IMD0033 - Probabilidade
Aula 23 - Probabilidade Condicional

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Agenda

- Solving Complex Probability Problems
- Conditional Probability
- Bayes' theorem

Atualizar o repositório

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git clone https://github.com/ivanovitchm/imd0033_2019_1.git
```

Ou ....

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git pull
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Complex Probability Problems

1. What is the probability that it takes three flips or more for a coin to land heads up?
2. What is the probability of a coin landing heads up 18 times in a row?
3. What is the probability of getting at least one 6 in four throws of a single six-sided die?
4. What is the probability of getting at least one double-six in 24 throws of two six-sided dice?
5. What is the probability of getting four aces in a row when drawing cards from a standard 52-card deck?

Previously on last lesson

Addition Rule

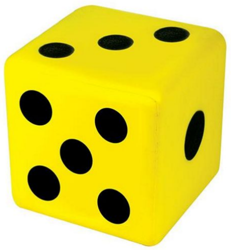
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

An advertisement company runs a quick test and shows two ads on the same web page (ad "A" and ad "B") to 100 users. At the end of the trial, they found:

- 12 users clicked on ad "A"
- 17 users clicked on ad "B"
- 3 users clicked on both ad "A" and ad "B"

Opposite Events

$$B = \{2\}$$
$$non\text{-}B = \{1, 3, 4, 5, 6\}$$

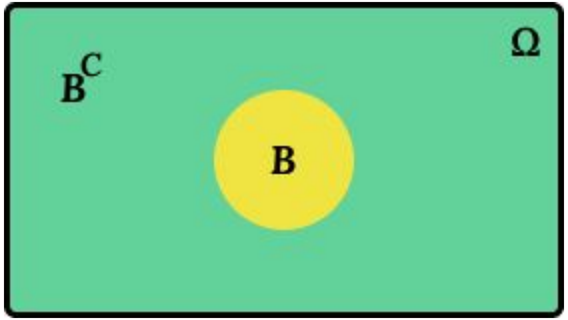


$$P(B \cup non\text{-}B) = P(B) + P(non\text{-}B) = \frac{1}{6} + \frac{5}{6} = 1$$

Opposite Events (Set Notation)

$$B = \{2\}$$

$$B^C = \{1, 3, 4, 5, 6\}$$



$$P(B \cup B^C) = P(B) + P(B^C) = 1$$

$$P(B) = 1 - P(B^C)$$

Example Walk-Through

- **What is the probability that it takes three flips or more for a coin to land heads up?**

Event A: corresponds to the following outcomes, where each number represents the number of flips it takes until we first get heads up

$$A = \{3, 4, 5, 6, \ldots, 100, 101, \ldots\}$$

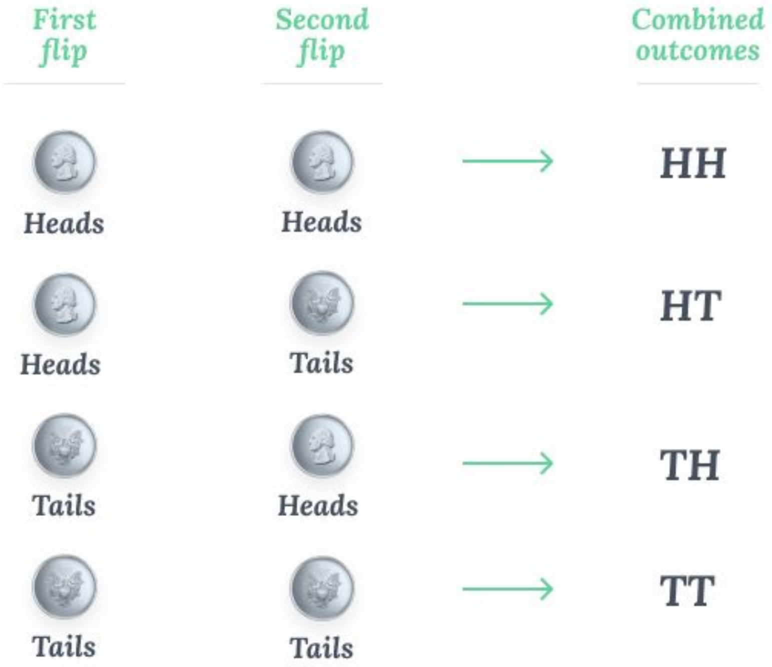
Example Walk-Through

$$A = \{3, 4, 5, 6, \ldots, 100, 101, \ldots\}$$

$$non\text{-}A = \{1, 2\}$$

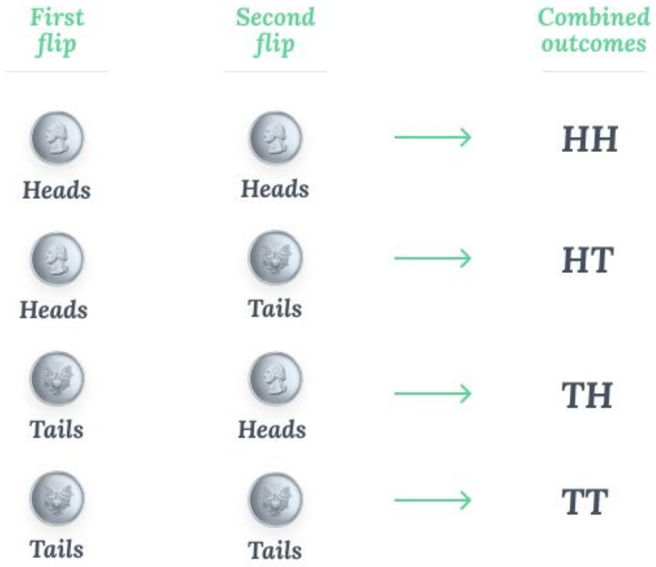
$$P(A) = 1 - P(non\text{-}A)$$

$$P(A) = 1 - 0.75 = 0.25$$



The Multiplication Rule

What is the probability of a coin landing heads up two times in a row?



$$P(HH) = \frac{1}{4} = 0.25$$

$$P(H_1 \cap H_2) = P(H_1) \times P(H_2)$$

$$P(H_1 \cap H_2) = 0.5 \times 0.5 = 0.25$$

Independent Events

Consider now the following two events, which are associated with rolling a fair six-sided die:

- A: we get a number less than 4; event A corresponds to the outcomes {1, 2, 3}
- B: we get an even number; event B corresponds to the outcomes {2, 4, 6}

$$P(A) = \frac{3}{6} \text{ and } P(B) = \frac{3}{6}$$

If event A happens: $P(B) = \frac{1}{3}$

Independent Events

$$P(E_1 \cap E_2 \cap E_3 \ldots E_n) = P(E_1) \times P(E_2) \times P(E_3) \times \ldots P(E_n)$$

For instance, this is how we find the probability that a coin lands heads up four times in a row:

$$P(H_1 \cap H_2 \cap H_3 \cap H_4) = H_1 \times H_2 \times H_3 \times H_4$$

$$P(H_1 \cap H_2 \cap H_3 \cap H_4) = 0.5 \times 0.5 \times 0.5 \times 0.5 = 0.5^4 = 0.0625$$

Independent Events

- **What is the probability of a coin landing heads up 18 times in a row?**

```
0.5**18
3.814697265625e-06
```

Combining formulas

$$P(E) = 1 - P(E^C)$$

$$P(E_1 \cap E_2 \cap E_3 \ldots E_n) = P(E_1) \times P(E_2) \times P(E_3) \times \ldots P(E_n)$$

What is the probability of getting at least one 6 in four throws of a single six-sided die?

Combining formulas

- Event A is getting at least one 6 in four throws
- Event A^C is not getting any 6 in four throws
- So event A^C is equivalent to getting any of the outcomes {1, 2, 3, 4, 5} four times in a row.

$$P(A^C) = \left(\frac{5}{6}\right)^4 = 0.4823 \qquad P(A) = 1 - \left(\frac{5}{6}\right)^4 = 0.5177$$

Combining formulas

What is the probability of getting at least one double-six in 24 throws of two six-sided dice?

First die

Second die

&	⚀	⚁	⚂	⚃	⚄	⚅
⚀	1,1	2,1	3,1	4,1	5,1	6,1
⚁	1,2	2,2	3,2	4,2	5,2	6,2
⚂	1,3	2,3	3,3	4,3	5,3	6,3
⚃	1,4	2,4	3,4	4,4	5,4	6,4
⚄	1,5	2,5	3,5	4,5	5,5	6,5
⚅	1,6	2,6	3,6	4,6	5,6	6,6

$$P(A) = 1 - \left(\frac{35}{36}\right)^{24}$$

Sampling With(out) Replacement

Sampling With(out) Replacement

$$P(ace) = \frac{4}{52}$$

$$P(AAAA) = \left(\frac{4}{52}\right)^4 = 0.000035$$?

$$P(AAAA) = \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49} = 0.00000369$$

Sampling Without Replacement

1. Getting two kings in a row
2. Getting a seven of hearts, followed by a queen of diamonds
3. Getting a jack, followed by a queen of diamonds, followed by a king, followed by another jack

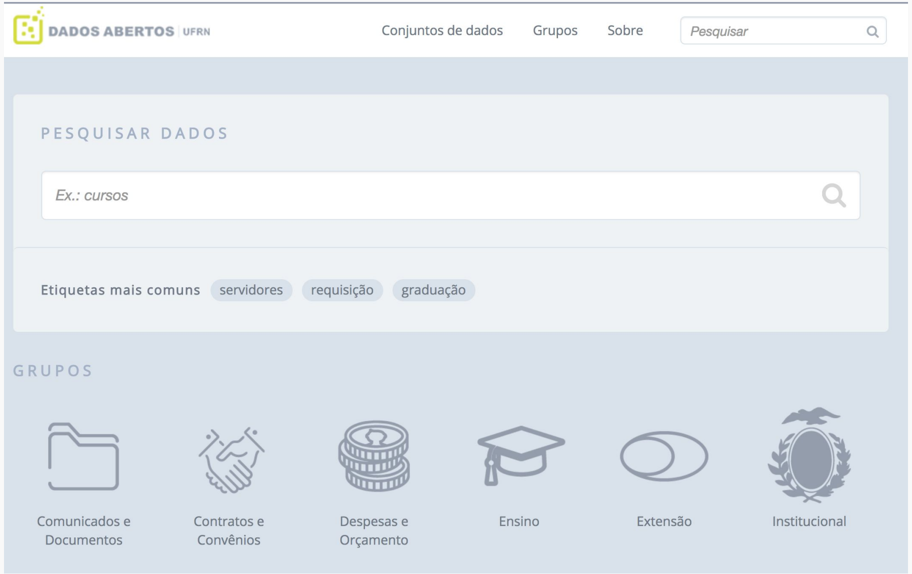
Conditional Probability

When new evidence comes our way, it helps us update our beliefs and create a new belief

Conditional Probability

How likely is a student to pass IMD0033?

Since a student took a grade higher than 5 (five) in the first unit of IMD0033, how likely is it to pass in the curricular component?

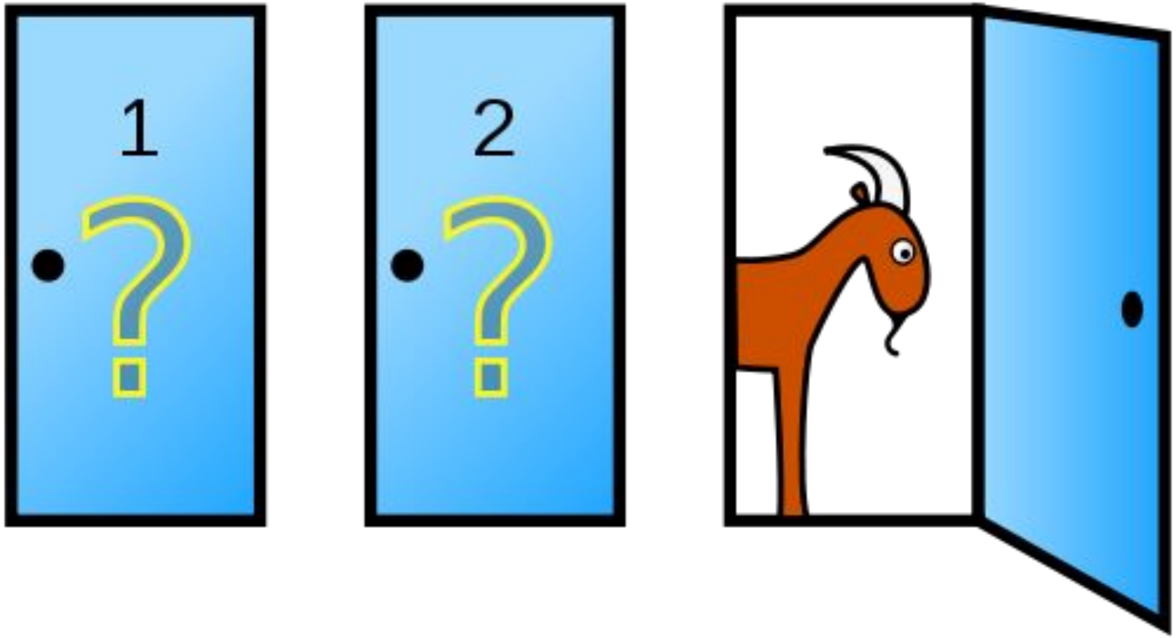


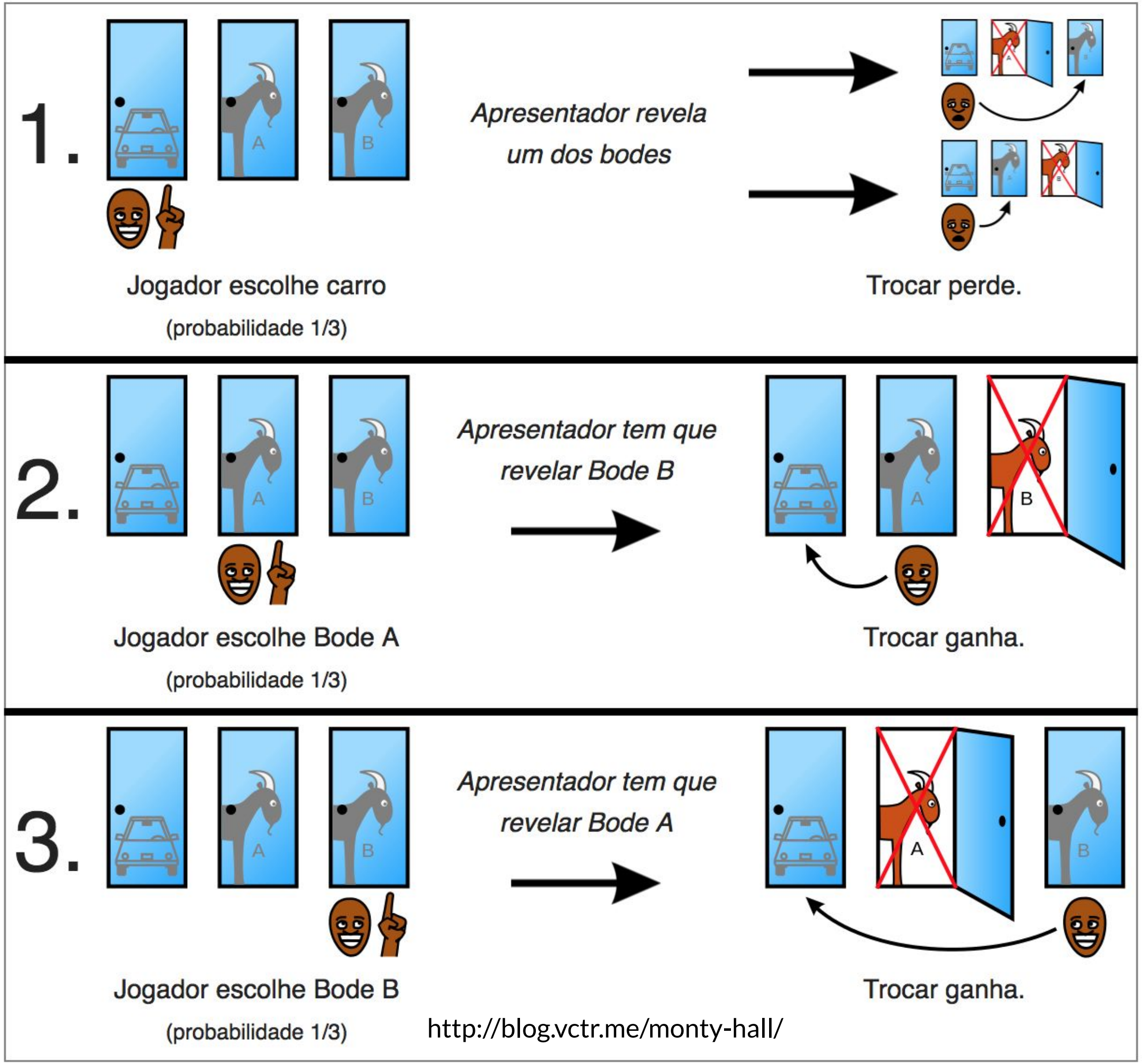
Conditional Probability

How likely is it to rain tomorrow?

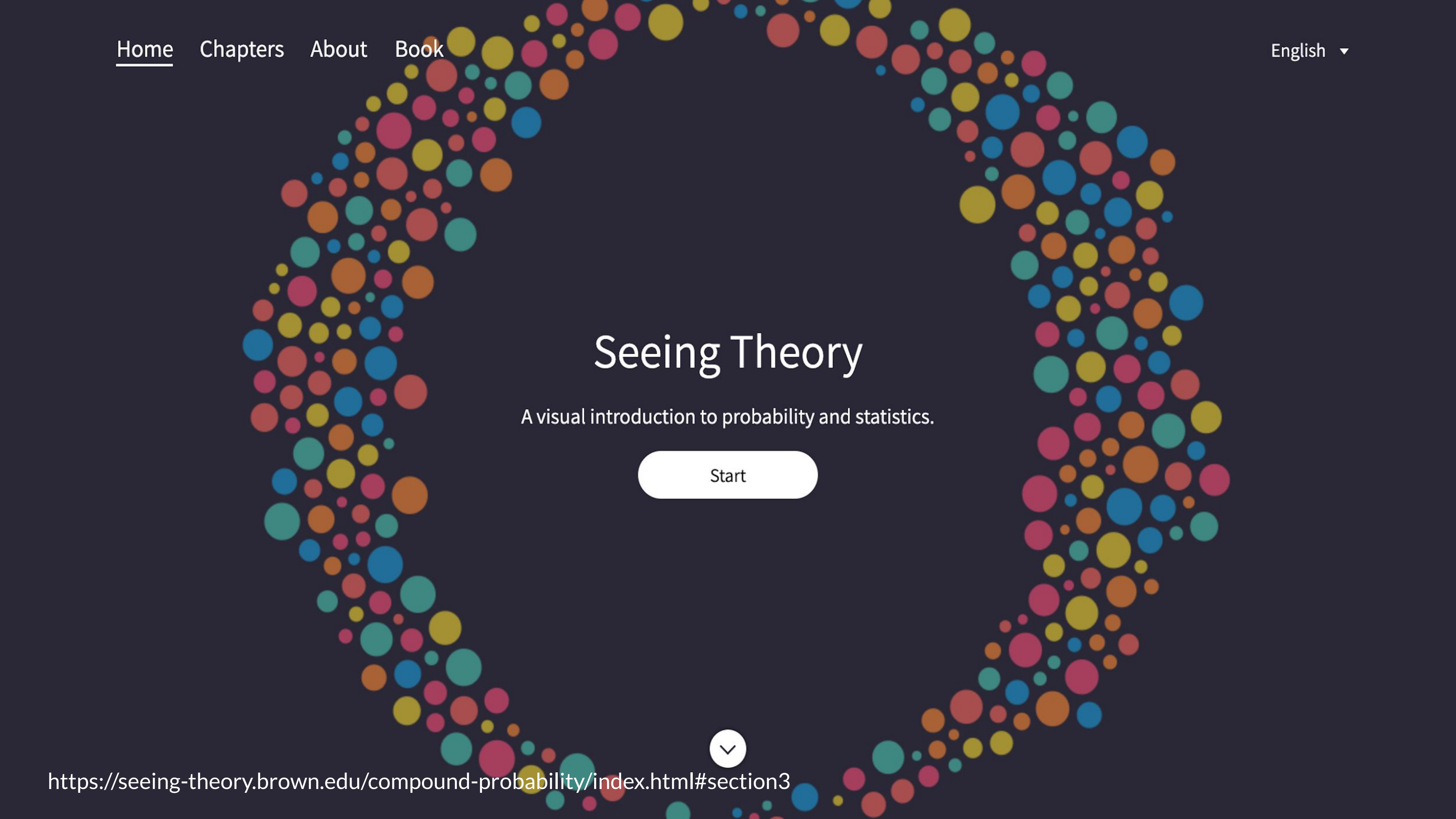
How likely is it to rain tomorrow since the night is cloudy?

Monty Hall Problem



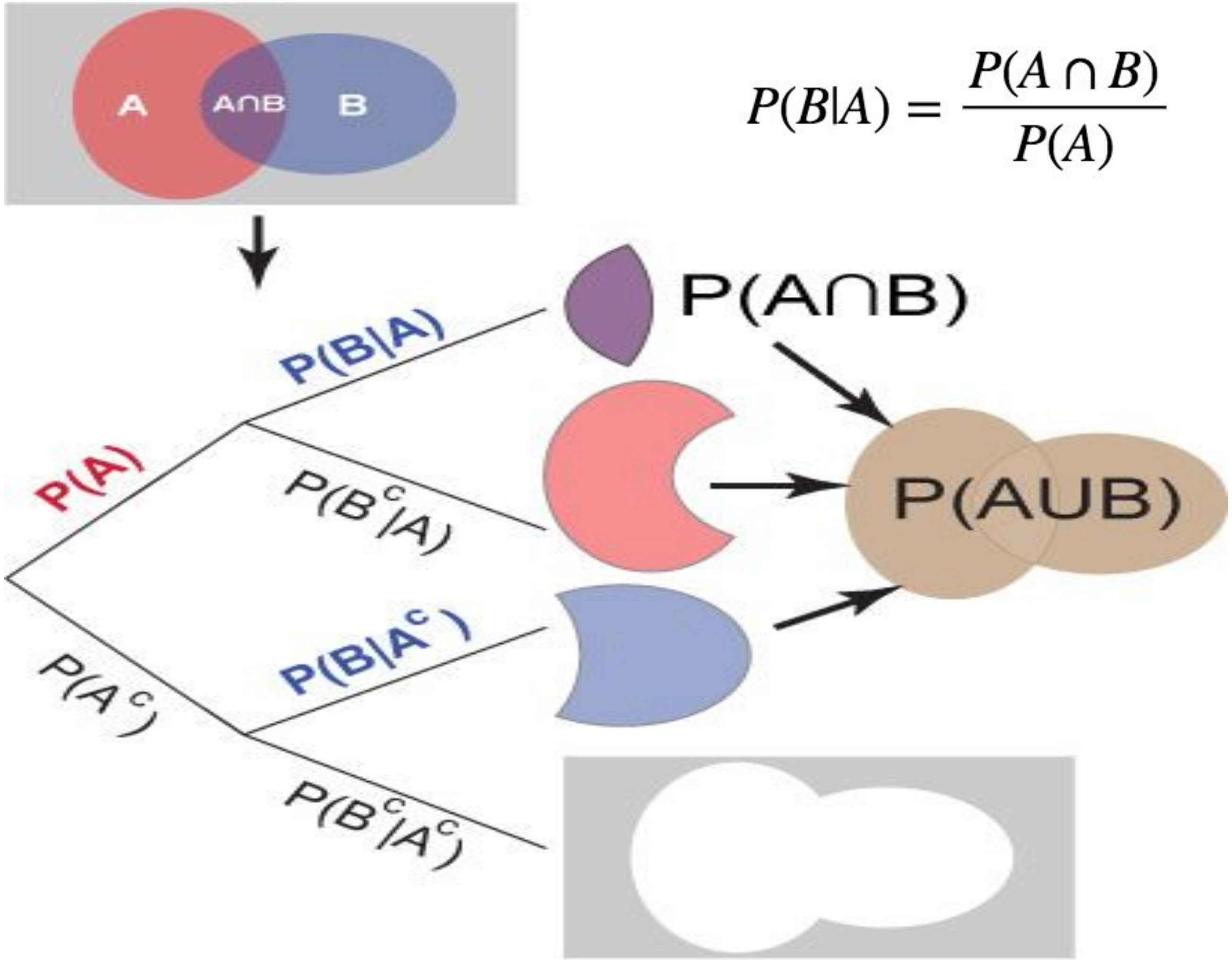


http://blog.vctr.me/monty-hall/



https://seeing-theory.brown.edu/compound-probability/index.html#section3

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$



Bayes' Rule or Bayes' Theorem

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A|B) = \frac{P(B \cap A)}{P(B)}$$

From the data

Prior Probability

Posterior probability

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Normalization

What if P(A) and P(B) were independent?

$$P(B|A) = P(B)$$

$$P(A|B) = P(A)$$

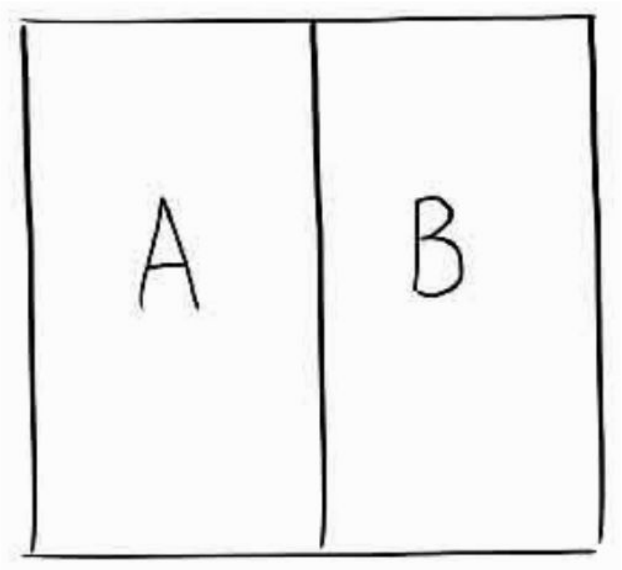
Bayes's Theorem Explained

- Bayes' Theorem helps us update a belief based on new evidence by creating a new belief.
- Bayes' Theorem helps us revise a probability when given new evidence.
- Bayes' Theorem helps us change our beliefs about a probability based on new evidence.
- Bayes' Theorem helps us update a hypothesis based on new evidence.”

Bayes's Theorem: here are few examples

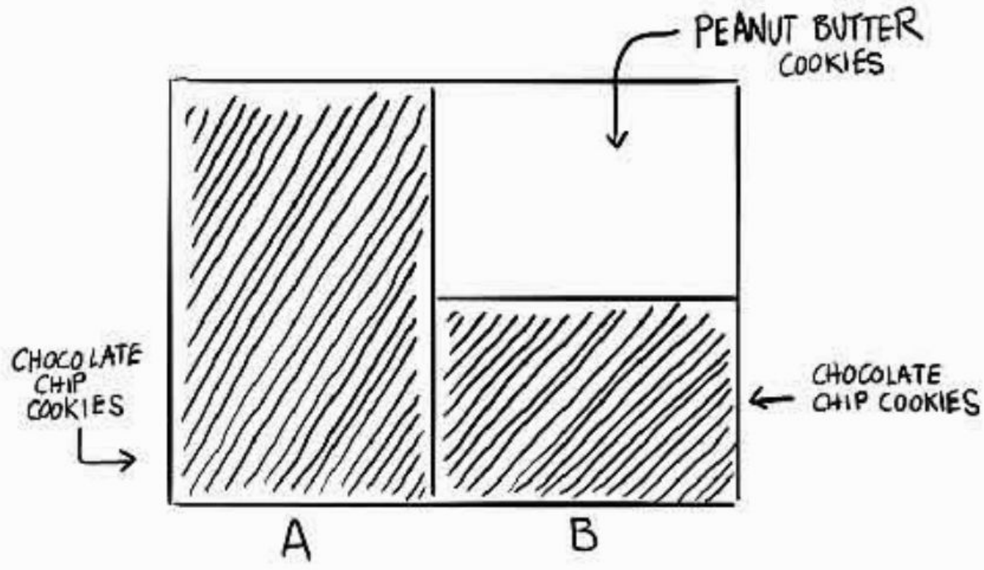
- You just had a test for cancer and it came back positive. What is the probability that you have cancer if the test is positive?
- Your friend claims that stock prices will decrease if interest rates increase. What is the probability stock prices will decrease if interest rates increase?
- You were just pulled over by the police and given a breathalyzer test. It came back positive. What is the probability you are truly drunk, given that the test is positive?

Bayes's Theorem: here are few examples



Box A is filled with 10 chocolate chip cookies
Box B is an even mix of 5 peanut butter cookies and 5 chocolate chip cookies

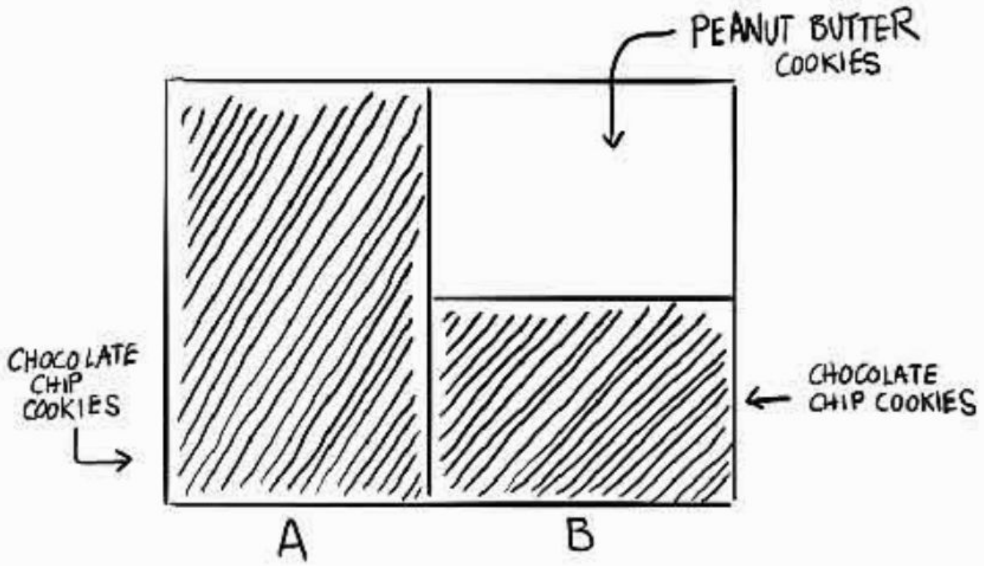
Bayes's Theorem: here are few examples



Now, what if you were to close your eyes and have both boxes shuffled, and then reach and select a cookie from one of them - and it was chocolate chip?

If you had to guess what box the cookie came from, what box would you select?

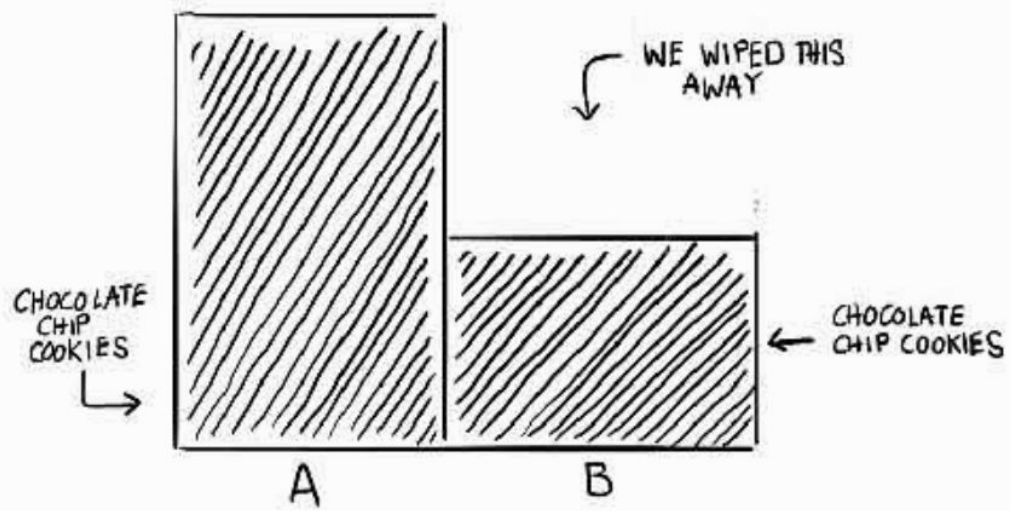
Bayes's Theorem: here are few examples



This calculation is a very basic, natural use of Bayes' Theorem.

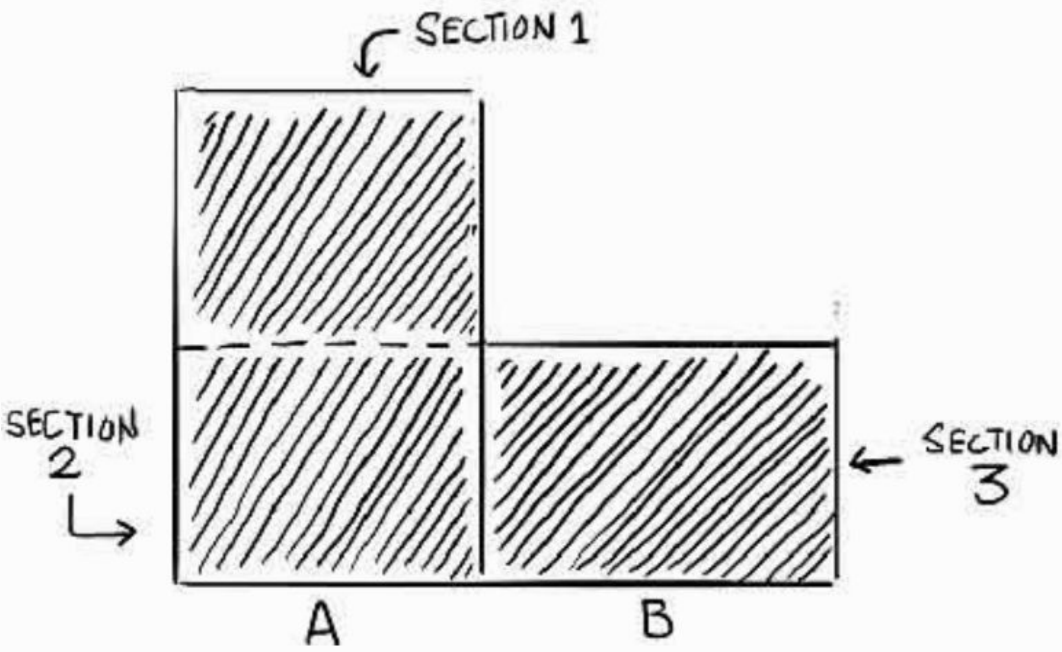
Given evidence (the amount and type of cookies in each box), you were able to quickly come to the conclusion that Box A has a greater probability of being selected than Box B.

Bayes's Theorem: here are few examples



When your hand selected a chocolate chip cookie, something disappeared: the probability of selecting a peanut butter cookie is now gone.

Bayes's Theorem: here are few examples



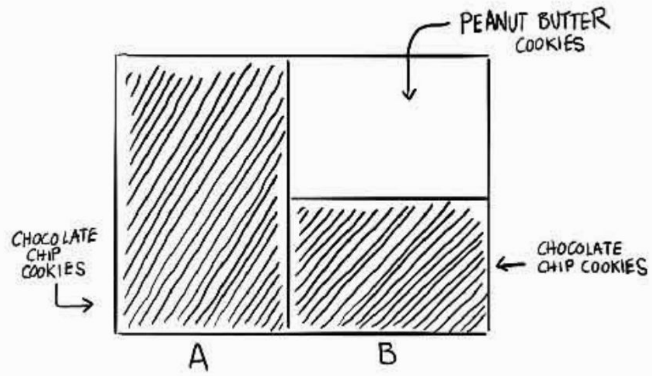
By looking at this, we can see that Box B has a probability of ⅓ , or ~33% of being selected.

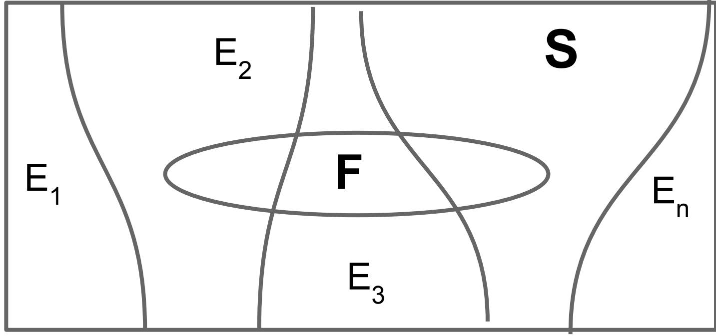
Box A has a probability of ⅔, ~66% of being selected.

Bayes's Theorem: here are few examples

$$P(\text{BOX A} \mid \text{CC COOKIE}) = \frac{P(\text{CC COOKIE} \mid \text{BOX A})\,P(\text{BOX A})}{P(\text{CC COOKIE})}$$

$$P(\text{BOX A} \mid \text{CC COOKIE}) = \frac{1 \times .5}{.75} = \frac{.5}{.75} = 0.6\overline{6} = 66\%$$

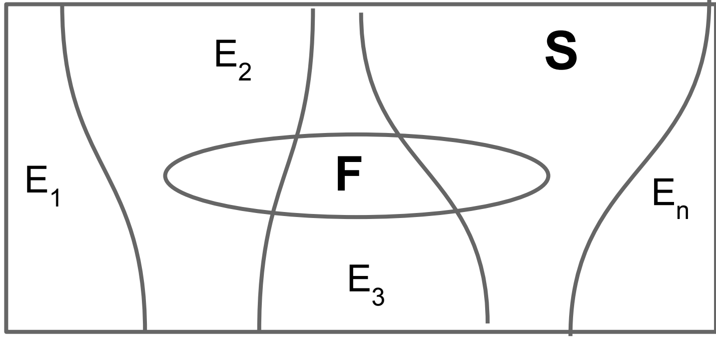




Law of total probability

Let F be an event of the sample space S, and E1, E2, ..., En a partition of S, we can calculate the probability of F as follows:

$$\Pr(F) = \sum_{i=0}^{n} \Pr(F \cap E_i) = \sum_{i=0}^{n} \Pr(F \mid E_i) \cdot \Pr(E_i)$$



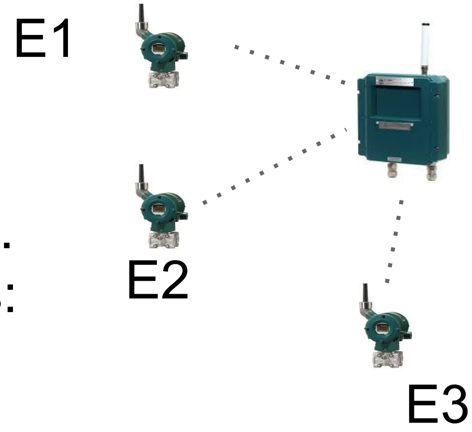
Complete Bayes's Rule

The Law of Total Probability describes that:

$$\Pr(F) = \sum_{i=0}^{n} \Pr(F \cap E_i) = \sum_{i=0}^{n} \Pr(F \mid E_i) \cdot \Pr(E_i)$$

$$\Pr(E_j \mid F) = \frac{\Pr(F \cap E_j)}{\Pr(F)} = \frac{\Pr(F \cap E_j)}{\sum_{i=0}^{n} \Pr(F \cap E_i)} = \frac{\Pr(F \mid E_j) \cdot \Pr(E_j)}{\sum_{i=0}^{n} \Pr(F \mid E_i) \cdot \Pr(E_i)}$$

Example

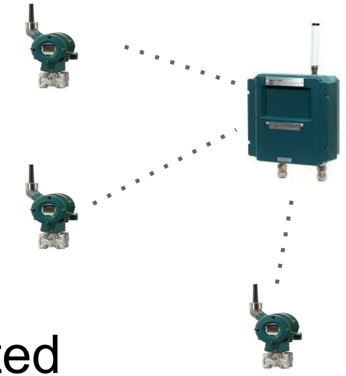


A gateway receives data from 3 devices (E1, E2, E3).
The proportion of packets generated by equipment is:
E1 = 20%, E2 = 30% and E3 = 50%.

The fraction of corrupted packets is: E1 = 5%, E2 = 3%, E3 = 1%

If a package is chosen at random, knowing that the package is corrupted, how likely is that the package originated from the E3 device?

Example



- Assume Ei to be the probability of a equipment is to be randomly selected.
- Assume C as the probability of a packet being corrupted

$Pr(E_1) = 0.20 \quad Pr(E_2) = 0.30 \quad Pr(E_3) = 0.50$

The conditional probabilities Pr (C | Ei) have already been given by the question
$Pr(C|E_1) = 0.05 \quad Pr(C|E_2) = 0.03 \quad Pr(C|E_3) = 0.01$

We need to find P(C):
$Pr(C) = \sum_i Pr(C|E_i)\, Pr(E_i) = 0.05x0.20 + 0.03x0.30 + 0.01x0.50 = 0.024 = 2.4\%$

Thus:
$Pr(E_3|C) = Pr(C|E_3) \times Pr(E_3) / Pr(C) = 0.01 \times 0.50 / 0.024 = 0.2083 = 20.83\%$

Data driven example

https://towardsdatascience.com/bayes-rule-applied-75965e4482ff