Fix coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ on $\mathbb{R}^{2n}$ and let ω_0 be the standard symplectic structure $\sum dx_i \wedge dy_i$. The ball with radius $r > 0$ is the set

$$B_{2n}(r) = \{(q,p) \in \mathbb{R}^{2n} : \sum (q_i^2 + p_i^2) < r^2\}$$

and the standard cylinder is the set

$$Z_{2n}(r) = \{(q,p) \in \mathbb{R}^{2n} : q_1^2 + p_1^2 < r^2\} = B_2(r) \times \mathbb{R}^{2(n-1)}.$$

Any open subset of $\mathbb{R}^{2n}$ with the restriction of ω_0 is a symplectic manifold. A symplectomorphism of $\mathbb{R}^{2n}$ is a diffeomorphism ϕ such that $\phi^* \omega_0 = \omega_0$.

Definition 1. *A* symplectic capacity *on $\mathbb{R}^{2n}$ is a function c defined on open subsets of $\mathbb{R}^{2n}$, taking values in the set $[0, \infty]$ with the following properties:*

1. *(Monotonicity) If $U \subset V$, then $c(U) \leq c(V)$,*
2. *(Invariance) If ϕ is a symplectomorphism then $c(\phi(U)) = c(U)$,*
3. *(Homogeneity) For all $\alpha > 0$, $c(\alpha U) = \alpha^2 c(U)$, and*
4. *(Non-triviality) $0 < c(B_{2n}(1))$ and $c_{2n}(Z(1)) < \infty$.*

A symplectic capacity c is said to be normalized *if $c(B_{2n}(1)) = c(Z_{2n}(1)) = \pi$.*

0.1. **The Eckland-Hofer-Zhender capacity.** The Eckland-Hofer-Zhender capacity, denoted c_{EHZ}, is a normalized symplectic capacity defined on compact, convex subsets of $\mathbb{R}^{2n}$. Let K be a compact, strictly convex set in $\mathbb{R}^{2n}$ such that the interior of K contains the origin and the boundary of K is smooth of class C^2 (i.e. locally it has C^2 parameterizations). Let

$$H(x) = ||x||_K^2 \ ,$$

where $\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}$ is the semi-norm with respect to the convex body K. The Hamiltonian vector field $X_H = J\nabla H$ defines a flow on ∂K which is the level set $H^{-1}(1)$. Consider the space of periodic orbits of X_H,

$$\mathcal{E}_K = \left\{ z \in C^2(\mathbb{R}, \partial K) : \dot{z} = X_H \circ z,\ z(0) = z(T) \text{ for some } T > 0 \right\}$$

Definition 2. *The* action *of an orbit $z \in \mathcal{E}_K$ with period T is*

$$\mathcal{A}(z) = \frac{1}{2} \int_0^T \langle -J\dot{z}, z \rangle \, dt.$$

The Eckland-Hofer-Zhender capacity *of K is the quantity*

$$c_{EHZ}(K) = \inf \{\mathcal{A}(z) : \, z \in \mathcal{E}_K\}$$

Let $\mathcal{K}$ be the set of all compact, strictly convex subsets of $\mathbb{R}^{2n}$ which have C^2 boundary.

Theorem 1. *The function c_{EHZ} satisfies the axioms of a symplectic capacity on the set $\mathcal{K}$.*

To prove this theorem, one considers an equivalent variational problem. Let

$$M = \left\{ x \in H^1([0,1], \mathbb{R}^{2n}) \ : \ \int_0^1 \dot{x}(t) dt = 0 \right\}$$

and consider the functional

$$\mathcal{F}_K(x) = \int_0^1 h_K^2(-J\dot{x}(t)) dt. \tag{1}$$

The constrained minimization problem

$$\lambda = \inf \left\{ \mathcal{F}_K(x) : x \in M, \int_0^1 \langle -J\dot{x}(t), x(t) \rangle dt = 1 \right\} \tag{2}$$

has at least one solution, denoted by u.

Proposition 1. *Let u be a minimizer for $\mathcal{F}_K$, consider the path*

$$z(t) = 2\sqrt{\lambda} u \left(\frac{t}{2\lambda} \right) + \frac{c}{\sqrt{\lambda}}, \tag{3}$$

$$c = \int_0^1 \nabla h_K^2(-J\dot{u})dt + 2\int_0^1 u(t)dt. \tag{4}$$

The path $z(t)$ is a 2λ-periodic solution of X_H which lies in ∂K and minimizes $\mathcal{A}$. Thus

$$c_{EHZ}(K) = \mathcal{A}(z) = 2\lambda.$$

Proof. □

0.2. **Generalized Characteristics.** Now we follow [?artstein-avidan-ostrover]. Let $\widetilde{\mathcal{K}}$ be the set of convex bodies in $\mathbb{R}^{2n}$ with non-empty interior. There is a unique extension of c_{EHZ} to $\widetilde{\mathcal{K}}$ that is homogeneous, monotone, and continuous with respect to the Hausdorff metric. By uniqueness, this extension can be given the same definition in terms of a dual minimization problem [?artstein-avidan-ostrover]

$$\tilde{c}_{EHZ}(K) = 2 \inf \left\{ \mathcal{F}_K(x) : x \in M \right\}.$$

It was shown in [?artstein-avidan-ostrover] that this extension has a geometric definition. The *normal cone* to a convex body K at $x \in \partial K$ is

$$N_{\partial K}(x) = \left\{ u \in \mathbb{R}^{2n} : \langle u, x - y \rangle \geq 0, \text{ for every } y \in K \right\}.$$

Definition 3. *A* generalized closed characteristic *is a periodic, piece-wise smooth path $z : \mathbb{R} \to \partial K$ such that*

$$\dot{z}_{\pm}(t) = JN_{\partial K}(z(t))$$

for all $t \in \mathbb{R}$, where $\dot{z}_{\pm}$ are the left and right derivatives of z.

Note that we have defined the Eckland-Hofer-Zhender capacity in terms of Hamilton's equation rather than solutions to the characteristic foliation, whereas this definition of a generalized closed characteristic coincides with the latter definition in the case where K has smooth boundary. This issue of parameterization does not effect the computed action.

If $\widetilde{\mathcal{E}_K}$ is the space of generalized closed characteristics for K, then

Theorem 2. [?artstein-avidan-ostrover]

$$\tilde{c}_{EHZ}(K) = 2 \inf \left\{ \mathcal{F}_K(x) : x \in M \right\} = \inf \left\{ |\mathcal{A}(z)| : z \in \widetilde{\mathcal{E}_K} \right\}.$$