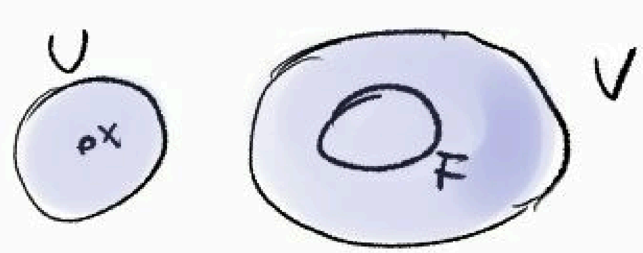
- T_0 : $\forall x,y \in X, \exists U \in \tau : (x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$

- T_1: $\forall x,y \in X \; \exists U \in \tau : x \in U \wedge y \notin U$

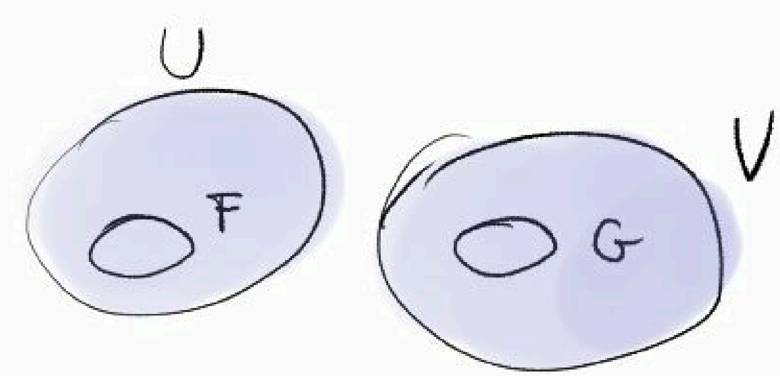
- T_2 (Hausdorff) : $\forall x, y \in X, \exists U, V \in \tau$ s.t. $x \in U, \; y \in V, \; U \cap V = \emptyset$



- T_3 (regular) $\forall x \in X, \forall F$ closed, s.t. $x \notin F$ $\exists U, V \in \tau$ s.t. $x \in U, \; F \subseteq V$



- T_4 (normal) $\forall$ closed F, G, $F \cap G = \emptyset$, $\exists U, V \in \tau$, $F \subseteq U$, $G \subseteq V$



Separation axioms in topological spaces	
Kolmogorov classification	
T_0	(Kolmogorov)
T_1	(Fréchet)
T_2	(Hausdorff)
$T_{2½}$	(Urysohn)
completely T_2	(completely Hausdorff)
T_3	(regular Hausdorff)
$T_{3½}$	(Tychonoff)
T_4	(normal Hausdorff)
T_5	(completely normal Hausdorff)
T_6	(perfectly normal Hausdorff)
History	

Why you should care about Hausdorff spaces?

Prop X is Hausdorff $\Leftrightarrow$ the diagonal map $\Delta_X : X \to X \times X;\ x \mapsto (x,x)$ satisfies $\Delta_X(X)$ closed in $X \times X$

Theorem: Let Y be Hausdorff, X an arbitrary top. space $f, g : X \to Y$ continuous. If $f|_S = g|_S$ on a dense $S \subseteq X$ then $f = g$ on X

Theorem : Limits of sequences are unique

$\searrow$ Can always isolate points!

Compactness

- A cover of (X,τ) is a collection $\mathcal{C} = \{B_i\}_{i\in I} \subset \mathcal{P}(X)$ s.t. $\bigcup_{i\in I} B_i = X$
  - open cover: $B_i \in \tau \ \forall i \in I$
  - subcover of $\mathcal{C}$: $\tilde{\mathcal{C}} \subset \mathcal{C}$ s.t. $\bigcup_{B\in\tilde{\mathcal{C}}} B = X$

- A topological space X is <u>compact</u> if every open cover of X has a <u>finite</u> subcover

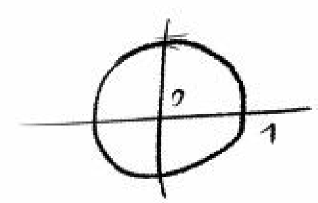
- <u>Finite Intersection Property</u>: $\mathcal{C} \subset \mathcal{P}(X)$ has the finite intersection property
$$\bigcap_{i=1}^{n} C_i \neq \emptyset \quad \forall C_1, \dots, C_n \in \mathcal{C} \quad \forall n \in \mathbb{N}$$

<u>Prop</u>: X compact $\Leftrightarrow$ $\forall \mathcal{C} \in \mathcal{P}(X)$ w/ FIP, $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$

Examples

- Any finite space is compact
- a trivial topological space is compact
- Any compact subspace of a Hausdorff space is closed
- $\boxed{X \subseteq \mathbb{R} \text{ compact} \Leftrightarrow \text{closed and bounded}}$

Example : the unit circle S^1 is compact



$[0, 1]$ is compact

Properties: . Let X be compact and $f: X \to Y$ continuous then $f(X)$ is a compact subspace of Y

. X compact, $A \subseteq X$ closed $\Rightarrow$ A is a compact subspace

. X compact, Y Hausdorff $f: X \to Y$ continuous bijection $\Rightarrow$ f is a homeomorphism

. Extremal Value Theorem: X Compact, $f: X \to \mathbb{R}$ continuous

$$\Rightarrow \exists\, a, b \in X \text{ s.t. } f(a) = \inf_{x \in X} f(x)$$
$$f(b) = \sup_{x \in X} f(x)$$

. Sequential compactness: X is limit point compact if every infinite subset has a limit point

X sequentially compact if every sequence $(x_n) \subset X$ has a convergent subsequence

. Finite products of compact spaces are compact (actually infinite products too!)

Local compactness

- **Local compactness**: (X, τ) is locally compact at $x \in X$ if there exist a compact subset $C \subseteq X$, which contains a neighbourhood of x.
  X is locally compact if it is locally compact at every $x \in X$

  Example: $\mathbb{R}^n$ is locally compact, but $\mathbb{Q}$ is not

Prop: Let X be a locally compact Hausdorff space, $x \in X$. Then each neighbourhood of x contains a compact neighbourhood of x

One-point compactification

Let (X, τ) be a locally compact Hausdorff space,

$$X^+ = X \cup \{\infty\}$$

↳ some arbitrary point not in X

topology on X^+: $U \subseteq X^+$ open if $\begin{cases} U \subseteq X \text{ and } U \in \tau \\ \text{or} \\ U = X^+ \setminus C \text{ where } C \subseteq X \text{ compact} \end{cases}$

↳ τ^+

Theorem: (X^+, τ^+) is a compact Hausdorff space and τ^+ is the only topology that makes X^+ compact and Hausdorff with $\tau^+|_X = \tau$

Example: One-point compactification of $\mathbb{R}$

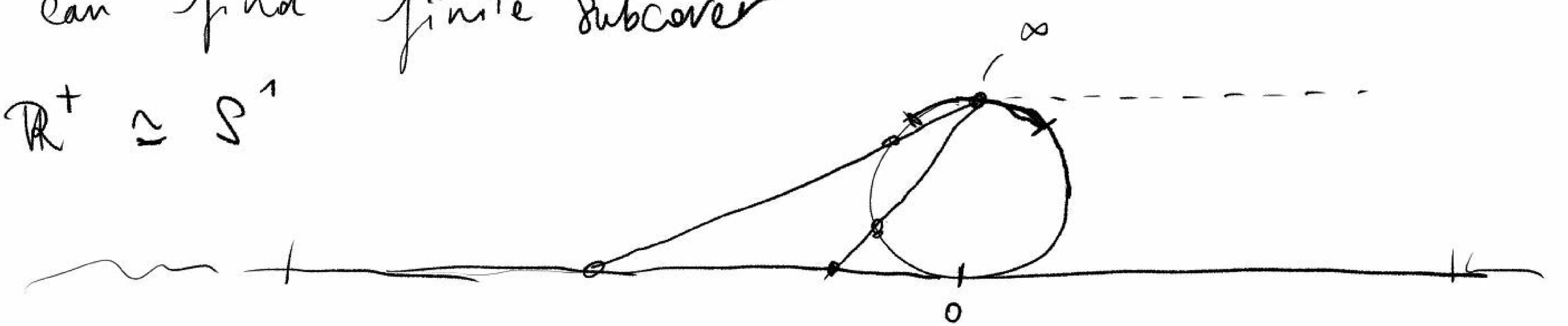
$\mathbb{R}^+ = \mathbb{R} \cup \{\infty\}$

Subbasis: $\{(a,b) \mid a < b\} \cup \{$"$[-\infty, a) \cup (b, +\infty]$"$\mid a < b\}$

Any open cover must contain at least one open set of the form "$[-\infty, a) \cup (b, +\infty]$"
and the rest must cover $[a, b]$ which is a compact subset

$\Rightarrow$ can find finite subcover

$\mathbb{R}^+ \simeq S^1$



Compactness in metric spaces

- **Cauchy sequence** $(x_n)_{n\in\mathbb{N}} \subset (X,d)$ is Cauchy $\Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t.

$$\forall m,n \in \mathbb{N},\ m,n \geq N_\varepsilon \Rightarrow d(x_n, x_m) < \varepsilon$$

(X,d) is complete if every Cauchy sequence is convergent

- **Total Boundedness**: (X,d) is totally bounded if $\forall \varepsilon > 0$,
  X can be covered by finitely many balls of radius ε

NB. In finite dimensional euclidian spaces, bounded $\Leftrightarrow$ totally bounded

Theorem: Let (X,d) be a metric space, then the following are equivalent

1) X is compact
2) X is limit point compact
3) X is sequentially compact
4) X is complete and totally bounded

<u>Corollary</u> If (X, d) is compact, then for any $\varepsilon > 0$, there can be only finitely many points $x_1, \dots, x_n$ such that

$$d(x_i, x_j) \geq \varepsilon \quad \forall i, j$$

<u>Other Properties</u>:

- Every isometry on a compact metric space is a homeomorphism
  $\hookrightarrow d_Y(f(x), f(x')) = d_X(x, x')$
- Every metric space is locally compact? [No!] (infinite-dim Banach/Hilbert spaces)

$\ell^2 \quad \overline{B(0,1)} \quad (1, 0, \dots)$

$(0, 1 \dots)$