Replication of Public Debt and Low Interest Rates (Blanchard 2019) - Appendix

Julien H. Acalin

Johns Hopkins University

First draft: July 1, 2019. This draft: November 10, 2019.

Abstract: This appendix derives the formulas used in the companion Jupyter notebook which fully replicates the analysis of the stochastic overlapping generations (OLG) model developed by Blanchard in his presidential address during the AEA meetings 2019.

Keywords: Stochastic OLG, Intergenerational transfers, Debt, Welfare, Python

Author's Note: The code is provided in a companion Jupyter notebook. I am particularly grateful to Christopher Carroll for guidance, to Olivier Blanchard for insightful conversations, and to Gonzalo Huertas and Thomas Pellet for sharing their Matlab codes. All errors are mine.

Appendix - Derivations

A) Description of the Model

Assume the representative agent maximizes an Epstein-Zin utility function:

$$\mathbb{U}_t = (1-\beta)u\left(C_t^y\right) + \frac{\beta}{1-\gamma}u(\mathbb{E}_t\left[(C_{t+1}^o)^{1-\gamma}\right])$$

Where: $u(C) = \log(C)$

With respect to:

$$\begin{aligned}
C_t^y + I_t + D_t &= W_t + X - T_t \\
K_t &= I_t + (1-\delta)K_{t-1} \\
C_{t+1}^o &= R_{t+1}K_t + R_t^f D_t + T_{t+1} \\
Y_t &= A_t F(K_{t-1}, L_t) \\
\log(A) &\sim \mathcal{N}(\mu, \sigma)
\end{aligned}$$

Where C_t^y and C_{t+1}^o respectively denote consumption when young and old, I_t is investment in physical capital, D_t is investment in the safe asset, W_t is the wage, X is an initial non-stochastic endowment, T_t and T_{t+1} denote intergenerational transfers (transfers can be stochastic or not), A_t is a log-normally distributed productivity shock, and R_{t+1} and R_t^f denote respectively the return to physical capital and risk-free asset. Assume that there is full depreciation after one period ($\delta = 1$) so that $K_t = I_t$.

Factors earn their marginal return:

$$\begin{aligned}
W_t &= A_t F_L(K_{t-1}, L_t) \\
R_t &= A_t F_K(K_{t-1}, L_t)
\end{aligned}$$

In the baseline scenario, there is no government intervention: $T_t = D_t = 0\ \forall t$. The paper discusses the welfare implications of two types of policy intervention. The government can start a social security system and sets the level of transfers T_t accordingly. Alternatively, the government can start to issue and rollover public debt D_t. Absent default, the debt dynamics equation is $D_{t+1} = R_t^f D_t$.[1]

[1] If the debt rollover fails, the government taxes the *young* so as to bring the debt back to its target value D^*. Formally, the *young* budget constraint is $C_t^y + I_t + D_t = W_t + X - T_t - \theta_t$ where $\theta_t = 0$ if $D_t < \bar{D}$ and $\theta_t = D_t - D^*$ if $D_t \geq \bar{D}$. For simplicity I assume $\theta_t = 0\ \forall t$ in the appendix. This does not change the main results but simplifies the exposition.

B) General solution with debt and transfers: $D_t \geq 0$; $T_t \neq 0$.

If the government runs a social security system and/or issues debt, the maximization problem can be rewritten:

$$\max_{I_t, D_t} \mathbb{U} = (1-\beta)\ \log(W_t + X - T_t - I_t - D_t) + \frac{\beta}{1-\gamma}\ \log(\mathbb{E}_t[(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{1-\gamma}]) \tag{1}$$

The first order condition with respect to I_t is:

$$\frac{1-\beta}{(W_t + X - T_t - I_t - D_t)} = \frac{\beta}{1-\gamma} \frac{(1-\gamma)\mathbb{E}_t[R_{t+1}(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{-\gamma}]}{\mathbb{E}_t[(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{1-\gamma}]} \tag{2}$$

Similarly, the first order condition with respect to D_t is:

$$\frac{1-\beta}{(W_t + X - T_t - I_t - D_t)} = \frac{\beta}{1-\gamma} \frac{(1-\gamma)\mathbb{E}_t[R_t^f(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{-\gamma}]}{\mathbb{E}_t[(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{1-\gamma}]} \tag{3}$$

As R_t^f is known at time t it can be taken out of the expectation term:

$$\frac{1-\beta}{(W_t + X - T_t - I_t - D_t)} = \beta \frac{R_t^f \mathbb{E}_t[(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{-\gamma}]}{\mathbb{E}_t[(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{1-\gamma}]} \tag{4}$$

The left hand side of equations (2) and (4) are equal, thus:

$$\beta \frac{R_t^f \mathbb{E}_t[(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{-\gamma}]}{\mathbb{E}_t[(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{1-\gamma}]} = \beta \frac{\mathbb{E}_t[R_{t+1}(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{-\gamma}]}{\mathbb{E}_t[(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{1-\gamma}]} \tag{5}$$

Diving by $\beta \frac{\mathbb{E}_t[(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{-\gamma}]}{\mathbb{E}_t[(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{1-\gamma}]}$ on both sides:

$$R_t^f = \frac{\mathbb{E}_t[R_{t+1}(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{-\gamma}]}{\mathbb{E}_t[(R_{t+1}I_t + R_t^f D_t + T_{t+1})^{-\gamma}]} \tag{6}$$

The environment is a closed economy. Market clearing requires that young households hold all the debt inelastically supplied by the government. Put differently, young households are constrained in their quantity of safe asset holdings, but their demand function is used to determine the equilibrium rate of return they require on this asset given the quantity D_t. If there is debt in the economy ($D_t > 0$), then the investment in physical capital I_t and the risk-free rate R_t^f consistent with a given debt level D_t are obtained by solving simultaneously equations (2) and (6) derived from the first order conditions.

C) Solution with transfers but no debt: $D_t = 0$; $T_t \neq 0$.

If there is no debt in the economy ($D_t = 0 \; \forall t$), then equation (2), which is used to solve for I_t, simplifies to:

$$\frac{1-\beta}{(W_t + X - T_t - I_t)} = \beta \frac{\mathbb{E}_t[R_{t+1}(R_{t+1}I_t + T_{t+1})^{-\gamma}]}{\mathbb{E}_t[(R_{t+1}I_t + T_{t+1})^{1-\gamma}]} \tag{7}$$

If there is no debt, the risk-free rate is not used to derive the optimal investment in physical capital. Yet, the shadow risk-free rate consistent with young agents being indifferent between investing at the margin or not in risk-free debt is given by (6), which simplifies to:

$$R_t^f = \frac{\mathbb{E}_t[R_{t+1}(R_{t+1}I_t + T_{t+1})^{-\gamma}]}{\mathbb{E}_t[(R_{t+1}I_t + T_{t+1})^{-\gamma}]} \tag{8}$$

D) Calibration without government: $D_t = 0$; $T_t = 0$.

The calibration follows Blanchard (2019). Assume for simplicity that $X = 0$. Again, this does not change the main results but simplifies the exposition. If there is no debt and no transfers in the economy ($D_t = T_t = 0 \; \forall t$), then equation (2), which is used to solve for I_t, simplifies to:

$$\frac{1-\beta}{(W_t - I_t)} = \beta \frac{\mathbb{E}_t[R_{t+1}(R_{t+1}I_t)^{-\gamma}]}{\mathbb{E}_t[(R_{t+1}I_t)^{1-\gamma}]} \tag{9}$$

After some algebra this leads to the following optimal investment decision:

$$I_t = \beta W_t \tag{10}$$

Similarly, given the separable log utility specification, the optimal consumption decision is:

$$C_t^y = (1-\beta)W_t \tag{11}$$

Absent government intervention, the risk-free rate is not used to derive the optimal investment in physical capital. Yet, the shadow risk-free rate consistent with young agents being indifferent between investing at the margin or not in risk-free debt is given by (6), which simplifies to:

$$R_t^f = \frac{\mathbb{E}_t[R_{t+1}(R_{t+1}I_t)^{-\gamma}]}{\mathbb{E}_t[(R_{t+1}I_t)^{-\gamma}]} \tag{12}$$

To make further analytic progress, we need to specify a production function.

i) Linear production function:

$$Y_t = A_t(\alpha K_{t-1} + (1-\alpha)L_t) \tag{13}$$

Factors earn their marginal product in equilibrium:

$$R_t = \alpha A_t \tag{14}$$

$$W_t = (1-\alpha)A_t \tag{15}$$

Equation (12) can be rewritten:

$$R_t^f = \frac{\mathbb{E}_t[(\alpha A_{t+1})^{1-\gamma}]}{\mathbb{E}_t[(\alpha A_{t+1})^{-\gamma}]} \tag{16}$$

Taking α out of the expectation operator, and using the fact that if X is log-normally distributed then $E[X^n] = e^{n\mu + \frac{1}{2}n^2\sigma^2}$, we obtain:

$$R_t^f = \alpha e^{\mu + \frac{1}{2}\sigma^2 - \gamma\sigma^2} \tag{17}$$

From equation (14), we obtain:

$$\mathbb{E}[R_{t+1}] = \alpha e^{\mu + \frac{1}{2}\sigma^2} \tag{18}$$

From equation (17) and (18), we obtain the log equity premium:

$$\ln(\mathbb{E}[R_{t+1}]) - \ln(R_t^f) = \gamma\sigma^2 \tag{19}$$

We obtain the steady state value of capital from equations (10) and (15):

$$\mathbb{E}[K_t] = \bar{K} = \beta(1-\alpha)e^{\mu + \sigma^2/2} \tag{20}$$

$\mathbb{E}[R_{t+1}]$ does not depend on K_t and thus does not depend on β, but can be calibrated with μ while the equity premium, and thus $\mathbb{E}[R_t^f]$, depends on γ.

ii) Cobb-Douglas production function:

$$Y_t = A_t K_{t-1}^{\alpha} L_t^{1-\alpha} \tag{21}$$

Factors earn their marginal product in equilibrium:

$$R_t = A_t \alpha K_{t-1}^{\alpha-1} \tag{22}$$

$$W_t = A_t (1-\alpha) K_{t-1}^{\alpha} \tag{23}$$

Equation (12) can be rewritten:

$$R_t^f = \frac{\mathbb{E}_t[(\alpha K_t^{\alpha-1} A_{t+1})^{1-\gamma}]}{\mathbb{E}_t[(\alpha K_t^{\alpha-1} A_{t+1})^{-\gamma}]} \tag{24}$$

Taking $\alpha K_t^{\alpha-1}$ out of the expectation operator, and using the fact that if X is log-normally distributed then $E[X^n] = e^{n\mu + \frac{1}{2}n^2\sigma^2}$, we obtain:

$$R_t^f = \alpha K_t^{\alpha-1} e^{\mu + \frac{1}{2}\sigma^2 - \gamma\sigma^2} \tag{25}$$

We derive the steady state value of capital from equations (10) and (23):

$$K_t = \beta A_t (1-\alpha) K_{t-1}^{\alpha} \tag{26}$$

By taking log on both sides:

$$k_{t+1} = \log[\beta(1-\alpha)] + \alpha k_t + \log(A_t) \tag{27}$$

Evaluating at $k_{t+1} = k_t$, the expectation and variance are:

$$\mathbb{E}[k_t] = \frac{\log[\beta(1-\alpha)] + \mu}{1-\alpha} \tag{28}$$

$$\mathbb{V}[k_t] = \frac{\sigma^2}{1-\alpha^2} \tag{29}$$

Thus, the steady state value of capital is:

$$\mathbb{E}[K_t] = \bar{K} = e^{\mathbb{E}k} e^{\frac{\mathbb{V}k}{2}} = e^{\left(\frac{\log[\beta(1-\alpha)]+\mu}{1-\alpha} + \frac{\sigma^2/2}{1-\alpha^2}\right)} \tag{30}$$

Taking log of equation (25):

$$r_t^f = \log(\alpha) + (\alpha - 1)k_t + \mu + (\frac{1}{2} - \gamma)\sigma^2 \tag{31}$$

Using (28), the expectation and variance are:

$$\mathbb{E}[r_t^f] = \log \frac{\alpha}{\beta(1-\alpha)} + (\frac{1}{2} - \gamma)\sigma^2 \tag{32}$$

$$\mathbb{V}[r_t^f] = \frac{1-\alpha}{1+\alpha}\sigma^2 \tag{33}$$

The unconditional expected value of the risk-free rate is:

$$\mathbb{E}[R_t^f] = \frac{\alpha}{\beta(1-\alpha)} e^{(\frac{\sigma^2}{1+\alpha} - \gamma\sigma^2)} \tag{34}$$

Similarly, the unconditional expected value of the risky rate is:

$$\mathbb{E}[R_{t+1}] = \frac{\alpha}{\beta(1-\alpha)} e^{(\frac{\sigma^2}{1+\alpha})} \tag{35}$$

From equation (34) and (35), we obtain the log equity premium:

$$\ln(\mathbb{E}[R_{t+1}]) - \ln(R_t^f) = \gamma\sigma^2 \tag{36}$$

$\mathbb{E}[R_{t+1}]$ does not depend on μ, but can be calibrated with β while the equity premium, and thus $\mathbb{E}[R_t^f]$, depends on γ.

E) Methodology

This paper evaluates the welfare implications of debt and transfers for different combinations of $\mathbb{E}[R_{t+1}]$ and $\mathbb{E}[R_t^f]$ absent government intervention. First, for every combination of both average rates, I find the corresponding parameters μ and γ from equations (17) and (18) if the production is linear, and the corresponding parameters β and γ from equations (34) and (35) if the production is Cobb-Douglas. Then, I use those parameters to simulate the economy for multiple periods, with and without policy intervention. The optimal investment decision and the shadow risk-free rate are computed numerically every period by solving simultaneously equations (2) and (6), which hold for any specification. Finally, I compare the welfare outcomes for every combination of parameters.