CSC 5350 Assignment 3

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1. (a)
   i. $N = \{1, 2\}$
   ii. $P(\{R\}) = P(\{G\}) = P(\{\varnothing\}) = 1$
   $P(\{(R, r), (G, r)\}) = P(\{(R, g), (G, g)\}) = 2$
   iii. $\mathcal{I}_1 = \{\{\varnothing\}, \{R\}, \{G\}\}$
   iv. $\mathcal{I}_2 = \{\{(R, r), (G, r)\}, \{(R, g), (G, g)\}\}$

   (b) Yes. For player 1, only one history in all of his information set, do not need to check X_1. For player 2,

$$X_2((R, r)) = X_2((G, r)) = \{(R, r), (G, r)\}$$
$$X_2((R, g)) = X_2((G, g)) = \{(R, g), (G, g)\}$$

   (c) Eight pure strategy for player 1: $Rrr, Rgr, Rrg, Rgg, Grr, Ggr, Grg, Ggg$.

   (d) Four pure strategy for player 2: yy, yn, ny, nn.

   (e) The game can be modeled as Figure 1.

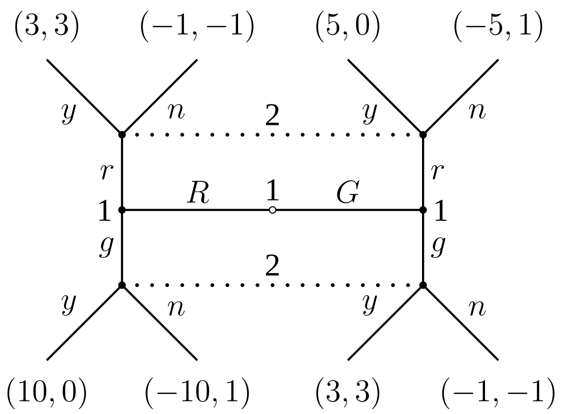

Figure 1: Illustration

Suppose player 2 believes that the probability player 1 chooses red box is α, and green box $1 - \alpha$, where $\alpha \in (0.5, 1]$.

- If player 1 proposes r, for player 2, the expected payoff to y is 3α and to n is $-\alpha + (1 - \alpha) = 1 - 2\alpha$. Hence player 2 should choose y if $3\alpha \geq 1 - 2\alpha$, i.e. $\alpha \geq 0.2$. Since $\alpha > 0.5 > 0.2$, player 2 will always choose y.
- If player 1 proposes g, for player 2, the expected payoff to y is $3(1 - \alpha)$ and to n is $\alpha - (1 - \alpha) = 2\alpha - 1$. Hence player 2 should choose y if $3(1 - \alpha) \geq 2\alpha - 1$, i.e. $\alpha \leq 0.8$, and he will choose n if $\alpha \geq 0.8$.

(f) The behavioral strategy of player 2 is

$$\begin{aligned} \beta_2(\{(R,r),(G,r)\}) &= (y(0.5), n(0.5)) \\ \beta_2(\{(R,g),(G,g)\}) &= (y(0.5), n(0.5)) \end{aligned}$$

Let the probability player 1 choose red box be p_1. Let the probability player 1 choose to tell the truth be p_t.

The expected payoff of player 1 is

$$0.5(2p_1p_t + 0(1-p_1)(1-p_t) + 0p_1(1-p_t) + 2(1-p_1)p_t) = p_t$$

Hence in order to maximize his payoff, he should choose $p_t = 1$, $p_1 = \alpha$ where $\alpha \in [0, 1]$, i.e. always tell the truth no matter in which color box he puts a ball. The best response β_1 is:

$$\begin{aligned} \beta_1(\{\varnothing\}) &= (R(\alpha), G(1-\alpha)), \alpha \in [0,1] \\ \beta_1(\{R\}) &= (r(1), g(0)) \\ \beta_1(\{G\}) &= (r(0), g(1)) \end{aligned}$$

(g) Let $\beta = (\beta_1, \beta_2)$, and

$$\begin{aligned} \mu = \quad \{ \quad & \{\varnothing\} \mapsto \{\varnothing\}(1) \\ & \{R\} \mapsto \{R\}(1) \\ & \{G\} \mapsto \{G\}(1) \\ & \{(R,r),(G,r)\}) \mapsto ((R,r)(\alpha),(G,r)(1-\alpha)) \\ & \{(R,g),(G,g)\}) \mapsto ((R,g)(\alpha),(G,g)(1-\alpha))\} \end{aligned}$$

Then (β, μ) is a consistent assessment, because we have β is the limit of $\varepsilon \to 0$ for

$$\begin{aligned} \beta_1^\varepsilon(\{\varnothing\}) &= (R(\alpha-\varepsilon), G(1-\alpha+\varepsilon)), \alpha \in [0,1] \\ \beta_1^\varepsilon(\{R\}) &= (r(1-\varepsilon), g(\varepsilon)) \\ \beta_1^\varepsilon(\{G\}) &= (r(\varepsilon), g(1-\varepsilon)) \\ \beta_2^\varepsilon(\{(R,r),(G,r)\}) &= (y(0.5\varepsilon), n(0.5\varepsilon)) \\ \beta_2^\varepsilon(\{(R,g),(G,g)\}) &= (y(0.5\varepsilon), n(0.5\varepsilon)) \end{aligned}$$

and for every ε,

$$\begin{aligned} \mu^\varepsilon = \quad \{ \quad & \{\varnothing\} \mapsto \{\varnothing\}(1) \\ & \{R\} \mapsto \{R\}(1) \\ & \{G\} \mapsto \{G\}(1) \\ & \{(R,r),(G,r)\} \mapsto ((R,r)(\alpha),(G,r)(1-\alpha)) \\ & \{(R,g),(G,g)\} \mapsto ((R,g)(\alpha),(G,g)(1-\alpha))\} \end{aligned}$$

(h) It is not a sequential equilibrium. We verify it for each information set:

- For $\{\varnothing\} \in \mathcal{I}_1$ we have already verified in (f) that β_1 is the best response.
- For $\{R\} \in \mathcal{I}_1$, $O(\beta, \mu|\{R\}) = 1$, let $\beta_1'(\{R\}) = (r(p_r), g(1-p_r))$, we can get $O((\beta_{-1}, \beta_1'), \mu|\{R\}) = p_r$, hence $O(\beta, \mu|\{R\}) \succsim_1 O((\beta_{-1}, \beta_1'), \mu|\{R\})$
- For $\{G\} \in \mathcal{I}_1$, it is similar as previous point.
- For $\{(R,r),(G,r)\} \in \mathcal{I}_2$, let $\beta_2'(\{(R,r),(G,r)\}) = (y(p_y), n(1-p_y))$, we can get $O((\beta_{-2}, \beta_2'), \mu|\{(R,r),(G,r)\}) = (3\alpha-1)p_y+1-\alpha$. The maximum 2 is achieved when $\alpha = p_y = 1$. If we fix p_y to 0.5, then the max is achieved when $\alpha = 1$. But when $\alpha = 1$, the maximum is achieved when $p_y = 1$, hence it is not sequentially rational here.

(i) Since the game has already been to the history (R, g), the subgame reduces to the game with only player 2 whose behavioral strategy is $\beta_2((R,g)) = (y(0.8), n(0.2))$. Hence the outcome of the game is $(Rgy(0.8), Rgn(0.2))$.

2. (a) As in Figure 2.

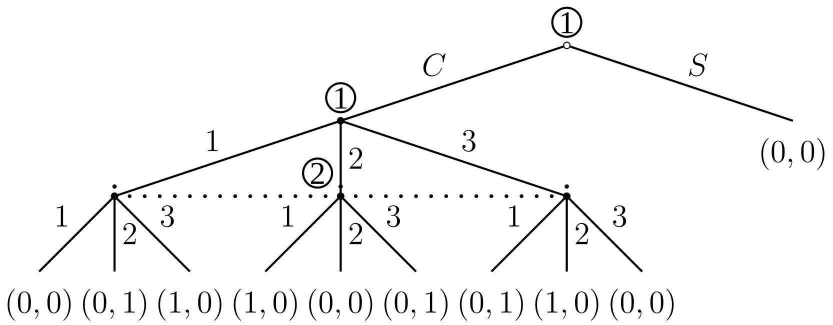

Figure 2: Game tree

(b) We have $\mathcal{I}_1 = \{\{\varnothing\}, \{C\}\}$, $\mathcal{I}_2 = \{\{(C,1),(C,2),(C,3)\}\}$. For player 1, each information set only contains a history. Let $I_2 = \{(C,1),(C,2),(C,3)\}$. For player 2, $X_2((C,1)) = X_2((C,2)) = X_2((C,3)) = I_2$. Hence it is a game with perfect recall.

(c) We have $\beta_2(I_2) = (1(\frac{1}{2}), 2(\frac{1}{8}), 3(\frac{3}{8}))$ in this case. Let:
$\beta_1(\{\varnothing\} = (C(p_c), S(1-p_c)))$,
$\beta_1(\{C\}) = (1(p_1), 2(p_2), 3(p_3))$, and $p_1 + p_2 + p_3 = 1$. Then the expected payoff of player 1 is:

$$\begin{aligned} & p_c p_1(\tfrac{1}{2} \cdot 0 + \tfrac{1}{8} \cdot 0 + \tfrac{3}{8} \cdot 1) + p_c p_2(\tfrac{1}{2} \cdot 1 + \tfrac{1}{8} \cdot 0 + \tfrac{3}{8} \cdot 0) + \\ & p_c p_3(\tfrac{1}{2} \cdot 0 + \tfrac{1}{8} \cdot 1 + \tfrac{3}{8} \cdot 0) + (1-p_c) \cdot 0 \\ = \; & \tfrac{1}{8} p_c(3p_1 + 4p_2 + p_3) \end{aligned}$$

In order to maximize this value, one should set $p_c = 1$ and $p_1 = p_3 = 0, p_2 = 1$, i.e.

$$\begin{aligned} \beta_1(\{\varnothing\} &= (C(1), S(0))) \\ \beta_1(\{C\}) &= (1(0), 2(1), 3(0)) \end{aligned}$$

(d) Let $\mu = \{\{\varnothing\} \mapsto \{\varnothing\}(1), \{C\} \mapsto \{C\}(1), I_2 \mapsto (1(0), 2(1), 3(0))\}$. Then (β, μ) is consistent assessment. We define:

$$\begin{aligned}
\beta_1^\varepsilon(\{\varnothing\}) &= (C(1-\varepsilon), S(\varepsilon)) \\
\beta_1^\varepsilon(\{C\}) &= (1(\varepsilon), 2(1-\varepsilon-\varepsilon^2), 3(\varepsilon^2)) \\
\beta_2^\varepsilon(I_2) &= (1(\tfrac{1}{2}\varepsilon), 2(\tfrac{1}{8}\varepsilon), 3(\tfrac{3}{8}\varepsilon))
\end{aligned}$$

We can always get $\mu^\varepsilon(\{\varnothing\}) = 1$, $\mu^\varepsilon(\{C\}) = 1 - \varepsilon \to 1$, $\mu^\varepsilon(I_2) = (1(\varepsilon), 2(1-\varepsilon-\varepsilon^2), 3(\varepsilon^2)) \to (1(0), 2(1), 3(0))$, for every $\varepsilon \to 0$.

(e) It is not a sequential equilibrium. We verify it for each information set:

- For $\{\varnothing\} \in \mathcal{I}_1$ we have already verified from (d) that β is best response.
- For $\{C\} \in \mathcal{I}_1$, let $\beta_1'(\{C\}) = (1(p_1), 2(p_2), 3(p_3)), p_1 + p_2 + p_3 = 1$. We can get $O((\beta_{-1}, \beta_1'), \mu|\{C\}) = \frac{1}{8}(3p_1 + 4p_2 + p_3)$, hence in order to maximize we set $p_1 = p_3 = 0, p_2 = 1$, $O(\beta_1, \mu|\{C\}) \succsim_1 O((\beta_{-1}, \beta_1'), \mu|\{C\})$.
- For $I_2 \in \mathcal{I}_2$, let $\beta_2'(I_2) = (1(q_1), 2(q_2), 3(q_3)), q_1 + q_2 + q_3 = 1$. We can get $O((\beta_{-2}, \beta_2'), \mu|I_2) = q_3$, set $q_3 = 1$ in order to maximize. However $O(\beta_2, \mu|\{C\}) = \frac{3}{8}$. It is not sequentially rational.

3. (a) i. $N = \{1, 2, ..., 7\}$

ii. For any coalition S, let $g_1 = \{1, 2\}$ and $g_2 = \{3, 4, 5, 6, 7\}$, then

$$v(S) = \min(|S \cap g_1|, |S \cap g_2|)$$

(b) Y_1 is a stable set. Prove:

- Internal stability: Suppose Y_1 does not hold internal stability. Then there must be some $z \in Y_1$ such that there exist an imputations $y \in Y_1$ and S for which $y \succ_S z$. Let $z = (x_1, x_1, y_1, y_1, y_1, y_1, y_1)$ and $y = (x_2, x_2, y_2, y_2, y_2, y_2, y_2)$. First note that $y(S) > 0$ for the coalition S to object z, otherwise $z(S) < y(S) <= v(S) = 0$. In order to let $y(S) > 0$, one in g_1 and one among g_2 must be in S. Then there must be $x_2 > x_1$ and $y_2 > y_1$, hence $2x_2 + 5y_2 > 2x_1 + 5y_1 = 2$. It is impossible because $v(N) = 2$.
- External stability: Without loss of generality, we rearrange player's position such that for an imputation $z \in X \backslash Y_1$, we have $z = (z_i)_{i \in N}$, where $z_1 \leq z_2$, $z_3 \leq ... \leq z_5$ (but there must be at least one strict inequality so that it is not in Y_1), i.e. rearrange g_1, g_2 in non-decreasing order. Hence z_1 and z_3 has the smallest payoff among two groups, respectively. Let A_1 and A_2 be the average payoff among g_1 and g_2, respectively. Then we have $2A_1 + 5A_2 = 2$ or $A_2 = 2(1 - A_1)/5$. Let y be the imputation in Y_1, where $y_1 = y_2 = A_1$ and $y_3 = y_4 = ... = y_7 = A_2$. Then there must be either $y_1 \geq x_1$ or $y_3 \geq x_3$ is strict inequality. If both are strict, then y is indeed an objection to z of $S = \{1, 3\}$. For $y_1 > x_1$ and $y_3 \geq x_3$, because $A_1 \leq 2$ and $y_1 + y_3 = A_1 + A_2 = (3A_1 + 2)/5 \leq 1$. By moving some fraction from y_1, y_2 and averaging to $y_3, ..., y_7$ which yields $y' \in Y_1$, we could make $y_1' > x_1$ and $y_3' > x_3$, then y' is an objection to z. Similarly it is true for the case $y_1 \geq x_1$ and $y_3 > x_3$.

In conclusion, Y_2 is externally stable.

(c) The following set is a stable set.

$$Y_2 = \{(t, t, 1-t, 1-t, 0, 0, 0)\}, \text{ for every } t \in [0, 1]$$

Prove:

- Internal Stability: Since player 5,6,7 always gets 0, it is impossible to find some coalition S to support an objection in Y_2. Now consider for any coalition S formed by players among 1 to 4 and $z \in S$, then $v(S) \leq z(S)$ hence it is impossible to find $y \in S$ such that $y_i > z_i$ for all $i \in S$, i.e. no $y \in Y_2$ can be an objection to $z \in Y_2$ for any coalition S.
- External Stability: For any $z \in X \backslash Y_2$, $\sum_{i=1}^{4} z_i \leq v(N) = 2$, and either one of $z_1 \neq z_2$ or $z_3 \neq z_4$ is true. If $\sum_{i=1}^{4} z_i = 2$, without loss of generality, if $z_1 + z_3 = 1$ then $z_2 + z_4 = 1$, by choosing the smallest from $z_i \in \{z_1, z_2\}$ and $z_j \in \{z_3, z_4\}$ we get $z_i + z_j < 1$ (otherwise $z_1 = z_2, z_3 = z_4$, then $z \in Y_2$), we can find an imputation $y \in Y_2$ where $y_i + y_j = 1$ such that $y_i > z_i, y_j > z_j$, to be an objection via $S = \{z_i, z_j\}$. If $z_1 + z_3 < 1$ then there is also objection $y \in Y_2$ for $S = \{z_1, z_3\}$. If $\sum_{i=1}^{4} z_i < 2$, similarly, also by finding $z_i \in \{z_1, z_2\}$ and $z_j \in \{z_3, z_4\}$ we get $z_i + z_j < 1$, an objection y can be found in Y_2 via $S = \{z_i, z_j\}$.

(d) This corresponds to the ‘standard of behavior’ that two pairs of green card holder and red card holder form a coalition, then implement a division of the profit(=2) they gain by giving player 1 & 2 the same amount t, and player 3 & 4 share the rest equally$(= 1 - t)$, while excluding other green card players(they get 0).

(e) $x = (1, 1, 0, 0, 0, 0, 0)$ is in the core. For player 1 or player 2, they can not increase their payoff by deviating, because for any S where either $1 \in S$ or $2 \in S$, $v(S) \leq 1$. They can not increase their payoff by both deviating and forming another coalition S. Similarly for any $v(S)$ where $1 \in S$ and $2 \in S$, $v(S) \leq 2$. And for players 3-7, they can only form coalition S and $v(S) = 0$. Hence for every coalition S there is $v(S) \leq x(S)$.

— End of Assignment—