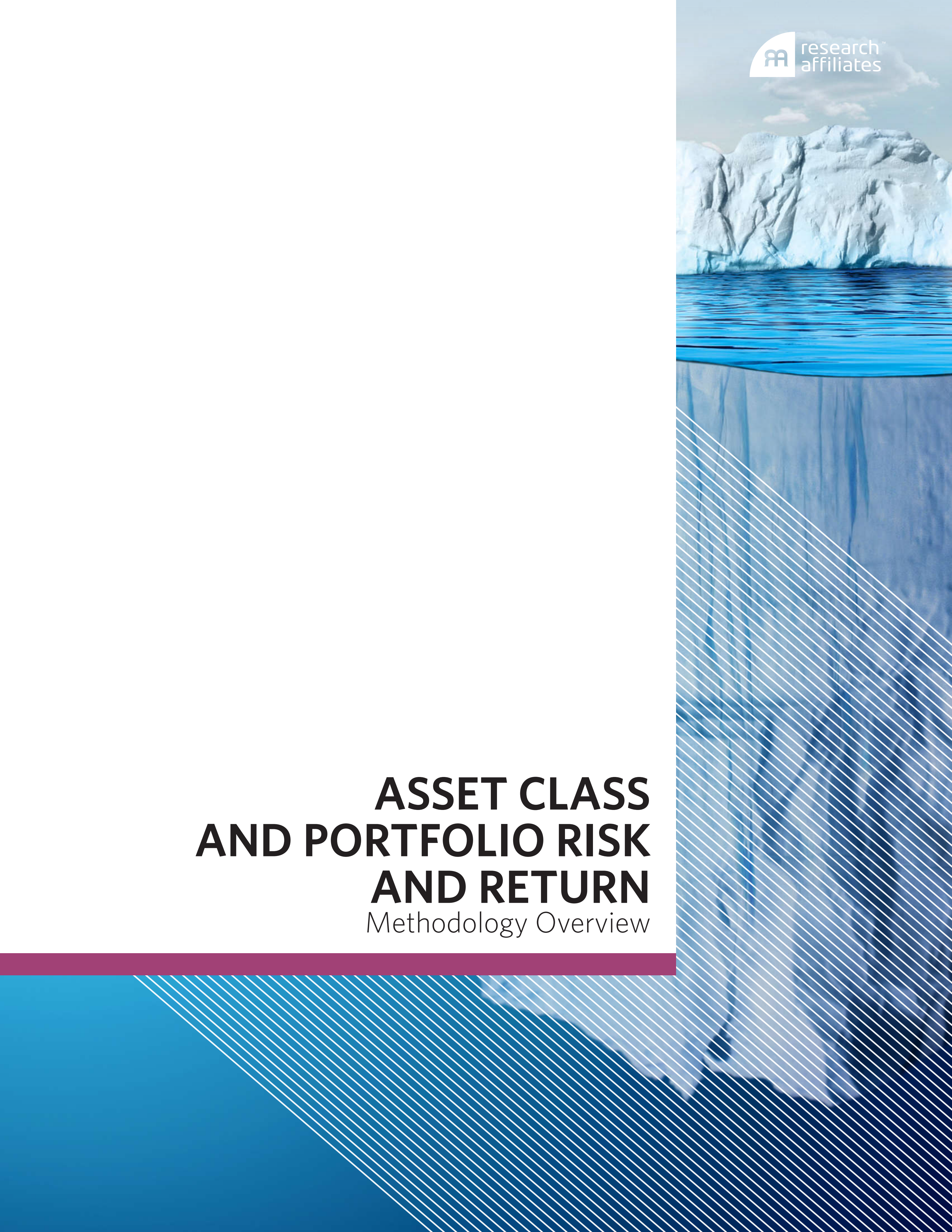
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ASSET CLASS AND PORTFOLIO RISK AND RETURN

Methodology Overview

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OVERVIEW

What is This Document?

> ***"We understand that some of our insights will never find their way into products, but we provide them in support of investors and the finance community."***
>
> **— ROB ARNOTT**
> CHAIRMAN & CEO

This is one in a series of plain-language white papers setting forth Research Affiliates' building block approach to developing long-term capital market expectations by asset class. (For information about the objectives and guiding principles of our asset allocation initiative, please refer to "Capital Market Expectations: Methodology Overview," the first of these white papers.) In working out our risk and return forecasts and making them publicly available, we keep three criteria in mind: transparency, robustness, and timeliness. By describing the conceptual framework and calculations behind the projected asset class risks, returns, and correlations in these papers, we hope to achieve a meaningful level of transparency without excessive details. By constructing simple, economically sound models for major asset classes, we strive to achieve a fitting standard of robustness for forecasting to a 10-year horizon. By initially refreshing our expectations on a quarterly basis, we seek to provide information that is updated with useful frequency. We will continue to refine our methods, extend the scope of our capital market expectations, and improve this documentation over time.The remainder of this document addresses how we think about asset class risk, and provides transparency into the methods employed to develop these risk forecasts.

This document outlines the process of estimating risk for asset classes, quantifying uncertainty in capital market expectations over short time horizons, and combining asset class risk-and-return expectations into portfolios that investors can use.

Asset Class Risk

Risk Model Overview

A corollary to expected returns forecasting is forecasting asset class risk. This is most commonly done by measuring volatility and correlations,[1, 2] either directly between the asset classes using historical time series or indirectly through the use of factor models.

The asset class approach entails directly forecasting the volatility and correlations of the asset class pairs and results in a covariance matrix that grows as the number of asset classes in the investment universe grows.

The factor model approach provides a layer of abstraction by modeling characteristics common to asset classes rather than modeling the asset classes themselves. Asset class risk is measured by mapping the asset classes to the factors via a set of exposures. If the factor set is comprehensive, the factor model framework also allows new assets to be easily added to the investment opportunity set by simply measuring exposures to the set of factors.

In this document, we describe the factor model approach.

FACTOR MODEL APPROACH

Factor models grew out of the research of Stephen Ross, who broached the topic in his description of arbitrage pricing theory (APT) in his article "The Arbitrage Theory of Capital Asset Pricing" published in 1976. He left the details of specific factor definition to other research, however.

The basic equation for calculating an asset-class covariance matrix (Σ) using a factor model is

$$\Sigma = \beta^T \Omega \, \beta + D \qquad (1)$$

In the equation, beta (β) represents the exposure of each investment to each factor; omega (Ω) represents the factor covariance matrix, which contains the variance of each factor and the covariance of each factor pair; and D represents the idiosyncratic risk component of each asset. This framework can be used to identify the risk of each asset and is used in portfolio construction, as described later.

The resultant asset-class covariance matrix contains, on the diagonal of the matrix, the variance of each asset, and in the off-diagonal terms, the asset's covariance with the other assets in the matrix.

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[1] *As used in this document, correlation refers to Pearson correlation.*

[2] *We focus on volatility and correlation as the measures of risk, but investors should also consider higher moments, such as skewness, as alternative measures of risk*

TYPES OF FACTOR MODELS

A large literature on factor models exists. The three types of factor models most commonly used are statistical, explicit, and implicit (Baturin, Cahan, and He, 2014). All three have the same functional form as described in the previous section, but are very different in nature.

A statistical model utilizes principal component analysis to identify the sources of risk in a set of asset classes as a set of purely statistical factors. The benefit of this type of model is that it requires, from a data perspective, nothing more than a time series of asset class returns. The challenge is that, because the factors are statistical in nature, they often lack economic intuitiveness. Therefore, although these models provide a robust mechanism for capturing risk, it may come at the expense of the intuitive understanding necessary to make investment decisions.

An explicit factor model is likewise convenient because it also only requires a time series of asset class returns. These models construct factors based on time series data, often by constructing long/short portfolios of two or more assets. A covariance matrix is then assembled based on the time series volatility of each factor and the correlation between each pair of factors. Exposures of each investment to the factors are then determined.

The third type of factor model, an implicit factor model, uses a cross-section of fundamental information. Unlike the explicit model, which directly measures factor returns and then determines, through regression or other means, exposures to those factors, implicit models use fundamental information to directly measure exposures. For example, price-to-book ratio could be used to measure the exposure of an asset to a value factor. Once the exposures of the asset universe are calculated, cross-sectional regression is used to generate the factor returns. The volatility and covariance of those factor returns over time is then used to generate a covariance matrix. Although implicit factor models have proven to be very useful, they can be very costly from a data and analytics perspective. Using an implicit factor model requires access not only to asset returns, but to fundamental information about each asset.

The remainder of this document focuses on constructing and using an explicit factor model.[3]

Model Creation

We now turn our attention to the process of factor selection and the mechanical construction of the factor and asset-class covariance matrices.

FACTOR SELECTION

In this model, factor selection involves identifying the set of characteristics that ultimately drive risk and return. Each of the factors is created as a zero investment (long/short) portfolio of asset classes with the goal of creating factors that are able to capture various sources of return premia. Modeling sources of return across a broad set of asset classes requires inclusion of a large list of factors, such as those specified in **Table 1**.

---

[3] *Some readers may wonder why we use a factor model at all. If the set of asset classes in question is diverse enough, the set of factors needed to model those assets will map one- to- one to the assets themselves. In this case, generating a factor model is nothing more than reorganizing the assets in a different form. We acknowledge this to be true, but creating a factor model allows us to easily replicate an ever expanding list of assets as well as to explain the process of constructing factor models to readers who are unfamiliar with the methodology.*

TABLE 1

Selected Factors Used to Model Asset Class Risk

Risk Factor	Long Index	Short Index
Real Cash	U.S. T-bills	U.S. CPI
Market	S&P 500	U.S. T-bills
Value	Russell 1000 Value	Russell 1000 Growth
Size	Russell 2000	S&P 500
Country	MSCI Country	S&P 500
U.S. Short Duration	Barclays US Treasury 1-3 Year	U.S. T-bills
U.S. Mid Duration	Barclays US Treasury 3-7 Year	Barclays US Treasury 1-3 Year
U.S. Long Duration	Barclays US Treasury 7-10 Year	Barclays US Treasury 3-7 Year
U.S. Credit Spread	Barclays Intermediate Credit	Barclays US Treasury 3-7 Year
U.S. HY Credit Spread	Barclays Corporate High Yield	Barclays Intermediate Credit
EM Hard Currency Spread	JPM EMBI+	Barclays US Treasury 3-7 Year
EM Local Currency Spread	JPM GBI-EM	Barclays US Treasury 3-7 Year
REITS	FTSE NAREIT	U.S. T-bills
Commodities	Bloomberg Commodity	U.S. T-bills

Source: Research Affiliates, LLC

Creating factors as long/short portfolios is beneficial because it isolates return premia and allows us to reduce collinearity between factors, which could otherwise cause spurious correlations to appear when measuring the exposure of securities to the factors.[4]

COVARIANCE MATRIX CONSTRUCTION

The covariance matrix is generated by calculating the covariance between each pair of factors. Two important considerations in constructing the covariance matrix are the amount of historical data to include and the method of weighting the data.

The simplest solution is to include all available data. In a perfect world, all factors would have an identical length and span of data history, and this would be a fine approach; in the real world, however, factors have very different time histories. For example, a U.S. market equity factor that uses the S&P 500 Index may have data for 100 years or more, whereas an emerging markets (EM) equity factor may have only 30 years of reliable history. When factors have different lengths of data history, using a full time series for each factor with all data weighted equally results in an unpredictable covariance matrix.

---

[4] *In creating the long/short portfolios, it is vital to make sure risk premia are isolated and are not corrupted by extraneous risks. For example, in creating a credit-spread factor, care should be taken to match the duration of the indices used in isolating a particular credit spread.*

In an effort to be consistent, it is possible to restrict the historical data used to construct the covariance matrix to a time interval shared by the factors. The obvious drawback of this approach is that a considerable amount of data could be ignored, such as in the previous example of 100 or more years of U.S. equity returns restricted to 30 years of EM equity returns. Restricting data to a common time window would cause decades of U.S. data to be discarded and along with it the information contained in that data. This is not an ideal approach, which leads us to consider another possible weighting scheme.

EWMA APPROACH

Instead of equally weighting data samples, as is done in a simple calculation of covariance, another approach is to create an exponentially weighted moving average (EWMA) as described by Jorion (2006). The EWMA weights the data so as to de-emphasize older time samples.

The EWMA methodology is based on a decay factor, lambda (λ), which dictates the rate at which older data is de-emphasized. The decay factor defines the half-life, the point at which the data are weighted at 50% of original value. The half-life we use is 120 months, which means that 10-year-old data are weighted one-half as much as current data, data that are 20 years old are weighted one-quarter as much, and so on. The EWMA equation, which follows, shows how recursively multiplying older returns by λ reduces their significance:

$$\sigma_t^2 = (1-\lambda)\times r_{t-1}^2 + \lambda \times \sigma_{t-1}^2$$

$$\text{where } \lambda = \frac{1}{2}^{\frac{1}{half-life}} \tag{2}$$

By de-emphasizing very old data, the EWMA approach addresses the issue of disparity in time histories across asset classes. Instead of, or even in addition to, weighting samples differently over time, it is also possible to infer historical data in order to extend shorter time series via the maximum likelihood estimator approach, which is discussed in the next section.

MLE APPROACH

The maximum likelihood estimator (MLE) approach, as described by Stambaugh (1997), uses ordinary least squares (OLS) regression to identify the relationship between two time series over the common period in which they both have data. The resultant regression equations are then used to estimate additional history for the shorter time series going back in time. Because this approach simulates data, care should be taken in examining the results of the regressions and the amount of history that is being estimated. Even with these precautions, however, the MLE approach is a strong method of addressing disparity in the histories of data series.

Once the issue of time-series length has been addressed, the covariance matrix can be easily generated using textbook equations for calculating variance and covariance.

VALIDATING THE MATRIX

It is important to test the covariance matrix that has been generated in order to verify that it is positive semi-definite. Because the diagonal of the covariance matrix represents variance values, which are the squares of volatilities, negative values on the diagonal would be nonsensical.

Testing that the matrix is positive semi-definite is done by ensuring that all the eigenvalues, or equivalently, all of the pivots, of the matrix are greater than or equal to zero.

MEASURING FACTOR EXPOSURES

Once the covariance matrix is constructed, the exposure of each asset in the investment universe must be measured versus each factor. This can be done by running an OLS regression of the returns of the asset class, the dependent variable, against the returns of each of the factors, the independent variables, as follows, where the regression coefficients are the exposures of the asset class to each factor:

$$r_t = \alpha + \sum_{i=1}^{N} \beta_i \times F_{i,t} + \varepsilon_t \tag{3}$$

Using an OLS regression to identify exposures does have a downside. First, because it can be difficult to construct a set of factors that are completely uncorrelated, the collinearity between factors can lead to noisy exposures that do not measure the true magnitude of the economic drivers of the risk and return of the asset class. Although it is easy to say that factors need to be uncorrelated, in practice this can be hard to achieve.

Second, although OLS will give the best fit of the historical data, the regression results may not be the best predictor of future outcomes. For example, consider an aggregate index, such as the Barclays U.S. Aggregate Bond Index. An OLS regression of this index on a set of bond factors will result in an exposure to sovereign, credit, and structured product factors. These exposures will do a good job of fitting the historical weights of the index to the factors, but they will not be effective in capturing a structural change in the composition of the index. Thus, if the index provider changes the structure of the index to better capture the distribution characteristics of the outstanding debt in an economy, such information would be missed in the OLS results and be unavailable to inform *ex ante* return forecasts.[5]

Before addressing these two issues—noisy exposures and unreliable forecasts—it is important to have an opinion regarding the exposure of each asset class to each factor. Establishing an opinion before running the OLS regression provides two sources of data for determining the best exposure set to use in the factor model. In some cases, a prior belief will be validated by the OLS results, and often the OLS results add useful information to the decision-making process.

MEASURING IDIOSYNCRATIC RISK

The idiosyncratic risk describes the volatility of the residual (ε) of the regression equation, or the difference between the actual and estimated returns ("hat" variables):

$$\begin{gathered} \sigma^2_{residual} = \sigma^2\left(r_t - \hat{r}_t\right) \\ where\ \hat{r}_t = \hat{\alpha} + \sum_{i=1}^{N} \hat{\beta}_i \times F_{i,t} \end{gathered} \tag{4}$$

---

[5] *The Barclays U.S. Aggregate is used as a hypothetical example and is not meant to imply that a structural change has recently occurred or is expected to occur in the index in the near future.*

The *D* term in Equation (1) is an "n × n" matrix, where "n" represents the number of asset classes. The diagonal of the matrix represents the variance of the residual of each asset, with the off-diagonal terms being equal to zero.

UNCERTAINTY IN ASSET CLASS EXPECTATIONS

With the information provided in this document, along with the information provided in the other documents in the Research Affiliates online library, it is possible to generate point forecasts for expected risk and return across asset classes. We now turn our attention to measuring the uncertainty that accompanies the use of long-term forecasts over shorter time horizons.

The foundation for the expected-return point forecast is the fact that if the time horizon is long enough, realized returns will average out to the expectation, and as the time horizon (in this case measured in years) shortens, the variability of returns around the point forecast increases.

To put this variability into context, take, for example, an equity index with a 10-year average expected return. Now consider that someone with perfect foresight gives you a bag of ping pong balls, one for each future year, and on each ball is written the return that will be earned in that year. If you pull 10 balls from the bag and average them, it is very likely that the result will differ from the 10-year average expected return. If, however, the experiment is repeated a large number of times and the results are plotted on a histogram, a distribution will emerge whose average will be equal to the expected return.[6]

CONFIDENCE INTERVALS

A simple estimation of the confidence interval is calculated as the expected-return point forecast, plus or minus the size of the interval, multiplied by the standard error of the estimate (standard deviation divided by the square root of the number of samples, *T*, in the investment horizon):[7]

$$\text{Confidence Interval}_{Annualized} = E\left[r\right]_{Annual} \pm Z \times SE = E\left[r\right]_{Annual} \pm Z \times \frac{\sigma}{\sqrt{T}} \tag{5}$$

From this equation, it is easy to see that as the number of samples, *T*, goes to infinity, the interval, as expected, collapses to the point forecast. Certain assumptions underlying this equation, which affect the results, are not applicable to asset class returns.

In particular, calculating the standard error in this way assumes that each annual return is both independent and identically distributed (i.i.d.) such that the covariance between annual returns is zero (Diebold *et al*., 1998).

MEAN REVERSION IN ASSET CLASS RETURNS

In real life, however, asset class returns experience both momentum and mean reversion (MMR), and the standard error should be enhanced to take this into account.

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[6] *In this situation, the expected return and volatility are assumed to be unbiased estimators (i.e., the true statistics) for the future return of each asset class (i.e., any model misspecification risk is assumed to be zero).*

[7] *We have parameterized the number of samples (years in the investment horizon or single-experiment ping pong balls pulled from the bag in the example in the previous section) as T = 10. Alternatively, the standard error could be calculated from an OLS regression of the expected and realized returns. Here we are using the z-distribution, but because of the size of T, others may decide to use the Student's t-distribution.*

STATISTICAL EVIDENCE OF MEAN REVERSION

As Alexander (2008) discusses, MMR can be measured by the autocorrelation of returns within each asset class using an AR(1) model; autocorrelation values greater than zero indicate momentum in returns, whereas values less than zero signify mean reversion.

In the following equation, T is the number of lagged periods, and δ is the AR(1) autocorrelation coefficient:

$$\text{MMR} = \left[T + 2\frac{\delta}{(1+\delta)^2}\left[(T-1)(1-\delta) - \delta\left(1-\delta^{T-1}\right)\right]\right]^{\frac{1}{2}} \times \frac{1}{\sqrt{T}} \tag{6}$$

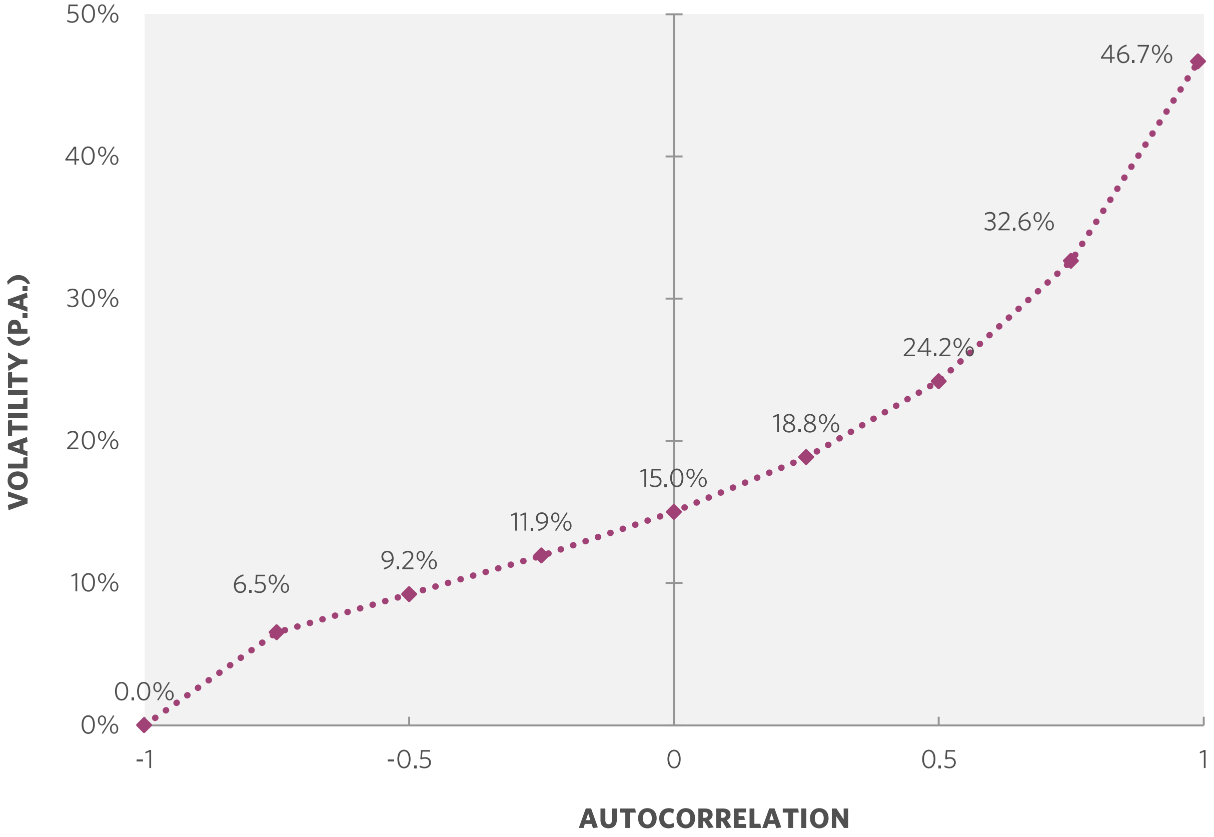
The MMR value can then be integrated into the confidence interval equation,

$$\text{MMR Confidence Interval}_{Annualized} = E[r]_{Annual} \pm Z \times \frac{\sigma}{\sqrt{T}} \times MMR \tag{7}$$

Figure 1 shows changes in per annum volatility for various autocorrelation values, based on an i.i.d. volatility of 15%. On the left side of the figure, autocorrelation values are below zero, denoting mean reversion and a reduction in volatility, whereas on the right side of the figure, the autocorrelation values are above zero, denoting momentum and a rise in volatility.

FIGURE 1

Per Annum Volatility for Autocorrelation Values of an Index with 15% Volatility



Source: Research Affiliates, LLC

Figure 2 shows annual, non-overlapping, autocorrelation results for the S&P 500. Unfortunately, the autocorrelation coefficients at a 10-year horizon are not statistically significant.[8] The same results are seen in the bond, currency, and commodity asset classes.

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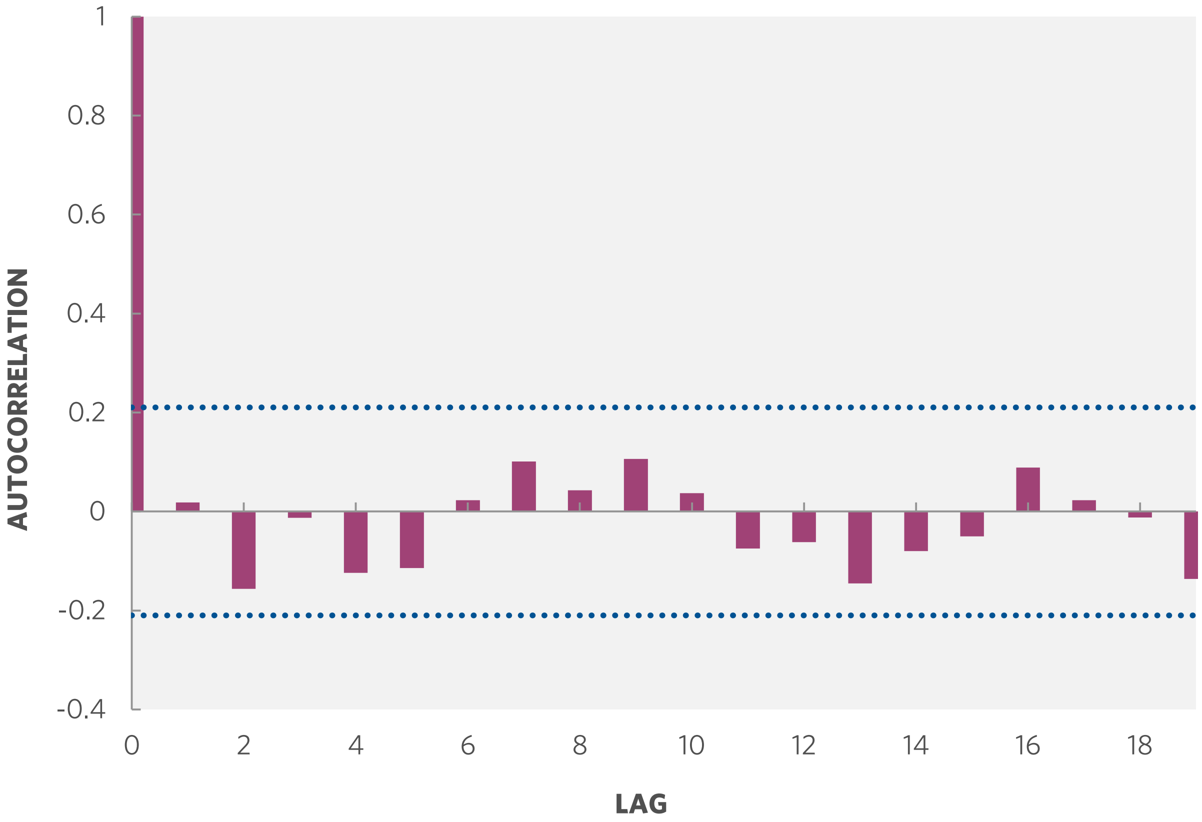
[8] *All references in this paper to statistical significance are at the 95% level.*

FIGURE 2

Autocorrelation of Annual Returns of S&P 500, 1926–2014 (p-value of Box–Pierce[9] = 0.86)



Source: Research Affiliates, LLC, based on data from Bloomberg.

In order to further test the statistical significance of mean reversion, we calculate variance ratios (Lo and MacKinlay, 1988) based on monthly periods out to 120 months.[10, 11] The tests show some amount of mean reversion at longer time periods, but the findings are not significant. Here we only include the variance ratio chart for equities, but other asset classes show similar results.

We analyze the variance ratio results because they, as well as other tests, are often included in articles on asset-class mean reversion. The lack of statistical significance of the tests is not unexpected. In fact, it is known that most statistical tests—including unit root, regression based, and maximum likelihood—are not powerful enough tools to identify mean reversion. Unfortunately, this is an example of when more data cannot improve the outcome.

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[9] *The Box–Pierce test measures the statistical significance of the autocorrelation residuals.*

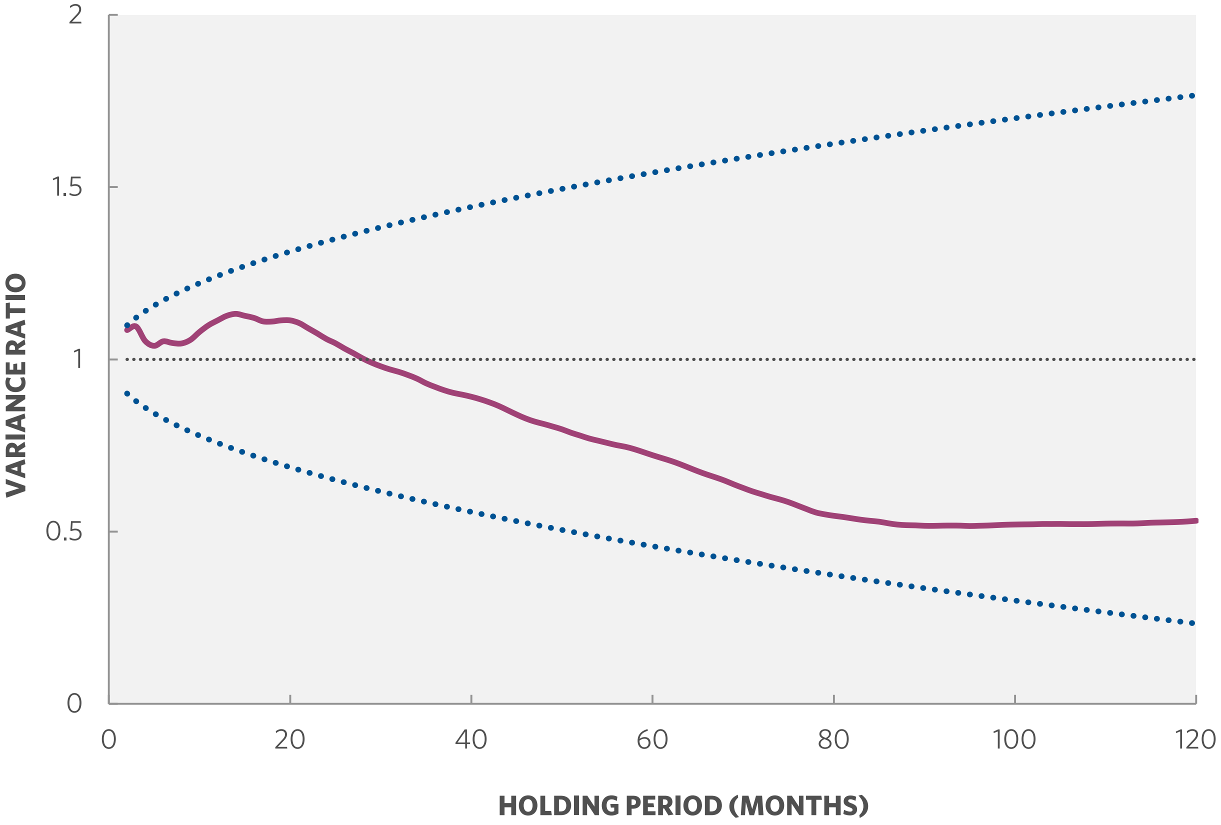
[10] *"Variance ratios are among the most powerful tests for detecting mean reversion," according to Poterba and Summers (1988). The variance ratio test is based on the idea that if a series is stationary, the variance of the series should not change over time; a series with a unit root, however, should have a changing variance. Comparing the variance of a time series of returns to the variance of lagged returns, $VR(q) = \sigma^2(q) / \sigma^2(1)$, will result in a value of one (random walk), values greater than one (momentum), or values less than one (mean reversion).*

[11] *We use monthly periods to show additional granularity in the figures, but the results using annual periods are consistent with the monthly results.*

Given the data shown in **Figure 3** and the lack of statistical significance in mean reversion, it would be easy to revert to the simple confidence interval as the best estimate of asset class returns, but doing so would ignore the fact that empirical mean reversion is observed in asset class returns. **Figures 4–6** show volatility of annualized returns for equity (S&P 500), bond (Barclays Global Aggregate), and commodity (S&P GSCI) indices for investment time horizons of 1–10 years[12] and compares the values to an estimated volatility computed by dividing one-year volatility by the square root of *T*.

FIGURE 3

S&P 500 Variance Ratio (1926–2015), Monthly



Source: Research Affiliates, LLC, based on data from Bloomberg.

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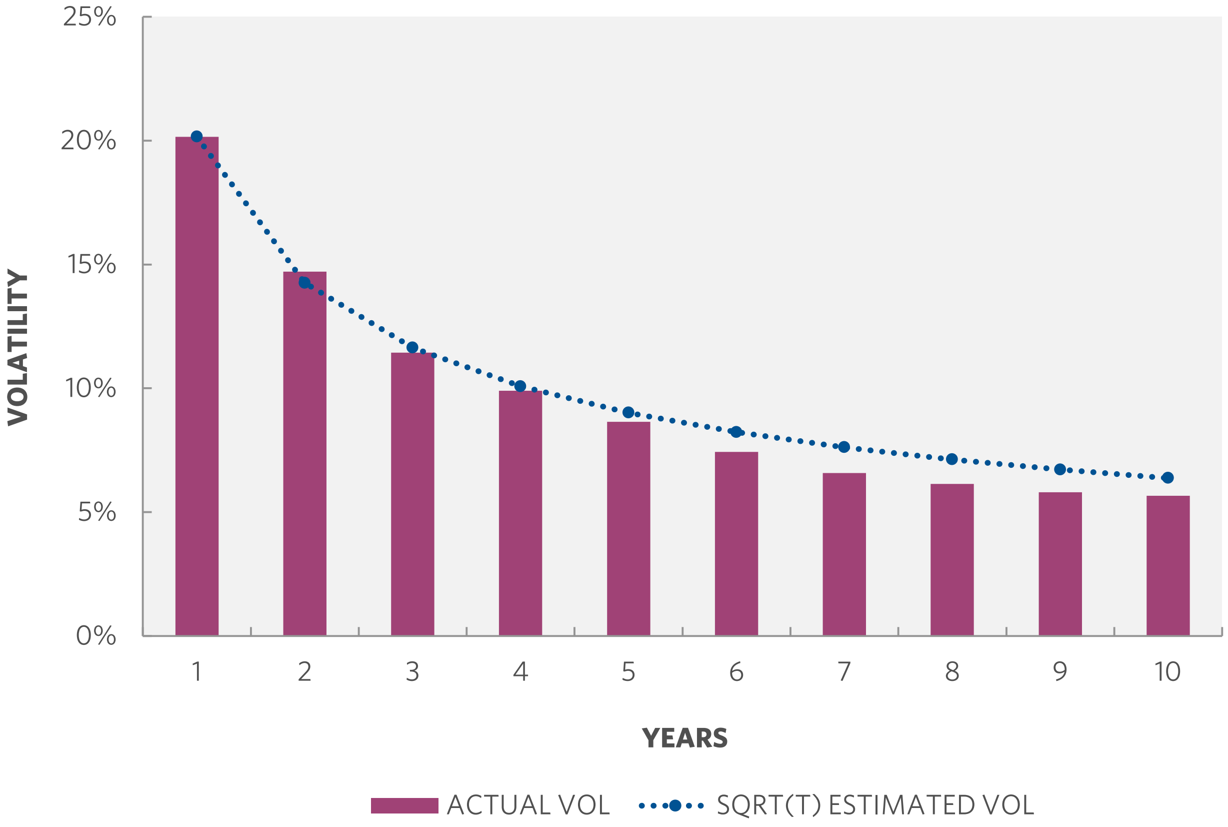
[12] *The x axis represents the time horizon. Therefore, plot points with an x coordinate of five show annualized five-year returns. These charts are based on year-end data.*

FIGURE 4

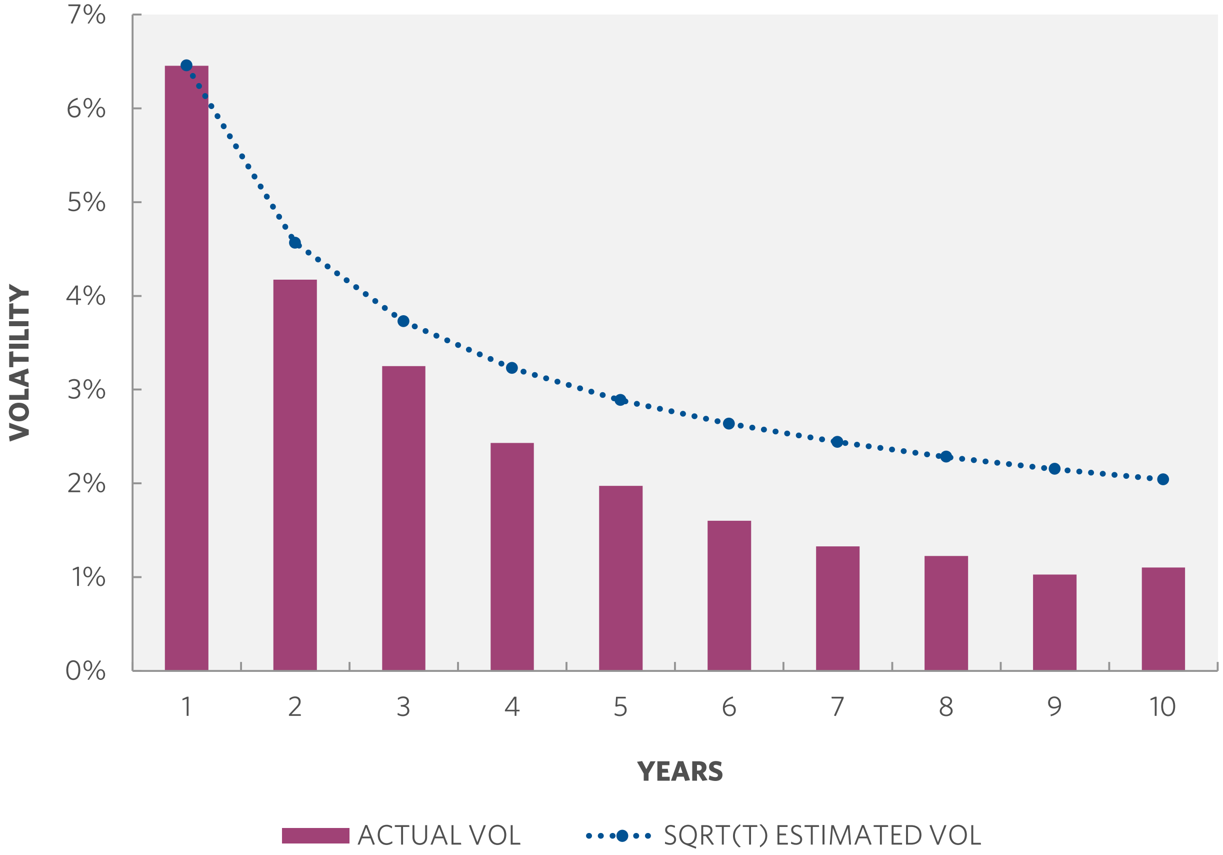
S&P 500 Volatility of Annualized Returns at 1–10 Year Horizons, 1926–2015



Source: Research Affiliates, LLC

FIGURE 5

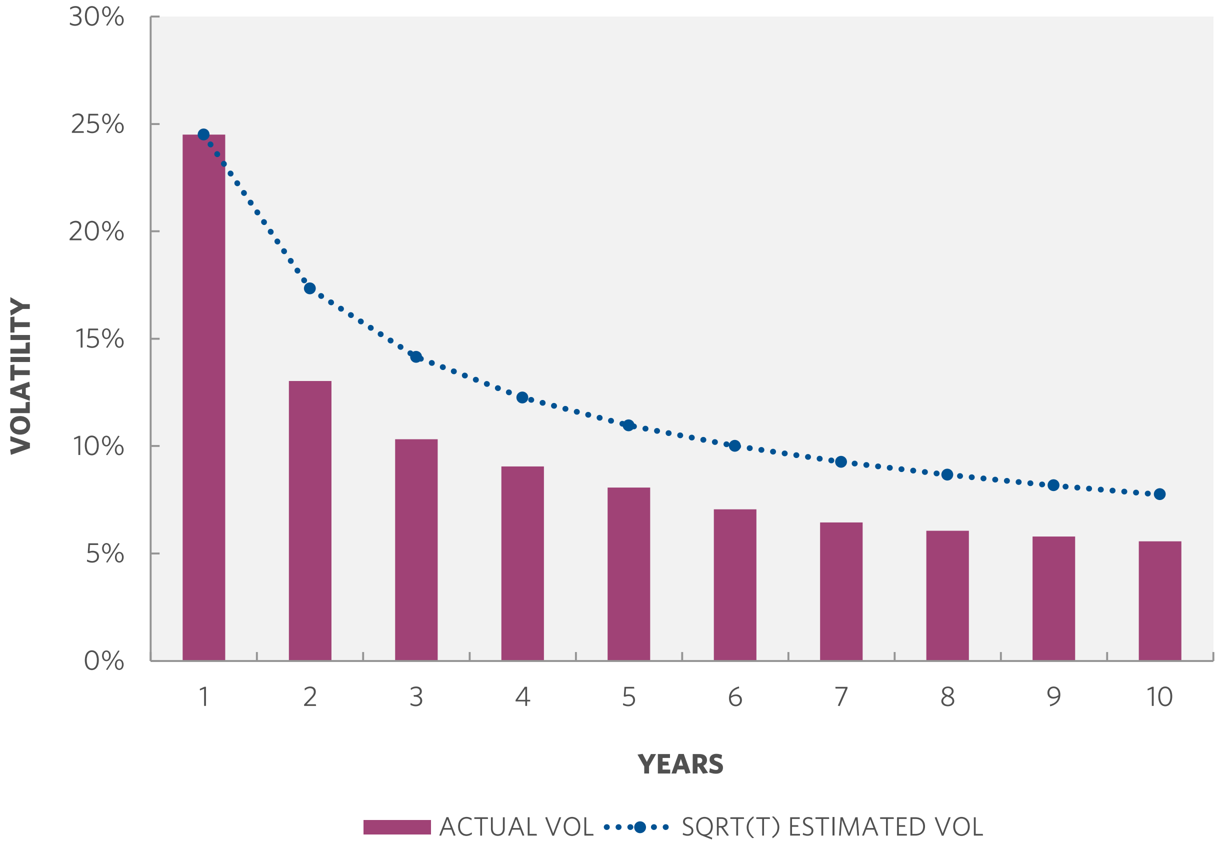
Barclays Global Aggregate Volatility of Annualized Returns at 1–10 Year Horizons, 1990–2015



Source: Research Affiliates, LLC

FIGURE 6

S&P GSCI Commodity Index Volatility of Annualized Returns at 1–10 Year Time Horizons, 1970–2015



Source: Research Affiliates, LLC

Across asset classes, at longer time horizons, the historically realized volatility of returns is lower than an estimate of volatility based on scaling the one-year volatility by $1/\sqrt{T}$, which implies mean reversion does exist. **Table 2** quantifies the difference at a 10-year investment horizon.

TABLE 2

Comparing the Square-Root-of-T Rule to Historical Returns

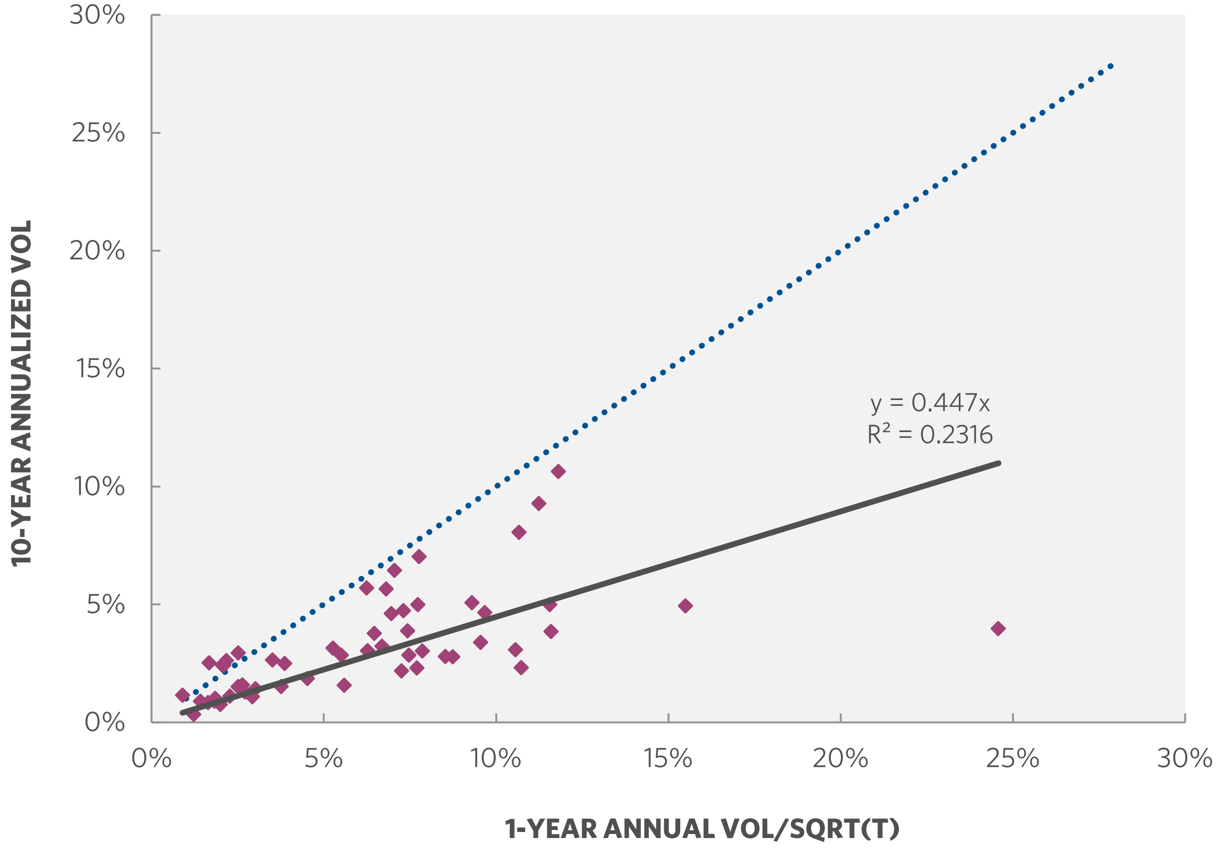
Asset Class	Scaled 1-Year Volatility	10-Year Volatility	Ratio
Equity	6.3%	5.5%	116%
Bond	2.0%	1.1%	185%
Commodity	5.9%	5.2%	115%

Source: Research Affiliates, LLC

Thus, if we ignore mean reversion, confidence intervals would be 15% larger for equities and commodities, and 85% larger for global bonds, than has been empirically observed. **Figure 7** reports the results when the sample is extended to a much larger set of 55 indices across equities, bonds, and commodities.

FIGURE 7

Comparing the Square-Root-of-T Rule to Historical Returns for Indices across Asset Classes



Source: Research Affiliates, LLC

If asset returns were i.i.d, in Figure 7 all the diamonds would line up on or near the 45-degree line. Instead, most of the diamonds—except those representing fixed income indices with very low volatility[13]—fall below the line. The R^2, which equals 23%, is being pulled lower by the outlier index (MSCI Turkey) in the lower far-right area of the figure. Without the data point of MSCI Turkey, the R^2 roughly doubles. Figure 7 illustrates that if we multiply the last term of Equation (6), $1/\sqrt{T}$, by $1/2$, the result is a more appropriate measure of the confidence interval. We, therefore, use this 1/2 scalar in our confidence interval calculations, as follows:

$$\text{Confidence Interval}_{Annualized} = E\left[r\right]_{Annual} \pm Z \times SE = E\left[r\right]_{Annual} \pm Z \times \frac{\sigma}{2\sqrt{T}}$$

Portfolio Analytics

Now, let's turn our attention to creating portfolios from individual asset class risk-and-return expectations.

PORTFOLIO RETURN AND RISK

In discussing future returns, the two metrics most often referenced are the geometric expected mean (average growth of compounded wealth) and the arithmetic expected mean (probability weighted average of all possible future states of return). Each is the forward-looking corollary of its respective historical counterpart: the geometric mean and the arithmetic mean. Ibbotson and Chen (2003) observe that the arithmetic average is the measure of returns most often used in portfolio optimization. Kaplan (2012) adds additional context to the interplay between arithmetic and geometric expectations when he observes that in the Markowitz investment model, the relevant measure of reward is the expected (arithmetic) return. He notes that the case is different for long-term investors, who are more concerned with the long-term rate of portfolio growth, or the geometric mean.

Up to now, we have described a methodology for calculating the geometric expected return. The reason the results thus far are geometric in nature goes back to the derivation of the dividend discount model, which is based on a geometric series.

The nice thing about the geometric expected mean is that it provides an apples-to-apples comparison with the commonly used geometric mean of historical returns, also known as the compound annual growth rate of an asset. The downside of the geometric expected mean is that these returns cannot directly be used to create portfolios. As described by McCulloch (2003), the geometric return of a portfolio is not equal to the weighted average of the geometric returns of the portfolio's constituents, as might be expected.

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[13] During the majority of the sample period, interest rates declined in a near-linear trend. This downward trajectory in rates likely impacted the risk attributes of the fixed income indices. Greater volatility in interest rates over the last 40 years would likely have produced different results.

In order to find the geometric return of a portfolio, we must first convert each asset's return to an arithmetic return, aggregate them based on a weighted average, and then convert the arithmetic portfolio return back to a geometric return. The following equation is an approximation, under the commonly held assumption of log normally distributed returns, which is used to convert back and forth between geometric and arithmetic returns:[14]

$$E\left[r_{arithmetic}\right]=E\left[r_{geometric}\right]+\frac{\sigma^2}{2}$$

Given that the arithmetic mean is almost always larger than the geometric mean, it is no surprise that converting a geometric return to arithmetic involves adding an additional term. In this case that term, $\sigma^2/2$, is one-half the variance of the asset class. The arithmetic return of each asset class is then aggregated into a single portfolio. This is done by multiplying the arithmetic return by the weight of each asset in the portfolio,

$$r_{p,arithmetic}=\sum_{i=1}^{N} w_i \times r_{i,arithmetic}$$

The asset-class covariance matrix can then be used to calculate the risk of the portfolio by multiplying the matrix by the weights of the individual assets in the portfolio, as follows:

$$\sigma_p^2 = w^T \Sigma w$$

Finally the geometric return of the portfolio can be determined by converting the arithmetic return using the calculated portfolio variance. From these portfolio point estimates, confidence intervals should also be calculated to account for the variability of expectations over shorter time horizons.

EFFICIENT FRONTIER

The efficient frontier is the only set of portfolios that a rational investor would choose, because, by definition, all other portfolios could be improved by either increasing return or decreasing risk. All that is required to create an efficient frontier is the set of expected risk-and-return forecasts for the asset class as well as a set of constraints, whose purpose is to prevent undesired outcomes, such as an overconcentration in a particular security or sector.

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[14] *A downside of using this approximation is that it assumes returns are independent over time, but that is not the case as previously discussed in the subsection on confidence intervals; that said, the formula is included here as a simple heuristic. McCulloch (2003) expounds on a more detailed relationship, as does Kaplan (2012), who shows that as the investment horizon grows longer.*

$$E\left[r_{geometric}\right]=\frac{\left(1+E\left[r_{arithmetic}\right]\right)^2}{\sqrt{\left(1+E\left[r_{arithmetic}\right]\right)^2+\sigma^2}}-1$$

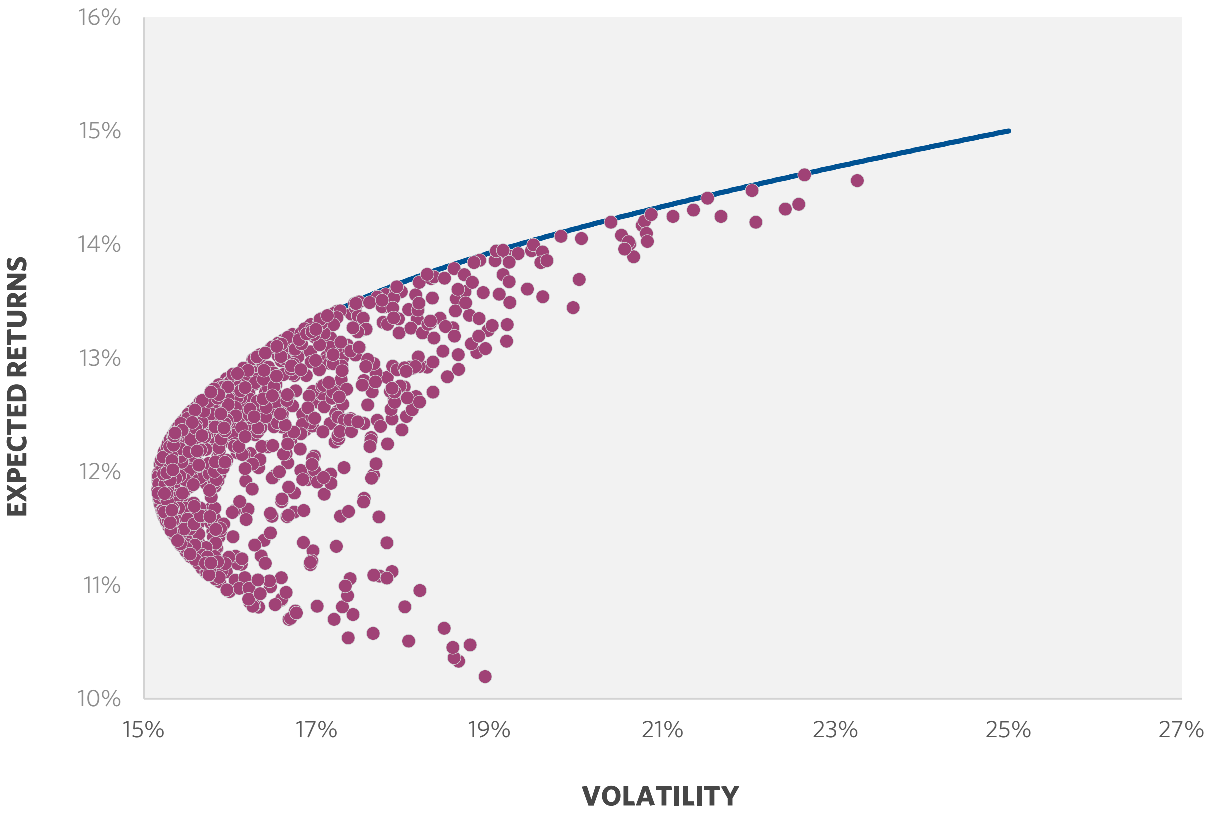
Two equally important questions govern the creation of portfolios based on expected risk-and-return metrics:

1. What is the maximum return for a given level of risk?
2. What is the minimum amount of risk that must be taken to ensure a specific level of return?

Answering these questions for all values of the risk or return dimension (for which a feasible solution exists) allows for the development of an efficient frontier of portfolios, as illustrated in **Figure 8**.

FIGURE 8

Mean Variance Efficient Frontier



Source: Research Affiliates, LLC

Several concerns surround the creation of efficient portfolios. One concern is that by using expectation point forecasts alone, the optimizer[15] used to generate the efficient frontier is forced to assume the inputs are exact and make the appropriate trade-offs. But as previously mentioned, the risk-and-return point forecasts are not exact; utilizing the confidence ranges around them would be beneficial in finding the efficient frontier.[16]

If we ignore the variability represented by the confidence ranges surrounding the point forecasts, we risk being overconfident in our resulting set of efficient portfolios. In addition, the optimizer could generate portfolios highly concentrated in a security, sector, or maturity. Minor, often insignificant, changes in the input parameters can also cause large swings in the constituents of the output portfolio. Investors who blindly follow the commensurate changes could incur large transaction costs, among other problems.

SIMULATED EFFICIENT FRONTIER

We adopt the process outlined by Jorion (1992) to include variability in the process of locating efficient portfolios. In this process, instead of calculating efficient portfolios using only expected risk-and-return point forecasts, we run a Monte Carlo simulation with 1,000 iterations to sample from the expected return distribution of each asset class and create optimal portfolios to meet the desired input condition. The optimal portfolios are calculated by averaging the asset class weights from each of the simulations.

Consider an example with only two runs—not a realistic simulation, but sufficient for illustrative purposes—and two asset classes: stocks and bonds, each with an expected return that is the mean of a distribution, quantified by confidence intervals. During each of the two Monte Carlo runs, an expected return for each asset will be selected from each respective asset class distribution for use in the optimization. Because the expected returns may change from run to run, the optimizer will create different portfolios each time. In this case, let's say the first run generates a portfolio that is 60% stocks/40% bonds, and the second run generates a portfolio that is 57% stocks/43% bonds. Due to the uncertainty in the initial expectations, it is impossible to know if one portfolio is more efficient than the other. The solution is therefore to average the two portfolios together to get a final efficient portfolio composed of 58.5% stocks and 41.5% bonds.

We follow exactly the same process, but use a much larger number of runs. **Figure 9** is an example histogram of a partial list of assets that make up a single efficient portfolio. The x-axis shows the weight of each asset in the portfolio, and the y-axis shows the number of simulations performed for each asset to determine that weight.

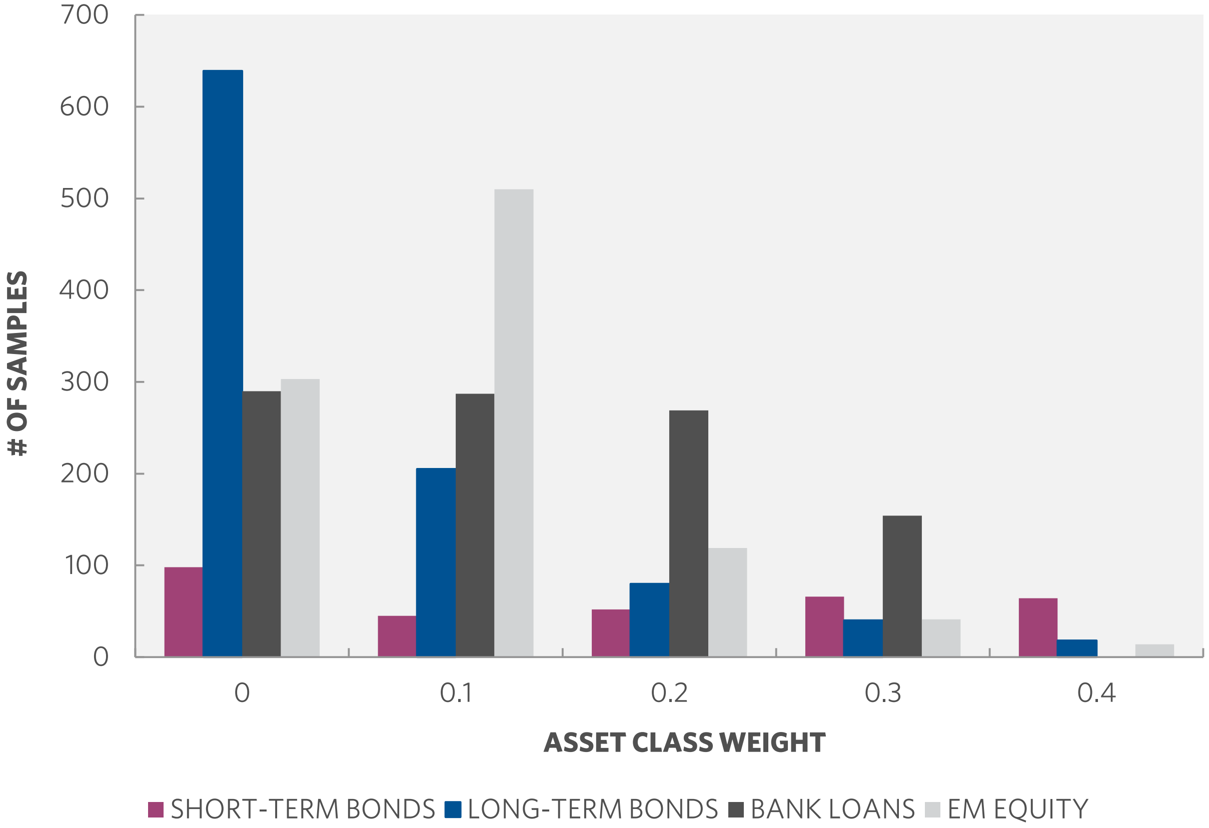
The same process is then run repeatedly for different levels of risk (or return) in order to generate a set of efficient portfolios. As the variability in the expected return inputs shrinks, these portfolios will approach the efficient frontier, based on the point forecasts alone, which can be thought of as the upper limit of the portfolios.

[15] *The standard optimization equation we use is* $\max w' \mu - 1/2\ \lambda w' \Sigma w$*, where w is the set of security weights,* μ *is the expected return, and* Σ *is the asset-class covariance matrix. The equation finds the maximum return for a given level of risk, which is scaled by a risk penalty defined by the risk aversion parameter lambda (* λ *). Investors with a higher risk aversion will have a higher risk penalty and thus will require a higher return per unit of risk taken.*

[16] *The same method described earlier is used to generate the confidence intervals, but in this case because we are dealing with portfolios, we multiply* $1/(2\sqrt{T})$ *by the entire covariance matrix.*

FIGURE 9

Example of Portfolio Weights from Multiple Iterations



Source: Research Affiliates, LLC

RESULTS

Putting all of this together—implementing the EWMA process to generate a factor covariance matrix and then multiplying that matrix by a matrix of exposures of the core asset classes—results in an asset-class covariance matrix, or a corollary correlation matrix. **Table 3** shows a sample of the correlation matrix derived from that covariance matrix. adopt the process outlined by Jorion (1992) to include variability in the process of locating efficient portfolios.

We can then generate portfolios, including a set of efficient portfolios by simulation, using the covariance matrix, the expected returns for each asset class, and a set of constraints. One constraint is the long-only weights have upper bounds, as described on the portfolio creation screen of the Asset Allocation site.

TABLE 3

Sample Correlation Matrix of Core Asset Classes

	U.S. Large-Cap	U.S. Small-Cap	EAFE Equity	EM Equity	REITS	Short-Term Bonds	Long Duration	High Yield	Core U.S. Bonds	TIPS	Global Core Bonds	HC EM Debt1	LC EM Debt	EM Currency	Commodities
U.S. Large-Cap	1.0														
U.S. Small-Cap	0.9	1.0													
EAFE Equity	0.9	0.8	1.0												
EM Equity	0.8	0.8	0.9	1.0											
REITS	0.7	0.8	0.7	0.6	1.0										
Short-Term Bonds	-0.4	-0.4	-0.3	-0.3	-0.1	1.0									
Long Duration	-0.3	-0.4	-0.3	-0.2	-0.1	0.6	1.0								
High Yield	0.7	0.7	0.7	0.7	0.7	-0.3	-0.2	1.0							
Core U.S. Bonds	-0.2	-0.2	-0.1	-0.1	0.0	0.7	0.9	0.0	1.0						
TIPS	0.1	0.0	0.2	0.2	0.3	0.5	0.6	0.3	0.7	1.0					
Global Core Bonds	-0.2	-0.3	-0.2	-0.2	0.0	0.6	0.8	-0.1	0.8	0.5	1.0				
HC EM Debt[17]	0.7	0.6	0.7	0.7	0.6	-0.3	-0.1	0.8	0.1	0.3	0.0	1.0			
LC EM Debt	0.7	0.6	0.8	0.8	0.6	0.0	0.0	0.6	0.2	0.4	0.1	0.7	1.0		
EM Currency	0.7	0.6	0.8	0.8	0.6	0.0	-0.1	0.6	0.1	0.4	0.0	0.6	0.9	1.0	
Commodities	0.4	0.4	0.6	0.6	0.3	-0.1	-0.2	0.4	-0.1	0.3	-0.2	0.4	0.5	0.6	1.0

Source: Research Affiliates, LLC, based on data from Bloomberg and Barclays

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[17] *HC EM Debt is hard currency debt, denominated in a foreign currency, such as USD, EUR, and so on. LC EM Debt is local currency debt issued in the home currency of the country issuing the debt.*

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