First, lets recall some basic facts about convex functions and Legendre transforms, which can be found in [?amann]. A function $H : \mathbb{R}^{2n} \to \mathbb{R}$ is *convex* if for all x, y,

$$H(tx + (1-t)y) \geq tH(x) + (1-t)H(y)$$

and H is strictly convex if this inequality is strict.

Proposition 1. *Let $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$. Then H is (strictly) convex if and only if the Hessian $D^2H(x)$ is positive (definite) semi-definite for all $x \in \mathbb{R}^{2n}$.*

The Hessian D^2H is *uniformly* positive definite if there is a constant $\alpha > 0$ such that

$$\langle D^2H(x)y, y\rangle \geq \alpha|y|^2, \forall x, y \in \mathbb{R}^{2n}$$

Proposition 2. *Let $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$. If D^2H is uniformly positive definite then the gradient $\nabla H : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is bijective.*

The *Legendre transform* of a function $H : \mathbb{R}^{2n} \to \mathbb{R}$ is defined as

$$\mathcal{L}H(y) = \sup_{x \in \mathbb{R}^{2n}} \left\{\langle y, x\rangle - H(x)\right\}.$$

Proposition 3. *Let $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ and assume that D^2H is uniformly positive definite. Then*

(1) *$\mathcal{L}H(y) = \langle y, x\rangle - H(x)$ for $x = (\nabla H)^{-1}(y)$.*
(2) *$\mathcal{L}H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, $\mathcal{L}H$ is strictly convex, and $\nabla(\mathcal{L}H) = (\nabla H)^{-1}$.*
(3) *$H(x) + \mathcal{L}H(y) \geq \langle x, y\rangle$, for all $x, y \in \mathbb{R}^{2n}$ with equality if and only if $x = y$.*
(4) *$\mathcal{L}^2H = H$.*

Now we follow [?hofer-zhender]. Suppose $K \subset \mathbb{R}^{2n}$ is a compact, convex set whose interior contains the origin. The *norm* of the convex body K is

$$||x||_K = \inf\left\{r > 0 : \frac{x}{r} \in K\right\}, \forall x \neq 0$$

and by definition,

$$K = \left\{x \in \mathbb{R}^{2n} : ||x||_K \leq 1\right\}.$$

Note that $H(x) = ||x||_K^2$ is homogeneous of degree two (i.e. $H(\alpha x) = \alpha^2 H(x)$ for all $\alpha > 0$) and has quadratic growth (i.e. there exist constants $c_1, c_2 > 0$ such that

$$c_1||x||^2 \leq H(x) \leq c_2||x||^2).$$

Proposition 4. *If K is strictly convex and ∂K is smooth of class C^2, then the function $H(x) = ||x||_K^2$ satisfies the following properties:*

(1) *$H \in C^2(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R})$.*
(2) *D^2H is uniformly positive definite (in particular, H is strictly convex).*

Proof. (1)
(2)

□

Given a (strictly) convex set K that contains the origin, its *dual* is the (strictly) convex set

$$K^\circ = \{y \in \mathbb{R}^{2n} : \langle x, y\rangle \leq 1\, \forall x \in K\}.$$

Observe that $(K^\circ)^\circ$.

Proposition 5. *K is C^2 and strictly convex if and only if K° is C^2 and strictly convex.*

In this paper, we often consider the supporting function

$$h_K(x) = \sup_{y \in K} \{\langle x, y\rangle\}.$$

Observe that $K^\circ = \{x : h_K(x) \leq 1\}$ and so $h_K(x) = ||x||_{K^\circ}$. If $K + K'$ is the Minkowski sum of two convex sets then it follows from the definition that

$$h_{K+K'} = h_K + h_{K'}$$

Proposition 6. *If K is a convex set then*

$$\mathcal{L} h_K^2 = \frac{1}{4} h_{K^\circ}^2$$

0.1. **Smoothed Dual Body Approach.** The following is one approach to running the algorithm with a smoothing of a convex polytope.

Suppose that K is a convex polytope and we want to run the algorithm for the smoothed body $K_\epsilon = (K^\circ + B(\epsilon))^\circ$. Then we need to express the function $h_{K_\epsilon}^2$ as well as the gradient $\nabla h_{K_\epsilon}^2$ in a computable way.

$$h_{K_\epsilon}^2 = 4\mathcal{L}\left(h_{K_\epsilon^\circ}\right)^2 = 4\mathcal{L} h_{K^\circ + B(\epsilon)}^2 = 4\mathcal{L}\left(h_{K^\circ} + h_{B(\epsilon)}\right)^2.$$

The function h_{K° can be computed combinatorially, and the Legendre transform can be computed using a software package. Since we would not like to evaluate the gradient of this Legendre transform, we note that

$$\nabla h_{K_\epsilon}^2(u) = v \text{ if and only if } u = \nabla\left(h_{K_\epsilon^\circ}\right)^2(v)$$

so to solve for the gradient of $h_{K_\epsilon}^2$ at u, we can evaluate the minimization problem

$$v \text{ minimizes } |u - \nabla\left(h_{K_\epsilon^\circ}\right)^2(v)|.$$

Now

$$\nabla\left(h_{K_\epsilon^\circ}\right)^2 = \nabla\left(h_K + h_{B(\epsilon)}\right)^2 = \nabla h_K^2 + 2\nabla\left(h_K h_{B(\epsilon)}\right) + \nabla h_{B(\epsilon)}^2$$

And every expression in this equation can be evaluated piece-wise analytically.