TACTICAL PROBLEMS INVOLVING SEVERAL ACTIONS

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§1. INTRODUCTION

A zero-sum, two-person game can be defined by a triplet (X, Y, Ψ) where X and Y are two closed sets, and Ψ is a real-valued, measurable function defined on $X \times Y$; Ψ is called the <u>pay-off</u> or <u>utility function</u>. The elements $\bar{x} \in X$ and $\bar{y} \in Y$ are called <u>pure strategies</u>; the positive measures with total measure 1 defined over X and Y are called (<u>mixed</u>) <u>strategies</u>. The game has a solution if there exist two strategies $F(\bar{x})$ and $G(\bar{y})$ such that

$$\int \Psi(\bar{x}, \bar{y})\, dF(\bar{x}) \geq v, \qquad \text{all} \qquad \bar{y} \in Y$$

and

$$\int \Psi(\bar{x}, \bar{y})\, dG(\bar{y}) \leq v, \qquad \text{all} \qquad \bar{x} \in X .$$

F and G are called <u>optimal strategies</u>; v is the <u>value</u> of the game.

We shall consider a class of games that can be interpreted as tactical problems. Each game will represent a contest between two players who are trying to achieve the same objective. When one of them succeeds, he wins one unit; his opponent loses the same amount, and the contest is over. Each player has limited resources and can make only a fixed number of attempts to reach his goal; these attempts must be made during the interval $0 \leq t \leq 1$, and each attempt may fail or may succeed. At $t = 0$, every attempt fails; at $t = 1$, every attempt succeeds; at any other time t, an attempt made by player I will be successful with probability $P(t)$, and will fail with probability $1 - P(t)$; an attempt made by player II succeeds with probability $Q(t)$, fails with probability $1 - Q(t)$; the functions P and Q increase continuously. Each player knows these functions and the total number of attempts that his opponent can make; however, after the contest begins, each player is unable to find out how many unsuccessful attempts have been made by his opponent. This description can be specialized to a combat between two airplanes: P and Q describe the accuracy of the firing machinery, and the initial resources correspond to the total amount of ammunition that each plane can carry; since it is assumed that each pilot is unable to find out how many times his opponent has fired and missed, this problem is often called a silent duel. In the formal description of the game, $\bar{x}$ and $\bar{y}$ will be vectors that describe the times when the attempts are made; $\Psi(\bar{x}, \bar{y})$ will be the expected gain for player I.

A special silent duel was solved by L. S. Shapley. A class of similar problems (called Games of Timing) has been studied by M. Shiffman [1] and S. Karlin [2]; the problems considered in [2] include a larger class of utility functions, but allow only one action by each player. It should be pointed out that the silent duels are essentially different from the noisy duels considered by D. Blackwell and M. A. Girshick [3]; apparently, the recursive method that they used cannot be adapted for the solution of our problems.

§2. DESCRIPTION OF THE OPTIMAL STRATEGIES

Each game is characterized by two numbers m and n that denote the total number of attempts that each player can make. The solutions are characterized by two sets of numbers $a_1, \dots, a_n$ and $b_1, \dots, b_m$ that depend only on P, Q, m and n. In every optimal strategy, the i^{th} action of player I and the j^{th} action of player II must be carried out during the intervals $[a_i, a_{i+1}]$, $[b_j, b_{j+1}]$; in every game, $a_1 = b_1$, and $a_{n+1} = b_{m+1} = 1$.

The optimal strategies are not necessarily unique, but each player has precisely one in which all attempts are made independently. In this special strategy, player I will make his i^{th} attempt at a time x_i chosen at random by means of a probability distribution $F_i(x_i)$; $F_n(x_n)$ may have a discrete mass α at $x_n = 1$; away from 1, each F_i is absolutely continuous and has a piecewise continuous density. The discontinuities of the densities occur at the points $b_1, \dots, b_m$; precisely,

$$dF_i(x_i) = \begin{cases} h_i\, f^*(x_i)\, dx_i & \text{if} \quad a_i < x_i < a_{i+1} , \\ 0 & \text{if} \quad x_i \notin [a_i, a_{i+1}] , \end{cases}$$

where

$$f^*(t) = \prod_{b_j > t} \left[1 - Q(b_j)\right] \frac{Q'(t)}{Q^2(t)\, P(t)} .$$

The constants h_i and h_{i+1} are related by the equation

$$h_i = [1 - D_i]\, h_{i+1} \quad ,$$

where

$$D_i = \int_{a_i}^{a_{i+1}} P(t)\, dF_i(t) \quad .$$

Player II has a similar strategy that can be described in an analogous way. It is sufficient to interchange the roles of P, Q, a_i and b_j in the previous description.

EXAMPLE. Symmetric Game: $P(t) \equiv Q(t)$, and $n = m$. In this case the two players have the same optimal strategies; $\alpha = 0$, and $a_k = b_k$, $k = 1, \ldots, n$. Furthermore

$$P(a_{n-k}) = \frac{1}{2k+3}, \qquad k = 0, 1, \ldots, n-1$$

$$dF_{n-k}(t) = \frac{1}{4(k+1)} \frac{P'(t)}{P^3(t)}\, dt, \qquad a_{n-k} < t < a_{n-k+1} \quad .$$

§3. DEFINITIONS OF X, Y, P, Q, AND Ψ

1) $X = \{\bar{x} \in E^n \mid 0 \le x_1 \le x_2 \le \cdots \le x_n \le 1\}$

$Y = \{\bar{y} \in E^m \mid 0 \le y_1 \le y_2 \le \cdots \le y_m \le 1\}$.

E^n and E^m denote the n and m-dimensional Euclidean spaces.

2) P and Q will be real-valued, continuously differentiable functions defined on the interval $[0, 1]$. They must satisfy the following conditions:

$$P(0) = 0; \quad P(1) = 1; \quad P'(t) > 0, \quad 0 < t < 1$$

$$Q(0) = 0; \quad Q(1) = 1; \quad Q'(t) > 0, \quad 0 < t < 1 \quad .$$

3) The symbols $\bar{z}$, $r(z_k)$, $s(z_k)$, $\Psi(\bar{z})$ are defined only when $\bar{x} \in X$ and $\bar{y} \in Y$ are two vectors such that $x_i \ne y_j$, all i, j. Then $\bar{z}$ denotes the vector whose components are the numbers $x_1, \ldots, y_m$ re-arranged in increasing order; for $k = 1, \ldots, n+m$, either $z_k = x_i$ (for some i), or $z_k = y_j$ (for some j), but not both. Thus

$$r(z_k) = \begin{cases} P(x_i) & \text{if } z_k = x_i \;, \\ -\,Q(y_j) & \text{if } z_k = y_j \;; \end{cases}$$

and

$$s(z_k) = \begin{cases} P(x_i) & \text{if } z_k = x_i \ , \\ Q(y_j) & \text{if } z_k = y_j \ ; \end{cases}$$

are well defined functions. Finally, $\Psi(\bar{z})$ is defined recursively as follows:

$$\Psi(z_1, \ldots, z_k) = r(z_1) + [1 - s(z_1)] \; \Psi(z_2, \ldots, z_k)$$
$$\Psi(z_2, \ldots, z_k) = r(z_2) + [1 - s(z_2)] \; \Psi(z_3, \ldots, z_k)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Psi(z_k) = r(z_k) \ .$$

The number of components in $\bar{x}$ and $\bar{y}$ is not essential, and the definition can be applied to a single vector $\bar{x} \in X$, or to a single vector $\bar{y} \in Y$. In this case one may say that the other vector has no components.

4) The pay-off function $\Psi(\bar{x}, \bar{y})$ is defined as follows: if each component of $\bar{x}$ is different from each component of $\bar{y}$, then

$$\Psi(\bar{x}, \bar{y}) = \Psi(\bar{z})$$

where $\Psi(\bar{z})$ is the number defined above. If the given condition is not satisfied, then

$$\Psi(\bar{x}, \bar{y}) = \frac{1}{2} \{\Psi(\overline{x + 0}, \overline{y - 0}) + \Psi(\overline{x - 0}, \overline{y + 0})\} \ .$$

§4. SOME PROPERTIES OF $\Psi(\bar{x}, \bar{y})$

$\Psi(\bar{x}, \bar{y})$ is continuous as long as the relative order of the components of $\bar{x}$ and $\bar{y}$ does not change. Furthermore, $\Psi(\bar{x}, \bar{y})$ is skew-symmetric in the following sense: if the roles of X and Y are interchanged, $\Psi(\bar{y}, \bar{x}) = - \Psi(\bar{x}, \bar{y})$. Other simple properties of Ψ are given in the next three lemmas.

LEMMA 1. If $\bar{z} = (z_1, \ldots, z_t, z_{t+1}, \ldots, z_k)$, then

$$\Psi(\bar{z}) = \Psi(z_1, \ldots, z_t) + \prod_{i=1}^{t} [1 - s(z_i)] \; \Psi(z_{t+1}, \ldots, z_k) \ .$$

PROOF. If $t = 1$, the formula reduces to the definition of

$\Psi(\bar{z})$. For $t > 1$, the formula is proved by induction. The details are omitted.

LEMMA 2. If

$$\bar{z} = (z_1, \ldots, z_{t-1}, z_t, z_{t+1}, \ldots, z_k),$$

$$\Psi(\bar{z}) = \Psi(z_1, \ldots, z_{t-1}, z_{t+1}, \ldots, z_k) + \prod_{i=1}^{t-1} [1 - s(z_i)] \{r(z_t) - s(z_t)\, \Psi(z_{t+1}, \ldots, z_k)\} .$$

PROOF. By Lemma 1,

$$\Psi(\bar{z}) = \Psi(z_1, \ldots, z_{t-1}) + \prod_{i=1}^{t-1} [1 - s(z_i)]\, \Psi(z_t, \ldots, z_k) ,$$

and

$$\Psi(z_t, \ldots, z_k) = r(z_t) + [1 - s(z_t)]\, \Psi(z_{t+1}, \ldots, z_k) .$$

Therefore,

$$\Psi(\bar{z}) = \Psi(z_1, \ldots, z_{t-1}) + \prod_{i=1}^{t-1} [1 - s(z_i)]\, \Psi(z_{t+1}, \ldots, z_k)$$

$$+ \prod_{i=1}^{t-1} [1 - s(z_i)] \{r(z_t) - s(z_t)\, \Psi(z_{t+1}, \ldots, z_k)\} .$$

Lemma 1 shows that the first two terms can be replaced by $\Psi(z_1, \ldots, z_{t-1}, z_{t+1}, \ldots, z_k)$.

LEMMA 3. For any fixed $\bar{y}$, $\Psi(\bar{x}, \bar{y})$ is a monotone increasing function of each component x_i of $\bar{x}$ as long as x_i ranges over an open interval that does not contain any components of $\bar{y}$. Similarly, for any fixed $\bar{x}$, $\Psi(\bar{x}, \bar{y})$ is a monotone decreasing function of each component y_j of $\bar{y}$ as long as y_j ranges over an open interval that does not contain any components of $\bar{x}$.

PROOF. Since $\Psi(\bar{x}, \bar{y})$ is skew-symmetric in $\bar{x}$ and $\bar{y}$, it is sufficient to prove only the first half of the lemma. The result is established by means of Lemma 2. Indeed, any component x_i appears only in the term

$$\prod_{k=1}^{i-1} [1 - P(x_k)] \prod_{y_j < x_i} [1 - Q(y_j)] \{P(x_i) - P(x_i) \Psi(\bar{z}*)\} ,$$

where $\bar{z}*$ is the vector constructed from those components of $\bar{x}$ and $\bar{y}$ that are larger than x_i. It is clear that the first two factors are positive; furthermore $-1 \leq \Psi(\bar{z}) \leq 1$, for all $\bar{z}$. Thus the coefficient of $P(x_i)$ is positive, and $\Psi(\bar{x}, \bar{y})$ is a monotone increasing function of x_i.

§5. STRATEGIES OF THE CLASS $\mathcal{O}$

DEFINITION. A strategy $F(\bar{x})$ belongs to the class $\mathcal{O}$ if it satisfies the following conditions:

1) F is separable, i.e.,

$$F(\bar{x}) = \prod_{i=1}^{n} F_i(x_i) .$$

2) Each F_i is a positive measure with total mass 1; furthermore, the support of each F_i (i.e., the complement of the largest open set on which F_i vanishes) is a non-degenerate interval $[a_i, a_{i+1}]$; also, $a_1 > 0$, and $a_{n+1} = 1$.

3) The first $n - 1$ measures are continuous; the last measure F_n may have only one discontinuity with mass α at $x_n = 1$.

Equivalent conditions are used when dealing with measures $G(\bar{y})$ defined over Y.

NOTATION. In the following sections, D_i and $R(\bar{y})$ will always denote the expected values of P and Ψ. That is,

$$D_i = \int_{a_i}^{a_{i+1}} P(t)\, dF_i(t)$$

$$R(\bar{y}) = \int \Psi(\bar{x}, \bar{y})\, dF(\bar{x}) .$$

The vector $\bar{D}^k$ and the function $\phi(\bar{D}^k)$ are defined as follows:

$$\bar{D}^k = (D_{k+1}, D_{k+2}, \ldots, D_n)$$
$$\phi(\bar{D}^{k-1}) = D_k + [1 - D_k]\, \phi(\bar{D}^k)$$
$$\cdots\cdots\cdots\cdots$$
$$\phi(\bar{D}^n) = 0 \quad .$$

Using the last definition, it is easy to see that

$$\int \Psi(x_{k+1}, \ldots, x_n)\, dF_{k+1}(x_{k+1}) \ldots dF_n(x_n) = \phi(\bar{D}^k) \quad .$$

LEMMA 4. Let $F(\bar{x})$ be in the class $\mathcal{O}$, and let $\bar{y} \in Y$ be a vector whose last component y_m is contained in the open interval (a_k, a_{k+1}). Then

$$R(y_1, \ldots, y_{m-1}, y_m) - R(y_1, \ldots, y_{m-1})$$
$$= \prod_{i=1}^{k-1} [1 - D_i] \prod_{j=1}^{m-1} [1 - Q(y_j)]\, [- Q(y_m)] \quad .$$
$$\cdot \left\{ 2 \int_{y_m}^{a_{k+1}} P(x_k)\, dF_k(x_k) + [1 - D_k]\, [1 + \phi(\bar{D}^k)] \right\} \quad .$$

PROOF. Let $\bar{y}$ be the given vector and let $\bar{x} = (x_1, \ldots, x_n)$ be any vector contained in the support of $F(\bar{x})$. Lemma 2 can be used to separate all the terms of $\Psi(\bar{x}, \bar{y})$ that depend on y_m; thus, if $a_k \leq x_k < y_m$,

$$\Psi(\bar{x}, (y_1, \ldots, y_{m-1}, y_m)) - \Psi(\bar{x}, (y_1, \ldots, y_{m-1}))$$
$$= \prod_{i=1}^{k} [1 - P(x_i)] \prod_{j=1}^{m-1} [1 - Q(y_j)]\, [- Q(y_m)]\, [1 + \Psi(x_{k+1}, \ldots, x_n)] \quad .$$

But if $y_m < x_k \leq a_{k+1}$,

$$\Psi(\bar{x}, (y_1, \ldots, y_{m-1}, y_m)) - \Psi(\bar{x}, (y_1, \ldots, y_{m-1}))$$
$$= \prod_{i=1}^{k-1} [1 - P(x_i)] \prod_{j=1}^{m-1} [1 - Q(y_j)]\, [- Q(y_m)]\, [1 + \Psi(x_k, x_{k+1}, \ldots, x_n)] \quad .$$

The vectors $\bar{x}$ with $x_k = y_m$ have F-measure zero. Therefore,

$$R(y_1, \ldots, y_{m-1}, y_m) - R(y_1, \ldots, y_{m-1})$$

$$= \int [\Psi(\bar{x}, (y_1, \ldots, y_{m-1}, y_m)) - \Psi(\bar{x}, (y_1, \ldots, y_{m-1}))] \, dF(\bar{x})$$

$$= \prod_{i=1}^{k-1} [1 - D_i] \prod_{j=1}^{m-1} [1 - Q(y_j)][-Q(y_m)] \left\{ \int_{a_k}^{y_m} [1 - P(x_k)][1 + \emptyset(\bar{D}^k)] \, dF_k(x_k) \right\}$$

$$+ \prod_{i=1}^{k-1} [1 - D_i] \prod_{j=1}^{m-1} [1 - Q(y_j)] \, [- Q(y_m)] \cdot$$

$$\cdot \left\{ \int_{y_m}^{a_{k+1}} \left\{ 1 + P(x_k) + [1 - P(x_k)] \, \emptyset(\bar{D}^k) \right\} dF_k(x_k) \right\}$$

$$= \prod_{i=1}^{k-1} [1 - D_i] \prod_{j=1}^{m-1} [1 - Q(y_j)] \, [- Q(y_m)] \cdot$$

$$\cdot \left\{ 2 \int_{y_m}^{a_{k+1}} P(x_k) \, dF_k(x_k) + [1 - D_k] \, [1 + \emptyset(\bar{D}^k)] \right\} .$$

LEMMA 5. If F belongs to the class 0, $R(\bar{y})$ is a continuous function of $\bar{y}$, except when $y_m = 1$.

PROOF. The result follows from the fact that $\Psi(\bar{x}, \bar{y})$ has only simple discontinuities located on the diagonals $x_i = y_j$; F is continuous except at $x_n = 1$. An alternate proof may be based on the fact that the right-hand side of the last equation varies continuously as y_m increases from $a_k - \epsilon$ to $a_k + \epsilon$.

§6. CORRESPONDING STRATEGIES

DEFINITION. Let $F(\bar{x})$ and $G(\bar{y})$ be two measures contained in the class 0, and let S_F and S_G denote the supports of these measures. F and G form a pair of corresponding strategies if

$$\int \Psi(\bar{x}, \bar{y}) \, dF(\bar{x}) \equiv \underline{v}, \quad \text{all} \quad \bar{y} \in S_G, \quad y_m \neq 1$$

and

$$\int \Psi(\bar{x}, \bar{y})\, dG(\bar{y}) \equiv \bar{v}, \quad \text{all} \quad \bar{x} \in S_F, \quad x_n \neq 1 \quad .$$

NOTATION. Since F and G belong to the class 0,

$$F(\bar{x}) = \prod_{i=1}^{n} F_i(x_i), \qquad G(\bar{y}) = \prod_{j=1}^{m} G_j(y_j) \quad ,$$

and the supports of F_i and G_j are intervals $[a_i, a_{i+1}]$, $[b_j, b_{j+1}]$. The numbers $b_1, \dots, b_m$ can be rearranged into subsets, one subset for each interval $[a_i, a_{i+1}]$; the resulting array can be written in the form

$$\begin{array}{l} a_1 \leq b_{11} < b_{12} < \cdots < b_{1r_1} < a_2 \\ a_2 \leq b_{21} < b_{22} < \cdots < b_{2r_2} < a_3 \\ \dots\dots\dots\dots\dots \\ a_n \leq b_{n1} < b_{n2} < \cdots < b_{nr_n} < a_{n+1} = 1 \quad . \end{array}$$

It is possible that there are no b's between two adjacent a's.

Every interval bounded by two adjacent b's must contain precisely one component of each vector $\bar{y} \in S_G$. It is possible to identify the component by means of the interval over which it ranges; for instance, y_{ij} will denote the component of $\bar{y}$ that is contained in the interval $[b_{ij}, b_{ij+1}]$. When two adjacent b's are separated by one of the a's, say a_i, it is convenient to write

$$a_i = b_{io} \quad .$$

In this case $y_{i-1,r_{i-1}}$ and y_{io} denote the same component of $\bar{y}$.

In the following lemma, α denotes the discrete mass that $F_n(x_n)$ may have at $x_n = 1$.

LEMMA 6. Let $F(\bar{x})$ and $G(\bar{y})$ be two strategies contained in the class 0. Then

$$\int \Psi(\bar{x}, \bar{y})\, dF(\bar{x}) \equiv \underline{v}, \quad \text{all} \quad \bar{y} \in S_G, \quad y_m \neq 1$$

if and only if the following conditions hold simultaneously:

1) In the open interval (b_{ij}, b_{ij+1}), the measure $F_i(x_i)$ is absolutely continuous and

$$dF_i(x_i) = h_{ij} \frac{Q'(x_i)}{Q^2(x_i)\, P(x_i)}\, dx_i \quad .$$

2) The coefficients h_{ij} satisfy the equations

$$1 + 2\alpha = D_n + 2h_{nr_n}$$

$$h_{ij-1} = [1 - Q(b_{ij})]\, h_{ij} \qquad j = 1, \ldots, r_i;\ i = 1, \ldots, n$$

$$h_{ir_i} = [1 - D_i]\, h_{i+1,o} \qquad i = 1, \ldots, n-1 \quad .$$

The first condition requires that $a_1 \leq b_1$.

PROOF. Let $\bar{y} = (y_{11}, \ldots, y_{nr_n})$ be any vector in S_G, and define

$$K_{i,j} = \prod_{s=1}^{i-1} [1 - D_s] \prod_{y_{st} < y_{ij}} [1 - Q(y_{st})] \quad ,$$

where the second product is taken over all the components of $\bar{y}$ that precede y_{ij}. By Lemma 4,

$$R(\bar{y}) = R(y_{11}, \ldots, y_{nr_n-1})$$

(1)
$$- K_{n,r_n} Q(y_{nr_n}) \left\{ 2 \int_{y_{nr_n}}^{1} P(x_n)\, dF_n(x_n) + [1 - D_n] \right\} .$$

Therefore, $R(\bar{y})$ is independent of y_{nr_n} if and only if

$$Q(y_{nr_n}) \left\{ 2 \int_{y_{nr_n}}^{1} P(x_n)\, dF_n(x_n) + 1 - D_n \right\} \equiv 2h_{nr_n} \quad ,$$

(a)
$$b_{nr_n} < y_{nr_n} < 1 \quad ,$$

for some constant h_{nr_n}; furthermore, if (a) holds, equation (1) becomes

(b)
$$R(\bar{y}) = R(y_{11}, \ldots, y_{nr_n-1}) - 2K_{n,r_n}\, h_{nr_n} \quad .$$

The dependence of $R(\bar{y})$ on the component y_{ij} is studied by successive applications of Lemma 4; two cases must be considered.

CASE 1. $j \neq r_i$. Let us assume that if $R(\bar{y})$ is independent of all the components of $\bar{y}$ that lie beyond y_{ij}, then

$$R(\bar{y}) = R(y_{11}, \ldots, y_{ij}) - 2K_{i,j+1}\,\gamma_{i,j+1} \tag{b}$$

for some constant $\gamma_{i,j+1}$. Then, by definition,

$$K_{i,j+1} = [1 - Q(y_{ij})]\,K_{i,j} \quad .$$

Furthermore, Lemma 4 can be applied to $R(y_{11}, \ldots, y_{ij})$. Then

$$R(\bar{y}) = R(y_{11}, \ldots, y_{ij-1}) - 2K_{i,j}\,\gamma_{i,j+1}$$

$$- K_{i,j}\,Q(y_{ij})\left\{2\int_{y_{ij}}^{a_{i+1}} P(x_i)\,dF_i(x_i) + [1 - D_i]\,[1 + \phi(\bar{D}^i)] - 2\gamma_{i,j+1}\right\} \quad .$$

Therefore, $R(\bar{y})$ is also independent of y_{ij} if and only if

$$Q(y_{ij})\left\{2\int_{y_{ij}}^{a_{i+1}} P(x_i)\,dF_i(x_i) + [1 - D_i]\,[1 + \phi(\bar{D}^i)] - 2\gamma_{i,j+1}\right\} \equiv 2h_{ij}, \qquad b_{ij} < y_{ij} < b_{ij+1} \quad , \tag{a}$$

for some constant h_{ij}; furthermore, if (b) is valid for the indices i, $j + 1$, and (a) is valid for the indices i, j, then

$$R(\bar{y}) = R(y_{11}, \ldots, y_{i,j-1}) - 2K_{i,j}\,\gamma_{i,j}$$

where

$$\gamma_{i,j} = h_{ij} + \gamma_{i,j+1} \quad . \tag{2}$$

This result justifies the assumption (b), provided that i is fixed.

CASE 2. $j = r_i$. In the present notation y_{i,r_i} and $y_{i+1,o}$ denote the same component of $\bar{y}$; as a matter of fact, if two adjacent b's are separated by $a_{i+1}, \ldots, a_{i+k}$, the component of $\bar{y}$ that follows

y_{ir_i} is $y_{i+k,1}$. The process outlined in Case 1 shows that if $R(\bar{y})$ is independent of the components of $\bar{y}$ that lie beyond y_{ir_i}, then

$$R(\bar{y}) = R(y_{11}, \ldots, y_{ir_i}) - 2K_{i+k,1}\, \gamma_{i+k,1} \quad .$$

The arguments that were used in the previous case can be applied again, but in this case

$$K_{i+k,1} = \prod_{t=i}^{i+k-1} [1 - D_t]\, [1 - Q(y_{ir_i})]\, K_{i,r_i}$$

and $R(\bar{y})$ is independent of y_{ir_i}, if and only if

$$\text{(a)} \quad Q(y_{ir_i}) \left\{ 2 \int_{y_{ir_i}}^{a_{i+1}} P(x_i)\, dF_i(x_i) + [1 - D_i]\, [1 + \emptyset(D^i)] - 2 \prod_{t=i}^{i+k-1} [1 - D_t]\, \gamma_{i+k,1} \right\} \equiv 2h_{ir_i}, \quad b_{ir_i} < y_{ir_i} < a_{i+1} ,$$

for some constant h_{ir_i}; when equation (a) holds $R(\bar{y})$ can be written in the form

$$\text{(b)} \quad R(\bar{y}) = R(y_{11}, \ldots, y_{ir_i-1}) - 2K_{i,r_i}\, \gamma_{i,r_i}$$

with

$$\text{(3)} \quad \gamma_{i,r_i} = h_{ir_i} + \prod_{t=i}^{i+k-1} [1 - D_t]\, \gamma_{i+k,1} \quad .$$

These considerations show that $R(\bar{y})$ is independent of $\bar{y}$ if and only if condition (a) holds for every pair of indices i, j. In the special case $i = n$, $j = r_n$, the given condition can also be written in the form

$$Q(y_{nr_n}) \left\{ 2 \int_{y_{nr_n}}^{1} P(x_n)\, dF_n^*(x_n) + 2\alpha + 1 - D_n \right\} \equiv 2h_{nr_n} \quad .$$

where F_n^* is the continuous part of F_n; every one of these conditions is of the form

$$2 \int_t^{a_{i+1}} P(x)\, dF_i(x) + C = \frac{2h_{ij}}{Q(t)}, \quad b_{ij} < t < b_{ij+1} \quad ,$$

with F_i continuous and C constant. An integration by parts shows that F_i must be absolutely continuous in the given interval, and then, by differentiation,

$$\text{(c)} \qquad dF_i(t) = h_{ij} \frac{Q'(t)}{Q^2(t)\,P(t)}\,dt, \qquad b_{ij} < t < b_{ij+1} .$$

The integral that appears in each one of the conditions (a) can be evaluated explicitly by means of (c); it is then seen that (a) holds if and only if (c) holds and also

$$\text{(d)} \quad \begin{cases} 2h_{nr_n} = 2\alpha + 1 - D_n \\[2ex] \dfrac{2h_{ij}}{Q(b_{ij+1})} = 2 \displaystyle\int_{b_{ij+1}}^{a_{i+1}} P(x_i)\,dF_i(x_i) + [1 - D_i]\,[1 + \phi(\bar{D}^i)] \\[1ex] \qquad\qquad\qquad\qquad - 2\gamma_{i,j+1}, \quad j \neq r_i \\[2ex] \dfrac{2h_{ir_i}}{Q(a_{i+1})} = [1 - D_i]\,[1 + \phi(\bar{D}^i)] \\[1ex] \qquad\qquad - 2 \displaystyle\prod_{t=i}^{i+k-1} [1 - D_t]\,\gamma_{i+k,1}, \quad j = r_i . \end{cases}$$

In order to complete the proof of the lemma it is sufficient to replace this system of equations by the equivalent system that is obtained by subtracting from each equation the preceding one; for instance, if the equation that contains h_{ij+1} is subtracted from the equation that contains h_{ij}, then

$$\frac{2h_{ij}}{Q(b_{ij+1})} - \frac{2h_{ij+1}}{Q(b_{ij+2})} = 2 \int_{b_{ij+1}}^{b_{ij+2}} P(x_i)\,dF_i(x_i) + 2\gamma_{i,j+2} - 2\gamma_{i,j+1}$$

$$= 2h_{ij+1} \left[\frac{1}{Q(b_{ij+1})} - \frac{1}{Q(b_{ij+2})} \right] - 2h_{ij+1} .$$

Simplifying,

$$h_{ij} = [1 - Q(b_{ij+1})]\,h_{ij+1} .$$

Similarly, the equations that contain h_{ir_i} and $h_{i+1,0}$ show that

$$h_{ir_i} = [1 - D_i]\, h_{i+1,o} \quad .$$

It may happen that one of the original a_i's coincides with one of the original b_j's, say $a_{i+1} = b_{i+1,1}$. In this case the system (d) does not contain an equation with $h_{i+1,o}$ and the last derivation is not valid. However, subtracting in this case the equation with h_{ir_i} from the equation with $h_{i+1,1}$ one can see that

$$h_{ir_i} = [1 - D_i]\, [1 - Q(a_{i+1})]\, h_{i+1,1} \quad .$$

In this case it is convenient to introduce the unnecessary parameter $h_{i+1,o}$ defined by

$$h_{i+1,o} = [1 - Q(a_{i+1})]\, h_{i+1,1} = [1 - Q(b_{i+1,1})]\, h_{i+1,1},$$

if

$$a_{i+1} = b_{i+1,1} \quad .$$

With this definition, h_{ir_i}, $h_{i+1,o}$, and $h_{i+1,1}$ are always related by the equations stated in the lemma.

LEMMA 7. Let $F(\bar{x})$ and $G(\bar{y})$ be contained in the class $\mathcal{O}$. If

$$\int \Psi(\bar{x}, \bar{y})\, dF(\bar{x}) = \underline{v}, \quad \text{all} \quad \bar{y} \in S_G, \quad y_m \neq 1$$

then

$$\int \Psi(\bar{x}, \bar{y})\, dF(\bar{x}) \geq \underline{v}, \quad \text{all} \quad \bar{y} \in Y \quad .$$

If

$$\int \Psi(\bar{x}, \bar{y})\, dG(\bar{y}) = \bar{v}, \quad \text{all} \quad \bar{x} \in S_F, \quad x_n \neq 1 \quad ,$$

then

$$\int \Psi(\bar{x}, \bar{y})\, dG(\bar{y}) \leq \bar{v}, \quad \text{all} \quad \bar{x} \in X \quad .$$

PROOF. First Inequality. The definition of $\Psi(\bar{x}, \bar{y})$ implies that

$$\Psi(\bar{x}, \bar{y}) \geq \Psi(\bar{x}, \overline{y - 0}), \quad \text{all} \quad \bar{x}, \bar{y} \quad .$$

Therefore, it is sufficient to consider only those vectors $\bar{y}$ with $y_m \neq 1$. The inequality is proved by induction, and we may assume that it has been established for all the vectors $\bar{y}$ whose last $m - q$ components coincide with the last $m - q$ components of some vector in the support of $G(\bar{y})$.

Let $\bar{y} = (y_1, \ldots, y_{q+1}, y^*_{q+2}, \ldots, y^*_m)$ be a vector whose last $m - (q + 1)$ components coincide with the last $m - (q + 1)$ components of some vector in the support of G. The assumptions of this lemma imply that $R(\bar{y})$ must satisfy all the equations derived in the proof of Lemma 6. In particular, $R(\bar{y})$ must be independent of all the starred components, and (by (b)),

$$R(\bar{y}) = R(y_1, \ldots, y_q, y_{q+1}) - 2 \prod_{s=1}^{i-1} [1 - D_s] \prod_{t=1}^{q+1} [1 - Q(y_t)] \gamma_{i,j+1} \quad ,$$

where $b_{ij+1} \le y^*_{q+2} \le b_{ij+2}$. Consider four cases.

CASE 1. $y_{q+1} > b_{ij+1}$. In this case we must have $b_{ij+1} < y_{q+1} \le b_{ij+2}$. Then, by Lemma 4,

$$R(\bar{y}) = R(y_1, \ldots, y_q) - 2 \prod_{s=1}^{i-1} [1 - D_s] \prod_{t=1}^{q+1} [1 - Q(y_t)] \gamma_{i,j+1}$$

$$- \prod_{s=1}^{i-1} [1 - D_s] \prod_{t=1}^{q} [1 - Q(y_t)] Q(y_{q+1}) \left\{ 2 \int_{y_{q+1}}^{a_{i+1}} P(t)\, dF_i(t) \right.$$

$$\left. + [1 - D_i] [1 + \phi(\bar{D}^i)] \right\} .$$

Furthermore, by condition (a) from Lemma 6, if $b_{ij+1} < y < b_{ij+2}$,

$$Q(y) \left\{ 2 \int_{y}^{a_{i+1}} P(t)\, dF_i(t) + [1 - D_i][1 + \phi(\bar{D}^i)] - 2\gamma_{i,j+2} \right\} \equiv 2h_{i,j+1} \quad .$$

Hence,

$$R(\bar{y}) = R(y_1, \ldots, y_q) - 2 \prod_{s=1}^{i-1} [1 - D_s] \prod_{t=1}^{q} [1 - Q(y_t)] \gamma_{i,j+1}$$

$$- \prod_{s=1}^{i-1} [1 - D_s] \prod_{t=1}^{q} [1 - Q(y_t)] \left\{ 2h_{ij+1} + 2[\gamma_{i,j+2} - \gamma_{i,j+1}] Q(y_{q+1}) \right\} .$$

By equation (2), $\gamma_{i,j+2} - \gamma_{i,j+1} = - h_{ij+1}$; clearly, $R(\bar{y})$ is a monotone increasing function of y_{q+1}. If the components $y_p, y_{p+1}, \ldots, y_q$ also lie in the interval $[b_{ij+1}, b_{ij+2}]$, the same argument will show that $R(\bar{y})$ is a monotone increasing function of these components, and $R(\bar{y})$ achieves a smaller value if all these components are replaced by

b_{ij+1}; we can write $y^*_{q+1} = b_{ij+1} \in [b_{ij}, b_{ij+1}]$. Then y^*_{q+1} coincides with the $q + 1$st component of some vector in the support of $G(\bar{y})$ and

$$R(\bar{y}) \geq R(y_1, \ldots, y_{p-1}, b_{ij+1}, \ldots, b_{ij+1}, y^*_{q+1}, y^*_{q+2}, \ldots, y^*_m) \geq v \ .$$

CASE 2. $b_{ij} \leq y_{q+1} \leq b_{ij+1}$. In this case we simply choose $y^*_{q+1} = y_{q+1}$.

CASE 3. $a_1 \leq y_{q+1} < b_{ij}$. For definiteness, assume that $b_{kl} \leq y_{q+1} < b_{kl+1}$. Applying Lemma 4 to $R(\bar{y})$ and simplifying the result by means of the appropriate condition (a) from Lemma 6,

$$R(\bar{y}) = R(y_1, \ldots, y_q) - \prod_{s=1}^{i-1} [1 - D_s] \prod_{t=1}^{q} [1 - Q(y_t)] \ \gamma_{i,j+1}$$

$$- \prod_{s=1}^{k-1} [1 - D_s] \prod_{t=1}^{q} [1 - Q(y_t)] \left\{ 2h_{kl} + 2Q(y_{q+1}) \left[\gamma_{k,l+1} - \prod_{\sigma=k}^{i-1} [1 - D_\sigma] \ \gamma_{i,j+1} \right] \right\} .$$

The definition of the constant $\gamma_{k,l+1}$ (equations (2) and (3)) shows that $R(\bar{y})$ is a monotone decreasing function of y_{q+1}. In this case we may take $y^*_{q+1} = b_{ij} \in [b_{ij}, b_{ij+1}]$.

CASE 4. $y_{q+1} < a_1$. By Lemma 3, if $\bar{x} \in S_F$, $\Psi(\bar{x}, \bar{y})$ is a monotone decreasing function of y_{q+1}. Then $R(\bar{y})$ has the same property, and

$$R(\bar{y}) \geq R(y_1, \ldots, y_q, a_1, y^*_{q+2}, \ldots, y^*_m) \ .$$

The problem is now reduced to Case 3.

<u>Second Inequality</u>. It is sufficient to interchange the roles of X and Y and to recall that $\Psi(\bar{y}, \bar{x}) = - \Psi(\bar{x}, \bar{y})$.. Then, by assumption

$$\int \Psi(\bar{y}, \bar{x}) \, dG(\bar{y}) = - \bar{v}, \quad \text{all } \bar{x} \in S_F, \quad x_n \neq 1 \ .$$

Then, by the first inequality (applied to $\Psi(\bar{y}, \bar{x})$)

$$\int \Psi(\bar{y}, \bar{x}) \, dG(\bar{y}) \geq - \bar{v} \ .$$

COROLLARY. If $\bar{y}$ is not contained in the support of $G(\bar{y})$, then

$$\int \Psi(\bar{x}, \bar{y})\, dF(\bar{x}) > \underline{v} \quad .$$

This result follows from the strict monotonicity of the functions that appear in the proof of Lemma 7.

LEMMA 8. Let $F(\bar{x})$ and $G(\bar{y})$ be two corresponding strategies such that at least one of them is continuous at 1. Then $\underline{v} = \bar{v}$, and F and G are optimal.

PROOF. For definiteness, we may assume that G is continuous. By definition of $\underline{v}$,

$$\int \Psi(\bar{x}, \bar{y})\, dF(\bar{x}) \equiv \underline{v}, \quad \text{all} \quad \bar{y} \in S_G, \quad y_m \neq 1 \quad .$$

Since the vectors with $y_m = 1$ have G-measure zero,

$$\iint \Psi(\bar{x}, \bar{y})\, dF(\bar{x})\, dG(\bar{y}) = \int \underline{v}\, dG(\bar{y}) = \underline{v} \quad . \tag{4}$$

Similarly, by definition of $\bar{v}$,

$$\int \Psi(\bar{x}, \bar{y})\, dG(\bar{y}) \equiv \bar{v}, \quad \text{all} \quad \bar{x} \in S_F, \quad x_n \neq 1 \quad .$$

Since G is continuous, the integral is continuous and the equation is valid for all $\bar{x} \in S_F$. Therefore,

$$\iint \Psi(\bar{x}, \bar{y})\, dG(\bar{y})\, dF(\bar{x}) = \int \bar{v}\, dF(\bar{x}) = \bar{v} \quad . \tag{5}$$

Equations (4) and (5) imply that $\underline{v} = \bar{v}$; this number is denoted by v. Lemma 7 asserts that v is the value of the game, and that F and G are optimal strategies.

REMARK. It is well known that if F and G are optimal, then

$$\int \Psi(\bar{x}, \bar{y})\, dF(\bar{x}) = v, \quad \text{all} \quad \bar{y} \in S_G,$$

provided that $\bar{y}$ is a point of continuity of the integral; a similar result is valid when F and G are interchanged. In particular, any pair of optimal strategies that belong to the class 0 must be a pair of corresponding strategies, and it is easy to show that one of them must be continuous at $t = 1$. This remark may be taken as the converse of Lemma 8.

CHARACTERIZATION OF CORRESPONDING STRATEGIES. Let $F(\bar{x})$ and $G(\bar{y})$ be two strategies contained in the class O, and let α and β denote the discrete masses that F_n and G_m may have at $x_n = 1$, $y_m = 1$. Let

$$D_i = \int_{a_i}^{a_{i+1}} P(t)\, dF_i(t), \quad E_j = \int_{b_j}^{b_{j+1}} Q(t)\, dG_j(t) \quad ,$$

and let $f^*(x)$ and $g^*(y)$ be two discontinuous functions defined by

(6) $$f^*(x) = \prod_{b_j > x} [1 - Q(b_j)] \frac{Q'(x)}{Q^2(x)\, P(x)}$$

(7) $$g^*(y) = \prod_{a_i > y} [1 - P(a_i)] \frac{P'(y)}{P^2(y)\, Q(y)}$$

The results of Lemma 6 can be stated as follows: $F(\bar{x})$ and $G(\bar{y})$ form a pair of corresponding strategies if and only if the following conditions hold:

(8) $$dF_i(x_i) = h_i\, f^*(x_i)\, dx_i, \quad a_i < x_i < a_{i+1}, \quad i = 1, \ldots, n$$

(9) $$dG_j(y_j) = k_j\, g^*(y_j)\, dy_j, \quad b_j < y_j < b_{j+1}, \quad j = 1, \ldots, m$$

(10) $$1 + 2\alpha = D_n + 2h_n$$

(11) $$1 + 2\beta = E_m + 2k_m$$

(12) $$h_i = [1 - D_i]\, h_{i+1}$$

(13) $$k_j = [1 - E_j]\, k_{j+1}$$

$$a_1 = b_1$$

If these equations are used to characterize the optimal strategies, it is necessary to include the continuity hypothesis $(\alpha\beta = 0)$ of Lemma 8 and the normalizing equations

$$\int dF_i = \int dG_j = 1, \quad i = 1, \ldots, n; \quad j = 1, \ldots, m.$$

§7. EXISTENCE OF A SOLUTION

We have shown that in order to find two optimal strategies it is sufficient to find a set of numbers $a_1, \ldots, b_m, h_1, \ldots, k_m, \alpha$ and β that satisfy the previous set of equations. In equations (10), (11), (12) and (13) it is convenient to write D_i and E_j in terms of f^* and g^*, and to eliminate h_n, k_m, h_i and k_j by means of the normalizing equations. In this form, the complete system is as follows:

Normalizing Equations

(14) $$h_n \int_{a_n}^{1} f^*(t)\, dt + \alpha = 1$$

(15) $$k_m \int_{b_m}^{1} g^*(t)\, dt + \beta = 1$$

(16) $$h_i \int_{a_i}^{a_{i+1}} f^*(t)\, dt = 1 \qquad i = 1, \ldots, n - 1$$

(17) $$k_j \int_{b_j}^{b_{j+1}} g^*(t)\, dt = 1 \qquad j = 1, \ldots, m - 1$$

Equations from Lemma 6 and Lemma 8

(18) $$\int_{a_n}^{1} [(1 + \alpha) - (1 - \alpha) P(t)]\, f^*(t)\, dt = 2(1 - \alpha)$$

(19) $$\int_{b_m}^{1} [(1 + \beta) - (1 - \beta) Q(t)]\, g^*(t)\, dt = 2(1 - \beta)$$

(20) $$\int_{a_i}^{a_{i+1}} [1 - P(t)]\, f^*(t)\, dt = \frac{1}{h_{i+1}}, \qquad i = 1, \ldots, n - 1$$

(21) $$\int_{b_j}^{b_{j+1}} [1 - Q(t)]\, g^*(t)\, dt = \frac{1}{k_{j+1}}, \qquad j = 1, \ldots, m - 1$$

$$a_1 = b_1 \tag{22}$$

$$\alpha\beta = 0 \ . \tag{23}$$

LEMMA 9. Let a be any number in the interval $(0, 1]$, and let $b_1, \ldots, b_m$ be any set of parameters, subject to the restriction $0 < b_1 < \cdots < b_m < 1$; let f^* be defined by equation (6). Then

$$\lim_{x \to 0} \int_x^a [1 - P(t)]\, f^*(t)\, dt = +\infty \ .$$

PROOF. Let

$$M = \prod_{j=1}^{m} [1 - Q(b_j)] \ ,$$

and choose c such that for $t \leq c$, $1 - P(t) \geq P(t)$. Then, for $x < c$,

$$\int_x^a [1 - P(t)]\, f^*(t)\, dt \geq \int_x^c M \frac{Q'(t)}{Q^2(t)}\, dt + \int_c^a [1 - P(t)]\, f^*(t)\, dt$$

$$= M\left[\frac{1}{Q(x)} - \frac{1}{Q(c)}\right] + \int_c^a [1 - P(t)]\, f^*(t)\, dt \ .$$

The first term tends to $+\infty$; the second one is finite.

LEMMA 10. Let α be any number in the interval $[0, 1)$, and let $b_1, \ldots, b_m$ be any set of parameters, subject to the restriction $0 < b_1, \ldots, b_m < 1$. Under these assumptions, equations (14), (16), (18) and (20) have a unique solution $a_1, \ldots, a_n, h_1, \ldots, h_n$. If the b's are fixed, a_n is a monotone increasing function of α, and $a_n \to 1$ as $\alpha \to 1$. If α is fixed, a_n is a decreasing function of b_m, and $a_n \to 0$ as $b_m \to 1$.

PROOF. Lemma 9 shows that equation (18) has a solution $a_n \in (0, 1]$; since the integrand is positive, the solution is unique, and it is clearly a continuous, monotone increasing function of α; setting $\alpha = 1$, one obtains $a_n = 1$; but if $\alpha < 1$, a_n must be in $(0, 1)$, and h_n can be computed by means of equation (14). When a_n and h_n are known, a_{n-1} can be obtained from equation (20), and h_{n-1} from equation (16). The process can be continued until all the a's and h's are found. Finally, it is necessary to consider a_n as a function of b_m. Let α and d be fixed, and consider

$$\int_d^{b_m} [(1 + \alpha) - (1 - \alpha) P(t)] f^*(t)\, dt$$

$$+ \int_{b_m}^{1} [(1 + \alpha) - (1 - \alpha) P(t)] f^*(t)\, dt \quad .$$

In the first term, the integrand contains the factor $[1 - Q(b_m)]$ and tends to zero uniformly as $b_m \longrightarrow 1$; in the second term the integrand is bounded, and the range of integration can be made arbitrarily small. Thus, as $b_m \longrightarrow 1$, the solution a_n of equation (18) must approach 0.

REMARK. The value of each a_i depends only on those parameters b_j that are larger than a_i. The remaining parameters can be changed, and new parameters may be added. Also, the process is symmetric: if $a_1, \ldots, a_n, \beta$ are given parameters, equations (15), (17), (19), and (21) have unique solutions $b_1, \ldots, b_m$. Results like those of Lemma 10 will hold.

THEOREM. The system of equations (14), ..., (23) has a unique solution. This solution determines two strategies $F(\bar{x})$ and $G(\bar{y})$ that are optimal, and these are the only optimal strategies that belong to the class 0.

PROOF. It is sufficient to show that the system has a unique solution $a_1, \ldots, a_n, b_1, \ldots, b_m, h_1, \ldots, h_n, k_1, \ldots, k_m, \alpha, \beta$. Then $F(\bar{x})$ and $G(\bar{y})$ can be defined by equations (8) and (9). The normalizing equations show that F and G are strategies, and the remaining equations show that F and G satisfy the hypothesis of Lemma 8. The remark that follows that lemma shows that no other strategies in the class 0 can be optimal.

EXISTENCE. Consider two numbers α and β such that

$$0 \leq \alpha < 1, \quad \alpha\beta = 0, \quad 0 \leq \beta < 1 \quad ,$$

and construct a set of numbers $a_1^*, \ldots, a_n^*, b_1^*, \ldots, b_m^*$ as follows: first, compute two numbers a_n and b_m by means of equations (18) and (19) using no parameters in the definitions of f^* and g^*; the resulting numbers are compared and the larger one is kept as a parameter; for definiteness, we assume that $a_n > b_m$, and define $a_n^* = a_n$; the smaller number is neglected. In the next step a number a_{n-1} is computed by means of equation (20) (without parameters), and a new b_m is computed from equation (19), using the single parameter a_n^*. It is clear that the new b_m is smaller than the one computed in the previous step, and therefore $b_m < a_n^*$; the two numbers a_{n-1} and b_m are compared, and the larger one is kept as a parameter. For definiteness, one may assume that $a_{n-1} > b_m$, and define $a_{n-1}^* = a_{n-1}$. The process is continued in this manner: at each step, a new a_i and a new b_j are computed, using as parameters the previously starred a's and b's; the two numbers are compared, and the larger one is kept as a parameter, denoted by a_i^* (or b_j^*); the corresponding h_i^* (or k_j^*) is computed by means of the appropriate normalizing equation. Since each parameter is smaller than those computed previously, it is clear that the resulting set is self-consistent: that is, if $b_1^*, \ldots, b_m^*$ are used as parameters, one obtains again $a_1^*, \ldots, a_n^*$, and conversely. Therefore, these numbers satisfy the system of equations, except for equation (22).

Now, consider a_1^* and b_1^* as functions of α and β. If the previous construction is carried out with $\alpha = \beta = 0$, it may happen that $a_1^* = b_1^*$, and all equations are satisfied; otherwise, either $a_1^* < b_1^*$ or $b_1^* < a_1^*$; for definiteness it may be assumed that $b_1^* < a_1^*$. In this case the same construction is applied with $\alpha = 0$, $\beta = 1 - \epsilon$, and ϵ is chosen so small that in the first step $a_n < b_m$, and $b_m^* = b_m$ is arbitrarily close to 1. In the remaining steps a_n is always computed with the parameter b_m^*, and $a_n \longrightarrow 0$; thus all the b*'s are computed without parameters and remain bounded away from zero; then $a_1^* < a_n^* < b_1^*$; the inequality has been reversed as β increased from 0 to 1. Since the a's and b's are limits of integration of strictly positive densities, it is clear that a_1^* and b_1^* are continuous functions of β, and there exists a positive β for which $a_1^* = b_1^*$. If the construction with $\alpha = \beta = 0$ leads to the inequality $a_1^* < b_1^*$, the system of equations has a solution with $\alpha > 0$, $\beta = 0$.

UNIQUENESS. The solution is unique. Indeed, suppose that $a_1, \ldots, a_n, b_1, \ldots, b_m$ and $a_1^*, \ldots, a_n^*, b_1^*, \ldots, b_m^*$ are two solutions. These solutions give rise to two pairs of optimal strategies $F(\bar{x}), G(\bar{y})$ and $F^*(\bar{x}), G^*(\bar{y})$. Since F and G^* are optimal

$$\int \Psi(\bar{x}, \bar{y})\, dF(\bar{x}) \equiv v, \quad \text{all} \quad \bar{y} \in S_{G^*}, \quad y_m \neq 1 \;.$$

The corollary to Lemma 7 shows that $S_{G^*} \subset S_G$. Since the argument is symmetric, the two measures must have the same supports, i.e., $b_j^* = b_j$, all j. Similarly, $a_i^* = a_i$.

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