Acta Mechanica 74, 107—122 (1988)

ACTA MECHANICA

Dispersion of Contaminant in Oscillatory Flows

By

A. Mukherjee and B. S. Mazumder, Calcutta, India

With 5 Figures

(*Received August 14, 1987; revised November 10, 1987*)

Summary

An extension of the Aris-Barton method of moments is presented for the study of statistical behaviour of dispersion of contaminant molecules in oscillatory flows inside uniform conduits. The main idea is to replace the constant pressure gradient which causes the flow simply by the time-dependent one, and then investigate the effects of the amplitude and frequency of the pressure pulsations on the dispersion process. The technique incorporates the case of the time-independent flow also, and gives an exact analysis of the central moments of the distribution of the cloud of contaminant, which are valid for all times after the injection. The general theory is applied to oscillatory laminar flows in tubes, and between parallel plates.

1. Introduction

The phenomenon of dispersion of a passive contaminant in a steady laminar shear flow within a tube was described by Taylor in his classic papers [16], [17]. He observed that the dispersion is due to the molecular diffusion of the contaminant, and the interaction between this diffusion and the velocity along the streamwise direction. These two influences develop respectively the molecular diffusion coefficient D, and the apparent diffusion coefficient D_a. At sufficiently large time the effect of D becomes much smaller than that of D_a, and consequently, the mean concentration of the contaminant over the cross-section becomes eventually a Gaussian distribution. Subsequently Aris [1] studied the dispersion process in terms of the first two central moments of the distribution of the contaminant with respect to the streamwise axis. He showed that the longitudinal effective diffusion coefficient D_e is proportional to $D + P_e^2 D_a$, where P_e denotes the Péclet number. Recently Barton [3] resolved certain technical difficulties in the

Aris method of moments, and obtained the solutions of the second and third moment equations of the distribution of the contaminant, which are valid for all times.

The present paper is a continuation of Barton [3], and its main objective is to extend the Aris-Barton theory for the study of all-time evolution of the second central moment of dispersion of a passive contaminant in an oscillatory flow within a conduit of uniform cross-section under a periodic pressure gradient. More precisely, we study, for all times, the effects of the amplitude and frequency of the pulsations in the pressure gradient on variance and dispersion coefficient, when the Péclet number is large, and when the contaminant is initially uniform over the cross-section. Our method of solution of the Aris moment equation for the oscillatory flow is derived by suitably modifying the treatment of Barton [3]. The method provides an algorithm for the exact analysis of the subsequent central moments; nevertheless, we restrict our attention only to the second central moment for its expediency. The higher moments involve long and tedious computations; moreover, unlike the second central moment, their significance in fluid mechanics is not well understood as yet.

Although the contaminant dispersion in an oscillatory flow undergoes the same physical mechanisms (see Bowden [4] and Caro et al. [5]), it is quite a different phenomenon. For example, in the steady case the apparent diffusion coefficient D_a is manifestly positive and becomes constant after certain (dimensionless) time (see Gill and Sankarasubramanian [8]), whereas in the unsteady case the fluctuations in the velocity profile induce similar effects in D_a which may even be negative for certain values of the parameters in the pressure gradient. As D_a is proportional to one-half of the growth rate of the variance, this means that the concentration of the contaminant may contract and expand periodically. Aris [2] observed by using his method of moments that D_a of a solute in an oscillatory flow contains terms proportional to the square of the amplitude of the pressure pulsations. However his analysis is valid for large time after the discharge of the solute. Chatwin [6] employed an exact solution of the diffusion equation, which is linear in the axial coordinates, to study the dispersion in oscillatory flow. While this assumption of linearity of concentration is adequate at large times, it is unrealistic at earlier times when the dispersion process has not attained its asymptotic state. As observed by him, a remarkable similarity, at least from the theoretical point of view, for both steady and unsteady flows is that the cross-sectionally averaged concentration approaches normality after a sufficiently large time. Other important characteristics of oscillatory laminar flows may be found in Jimenez and Sullivan [10] who studied the growth rate of variance by modifying a probabilistic model proposed earlier by Taylor [15].

In our analysis of the oscillatory problem, we have used the mathematical model suggested by Aris [1], [2], where the diffusion equation involves the constant molecular diffusivity D. The more general diffusion equatio nwith the time-dependent effective diffusivity D_e, as considered by Gill and Sankarasubramanian [9], is inappropriate for our purpose. As remarked by Smith [14] and others, in this model the change of concentration across the flow is not determined by the local streamwise concentration gradient for small values of time. Moreover, the general diffusion equation presents spontaneous singularities because of the fluctuations of D_a from positive to negative values. It may be noted that although Gill and Sankarasubramanian [9] extended the scope of the steady state model to study the dispersion of a non-uniform slug in an oscillatory laminar flow, they confined their analysis only to the case of steady flow. Smith [14] proposed an alternative model equation to avoid the undesirable singularities. He reduced the general diffusion equation (with an extra term of source of strength) to a truncated delay-diffusion equation by replacing the apparent diffusivity by an advected memory function. The resulting equation does not present any singularity at small times after discharge, because then the memory function is always positive. However the model is inadequate at large times as the singularities may recur by the influence of reversed flow at earlier times. Yasuda [18] also suggested a method to escape the negative diffusion coefficient by proposing a new definition of the vertical average of the dispersion. His new dispersion coefficient is always positive during initial stages after discharge, but may be negative at large times when the contaminant is well-mixed vertically, because then there exists little difference between the new diffusion coefficient and the usual one.

The motivation for the study of oscillatory flows stems mainly from two important applications, namely the injection of a chemical substance in pulsatile blood flow, and the discharge of outfalls in tidal estuaries (see Caro et al. [5], Chatwin and Allen [7]). These two flow situations may be realized, as first approximations, in the mathematical models of oscillatory laminar flows in a tube and between parallel plates. We have applied our general principle to these two models for providing the theoretical frameworks for the interpretations of contaminant spreading in the more intricate dispersive systems. It has been assumed in both cases that the fluid is subjected to the combined influence of a mean pressure gradient and an imposed perturbation. This flow condition is more general than that assumed by Chatwin [6], Smith [14], and others. Each of the models has been explored in two specific cases when the Schmidt number S is 1, and when it is 10^3. The Schmidt number lies between 0.2 and 5.0 for the exchange of most gas pairs, and it is very large for diffusion in liquids. For example, for the diffusion of hydrogen in carbon dioxide S is of the order unity, and it is of the order 10^3 for diffusion in blood. The case $S = 1$ is interesting, for then the molecular diffusivity equals the kinematic viscosity. In these studies the contaminant itself may be thought of as a simulation of an accidental spill, or a controlled release of a low-level pollutant quantity into the oscillatory flow.

Our analysis starts in Section 2 with the formulation of the problem, and the solutions are given in Section 3. Then the results are applied to tube flow in Section 4, and to channel flow in Section 5. Finally the conclusions are given in Section 6.

2. Mathematical Formulation

We consider a fully developed unsteady laminar flow in a straight circular tube with uniform cross-section and the axis in the z^*-direction of a cylindrical polar coordinate system (r^*, θ, z^*), where as usual an asterisk denotes a dimensional quantity. We suppose that the flow is along the z^*-direction with velocity $u^*(r^*, t^*)$. Then the dimensionless convection diffusion equation, which describes the local concentration C of an injected contaminant with constant molecular diffusivity D, is given by

$$\partial_t C + u\, \partial_z C = \nabla C + \partial_z{}^2 C, \tag{2.1}$$

where the dimensionless quantities are given by

$$r = r^*/R, \qquad z = z^*/R, \qquad t = Dt^*/R^2, \qquad u = u^*/u_0, \tag{2.2}$$

R being the radius of the tube, and u_0 the reference velocity. Note that ∇ stands for the operator $r^{-1}\, \partial_r(r\, \partial_r)$, and P_e for the Péclet number Ru_0/D. The Péclet number measures the relative characteristic time of the diffusion process (R^2/D) to the convection process (R/u_0). Also note that the angular coordinate θ does not appear in (2.1) because of the radial symmetry. The solution of (2.1) is subjected to the conditions

$$\begin{gathered} C(r, z, 0) = \mathscr{C}(r, z), \\ \partial_r C = 0 \quad \text{at} \quad r = 1, \\ C \text{ is finite at all points}, \\ z^m\, \partial_z{}^m C \to 0 \text{ as } z \to \infty \text{ for all integers } m, n \geqq 0, \\ \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \int_{-\infty}^{\infty} rC\, dr\, d\theta\, dz = 1. \end{gathered} \tag{2.3}$$

As in Aris [1], [2] we define the nth moment of the distribution of the solute in the filament through r at time t

$$C_n(r, t) = \int_{-\infty}^{\infty} z^n C(r, z, t)\, dz, \tag{2.4}$$

and the nth moment of the distribution of the solute over the cross-section of the tube

$$M_n(t) = \overline{C_n} = \frac{1}{\pi} \int_0^{2\pi} d\theta \int_0^1 rC_n(r, t)\, dr. \tag{2.5}$$

Then the equations governing C_n and M_n are given respectively by

$$\partial_t C_n - \nabla C_n = P_e n u C_{n-1} + n(n-1)\, C_{n-2} \tag{2.6}$$

$$C_n(r, 0) = \mathscr{C}_n(r),$$

$$\partial_r C_n = 0 \quad \text{at} \quad r = 1, \tag{2.7}$$

$$C_n \text{ is finite over the cross-section},$$

and

$$\partial_t M_n = P_e n \overline{u C_{n-1}} + n(n-1) M_{n-2}, \tag{2.8}$$

$$M_n(0) = \overline{\mathscr{C}}_n. \tag{2.9}$$

Here (and throughout the paper) the bar over a quantity denotes its mean over the cross-section.

Note that $M_0(t) = 1$ since C has cross-sectional mean unity for all times (see the last condition of (2.3)), and that $M_1(t)$ is the mean $\bar{z}$ of the distribution. Moreover $M_m(t)$, for $m \leqq n$, determine the nth central moment $\nu_n(t)$, which is defined by

$$\nu_n(t) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \int_{-\infty}^{\infty} r(z - \bar{z})^n \, C \, dr \, d\theta \, dz, \tag{2.10}$$

for example

$$\begin{aligned} \nu_2(t) &= M_2(t) - M_1(t)^2, \\ \nu_3(t) &= M_3(t) - 3M_1(t)\,\nu_2(t) - M_1(t)^3, \\ \nu_4(t) &= M_4(t) - 4M_1(t)\,\nu_3(t) - 6M_1(t)^2\,\nu_2(t) - M_1(t)^4, \end{aligned} \tag{2.11}$$

etc.

The aim of the analysis which follows is to obtain explicit analytic expression for $\nu_2(t)$, and lay the foundation for further computations of higher moments.

3. Solutions of Moment Equations

Our starting point is similar to that considered by Barton [3]. We consider the eigenvalue problem

$$(\nabla - \partial_t + \mu_i) f_i = 0, \tag{3.1}$$

$$\partial_r f_i = 0 \quad \text{at} \quad r = 1, \qquad f_i \text{ finite}, \tag{3.2}$$

where i runs over the positive integral values, such that the set of eigenvalues $\{\mu_i\}$ is discrete, and the corresponding eigenfunctions $f_i(r, t)$ satisfy

$$\overline{f_i} = 0, \qquad \overline{f_i f_j} = 0 \quad \text{if} \quad i \neq j, \qquad \overline{f_i f_i} = h_i, \tag{3.3}$$

where h_i is a function of t with $h_i(0) = 1$. We augment the set of eigenfunctions by setting $f_0 = 1$ (with eigenvalue $\mu_0 = 0$) in order to get a complete set.

Then the solution C_0 of (2.6), (2.7) for $n = 0$ is given by

$$C_0 = A_{00} + \sum_i A_{0i} f_i e^{-\mu_i t}, \tag{3.4}$$

where the constants A_{00}, A_{0i} are given in terms of the initial conditions as

$$A_{00} = \overline{\mathscr{C}_0} = 1, \qquad A_{0i} = \overline{\mathscr{C}_0 f_i}. \tag{3.5}$$

Note that $\overline{\mathscr{C}_0} = 1$ by (2.9), since, as noted earlier, $M_0(t) = 1$.

For the solution C_1 we rearrange the Eq. (2.6) for $n = 1$ as

$$\begin{aligned} \partial_t C_1 - \nabla C_1 = P_e(u - \partial_t \lambda_{10}) + P_e \sum_i A_{0i}(u - \partial_t \lambda_{1i}) f_i e^{-\mu_i t} \\ + P_e\, \partial_t \lambda_{10} + P_e \sum_i A_{0i}(\partial_t \lambda_{1i}) f_i e^{-\mu_i t}, \end{aligned} \tag{3.6}$$

where λ_{10}, λ_{1i} are certain functions of t to be determined. Then the solution C_1 is given by

$$\begin{aligned} C_1 = A_{10} + \sum_i A_{1i} f_i e^{-\mu_i t} + P_e \varphi_{10} + P_e \sum_i A_{1i} \varphi_{1i} e^{-\mu_i t} \\ + P_e \lambda_{10} + P_e \sum_i A_{0i} \lambda_{1i} f_i e^{-\mu_i t}, \end{aligned} \tag{3.7}$$

where A_{10}, A_{1i} are constants, and φ_{10}, φ_{1i} are particular solutions of

$$\begin{aligned} (\nabla - \partial_t)\, \varphi_{10} &= -(u - \partial_t \lambda_{10}), \\ (\nabla - \partial_t + \mu_i)\, \varphi_{1i} &= -(u - \partial_t \lambda_{1i})\, f_i, \end{aligned} \tag{3.8}$$

$$\partial_r \varphi_{10} = \partial_r \varphi_{1i} = 0 \quad \text{at} \quad r = 1; \qquad \varphi_{10}, \varphi_{1i} \text{ finite}. \tag{3.9}$$

The integrability condition for each of these inhomogeneous equations is that the right hand side of the ith equation in (3.8) is orthogonal to the eigenfunction f_i of the corresponding homogeneous Eq. (3.1), $i \geqq 0$. Using these conditions, we find

$$\lambda_{10} = \int_0^t \overline{u}\, dt, \tag{3.10}$$

$$\lambda_{1i} = \int_0^t \overline{u f_i f_i} h_i^{-1}\, dt. \tag{3.11}$$

The solutions φ_{10}, φ_{1i} can be expressed as linear combinations of the complete set of eigenfunctions $f_0 = 1$, f_i, $i = 1, 2, \ldots$,

$$\varphi_{10} = a_{00}^1 + \sum_j a_{0j}^1 f_j, \qquad \varphi_{1i} = a_{i0}^1 + \sum_j a_{ij}^1 f_j, \tag{3.12}$$

where the a's are determined by substitution in (3.8) followed by integration with respect to r. It is found that a_{00}^1 and a_{ii}^1 are arbitrary constants and so may be taken to be zero, and the other a's are functions of t

$$a^1_{0j} = e^{-\mu_j t} \int_0^t e^{\mu_j t} \overline{u f_j} h_j^{-1}\, dt, \tag{3.13}$$

$$a^1_{i0} = e^{\mu_i t} \int_0^t e^{-\mu_i t} \overline{u f_i}\, dt, \tag{3.14}$$

$$a^1_{ij} = e^{(\mu_i - \mu_j)t} \int_0^t e^{-(\mu_i - \mu_j)t} \overline{u f_i f_j} h_j^{-1}\, dt. \tag{3.15}$$

Then the constants A_{10}, A_{1i} are evaluated, using the initial condition $C_1(r, 0) = \mathscr{C}_1(r)$, as

$$A_{10} = \overline{\mathscr{C}_1(r)} - P_e \sum_i A_{0i} a^1_{i0}(0) - P_e \lambda_{10}(0), \tag{3.16}$$

$$A_{1i} = \overline{\mathscr{C}_1(r)\, f_i(r, 0)} - P_e a^1_{0i}(0) - P_e \sum_k A_{0k} a^1_{ki}(0) - P_e A_{0i} \lambda_{1i}(0). \tag{3.17}$$

Then substitutions in Eq. (2.8) followed by integrations give

$$M_1 = P_e \int \overline{u}\, dt + P_e \sum_i A_{i1} \int \overline{u f_i} e^{-\mu_i t}\, dt + p_1 \tag{3.18}$$

$$\begin{aligned} M_2 = 2t &+ 2P_e A_{10} \int \overline{u}\, dt + 2P_e \sum_i A_{1i} \int \overline{u f_i} e^{-\mu_i t}\, dt + 2P_e^2 \int \overline{u \varphi_{10}}\, dt \\ &+ 2P_e^2 \sum_i A_{0i} \int \lambda_{1i} \overline{u f_i} e^{-\mu_i t}\, dt + 2P_e^2 \int \lambda_{10} \overline{u}\, dt \\ &+ 2P_e^2 \sum_i A_{0i} \int \overline{u \varphi_{1i}} e^{-\mu_i t}\, dt + p_2. \end{aligned} \tag{3.19}$$

The constants p_i are chosen so that $M_i(0) = 0$, $i = 1, 2$. This is equivalent to taking the initial conditions as $\mathscr{C}_i = 0$. We shall assume further that $\mathscr{C}_0 =$ constant (and this must be 1), in other words the cloud of contaminant is uniform over the cross-section. In this case ν_2 may be found to be

$$\begin{aligned} \nu_2 = 2t &- 2P_e^2 \sum_i a^1_{0i}(0) \int \overline{u f_i} e^{-\mu_i t}\, dt + 2P_e^2 \int \overline{u \varphi_{10}}\, dt \\ &+ 2P_e^2 \sum_i a^1_{0i}(0) \left[\int \overline{u f_i} e^{-\mu_i t}\, dt \right]_{t=0} - 2P_e^2 \left[\int \overline{u \varphi_{10}}\, dt \right]_{t=0}. \end{aligned} \tag{3.20}$$

The corresponding results for the steady flow may be retrieved from the above theory easily enough. In this case the eigenvalue problem (3.1) does not involve the term $\partial_t f_j$. Consequently, the inhomogeneous Eq. (3.8) become independent of t, and therefore we may write λ_{1i} for $\partial_t \lambda_{1i}$, $i \geqq 0$, in these equations. The constants λ_{1i} and a^1_{ij} can then be evaluated simply by using the integrability conditions in the related eigenvalue problem in an obvious way (integrations with respect to t are not necessary here). Our computations show that the expression for ν_2 for steady flow ($\varepsilon = 0$) is in complete agreement with that of Barton {[3, Eq. (3.15)] for $K =$ constant, note that his K is actually our D}.

The expressions for higher central moments may be derived in an analogous way. For example, the third central moment ν_3 can be calculated by the same reasoning from C_2, which can be obtained in terms of particular solutions of certain obvious inhomogeneous equations corresponding to (2.8). The key points to note are that these particular solutions are linear combinations of the complete set of eigenfunctions $\{1, f_i\}$, and the constants are evaluated by using certain integrability conditions and solving certain first order linear equations. We omit the details which are straightforward.

4. Dispersion in Oscillatory Tube Flow

For a radially symmetric unsteady flow in a straight circular tube the Navier-Stokes equations reduce to the following dimensional equation

$$\partial_{t*} u^* = -\varrho^{-1}\, \partial_{z*} p + \nu \nabla^* u^* . \tag{4.1}$$

Here ∇^* is the operator $r^{*-1}\, \partial_r{}^*(r^*\, \partial_r{}^*)$, $u^*(r^*, t^*)$ is the velocity parallel to the axis $r^* = 0$ of the tube, z^* is the axial coordinate, ϱ is the density, and ν is the kinematic viscosity of the fluid. Let the mean pressure gradient P_{z*} be perturbed by a fluctuation in which the longitudinal pressure gradient $\partial_{z*} p$ satisfy

$$-\varrho^{-1}\, \partial_{z*} p = P_{z*}(1 + \varepsilon e^{i\omega t^*}) , \tag{4.2}$$

where εP_{z*} and ω are respectively the amplitude and frequency of the pressure fluctuation. The solution of (4.1) with (4.2) satisfying the no-slip condition ($u^* = 0$ at the boundary of the tube $r^* = R$) was given by Sexl [13] and Schlichting [12] as

$$u(r, t) = (1 - r^2) - \frac{4i\varepsilon}{\alpha} \left[1 - \frac{J_0\left(r\sqrt{-i\alpha}\right)}{J_0\left(\sqrt{-i\alpha}\right)} \right] e^{i\alpha S t} , \tag{4.3}$$

where $u = u^*/u_0$ is the dimensionless axial velocity (u_0 being the time averaged axial velocity $P_{z*}R^2/4\nu$), $\alpha = \omega R^2/\nu$ is the dimensionless frequency parameter, $S = \nu/D$ is the Schmidt number, and J_0 denotes the Bessel function of order zero. The first term on the right hand side is due to the mean pressure gradient, and the second term corresponds to the imposed perturbation. Of course here (and throughout the rest of the paper) the physical significance is attributed only to the real part.

The eigenvalues and corresponding eigenfunctions of the eigenvalue problem (3.1), (3.2) are

$$\mu_j = \alpha_j{}^2(1 - \varepsilon/2) , \tag{4.4}$$

$$f_j = g_j(r)\, e^{-\alpha_j{}^2 \varepsilon t/2} , \tag{4.5}$$

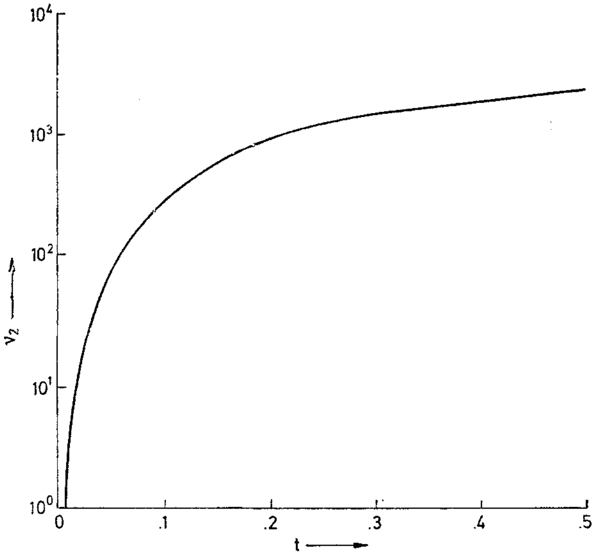

Fig. 1. The change of variance with time for the steady tube flow

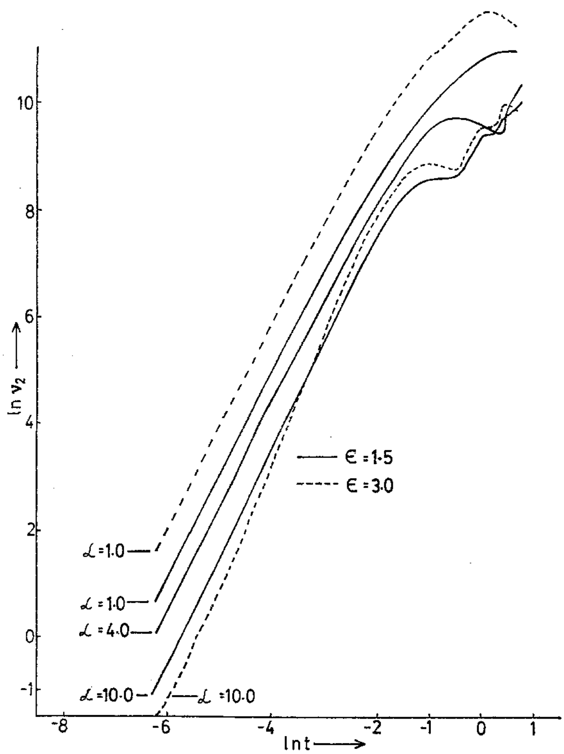

Fig. 2. The change of variance ($\ln \nu_2$) with time ($\ln t$) when $S = 1$ and $P_e = 10^3$ for various values of ε and α

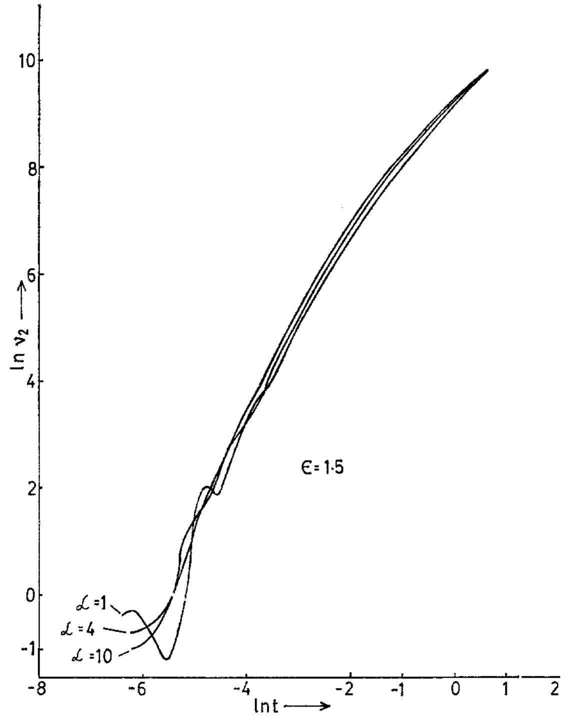

Fig. 3. The plot of $\ln \nu_2$ against $\ln t$ for $S = 10^3$ and $P_e = 10^3$

where α_j, $j = 1, 2, \ldots$, are the roots of the Bessel function of order one J_1, and $g_j(r) = J_0(\alpha_j r)/J_0(\alpha_j)$. It is easily checked that f_j's satisfy (3.3) with $h_j = \overline{f_j f_j} = e^{-\alpha_j^2 \varepsilon t}$. Note that g_j's are the orthonormal eigenfunctions of the eigenvalue problem corresponding to the Poiseuille flow ($\varepsilon = 0$).

Matching (4.3)—(4.5) in the various formulae in Section 3, their explicit expressions can be derived easily. Substituting these results in (3.20) we obtain

$$\begin{aligned} \nu_2 = 2t + 2P_e^2 \sum_j \Big[&(16t/\alpha_j^6) + \Big\{\big(A_j/(\alpha_j^2 + i\alpha S)\big) - 4/\alpha_j^4\Big\} \\ &\times \Big\{\big(4(1 - e^{-\alpha_j^2 t})/\alpha_j^4\big) + \big(A_j(1 - e^{(i\alpha S - \alpha_j^2)t})/(i\alpha S - \alpha_j^2)\Big\} \\ &- \{4i(2\alpha_j^2 + i\alpha S)\, A_j(1 - e^{i\alpha S t})/\alpha\alpha_j^4 S(\alpha_j^2 + i\alpha S)\} \\ &+ iA_j^2(1 - e^{2i\alpha S t})/2\alpha S(\alpha_j^2 + i\alpha S)\Big], \end{aligned} \tag{4.6}$$

where $A_j = -8i\varepsilon \sqrt{-i\alpha}\, J_1\big(\sqrt{-i\alpha}\big)\big/\alpha(\alpha_j^2 + i\alpha)\, J_0\big(\sqrt{-i\alpha}\big)$.

The plot of ν_2 for the steady flow is presented in Fig. 1. In Fig. 2, $\ln \nu_2$ is plotted against $\ln t$ for various values of ε and α, when $S = 1$ and $P_e = 10^3$. By inspection of these plots, we see that, for each values of ε and α, ν_2 increases monotonically up to a certain lapse of time, and then oscillates increasingly. This observation is in general agreement with the results of Smith [14, Fig. 2]

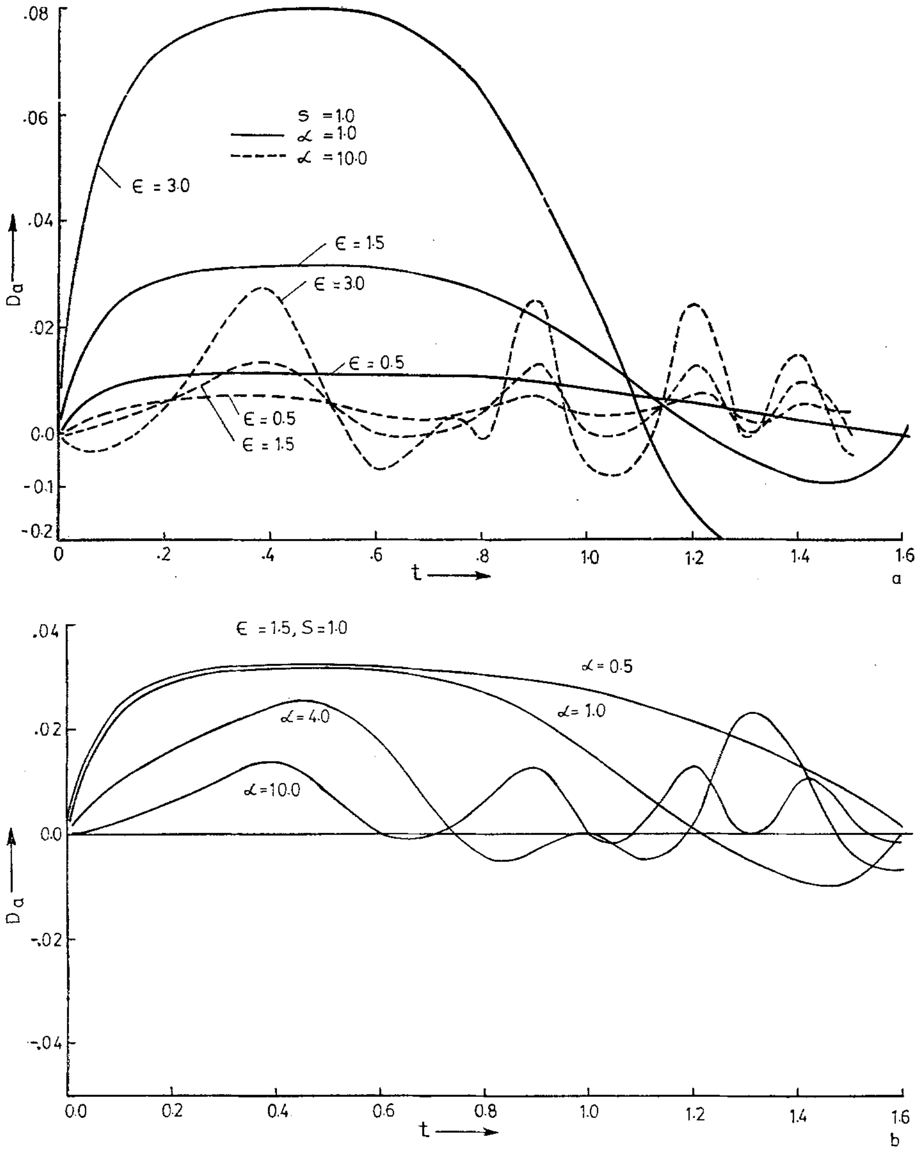

Fig. 4. The variation of dispersion coefficient D_a when $S = 1$,

(*a*) for various ε and α,

(*b*) for $\varepsilon = 1.5$ and various α

and Yasuda [18, Fig. 7 — dotted lines]. Moreover, for a given time, ν_2 decreases with α. Figure 3 shows the similar increasing behaviour of variance when $S = 10^3$ and $P_e = 10^3$.

The rate of growth of variance is given by

$$d\nu_2/dt = 2 + 2P_e^2 D_a, \tag{4.7}$$

where the apparent diffusion coefficient D_a has the expression

$$D_a = \sum_j \left[(16/\alpha_j^6) + \left\{\left(A_j/(\alpha_j^2 + i\alpha S)\right) - 4/\alpha_j^4\right\} \times \{(4/\alpha_j^2) - A_j e^{i\alpha St}\}\, e^{-\alpha_j^2 t} - \left(4(2\alpha_j^2 + i\alpha S)\, A_j e^{i\alpha St}/\alpha_j^4(\alpha_j^2 + i\alpha S)\right) + A_j^2 e^{2i\alpha St}/(\alpha_j^2 + i\alpha S)\right]. \tag{4.8}$$

The variation of D_a with respect to t for various values of ε and α is shown in Figs. 4a, 4b (when $S = 1$), and in Figs. 5a, 5b, 5c (when $S = 10^3$). It is seen from Figs. 4a, 4b that D_a undergoes fluctuation with ε and α. The scale of this fluctuation for low frequency is larger than that of high frequency, and, for a fixed ε, the amplitude of fluctuation reduces as α increases. Therefore, it is seen that the contribution to D_a caused by the reversed flow is negative in a certain interval of time. It is interesting to note that for each ε there is a critical time for which D_a is zero, and the critical time decreases with the increase of ε.

Figures 5a, 5b, 5c show that, for $S = 10^3$, D_a oscillates rapidly for various values of ε and α. For a fixed α, the amplitude of oscillation of D_a increases up to a certain extent of time and then it becomes stationary. Moreover, this oscillatory behaviour is sometimes positive and sometimes negative, but the scale of the oscillation is almost the same for all times. Also the amplitude of oscillation of D_a decreases with α, but increases with ε. It follows then that immediately after the injection of the contaminant, the slug tends to disperse longitudinally at a rate which increases with time. However, after a (dimensionless) time t of order 0.2 to 0.3 the above situation stabilizes and the slug disperses longitudinally at a fairly uniform rate. Another important fact is that if the cross-sectional mixing of the contaminant takes place on a long time scale than the flow oscillation, then there can be certain span of time after each flow reversal during which the contaminant contracts and D_a becomes negative.

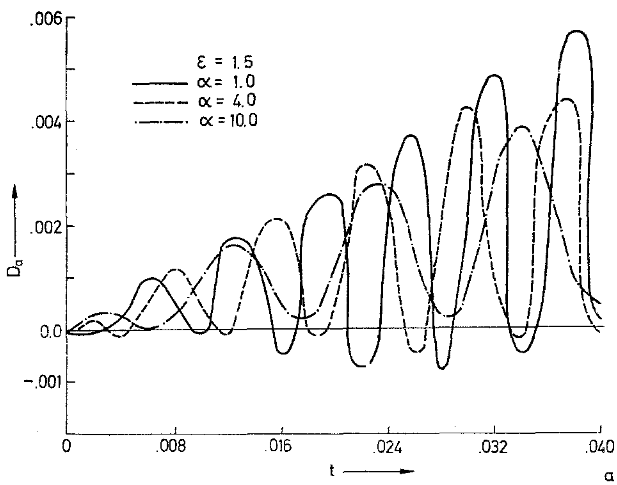

Fig. 5 (a)

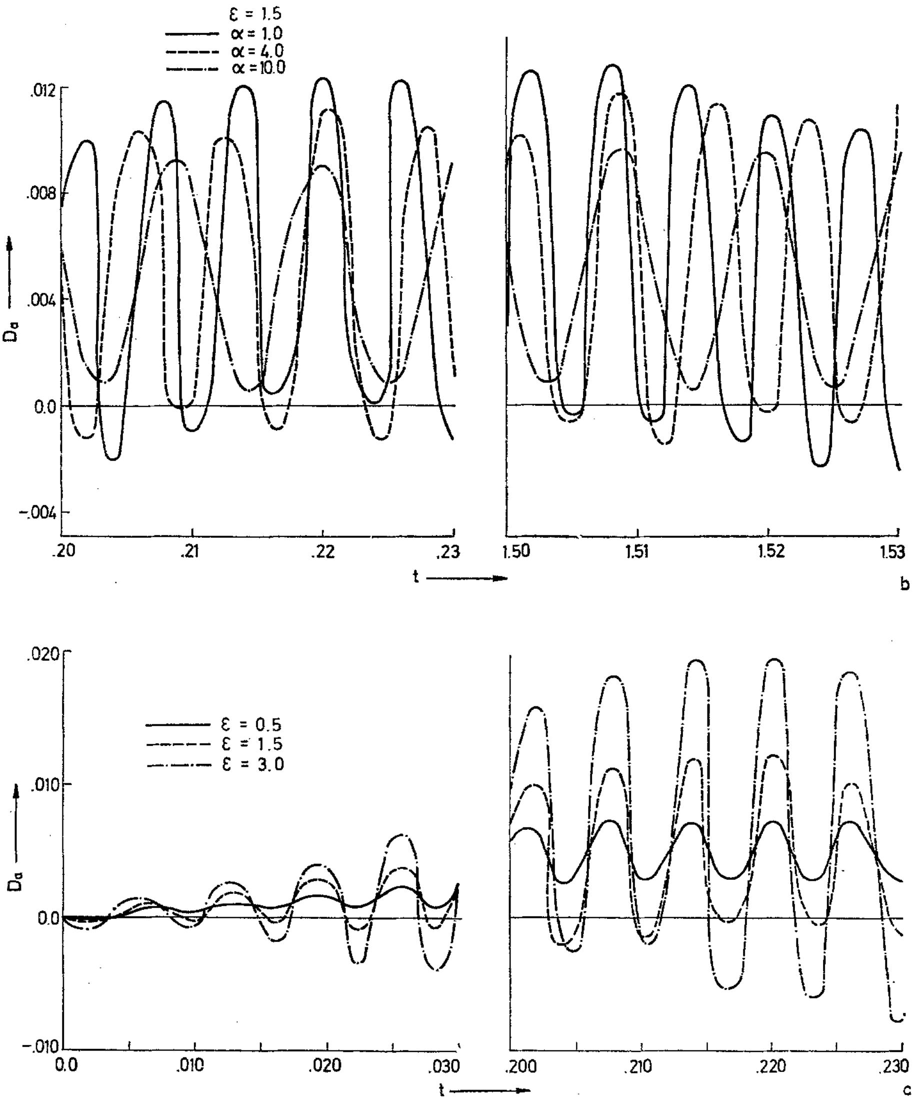

Fig. 5. The temporal changes of dispersion coefficient D_a when $S = 10^3$,

(*a*) for $\varepsilon = 1.5$ (against small time),
(*b*) for $\varepsilon = 1.5$ (against moderate and large time),
(*c*) for $\alpha = 1$ (against all time)

For a large time after release the expression (4.8) reduces to

$$D_a \approx P_0 + P_1 \cos \alpha St + P_2 \sin \alpha St + P_3 \cos 2\alpha St + P_4 \sin 2\alpha St \tag{4.9}$$

where the P's are real constants that depend on ε, α, and S. This result is more general than that obtained by Chatwin [6]. However, if we ignore the first term, or the steady part corresponding to the tube Poiseuille flow, in the velocity profile (4.3), then D_a may be obtained as

$$D_a = \sum_i A_i^2(e^{2i\alpha St} - e^{(i\alpha S - \alpha_i^2)t})/(\alpha_i^2 + i\alpha S) \tag{4.10}$$

which reduces, following a long time after release, to the form

$$D_a \approx P \cos 2\alpha St + Q \sin 2\alpha St \tag{4.11}$$

where the real constants P and Q depend on ε, α, and S. This result is consistent with the work of Chatwin [6].

5. Dispersion in Oscillatory Channel Flow

If a solute diffuses in an unsteady laminar flow between two parallel plates $y = \pm 1$, then the concentration C of the solute satisfies the Eq. (2.1) in cartesian coordinates (x, y, t), with the initial and boundary conditions

$$\begin{aligned} C(x, y, 0) &= \mathscr{C}(x, y) \\ \partial_y C = 0 \quad &\text{at} \quad y = \pm 1. \end{aligned} \tag{5.1}$$

The velocity distribution is obtained from the Navier-Stokes Eqs. (4.1) and (4.2) in cartesian form with no-slip conditions at the boundary $y = \pm 1$ as

$$u = \frac{1}{2}(1 - y^2) - \mathrm{Re}\left[\frac{i\varepsilon}{\alpha}\left(1 - \frac{\cosh\sqrt{i\alpha}\,y}{\cosh\sqrt{i\alpha}}\right) e^{i\alpha St}\right], \tag{5.2}$$

where the first and second term corresponds respectively to the steady and unsteady flow.

In order to obtain the solution, the eigenfunctions and corresponding eigenvalues of the problem (3.1) and (3.2) in cartesian form become

$$f_k = \sqrt{2} \cos k\pi y e^{-\mu_k \varepsilon t} \tag{5.3}$$

$$\mu_k = (k\pi)^2/(1 + \varepsilon), \tag{5.4}$$

where $k = 1, 2, \ldots$

As before, one may get, using the expressions (5.2)—(5.4), the following formula for $d\nu_2/dt$:

$$\begin{aligned} d\nu_2/dt = 2 + 2P_e^2 \Big[\sum_k a_k a_k' &+ e^{2i\alpha St} \sum_k b_k b_k' \\ &+ e^{i\alpha St} \sum_k \{a_k b_k' + a_k' b_k - (a_k' + b_k')\, b_k e^{-k^2\pi^2 t}\} \\ &- \sum_k (a_k' + b_k')\, a_k e^{-k^2\pi^2 t}\Big], \end{aligned} \tag{5.5}$$

where

$$a_k = (-1)^{k+1} \sqrt{2}/(k\pi)^2, \qquad a_k' = a_k/(k\pi)^2$$

$$b_k = (-1)^k i \sqrt{2i}\, \varepsilon \tanh \overline{i\alpha}\Big/\left(\sqrt{\alpha}\,(i\alpha + k^2\pi^2)\right), \qquad b_k' = b_k/(i\alpha S + k^2\pi^2).$$

The corresponding formula for the flow without the steady part in the velocity profile (5.2) may be deduced from (5.5) by putting $a_k = a_k' = 0$. As may be seen, the asymptotic behaviour of the result thus obtained is at per with Chatwin [6].

The expression within the square brackets in (5.5) represents the apparent diffusion coefficient D_a. The plots of D_a against t for various ε and α may be found in Mukherjee and Mazumder [11, Fig. 1a, b] (when $S = 1$) and [11, Fig. 2] (when $S = 10^3$). It may be observed that the effects of the parameters on D_a in this case are similar to that of the tube flow.

6. Conclusions

The method of the present paper is different from those employed earlier, and gives complete results which are valid for all times after the release of the contaminant. However, the comparison of our results agree reasonably with those of the earlier works. For example, the asymptotic behaviour of our second moment agrees fairly consistently with the work of Chatwin [6]. It may be seen from Caro et al. [5], and others that at present very little experimental information exists on our results, nor is the theory of oscillatory flow far enough developed to present them with any confidence.

Acknowledgement

The computations of the paper were carried out in double-precision arithmetic on a EC 1033 computer at the Indian Statistical Institute. The authors are grateful to Professors J. K. Ghosh, and A. S. Gupta for discussion and encouragement.

References

[1] Aris, R.: On the dispersion of a solute in a fluid flowing through a tube. Proc. R. Soc. Lond. A **235**, 67–77 (1956).

[2] Aris, R.: On the dispersion of a solute in pulsating flow through a tube. Proc. R. Soc. Lond. A **259**, 370–376 (1960).

[3] Barton, N. G.: On the method of moments for solute dispersion. J. Fluid Mech. **126**, 205–218 (1983).

[4] Bowden, K. F.: Horizontal mixing in the sea due to shearing current. J. Fluid Mech. **21**, 83–95 (1965).

[5] Caro, C. G., Pedley, T. J., Schroter, R. C., Seed, W. A.: The mechanics of the circulation. Oxford: University Press 1978.

[6] Chatwin, P. C.: On the longitudinal dispersion of passive contaminant in oscillatory flows in tubes. J. Fluid Mech. **71**, 513–527 (1975).
[7] Chatwin, P. C., Allen, C. M.: Mathematical models of dispersion in rivers and estuaries. Ann. Rev. Fluid Mech. **17**, 119–149 (1985).
[8] Gill, W. N., Sankarasubramanian, R.: Exact analysis of unsteady convective diffusions. Proc. R. Soc. Lond. **A 316**, 341–350 (1970).
[9] Gill, W. N., Sankarasubramanian, R.: Dispersion of a nonuniform slug in time-dependent flow. Proc. R. Soc. Lond. **A 322**, 101–117 (1971).
[10] Jimenez, C., Sullivan, P. J.: Contaminant dispersion in some time-dependent laminar flows. J. Fluid Mech. **142**, 57–77 (1984).
[11] Mukherjee, A., Mazumder, B. S.: Diffusion in an oscillatory channel flow. In: Proc. Third Asian Congress Fluid Mech., Sept. 1–5, 1986, Tokyo, Japan; (Matsui, T., ed.), pp. 252–255.
[12] Schlichting, H.: Boundary layer theory. New York: McGraw-Hill 1966.
[13] Sexl, Th.: Über den von E. G. Richardson entdeckten „Annulareffekt". Z. Phys. **61**, 349 (1930).
[14] Smith, R.: Contaminant dispersion in oscillatory flows. J. Fluid Mech. **114**, 379–398 (1982).
[15] Taylor, G. I.: Diffusion by continuous movements. Proc. Lond. Math. Soc. **20**, 196–211 (1921).
[16] Taylor, G. I.: Dispersion of soluble matter in solvent flowing slowly through a tube. Proc. R. Soc. Lond. **A 219**, 186–203 (1953).
[17] Taylor, G. I.: The dispersion of matter in turbulent flow through a pipe. Proc. R. Soc. Lond. **A 223**, 446–468 (1954).
[18] Yasuda, H.: Longitudinal dispersion of matter due to the shear effect of steady and oscillatory currents. J. Fluid Mech. **148**, 383–403 (1984).

A. Mukherjee and B. S. Mazumder
Stat.-Math. Division
Indian Statistical Institute
Calcutta 700 035
India