

LECTURE 22

MATH 229

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1. MODEL THEORETIC PROPERTIES OF STRUCTURES

Today we will mostly state things and not prove much, and then we will do the NIP case in more detail.

1.1. Counting substructures. To motivate the introduction we start with the combinatorial question from last time of counting the number of substructures.

Assume the language is relational. Let M be a ω -categorical and $G = \text{Aut}(M)$. Define $f_n(M)$ to be the number of substructures of M of size n up to isomorphism. We could also define some other function (which we will not be using) that we just mention. First $F_n(M)$ is the number of n -types over \emptyset , and $\tilde{F}_n(M)$ is the number of n -types of pairwise distinct elements. The difference between the two is somehow that in f we somehow don't have an ordering on the things we are counting so we have

$$f_n(M) \leq \tilde{F}_n(M) < F_n(M) .$$

If substructures are rigid (e.g. we have a linear order) then

$$f_n(M) = \frac{\tilde{F}_n(M)}{n!} .$$

The problem of $\tilde{F}_n(M)$ is that this thing somehow grows too quickly whereas f_n has a slower growth rate so it's somehow more interesting to the combinatorialists.

Example 1. For DLO, $f_n(M) = 1$.

Example 2. For M the random graph we roughly have that

$$f_n(M) \approx F_n(M) \approx 2^{n^2}$$

since this is somehow asking for graphs with n vertices.

Example 3. Let $L = \{\leq\}$ be a partial order then a tree is when this is actually a linear order. Then $f_n(M)$ is the number of trees on n vertices, so it's c^n for some constant c which is between $2 < c < 3$. This is some kind of Catalan number. Note that this is smaller than a factorial.

If we take two linear orders $L = \{\leq_1, \leq_2\}$ then $f_n(M) = n!$. This is why these things are sometimes called permutation structures.

So we have seen constant growth rates, exponential, and factorial growth rates. Now we have some basic facts.

Fact 1 (Cameron). *(i) f_n is non-decreasing.*

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- (ii) f_n can grow as fast as one wants (with an infinite language).
- (iii) If the language has max arity k and M is homogeneous in L then $f_n(M) = \mathcal{O}(e^{n^k})$.

Question 1. Understand structures with slow growth rate of $f_n(M)$.

Fact 2 (Cameron). If $f_n(M) = 1$ for all n then M is one of the five reducts of DLO, i.e. one of the following:

- (i) (\leftarrow) : DLO
- (ii) (\leftrightarrow) : Betweenness relation $B(x, y, z)$
- (iii) (\odot) : Circular order $C(x, y, z)$
- (iv) (\oplus) : Separation relation $S(x, y, u, r)$
- (v) Pure equality

Remark 1. This class is a priori included in the reducts of DLO (since f_n only decreases when one takes reducts) so this fact was shown by showing the other inclusion manually on each reduct.

Fact 3 (Macpherson). If M is primitive, either $f_n(M) = 1$ for all n or

$$f_n(M) \geq \frac{c^n}{p(n)}$$

for $c = 2^{1/5} \approx 1.148$ and p is some polynomial depending on M .

Conjecture 1. In fact, it is true with $c = 2$.

Example 4. Note that 2 is possible. Take a circular order and add a relation where from every point we add an arrow to the other half of the circle. This should be thought of as a topological 2-cover of the circle.

1.2. Model theoretical notions. There is a model theoretic property (NIP) which is implied by a small growth rate of f_n .

Definition 1. A formula $\varphi(\bar{x}, \bar{y})$ has IP (in M) if for all $n < \omega$ we can find tuples

$$(\bar{a}_i \mid i < n) \quad (\bar{b}_j \mid j \in \mathcal{P}(n))$$

in M such that

$$M \models \varphi(\bar{a}_i, \bar{b}_j) \iff i \in j.$$

Definition 2. We say M is NIP if no formula has IP.

So IP somehow gives us something complicated and it has NIP if this doesn't happen.

Example 5. The graph relation $R(x, y)$ has IP in M the random graph.

Example 6. Higher order things like DLO and trees are NIP. NIP formulas are closed under Boolean combinations.

Lemma 1 (Sauer-Shelah). If $\varphi(\bar{x}, \bar{y})$ is NIP then for finite $A \subseteq M^{\bar{y}}$ there is some k such that

$$|S_\varphi(A)| = \mathcal{O}(|A|^k)$$

where $S_\varphi(A)$ consists of the φ -types over A , i.e. the maximal consistent set of formulas of the form $\varphi(x, \bar{b})$ where $\bar{b} \in A$.

Note if φ has IP

$$|S_\varphi(A)| = 2^{|A|}$$

for some A of arbitrarily large size.

This has the following consequence. Assume M is homogeneous in a finite relational language and every relation in L is NIP (for every partition of the variables) then

$$f_n(M) = \mathcal{O}(e^{cn \ln n})$$

for some $c > 0$ so its growth rate is at most factorial.

For simplicity assume we have 1-formula. Consider some n -elements a_1, \dots, a_n . We might as well count $F_n(M)$ since the factorial won't change whether or not we satisfy this bound. Then the number of n -types over \emptyset is at most

$$S_n(\emptyset) \leq S_1(\emptyset) \cdot S_1(\{a_0\}) \cdot S_1(\{a_0, a_1\}) \cdot \dots$$

and by NIP this is a product which looks like:

$$S_n(\emptyset) \leq 1^k \cdot 2^k \cdot 3^k \cdot \dots \cdot n^k \leq (n!)^c$$

for some c .

Fact 4 (Macpherson). *If some $\varphi(\bar{x}, \bar{y})$ has IP then*

$$f_n(M) \geq 2^{p(n)}$$

for some polynomial p of degree at least 2.

For finitely homogeneous structures we get a gap that either

$$f_n(M) = \mathcal{O}(e^{cn \ln n})$$

where M is NIP, and

$$f_n(M) \geq e^{p(n)}$$

(for $\deg p \geq 2$) in the IP case. So the picture is:

$$\underbrace{1 \quad |\text{gap}| \quad \underbrace{e^{cn}}_{\text{trees}} \quad \dots \quad \underbrace{e^{cn \ln n}}_{2 \text{ lin ord}} \quad |\text{gap}| \quad e^{cn^2} \dots}_{\text{NIP}}$$

New goal: understand NIP for finite homogeneous (or ω -categorical) structures.

1.3. Stability.

Definition 3. $\varphi(\bar{x}, \bar{y})$ is unstable (in M) if for every n there are $(\bar{a}_i \mid i < n)$ $(\bar{b}_j \mid j < n)$ such that

$$M \models \varphi(\bar{a}_i, \bar{b}_j) \iff i \leq j.$$

Definition 4. M is stable if all formulas are stable.

Note that stable implies NIP.

Example 7. $L = \{E\}$ where E is an equivalence relation then M is stable. DLO is NIP but not stable.

Example 8. \mathbb{F}_p vector spaces are stable ω -categorical but not finitely homogeneous.

- (1) If M is finitely homogeneous and stable then it is ω -stable.
- (2) ω -categorical and ω -stable structures are very understood. They are all build out of $(M, =)$, \mathbb{F}_q vector spaces by families of covers. More precisely:

- Fact 5.**
- *A primitive ω -categorical ω -stable is a Grassmannian over one of:*
 - $(M, =)$
 - *Affine or projective space over a finite field.*
 - *A finite homogeneous stable primitive structure is interdefinable with unordered k -tuples of distinct elements of $(M, =)$.*

Once we know that these are the primitive ones, one can just check what happens with f_n . For $(M, =)$, $f_n = 1$. The 2-Grassmannian of $(M, =)$, then f_n is roughly the set of graphs with n edges. This is hard to compute, but certainly it grows faster than c^n but the actual asymptotic behavior is not known.

In the case of the \mathbb{F}_p vector space f_n looks like the following. This has at least the number of partitions, so f_n also grows faster than c^n .

The conclusion is that if \mathcal{M} is ω -stable, primitive, and $f_n = \mathcal{O}(c^n)$, then $\mathcal{M} \cong (M, =)$. So then the unstable case is left which we will deal with next time.

Stable not ω -stable things are not understood at all. Pseudo-planes can get some lower bounds of f_n by powers of \sqrt{n} but even powers of n hasn't been done.