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On the longitudinal dispersion of passive contaminant in oscillatory flows in tubes

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The paper examines how a passive contaminant disperses along the axis of a tube in which the flow is driven by a longitudinal pressure gradient varying harmonically with time. This problem is of intrinsic interest and is relevant to some important practical problems. Two examples are dispersion in estuaries and in the blood stream. By means both of statistical arguments and an analysis like that used by Taylor (1953) in the case of a steady pressure gradient it is shown that eventually the mean distribution of concentration satisfies a diffusion equation (and is therefore a Gaussian function of distance along the axis) with an effective longitudinal diffusion coefficient $K(t)$ which is a harmonic function of time with a period equal to one half of that of the imposed pressure gradient. Contrary to the supposition made in most previous work on this problem it is shown by examining some special cases that the harmonic terms in $K(t)$ may have a noticeable effect on the dispersion of the contaminant and in particular on the rate at which it is spreading axially. The size of the effect depends on both the frequency and the Schmidt number and is particularly large at low frequencies. The paper concludes with an analysis of a model of dispersion in estuaries which has been used frequently and it is concluded that here too oscillatory effects may often be noticeable.

1. Introduction

When a cloud of dye or other passive contaminant is injected into fluid flowing along a tube or channel it spreads out along the tube in the direction of flow owing to the two influences (i) of longitudinal diffusion and (ii) of the interaction between advection and lateral diffusion. Taylor (1953, 1954) showed that in steady laminar flow, or statistically steady turbulent flow, in a tube of uniform cross-section the result of these processes is that sufficiently long after the injection of dye the distribution of concentration $C(x,y,z,t)$ has a mean over the cross-section $\overline{C}(x,t)$ (where x measures longitudinal distance) which satisfies a diffusion equation of the form

$$\frac{\partial \overline{C}}{\partial t}+\overline{u}\frac{\partial \overline{C}}{\partial x}=K\frac{\partial^2 \overline{C}}{\partial x^2}, \tag{1.1}$$

where $\overline{u}$ is the discharge velocity and K is a constant 'effective longitudinal diffusion coefficient'. Aris (1956) showed that K is the sum of two terms each describing one of the influences mentioned above. In laminar flow with molecular diffusivity κ, K has the form

$$K = \kappa + D, \quad \text{where} \quad D = \gamma \bar{u}^2 a^2/\kappa; \tag{1.2}$$

here a is a length representative of the dimensions of the channel cross-section and γ is a constant depending on the shapes of the channel cross-section and of the velocity profile. Taylor (1953) showed that for Poiseuille flow in a circular tube of radius a the value of γ is $\frac{1}{48}$.

Equation (1.1) holds with increasing accuracy as the ratio of the time after injection to the time taken for a molecule of dye to wander over the cross-section increases, i.e. as $\kappa t/a^2$ increases (Chatwin 1970). For practical purposes (1.1) can usually be assumed correct provided that $\kappa t/a^2$ is of order unity or greater. In turbulent flow the value of K given in (1.2), and the parameter $\kappa t/a^2$ in the above discussion, have to be amended by replacing κ by other variables describing the effects of turbulent and molecular diffusion.

For a cloud of dye the appropriate solution of (1.1) is (Taylor 1953)

$$\bar{C}(x,t) = (4\pi K t)^{-\frac{1}{2}} \exp\{-(x-\bar{u}t)^2/4Kt\}, \tag{1.3}$$

where units of concentration have, for convenience, been chosen so that the integral of $\bar{C}$ over all x is unity†. Thus the cloud of dye spreads out symmetrically about a point moving at the discharge velocity, so that its centre of mass x_g and its variance σ^2 satisfy

$$\left.\begin{aligned} x_g &= \int_{-\infty}^{\infty} x\bar{C}\,dx = \bar{u}t, \\ \sigma^2 &= \int_{-\infty}^{\infty} (x-x_g)^2\,\bar{C}\,dx = 2Kt. \end{aligned}\right\} \tag{1.4}$$

The description that has just been given is of dispersion in a steady flow. However the same physical mechanisms also cause dispersion in unsteady flows. Two such problems which have attracted some attention are the diffusion of substances in homogeneous tidal estuaries (see, for example, Bowden 1965), and the spreading of tracers injected into the blood (see, for example, Caro 1966). The first of these problems will be considered in more detail in §4 of this paper. The second supplied the motivation for work by Watson (1975), whose analysis provided many ideas for the work presented in §§2 and 3 of the present paper.

The primary problem studied in this paper is dispersion in a flow in which the longitudinal pressure gradient $\partial p/\partial x$ satisfies

$$-\rho^{-1}\,\partial p/\partial x = G\cos\omega t, \tag{1.5}$$

† Strictly speaking the variables x and t in (1.3) should be replaced by $x-x_0$ and $t-t_0$, where x_0 and t_0 are constants depending on the initial distribution of concentration and on the flow properties. However (1.3) can then be recovered by a new choice of the origins of space and time and it can be assumed that this has been done. But this point is of little practical importance since calculations (Chatwin 1970) show that $\kappa t_0/a^2$ is normally much less than unity whereas $\kappa t/a^2$ must be greater than unity for (1.1), and thus (1.3), to be valid.

where ρ is the fluid density and G and ω are constants. In order to keep the analysis well founded the flow will be assumed to be laminar, although, if dispersion in steady flows is a reliable guide, the way in which dye disperses in a turbulent flow caused by a mean pressure gradient satisfying (1.5) will be similar in type, if not in scale. In order to compare the results with those described above for steady flows the main aim will be to describe the dispersion a long time after release, when, as will be shown later, the effect of a general unsteady longitudinal pressure gradient can be predicted from the results by means of Fourier analysis. With the exception of some of the results in Holley, Harleman & Fischer (1970), previous work has concentrated on the properties of the dispersion (i.e. parameters like x_g and σ^2) when averaged over the period of the imposed oscillation but it will be shown in this paper that several potentially important oscillatory effects are thereby excluded.

Inferences from statistical arguments

It is well known that $\bar{C}(x,t)$ can be interpreted as the probability density function of the longitudinal displacement of a molecule of dye. Let the longitudinal displacement and velocity of a molecule of dye at time t be $X(t)$ and $U(t)$, so that, assuming that the molecule is released at $t = 0$,

$$X(t) = \int_0^t U(t')\,dt'. \tag{1.6}$$

Because of Brownian motion both $X(t)$ and $U(t)$ are random functions of time. Now when t is large the integral in (1.6) can be written as the sum of a large number of integrals over a small fixed interval, and a generally accepted form of the central limit theorem (which seems to be valid except for some pathological random functions which do not arise in dispersion problems) shows that the probability density function of X, and hence $\bar{C}$, approaches a Gaussian form (Batchelor & Townsend 1956). This means that $\bar{C}$ has the form [cf. (1.3)]

$$\bar{C}(x,t) = (2\pi\sigma^2)^{-\frac{1}{2}}\exp\{-(x-x_g)^2/2\sigma^2\}, \tag{1.7}$$

where, using angular brackets to denote a probability average,

$$\left.\begin{aligned} x_g(t) &= \langle X(t)\rangle = \int_{-\infty}^{\infty} x\bar{C}\,dx, \\ \sigma^2(t) &= \langle\{X(t)-\langle X(t)\rangle\}^2\rangle = \int_{-\infty}^{\infty}(x-x_g)^2\,\bar{C}\,dx. \end{aligned}\right\} \tag{1.8}$$

This argument does not rely on the detailed statistical properties of $U(t)$, but these do affect the behaviour of x_g and σ^2 with time. Since the operations of taking a probability average and integrating commute it follows from (1.6) and (1.8) that

$$x_g(t) = \int_0^t \langle U(t')\rangle\,dt'. \tag{1.9}$$

When $U(t)$ is a stationary random function of time $\langle U(t')\rangle$ is equal to the discharge velocity $\bar{u}$ (Batchelor, Binnie & Phillips 1955), so that (1.9) is consistent with (1.4). In a flow caused by a pressure gradient of the form (1.5), $U(t)$ is not a stationary random function of time but it seems obvious, since molecules of dye wander over the cross-section and are eventually indistinguishable dynamically from molecules of the ambient fluid, that $\langle U(t')\rangle$ is eventually equal to the average over the cross-section of the longitudinal fluid velocity at time t'. This is certainly a harmonic function of t' with the same period as the imposed pressure gradient, and will be denoted by $\overline{u}(t')$, the overbar notation being consistent with its use earlier. Thus in both the steady and the oscillatory flow (1.9) becomes

$$x_g(t) = \int_0^t \overline{u}(t')\,dt'.$$

From the definition of σ^2 in (1.8), it follows that (Taylor 1921)

$$\frac{d\sigma^2}{dt} = 2\int_0^t \langle\{U(t)-\langle U(t)\rangle\}\{U(t')-\langle U(t')\rangle\}\rangle\,dt'. \tag{1.10}$$

In the case when $U(t)$ is a stationary random function of time the integrand in (1.10) is a function only of $t-t'$, say $R(t-t')$. Thus assuming that the integral converges it follows that, for large t,

$$\frac{d\sigma^2}{dt} \approx 2\int_0^\infty R(\xi)\,d\xi,$$

which is consistent with (1.4) provided K is identified as the integral of $R(\xi)$ from zero to infinity. However when $U(t)$ is not a stationary random function of time the integrand in (1.10) is a function both of t and t' separately and it is difficult to proceed further with certainty. Since the longitudinal fluid velocity is harmonic in time and since the integrand in (1.10) is quadratic in $U(t)$, it can be reasonably conjectured that the right-hand side of (1.10) contains terms proportional to $\cos^2\omega t$, $\sin^2\omega t$ and $\cos\omega t\sin\omega t$, or equivalently terms which are constant, proportional to $\cos 2\omega t$ and proportional to $\sin 2\omega t$.

From (1.7) it follows by direct differentiation that

$$\frac{\partial\overline{C}}{\partial t} + \overline{u}(t)\frac{\partial\overline{C}}{\partial x} = K(t)\frac{\partial^2\overline{C}}{\partial x^2}, \tag{1.11}$$

where $$\overline{u}(t) = dx_g/dt, \quad K(t) = \tfrac{1}{2}\,d\sigma^2/dt. \tag{1.12}$$

Thus the statistical arguments lead, in the steady case, to the results (1.1), (1.3) and (1.4) (Batchelor & Townsend 1956). In the oscillatory case considered in this paper they suggest that the only difference is that $\overline{u}$ and K are periodic functions of time, with periods $2\pi/\omega$ and π/ω respectively. Thus, according to (1.7) and (1.12), the evolution in time of the Gaussian profile of $\overline{C}$ as a function of x in the oscillatory case is different from its evolution in the steady case.

2. General theory

In this section the conjectures made at the end of § 1 will be supported by an analysis similar to that used by Taylor (1953) to establish (1.1) in the steady case, and by Watson (1975) in the present case. This analysis has the advantage that it gives explicit expressions for $\bar{u}(t)$ and $K(t)$ in (1.12), which is not otherwise possible because of ignorance about the detailed statistical behaviour of $U(t)$.

Under the imposed pressure gradient (1.5) the velocity of the fluid in laminar flow in a straight tube of uniform cross-section is in the longitudinal direction when any entry effects have died away. Let the value of this velocity be $u(y,z,t)$, where y and z measure position in the cross-section. From the Navier–Stokes equations

$$\frac{\partial u}{\partial t} = G\cos\omega t + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{2.1}$$

with

$$u = 0 \quad \text{on the tube boundary.} \tag{2.2}$$

The equation governing the distribution of concentration $C(x,y,z,t)$ is

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = \kappa\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right), \tag{2.3}$$

where, with impermeable walls,

$$\partial C/\partial n = 0 \quad \text{on the tube boundary.} \tag{2.4}$$

By taking the mean of (2.3) over the cross-section and using (2.4), an equation is obtained for $\bar{C}$:

$$\frac{\partial \bar{C}}{\partial t} + \frac{\partial}{\partial x}(\overline{uC}) = \kappa\frac{\partial^2 \bar{C}}{\partial x^2}, \tag{2.5}$$

where (as in § 1 and throughout this paper) an overbar over a quantity denotes its mean over the cross-section.

The aim of the analysis which follows is to obtain an expression for $\overline{uC}$ in terms of $\bar{u}$ and $\bar{C}$ which can be substituted into (2.5). Now (2.3) has an exact solution of the form

$$C = \alpha x + \alpha f(y,z,t), \tag{2.6}$$

where α is a constant†. This solution is exactly linear in x whereas a long time after release the cloud of dye will be spread over a distance much greater than a and C will therefore be approximately linear in x in the neighbourhood of any point. Hence a good approximation to the value of $\overline{uC}$ ought to be given by calculating it from the exact solution (2.6) (Taylor 1953).

Define non-dimensional variables by

$$T = \omega t, \quad Y = y/a, \quad Z = z/a. \tag{2.7}$$

Then u and C have the forms (using (2.6) for C)

$$\left.\begin{aligned} u &= (G/\omega)\,\mathscr{R}[V(Y,Z)\,e^{-iT}], \\ C &= \alpha x + (\alpha G/\omega^2)\,\mathscr{R}[H(Y,Z)\,e^{-iT}]. \end{aligned}\right\} \tag{2.8}$$

† Note that a solution of the form (2.6) exists however u depends on time and therefore for any unsteady longitudinal pressure gradient. Thus, as stated earlier following (1.5), the dispersion a long time after release in a flow driven by an arbitrary unsteady pressure gradient can be described using Fourier analysis and the results of this paper.

Using the governing equations and boundary conditions for u and C gives the following equations for V and H:

$$\nabla^2 V + \beta^2 V = i\beta^2; \quad V = 0 \text{ on the tube boundary}; \tag{2.9a}$$

$$\nabla^2 H + \eta\beta^2 H = -i\beta^2\eta V; \quad \partial H/\partial n = 0 \text{ on the tube boundary}; \tag{2.9b}$$

where
$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right), \quad \beta^2 = \frac{i\omega a^2}{\nu} \quad (\mathscr{R}\beta > 0), \quad \eta = \frac{\nu}{\kappa}. \tag{2.10}$$

These equations are due to Watson (1975), who gives solutions for some special cross-sections (there are some trivial changes in notation).

Using (2.8) and noting that

$$\alpha x = \bar{C} - \left(\frac{G}{\omega^2}\right)\left(\frac{\partial \bar{C}}{\partial x}\right)\mathscr{R}[\bar{H}\,e^{-iT}],$$

it follows after some manipulation that

$$\overline{uC} = \bar{u}\bar{C} - \left(\frac{G^2}{2\omega^3}\right)\left(\frac{\partial \bar{C}}{\partial x}\right)\mathscr{R}[\overline{V^*(\bar{H}-H)} + \overline{V(\bar{H}-H)}\,e^{-2iT}], \tag{2.11}$$

where an asterisk denotes a complex conjugate, and $\bar{u} = \bar{u}(t)$ is the mean of u, given by (2.8), over the cross-section, this notation being consistent with that introduced in § 1 and used in (1.12). Explicitly

$$\bar{u}(t) = (G/\omega)\,\mathscr{R}[\bar{V}\,e^{-i\omega t}]. \tag{2.12}$$

It is useful to denote by D the coefficient of $-\partial\bar{C}/\partial x$ that would be obtained in (2.11) were the flow steady and driven by the pressure gradient for which $-\rho^{-1}\,\partial p/\partial x = G$. (This usage of D is of course consistent with its usage in (1.2).) Then substituting (2.11) into (2.5) gives the diffusion equation (1.11) with $\bar{u}(t)$ given by (2.12) and $K(t)$ satisfying

$$K(t) = \kappa + D[A_0 + A_1\cos 2\omega t + A_2 \sin 2\omega t], \tag{2.13}$$

where

$$A_0 = (G^2/2\omega^3 D)\,\mathscr{R}[\overline{V^*(\bar{H}-H)}], \quad A_1 + iA_2 = (G^2/2\omega^3 D)\,[\overline{V(\bar{H}-H)}]. \tag{2.14}$$

The value of A_0 is in effect given by Watson (1975). In the steady case $\bar{u}$ is proportional to Ga^2/ν, so that, using (1.2),

$$G^2/2\omega^3 D = \Gamma/\eta\,|\beta|^6, \tag{2.15}$$

where Γ is a constant depending only on the shape of the tube cross-section. Hence the relations (2.14) become

$$A_0 = (\Gamma/\eta\,|\beta|^6)\,\mathscr{R}[\overline{V^*(\bar{H}-H)}], \quad A_1 + iA_2 = (\Gamma/\eta\,|\beta|^6)\,[\overline{V(\bar{H}-H)}]. \tag{2.16}$$

Thus the analysis in this section has supported the conjectures made in § 1 and in particular the eventual validity of the diffusion equation for $\bar{C}$, (1.11). In the steady case this diffusion equation describes $\bar{C}$ for times greater than a^2/κ, the time taken for a molecule of dye to wander over the cross-section, and it seems reasonable that this should also be a sufficient condition in oscillatory flows since the transverse mixing is not directly affected by a change in the longitudinal pressure gradient. This conclusion is supported by calculations of the integral moments of C in the way described by Aris (1956) but these will not be presented here. Hence (1.11) will be assumed valid provided

$$\kappa t/a^2 \gtrsim 1. \tag{2.17}$$

Furthermore, when (2.17) holds, the appropriate solution of (1.11) is the Gaussian curve (1.7) (Chatwin 1972). In this Gaussian curve, according to (1.12), (2.12) and (2.13),

$$\left.\begin{aligned} x_g &\approx \text{constant} + (G/\omega^2)\,\mathscr{R}[i\bar{V}\,e^{-i\omega t}], \\ d\sigma^2/dt &\approx 2\kappa + 2D[A_0 + A_1\cos 2\omega t + A_2 \sin 2\omega t], \\ \sigma^2 &\approx \text{constant} + 2\kappa t + (D/\omega)[2A_0\omega t + A_1 \sin 2\omega t - A_2\cos 2\omega t], \end{aligned}\right\} \tag{2.18}$$

where the constants in the expressions for x_g and σ^2 depend both on the initial distribution of C and on the shape of the tube cross-section.

3. The effect of frequency and Schmidt number on the dispersion

As explained in § 1, previous work has mostly assumed the eventual validity of the diffusion equation (1.11) but with the important difference from the present work that $\bar{u}(t)$ and $K(t)$ have been replaced by their time averages, which from (2.12) and (2.13) are zero and $\kappa + DA_0$ respectively. It is difficult to see the merit of this procedure since the profile of $\bar{C}$ is Gaussian in both cases but it is only the time-dependent theory of the present paper which correctly predicts the variation in x_g and σ^2 with time.

For very large times the value of σ given by (2.18) becomes indistinguishable from that given by the time-averaged procedure and is much greater than the amplitude of the motion of the centre of the cloud. Thus the time-averaged procedure is asymptotically correct as $t \to \infty$. But under certain circumstances noticeable oscillatory effects not predicted by the time-averaged procedure occur for intermediate times. Here attention will be confined to the behaviour of $d\sigma^2/dt$ and σ^2 with time. As (2.18) shows, the overall increase in time of the longitudinal extent of the cloud is modulated by terms harmonic in time whose amplitudes may be such that the cloud appears to be periodically expanding and contracting, with the secular terms having little apparent effect over intervals of the order of one period. Even if the amplitudes of the harmonic terms are not large enough for σ^2 to behave in this way the modulation may still be readily visible (so that $d\sigma^2/dt$ is noticeably oscillatory) if its frequency is sufficiently large. From (2.17) and (2.18) it can be seen that there is a range of values of t for which the theory of the present paper is valid and for which the oscillatory terms in the expressions for $d\sigma^2/dt$ and σ^2 make noticeable contributions of the sort just described provided the following conditions hold:

$$d\sigma^2/dt \text{ noticeably oscillatory if } N_1 = (A_1^2 + A_2^2)^{\frac{1}{2}}/A_0 \geqslant O(1); \tag{3.1}$$

$$\sigma^2 \text{ noticeably oscillatory if } N_2 = \frac{(A_1^2 + A_2^2)^{\frac{1}{2}}}{A_0\eta\,|\beta|^2} \geqslant O(1). \tag{3.2}$$

As is evident from the analysis of § 2 the values of A_0, A_1 and A_2, and thus of N_1 and N_2, depend only on $|\beta|$, η and the shape of the cross-section of the tube. Now $|\beta|^2 = \omega a^2/\nu = (a^2/\nu):(1/\omega)$, so that $|\beta|^2$ is a measure of the ratio of the time taken for viscosity to smooth out transverse variations in vorticity to the period of the imposed oscillation. The Schmidt number η is a measure of the ratio of the intensities of viscous diffusion and molecular diffusion, so that $\eta\,|\beta|^2$ is a measure of the ratio of the time taken for transverse variations in concentration to be smoothed out by molecular diffusion to the period of the imposed oscillation.

The values of $|\beta|$ in the principal mammalian arteries range from about 1 in the aorta of the mouse to about 50 in the aorta of the elephant (McDonald 1960, p. 90), and the Schmidt number is of order 10^3. However the theory developed in this paper may not have great significance in these arteries because their lengths are so short that blood passes right through them before thorough transverse mixing has occurred, i.e. before (2.17) is satisfied (Lighthill 1966). On the other hand in capillaries and arterioles where the flows are slow enough for (2.17) to be satisfied blood does not behave as a Newtonian fluid. It should be noted that the effect of both curvature and turbulence, present in many blood vessels, is to enhance lateral mixing and so reduce both $K(t)$ and the time before the longitudinal dispersion can be described by a diffusion equation. See the discussion in Erdogan & Chatwin (1967).

In estuaries where the flows are turbulent the theory developed in this paper can be quantitatively applied, but then only empirically, when ν and κ are taken as an eddy viscosity and eddy diffusivity respectively. Thus the effective value of η is of order unity and the value of $|\beta|$ based on the depth is normally less than unity (Holley *et al.* 1970). However the value of $|\beta|$ based on the width is usually large and, as in the almost steady flows in canals and rivers, the shape of the estuarine cross-section is very important when calculating the value of $K(t)$ (Fischer 1966, 1972).

Thus, although there are difficulties in applying the theory of this paper to give precise predictions in important practical problems, it does appear that values of $|\beta|$ from very small to very large do occur and it is intrinsically interesting to see how the properties of the dispersing cloud vary as the values of $|\beta|$ and η vary.

The case of low frequency

For frequencies which are such that $|\beta| = (\omega a^2/\nu)^{\frac{1}{2}}$ and $\eta^{\frac{1}{2}}\,|\beta| = (\omega a^2/\kappa)^{\frac{1}{2}}$ are both small the times taken for transverse variations in both vorticity and concentration to be smoothed out by viscosity and molecular diffusivity respectively are much less than the period of the imposed pressure gradient. Thus the time derivatives in the equations (2.1) and (2.3) for u and C are much less than the other terms, so that u and C are unaffected by the fact that the pressure gradient is *changing* with time. Hence V and H in (2.8) are such that (2.8) become

$$\left.\begin{aligned} u &\approx u_0(Y,Z)\cos\omega t,\\ C &\approx \alpha x+\alpha C_0(Y,Z)\cos\omega t,\end{aligned}\right\} \tag{3.3}$$

where u_0 and $\alpha x+\alpha C_0$ are the values of u and C that would be obtained with a steady pressure gradient satisfying $-\rho^{-1}\partial p/\partial x = G$. Thus (2.11) becomes, by virtue of the definition of D in (2.13),

$$\overline{uC} \approx \bar{u}\bar{C} - D\cos^2\omega t(\partial\bar{C}/\partial x).$$

It follows that (2.13) becomes

$$K(t) = \kappa + D[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\omega t], \tag{3.4}$$

so that (in tubes of any cross-section)

$$A_0 = A_1 = \tfrac{1}{2}, \quad A_2 = 0. \tag{3.5}$$

The result that $A_0 = \frac{1}{2}$ is equivalent to that given by Bowden (1965) for a slightly different situation which will be discussed further in §4. Note that, according to (3.4), $K(t)$ correctly tends to the steady value $\kappa + D$ [see (1.2)] when $\omega \to 0$ for fixed t, and that, for all values of t such that $\kappa t/a^2 \gtrsim 1$, $d\sigma^2/dt$ is noticeably oscillatory. Also, according to (3.2) there is a range of values of t for which the oscillatory terms in the expression (2.18) for σ^2 make a noticeable contribution.

The case of high frequency

When the frequency is such that $|\beta| = (\omega a^2/\nu)^{\frac{1}{2}}$ and $\eta^{\frac{1}{2}}|\beta| = (\omega a^2/\kappa)^{\frac{1}{2}}$ are both large then, in contrast to the case considered above, the time derivatives in the equations (2.1) and (2.3) for u and C are much greater than the diffusion terms, so that the velocity and concentration vary very little in the transverse direction except near the tube boundary, where thin boundary layers (layers of thicknesses of order $(\nu/\omega)^{\frac{1}{2}}$ and $(\kappa/\omega)^{\frac{1}{2}}$ for velocity and concentration respectively) are formed in which the uniform values of u and C in the core change rapidly so that the boundary conditions (2.2) and (2.4) can be satisfied. From (2.9) it follows that the core values of V and H are i and 1 respectively, so that

$$V \approx i + V_1, \quad H \approx 1 + H_1, \tag{3.6}$$

where V_1 and H_1 are negligible away from the boundary. Watson (1975) has determined V_1 and H_1 by means of a boundary-layer analysis and on substitution of (3.6) into (2.16) these values lead to

$$\left.\begin{aligned} A_0 &= \left(\frac{\Gamma l}{S\sqrt{2}}\right)\left(\frac{1}{|\beta|^7}\right)\frac{1}{\eta^{\frac{1}{2}}(\eta^{\frac{1}{2}}+1)(\eta+1)}, \\ A_1 = -A_2 &= \left(\frac{\Gamma l}{2S\sqrt{2}}\right)\left(\frac{1}{|\beta|^7}\right)\frac{(\eta^{\frac{1}{2}}+2)}{\eta^{\frac{1}{2}}(\eta^{\frac{1}{2}}+1)^2}, \end{aligned}\right\} \tag{3.7}$$

where la and Sa^2 are respectively the length of the boundary of the cross-section and the area of the cross-section. The value of A_0 in (3.7) is given by Watson (1975).

Since $\Gamma l/S$ is a number nominally of order unity, (3.7) shows that A_0, A_1 and A_2 are numerically very small. All are inversely proportional to the seventh power of the large number $|\beta|$ and, additionally, for the high values of η that are common in liquids, A_0 is proportional to η^{-2} and A_1 and A_2 are proportional to η^{-1}. These low values are a consequence of the flatness of the profiles of velocity

$\lvert\beta\rvert$	$\eta^2 A_0$	ηA_1	ηA_2	N_1	N_2
0·1	$1{\cdot}92\times10^{6}$	$-3{\cdot}00$	$8{\cdot}00\times10^{2}$	$4{\cdot}16\eta\times10^{-4}$	$4{\cdot}16\times10^{-2}$
0·3	$2{\cdot}37\times10^{4}$	$-3{\cdot}00$	$8{\cdot}88\times10^{1}$	$3{\cdot}74\eta\times10^{-3}$	$4{\cdot}16\times10^{-2}$
0·5	$3{\cdot}07\times10^{3}$	$-2{\cdot}99$	$3{\cdot}18\times10^{1}$	$1{\cdot}04\eta\times10^{-3}$	$4{\cdot}16\times10^{-2}$
0·75	$6{\cdot}01\times10^{2}$	$-2{\cdot}94$	$1{\cdot}38\times10^{1}$	$2{\cdot}35\eta\times10^{-2}$	$4{\cdot}16\times10^{-2}$
1·0	$1{\cdot}87\times10^{2}$	$-2{\cdot}82$	$7{\cdot}23$	$4{\cdot}16\eta\times10^{-2}$	$4{\cdot}16\times10^{-2}$
1·3	$6{\cdot}22\times10^{1}$	$-2{\cdot}52$	$3{\cdot}55$	$7{\cdot}00\eta\times10^{-2}$	$4{\cdot}14\times10^{-2}$
1·7	$1{\cdot}86\times10^{1}$	$-1{\cdot}86$	$1{\cdot}17$	$1{\cdot}18\eta\times10^{-1}$	$4{\cdot}08\times10^{-2}$
2·0	$8{\cdot}28$	$-1{\cdot}30$	$3{\cdot}25\times10^{-1}$	$1{\cdot}62\eta\times10^{-1}$	$4{\cdot}04\times10^{-2}$
3·0	$7{\cdot}58\times10^{-1}$	$-1{\cdot}91\times10^{-1}$	$-1{\cdot}52\times10^{-1}$	$3{\cdot}22\eta\times10^{-1}$	$3{\cdot}58\times10^{-2}$
5·0	$2{\cdot}38\times10^{-2}$	$2{\cdot}19\times10^{-4}$	$-1{\cdot}18\times10^{-2}$	$4{\cdot}96\eta\times10^{-1}$	$1{\cdot}98\times10^{-2}$
10·0	$2{\cdot}02\times10^{-4}$	$5{\cdot}16\times10^{-5}$	$-1{\cdot}04\times10^{-4}$	$5{\cdot}75\eta\times10^{-1}$	$0{\cdot}58\times10^{-2}$

TABLE 1. Values of constants for the case of dispersion in a tube of circular cross-section of radius a when $\eta^{\frac{1}{2}}\,\lvert\beta\rvert \gg 1$, where η is the Schmidt number and $\lvert\beta\rvert = (\omega a^2/\nu)^{\frac{1}{2}}$.

and concentration across the cross-section, so that even the small amount of lateral diffusion that does take place does not lead to axial spreading of the cloud of contaminant. Indeed for sufficiently large values of $\lvert\beta\rvert$ the value of σ^2 in (2.18) is dominated by $2\kappa t$, just as it would be were there no motion. It is therefore of passing interest only to note that, for large η, $N_1 \approx \eta/\sqrt{2} \gg 1$ but that

$$N_2 \approx (\lvert\beta\rvert^2\sqrt{2})^{-1} \ll 1$$

[see (3.1) and (3.2)]. Thus the part of $d\sigma^2/dt$ arising from the motion is always highly oscillatory, whereas the oscillatory terms in σ^2 are never noticeable.

The case of moderate frequency but high Schmidt number

Values of $\lvert\beta\rvert$ in the arteries of small animals, and in some human arteries, are of order unity, so that neither of the above approximate analyses is useful. However the value of the Schmidt number is very large in blood, so that, although the velocity profile does not have a significant boundary layer, the concentration profile does, since $\eta^{\frac{1}{2}}\,\lvert\beta\rvert \gg 1$ implies that the time taken for lateral mixing is much greater than the period of oscillation. Thus, as in the case of high frequency, the time-derivative in equation (2.3) for C is much greater than the diffusion term except in a thin boundary layer. From (2.9b) it now follows that, except near the walls,

$$H \approx -iV. \tag{3.8}$$

Use of (3.8) with (2.16) leads immediately to

$$A_0 \approx 0, \quad A_1 + iA_2 \approx (i\Gamma/\eta\,\lvert\beta\rvert^6)\,[\overline{V^2} - \overline{V}^2]. \tag{3.9}$$

The result for A_0 reflects the fact that in the bulk of the flow the velocity and concentration are exactly out of phase as shown in (3.8); however the boundary layer does give a smaller-order non-zero contribution to A_0, which is shown in the appendix to be given by

$$A_0 = (\Gamma/\eta^2\,\lvert\beta\rvert^6)\,\mathscr{R}[\overline{V}]. \tag{3.10}$$

From (3.1) and (3.2) it now follows that

$$N_1 = O(\eta), \quad N_2 = O(1), \tag{3.11}$$

suggesting that both $d\sigma^2/dt$ and, to a lesser extent, σ^2 are noticeably oscillatory. However this conclusion is weakened to some extent by calculations for the case of a tube of circular cross-section of radius a, for which (after Watson 1975)

$$V = i\{1 - J_0(\beta R)/J_0(\beta)\}, \quad \Gamma = 1536. \tag{3.12}$$

The results of the calculations are given in table 1 and it is immediately apparent that the constants of proportionality in (3.11) are numerically small essentially because the dominant term in $\mathscr{R}[\bar{V}]$ in (3.10) cancels out in $\overline{V^2} - \bar{V}^2$ in (3.9); this seems likely to happen in tubes of other cross-sections. But the calculations do show nevertheless that for the high values of η common in laminar flows in liquids the value of $d\sigma^2/dt$ is noticeably oscillatory, but the value of σ^2 is not.

4. A model of dispersion in homogeneous estuaries

Several authors (Bowden 1965; Holley & Harleman 1965; Okubo 1967; Holley *et al.* 1970; Fukuoka 1973) have modelled the dispersion of salt and other materials in estuaries by considering the dispersion of a passive contaminant in a two-dimensional open channel in which the longitudinal velocity is prescribed as a harmonic function of time. Although this model ignores the important effects due to variation of flow properties across the estuary (Fischer 1966) and buoyancy (Fischer 1972), its simplicity makes possible the calculation of the effect of flow oscillation alone. Suppose therefore that in the open channel the longitudinal velocity u is given by

$$u = (Uy/h)\cos\omega t, \tag{4.1}$$

where the y co-ordinate is measured vertically upwards with $y = 0$ and $y = h$ being the bottom of the channel and the free surface respectively. Then the constant U is the velocity amplitude at the free surface. The form (4.1) has been chosen for comparison with the work of the authors mentioned above, all of whom considered it; othermore realistic profiles could be dealt with by the methods of this paper. Assume further, with the above authors, that (2.3) holds; then (2.3) has an exact solution of the form [cf. (2.8)]

$$C = \alpha x + (\alpha U/\omega)\mathscr{R}[F(Y)\,e^{-iT}],$$

where Y is now y/h and the other variables are those used previously. Then F satisfies [cf. (2.9)]

$$F'' + \mu^2 F = -i\mu^2 Y \quad \text{with } F' = 0 \text{ at } Y = 0, 1, \tag{4.2}$$

where

$$\mu^2 = i\omega h^2/\kappa. \tag{4.3}$$

Calculations analogous to those described in §2 lead to the form of $K(t)$ given in (2.13), where now

$$D = U^2h^2/120\kappa, \tag{4.4}$$

and

$$A_0 = A_1 = \frac{60\kappa}{\omega h^2}X_1, \quad A_2 = \frac{60\kappa}{\omega h^2}X_2, \quad X_1 + iX_2 = i\left[\frac{1}{\mu^2} + \frac{1}{12} - \frac{2(1-\cos\mu)}{\mu^3\sin\mu}\right]. \tag{4.5}$$

That $K(t)$ is oscillatory in this model of estuary dispersion was pointed out by Holley & Harleman (1965), and the value of A_0 is equivalent to that given by

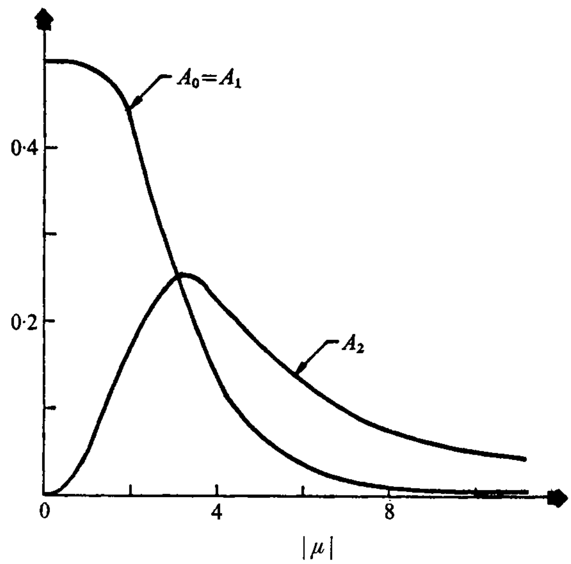

FIGURE 1. Graphs of A_0, A_1 and A_2 given by (4.5), where $|\mu| = (\omega h^2/\kappa)^{\frac{1}{2}}$.

Okubo (1967) and Holley *et al.* (1970), although these authors used different methods and obtained the result in other forms.

For low frequencies when $|\mu| \ll 1$ the arguments in § 3, or direct expansion of (4.5), give $A_0 \approx \frac{1}{2}$ (Bowden 1965), $A_1 \approx \frac{1}{2}$ and $A_2 \approx 0$, so that (3.4) again holds. Now one difference between the model of this section, in which the longitudinal velocity is prescribed, and that of § 2, in which the longitudinal pressure gradient is prescribed, is that here the velocity amplitude is independent of frequency (and viscosity), so that A_0, A_1 and A_2 decay to zero as the frequency increases less rapidly than they do according to (3.7). From (4.5) it follows by direct expansion that, when $|\mu| \gg 1$, $A_0 \approx 60/|\mu|^4$ (Okubo 1967; Holley *et al.* 1970), $A_1 \approx 60/|\mu|^4$ and $A_2 \approx 5/|\mu|^2$. Detailed graphs of A_0, A_1 and A_2 are given in figure 1. From (4.5) it follows that, according to this model, $d\sigma^2/dt$ is always noticeably oscillatory and similar arguments to those in § 3 show that there is a period when σ^2 is noticeably oscillatory provided $|\mu| \lesssim 1$. According to figures in Holley *et al.* (1970) the values of $|\mu|$ in most estuaries are somewhat less than unity.

One feature of the velocity profile in real estuaries is that there is a variation of phase across the cross-section. Such a variation is present in the velocity profiles arising in the earlier sections of this paper and partly explains the differences between the results in those sections and those in the present section. In order to include such an effect here the velocity profile given by (4.1) may be modified such that

$$u(y) = (Uy/h)\cos[\omega t + \phi(y/h)]. \tag{4.6}$$

Provided that ϕ is known calculations of the type described in this section can be carried out. A simple example occurs with

$$\phi(y/h) = \delta y/h, \tag{4.7}$$

where δ is a constant. The detailed results will not be given here because they are rather long and it is not known whether (4.7) is realistic, but it is interesting to note that, for the values of $|\mu|$ typical of real estuaries for which, when $\delta = 0$, $A_0 \approx \frac{1}{2}$, the calculations give

$$A_0 \approx (10/\delta^6)\left[(4\delta^4 - 14\delta^2 - 72) + (2\delta^2 + 72)\cos\delta + (2\delta^2 + 48)\,\delta\sin\delta\right]. \tag{4.8}$$

Numerical evaluation of (4.8) shows that as δ increases from zero the value of A_0 rises from $\frac{1}{2}$ to a maximum value of about 1·06 near $\delta = 5$, after which A_0 falls to zero. For a value of δ near unity (which means that the velocity reverses direction about 2 h sooner at the free surface than at the bed) the increase in A_0 is about 12 %. The increase in A_0 for low and moderate values of δ means an increase in the time-averaged longitudinal diffusion coefficient, which is to be expected because of the increased shear when $\delta \neq 0$.

The results of this section have been presented simply to show that here also the methods used in the main part of the paper lead to the Gaussian form of $\bar{C}$ given by (1.7) with a value of $K(t)$ given by (2.13) and (4.5). Other velocity profiles than (4.1) or (4.6) will lead to the same conclusions. There seems little point in pursuing the model further in view of its obvious limitations.

I wish to thank E. J. Watson for showing me his work on this topic and for discussing it with me. I am also grateful to Professor K. F. Bowden for his help.

Appendix

Here is derived the approximate result (3.10) for A_0 valid for moderate frequency and high Schmidt number with $\eta^{\frac{1}{2}}|\beta| \gg 1$. Note first that on applying Gauss's theorem to the exact equation (2.9) for H and using the boundary conditions it follows that

$$\bar{H} = -i\bar{V}, \tag{A 1}$$

so that from (2.16)

$$A_0 = -(\Gamma/\eta\,|\beta|^6)\,\mathscr{R}[\overline{V^*H}]. \tag{A 2}$$

Furthermore from (2.9) it follows after some obvious manipulation that

$$\nabla.\{V^*\nabla H - H\nabla V^*\} + (\eta+1)\,\beta^2 V^*H = -i\eta\beta^2 VV^* - i\beta^2 H,$$

so that from (A 1), (A 2) and the boundary conditions in (2.9) it can be seen that

$$A_0 = \left(\frac{\Gamma}{\eta(\eta+1)\,|\beta|^6}\right)\mathscr{R}[\bar{V}] - \left(\frac{\Gamma}{\eta(\eta+1)\,|\beta|^6 S}\right)\mathscr{R}\left[\frac{1}{\beta^2}\oint H\frac{\partial V^*}{\partial n}\,ds\right], \tag{A 3}$$

where $\mathbf{n}$ is the outward normal and, as in the text, S is the non-dimensional area of the cross-section. This result is exact.

It is explained in the text that for moderate frequency and high Schmidt number with $\eta^{\frac{1}{2}}|\beta| \gg 1$ the profile of concentration H is approximately equal to $-iV$ everywhere except in a thin boundary layer, so write

$$H = -iV + H_1. \tag{A 4}$$

The equation satisfied by H_1 follows from (2.9) and is

$$\nabla^2 H_1 + \eta\beta^2 H_1 = i\beta^2(i - V), \tag{A 5}$$

with the boundary condition

$$\partial H_1/\partial n = i\,\partial V/\partial n \quad \text{on the tube boundary.} \tag{A 6}$$

The terms on the left-hand side of (A 5) are of the same order in the boundary layer, which therefore has a thickness of order $\eta^{-\frac{1}{2}}|\beta|^{-1}$. Since $\partial V/\partial n$ is of order $|\beta|$ on the boundary it follows from (A 6) that H_1 is of order $\eta^{-\frac{1}{2}}$ in the boundary layer. Since V is zero on the boundary it now follows from (A 4) that H is also of order $\eta^{-\frac{1}{2}}$ in the boundary layer. Thus (A 3) reduces to

$$A_0 = (\Gamma/\eta^2|\beta|^6)\{\mathscr{R}[\overline{V}] + O(\eta^{-\frac{1}{2}}|\beta|^{-1})\},$$

which is equivalent, as required, to (3.10) since $\eta^{-\frac{1}{2}}|\beta|^{-1} \ll 1$ and $\overline{V}$ is independent of η.

It can be verified that when $\eta^{-\frac{1}{2}}|\beta|^{-1} \ll 1$ equations (A 5) and (A 6) lead to a value of H_1 which is of order η^{-1} in the core, so that the boundary-layer assumption used in the above argument is consistent to the degree of approximation required.

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