1 Potential Gap Theorem

Theorem 1. *Assume that a point-mass robot with a planar, first-order, fully controlled motion model and a 360° scanner operates in a planar environment. Let the latest scan measurement from the robot provide a set of candidate gaps $\mathcal{G}_{SGP}$ to travel through based on the* Swept Gap Prioritization *algorithm followed by gap convexification. The trajectory $\mathcal{T}$, generated by forward integrating along the vector field $\mathbf{D}$ constructed from the gap $G^* \in \mathcal{G}$ when starting from the robot position x located at the source position of the scan, is guaranteed to result in collision-free passage across the gap boundary.*

For this document to be standalone, the important constructions associated to the motion vector field will be reproduced here, followed by the proof proper.

1.1 Potential Gap Gradient Fields

The premise behind gaps is that there is are obstacles in the world that must be avoided by staying *within* the gap region and only leaving by moving across the gap curve. The gap region is known to be collision-free due to line-of-sight visibility to all points inside of it and to a non-trivial connected region on the other side of it. Based on the construction of $\mathcal{G}_{\text{SGP}}$, there exists a set of local goals on the other side of the gap (relative to the robot position) that are line-of-sight visible from any point within the gap region.

Let the line-of-sight visible local goal point be x^*_{LG} as determined from the chosen gap $G^* \in \mathcal{G}_{\text{SGP}}$. The attractive potential is

$$\Phi(x) = \mathrm{d}(x, x^*_{\text{LG}}) + \mathrm{d}_{\text{H}}(x, G^*), \tag{1}$$

where the first distance is to the local goal point and the second is the hinge distance to the gap curve. The hinge distance is the signed distance clipped to zero out negative values (a composition of the Heaviside and signed distance functions). In this case negative distances to the gap lie on the other side of the gap. Thus, the hinge distance is positive on the robot side of the gap and vanishes on the other side of the gap. These potentials attract the robot to the gap curve then through to the local goal.

Rather than impose an obstacle avoiding potential, which could create a fixed point in the resulting vector field, a purely rotational vector field is created

$$\Theta(x) = \mathbb{J}e^{-\mathrm{d}_\theta(x,p_l)/\sigma}\frac{p_l - x}{||p_l - x||} - \mathbb{J}e^{-\mathrm{d}_\theta(x,p_r)/\sigma}\frac{p_r - x}{||p_r - x||}, \tag{2}$$

where $\mathbb{J} = R(-\pi/2)$ is skew-symmetric, and $\mathrm{d}_\theta(\cdot,\cdot)$ is the angular distance. p_l and p_r are points of the left and right sides of the gaps. The vector fields are two rotational fields anchored at the left and right gap points. Figure 1 shows an example circulation vector field.

Definition 1.1 (Gap Gradient Field)**.** The potential field $\Phi(\mathbf{x})$ and circulation field $\Theta(\mathbf{x})$ given by (1) (2) define the gradient field $\mathbf{D}(\mathbf{x}) = \hat{\nabla}\Phi(\mathbf{x}) + \Theta(\mathbf{x})$.

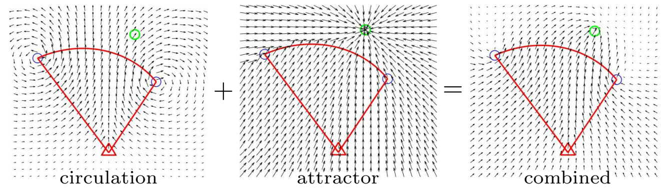

Figure 1: Gap gradient field construction. Red triangle is robot location, blue circles are gap curve endpoints, and green circle is goal point.

Definition 1.2. $\hat{\nabla}F$ is the gradient of F with normalization to unit length for non-zero magnitude gradients.

1.2 Potential Gap Proof of Passage

Proof of Theorem 1. Proving that passage through the gap for the robot must happen involves showing that the boundary of the gap navigation region points inwards along the robot-to-gap-endpoint edges (or simply *gap sides*) and that there is no fixed point interior to the region. The only reasonable flow for any point in the region is to exit via the gap curve, e.g., *gap passage*.

Along the gap edges, the gradients of (1) either point inwards or parallel to it, never out by virtue of gap convexity, the location of the local goal relative to the gap, and by definition of the gap region. Thus, what must be shown is that the circulation components also point inwards. On the gap edge, one circulation term has d_θ vanishing; the circulation is purely perpendicular and inward pointing. Let this vector be $e_\perp$. Let the other circulation term contribute the vector f_ϕ. It satisfies

$$(e_\perp + f_\phi) \cdot e_\perp = (1 - \cos(\pi - \phi)) \geq 0, \text{ for some } \phi > 0, \tag{3}$$

which means that $\Theta(\cdot)$ restricted to gap edges is inward pointing. The only outward flow can be on the gap curve.

A similar argument as above applies to show that the interior gap region points have a non-trivial outward pointing flow, which means that there cannot be a fixed point for $\mathbf{D}$ interior to the gap region. Define $e_\rho(x)$ to be the radially directed outward vector for a point $x \in G$. The following properties hold:

$$e_\rho(x) \cdot \Theta(x) \geq 0, \quad \text{and} \quad e_\rho(x) \cdot \hat{\nabla}\Phi(x) > 0 \tag{4}$$

by virtue of the gap angular extent α_g being 90°; this angle is the polar space angular difference between the two gap points relative to the robot reference frame. Since the definition of $\hat{\nabla}$ is that it computes the gradient in (1) then makes it unit length when non-zero, the gradients should not vanish. Furthermore, as the vector field interior to the gap region always has positive outward pointing contributions (and therefore no vanishing gradients), there can be no fixed points. Positivity implies that any initial point in G^* will flow out through the gap curve, see vector field in Figure 1. All planar trajectories starting in the polar triangle defined by G^* and the robot location. and following the constructed vector field are guaranteed to exit the gap region through the *gap curve*, to be attracted to the local goal, and to be non-colliding.

The potential gap algorithm generates the trajectory $\mathcal{T}$ by starting at the current robot position and following the flow field until passage through the gap to the local goal point. $\square$