Appendix A
Calculation of the Number of Degrees of Freedom of Robots with Closed Chains

A.1 Introduction

Let us recall first that the mobility, or number of *DOF*, of a robot is defined as the number of independent joint variables required to specify the location of all the links of the robot in space. It is equal to the minimal number of actuated joints to control the system.

The number of degrees of freedom N_{dof} of a robot is equal to the number of joints in the case of tree structure system L. In the case of a closed-loop mechanism, the calculation of the mobility N_{dof} can be expressed by the following relation:

$$N_{dof} = L - c \tag{A.1}$$

where L is the number of joints of the structure and c is the number of independent relationships (constraints) between the joint variables, i.e. the number of dependent joints.

Since 1854 with the work of P.A. Chebychev, several researchers have proposed different formulas that can be used to find the mobility of complex systems. Recently, Gogu (2008) has evaluated 35 methods that have been proposed to calculate the mobility of complex systems. He concluded that the majority of methods cannot properly calculate mobility for all mechanisms, and only those that require the construction of the kinematic constraint equations can give a good result.

In case of a single loop, c represents the number of independent kinematic constraint equations of the loop. Consequently $c \leq 6$ for a spatial loop and $c \leq 3$ for a planar loop. Consequently, N_{dof} gives the dimension of the space in which the situation of all the links belong. It is possible to calculate it by calculating the maximum rank of the Jacobian matrix of the serial structure constructed by cutting one link in the loop. This result can be interpreted by the fact that the open structure has L degrees of freedom, and since the c degrees of freedom of the terminal link will be lost when closing the loop, thus the number of remaining degrees of freedom is equal to $L - c$.

S. Briot and W. Khalil, *Dynamics of Parallel Robots*,
Mechanisms and Machine Science 35, DOI 10.1007/978-3-319-19788-3

In case of a system composed of B independent closed loops, the mobility of the system may be calculated by:

$$N_{dof} = L - \sum_{j=1}^{B} c_j. \tag{A.2}$$

This simple formula gives good results for most robot structures but it can yield bad results for certain complex systems and does not give information about the type of motion of the system. However, for some robots, the exact solution is obtained by analyzing the kinematic constraints and taking into account the coupling between the loops (Hervé 1978; Le Borzec and Lotterie 1975). In the following, we present two methods: the Morokine's method and the Gogu's method.

A.2 Moroskine's Method

The mobility can be calculated correctly using the rank of the matrix $\mathbf{J}_c$ defined in the following equation:

$$c = \max_{\mathbf{q}}(\text{rank}(\mathbf{J}_c(\mathbf{q}))) \tag{A.3}$$

where $\mathbf{J}_c$ is the Jacobian of the constraint equations between the joint variables such that:

$$\mathbf{J}_c(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}. \tag{A.4}$$

$\mathbf{J}_c(\mathbf{q})$ can be calculated by derivation of the geometric constraint equations of the loops or by constructing the constraint equations of the velocities through the loop. In fact, from this equation, we deduce that $\dot{\mathbf{q}}$ belongs to the null space of $\mathbf{J}_c$. Therefore, at a given configuration, the number of degrees of freedom is equal to the dimension of the null space of $\mathbf{J}_c$. Consequently:

$$N_{dof} = \min_{\mathbf{q}}(\dim(\mathscr{N}(\mathbf{J}_c(\mathbf{q})))) \tag{A.5}$$

where $\mathscr{N}(\mathbf{J}_c(\mathbf{q}))$ is the null space of the matrix $\mathbf{J}_c$.

In general, the rank of $\mathbf{J}_c(\mathbf{q})$ must be calculated for $\mathbf{q}$ satisfying the closure constraints of the loops. For a single closed loop, the rank can be calculated using random values of its joints.

This method yields the correct result, but it is significantly more difficult to execute. In order to find the rank of the Jacobian matrix defining a mechanism with closed loops, the kinematic constraint equations must be solved. The kinematic constraint equations display the relationship between the joint variables in the mechanism in order to ensure loop closure. In general it is difficult to solve these equations symbolically though it is possible to solve them numerically in order to obtain some random configurations satisfying the closure conditions.

A.3 Gogu's Method

In order to overcome the drawbacks of the previous method, Gogu proposed a method that does not require the construction of the kinematic constraint equations, but will yield the correct mobility for all mechanisms including complex parallel mechanisms. We present here how to use Gogu's method to calculate the mobility for:

- single loop kinematic chains,
- parallel mechanisms with simple legs, and
- parallel mechanisms with complex legs.

The proposed solution makes use of the mobility of the terminal link of simple open loop which is equal to the rank of the Jacobian matrix between the terminal link or the mobile platform in the case of a *PKM* and the base. In fact the mobility $M_{n/0}$ of the terminal link with respect to the base of an open loop chain is equal to the dimension of the task (Cartesian) space $\dim(E(\mathbf{x}))$.

$$M_{n/0} = M = \mathrm{rank}(\mathbf{J}_n) = \dim(E(\mathbf{x})). \tag{A.6}$$

A.3.1 Mobility of Single Loop Kinematic Chains

A link must be opened to obtain an equivalent simple open loop. If the number of joints is equal to L, and the chain is opened around joint L such that the links $1, \ldots, L$ constitute a serial structure, then the rank of the Jacobian matrix J_L gives the number of joint variables that lose their independence after loop closure. Thus the mobility of the closed loop is given by:

$$N_{dof} = L - \mathrm{rank}(\mathbf{J}_L) \tag{A.7}$$

where $\mathrm{rank}(\mathbf{J}_L) = M_{L/0}$ is the mobility of link L w.r.t. link 0 in the open chain.

A second method to calculate the mobility is to open the structure around a joint $k < L$, in order to obtain two serial branches with n_1 and n_2 joints respectively.

Supposing $\mathrm{rank}(\mathbf{J}_{n_i}) = \dim(E(\mathbf{x}_i))$ is the dimension of the task (Cartesian) space of branch i, with $i = 1, 2$, and $\dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2))$ gives the dimension of the common task space that the two branches share. Thus, the number c of joints losing their independence will be obtained as follows:

$$c = \text{rank}(\mathbf{J}_L) = \dim(E(\mathbf{x}_1)) + \dim(E(\mathbf{x}_2)) - \dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2)) \\ = \text{rank}(\mathbf{J}_{n_1}) + \text{rank}(\mathbf{J}_{n_2}) - \dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2)). \tag{A.8}$$

The mobility can be calculated using $N_{dof} = L - c$.

If there is no kinematic redundancies such that $\text{rank}(\mathbf{J}_{n_1}) = N_1$ and $\text{rank}(\mathbf{J}_{n_2}) = N_2$, the mobility of the mechanism is given by:

$$N_{dof} = L - \text{rank}(\mathbf{J}_L) = \dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2)). \tag{A.9}$$

In case of redundancy, the mobility will be:

$$N_{dof} = \dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2)) + \text{number of redundant joints in both branches.} \tag{A.10}$$

A.3.2 Mobility of Parallel Mechanisms with Serial Legs

Let us consider a parallel mechanism with a base platform and a mobile platform that are connected together with m simple open kinematic chains. The number of joints of each chain is denoted by n_i for $i = 1, \ldots, m$. The mobility of the platform M with respect to the base is given by the dimension of the common task spaces of the simple legs associated with the parallel mechanism, as seen in the following equations as long as there are no redundancy:

$$N_{P/0} = \dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2) \cdots \cap E(\mathbf{x}_m)). \tag{A.11}$$

The number of joints that lose their independence after loop closure is equal to the difference between the sum of mobilities of the terminal links of simple chains and the mobility of the platform:

$$c = \sum_{j=1}^{m} \dim(E(\mathbf{x}_j)) - \dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2) \cdots \cap E(\mathbf{x}_m)). \tag{A.12}$$

Thus the mobility of the structure is given by

$$N_{dof} = \sum_{j=1}^{m} n_j - \sum_{j=1}^{m} \dim(E(\mathbf{x}_j)) + \dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2) \ldots \cap E(\mathbf{x}_m)) \tag{A.13}$$

where $\sum_{j=1}^{m} n_j = L$, the number of total joints.

Note that in case of non-redundant legs, $n_j = \dim(E(\mathbf{x}_j))$, thus leading to mobility as the dimension of the common space between all the legs.

A.3.3 Mobility of Parallel Mechanisms with Complex Legs

A parallel mechanism with complex legs is a complex mechanism with L joints in which the mobile platform is connected to the base by $m \geq 2$ legs, of which at least one leg contains at least one closed loop. Theoretically in this case the platform is not uniquely defined, but in practice it is easy to select an appropriate one. The mobility of the mechanism in this case is calculated by:

$$N_{dof} = L - \sum_{j=1}^{m} \dim(E(\mathbf{x}_j)) + \dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2) \ldots \cap E(\mathbf{x}_m)) - C_{cl} \quad \text{(A.14)}$$

where $C_{cl} = \sum_{k=1}^{n_c} c_k$ is sum of the additional joint variables that lose their independence in the closed loops belonging to the complex legs. C_{cl} can be calculated using the previous cases in Sects. A.3.1 and A.3.2 depending on whether the leg contains loops connected serially or in parallel respectively.

A.4 Examples

In this section we calculate the mobility of some robot architectures treated in the different chapters of this book.

A.4.1 The Planar Five-Bar Mechanism

This structure is shown in Fig. 7.3. It consists of five revolute joints with parallel axes. The system ensures motion in the plane perpendicular to the joint axes. The mobility can be calculated using different interpretation:

1. The number of independent constraint equations around the loop is equal to 3, specifying the equality of x, y and the orientation ϕ. Thus, the mobility is equal to 2.
2. We open the loop at the terminal joint L, to obtain a serial architecture with five revolute joints. It is intuitive to deduce that the rank of this serial system is equal to the mobility of its terminal link, i.e. 3. Thus, connecting the terminal link with the base leads to a loss of these three degrees of freedom. Consequently, the mobility of the system is equal to 2.
3. We open the structure to obtain two serial chains connected with the base, such that one chain contains 3 joints and the other contains 2 joints. The dimension of the terminal link spaces of these chains are respectively 3 and 2. The motion of the first chain can be classified as: Tx (translation along $\boldsymbol{x}_0$), Ty (translation along $\boldsymbol{y}_0$) and Rz (rotation about z_0), whereas the motion of the terminal link of

the second chain, whose mobility is equal to 2, can be any two degrees of freedom among: Tx, Ty and Rz.

Consequently the common motion when connecting the two terminal links together will be of dimension 2, and can be represented by any two variables among (Tx, Ty, Rz). Thus the number of degrees of freedom of the terminal link (point) and the number of degrees of freedom of the system are respectively:

$$n_{dof} = 2$$

$$N_{dof} = L - \sum_{j=1}^{2} \dim(E(\mathbf{x}_j)) + \dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2)) = 2.$$

Consequently, two axes should be actuated. We selected the actuators near the base. In case of actuating more than two joints, the system will be redundantly actuated.

A.4.2 *The Planar 3–*R$\underline{\mathrm{P}}$R *Parallel Robot*

This *PKM* is shown in Fig. 7.5. It has three legs, each being composed of three joints (*R*, *P*, and *R* joints) ensuring planar motions.

The mobility of each leg is composed of Tx, Ty and Rz.

Thus

$$E(\mathbf{x}_i) = \text{Tx, Ty, Rz for } i = 1, 2, 3$$

and thus the common space consists of: Tx, Ty and Rz.

Consequently, the mobility is equal to 3. The three prismatic joints are selected to be the actuated joints.

A.4.3 *The Orthoglide*

The Orthoglide (Fig. 7.7) is composed of three legs, each of them having a prismatic joint and a spatial parallelogram. The prismatic joints are perpendicular. The mobility of the Orthoglide will be analyzed using the equivalent kinematic chain of each leg, which is considered to be composed of PUU architecture, with U a universal joint represented by two intersecting revolute axes. Thus the motion of terminal link of each leg is composed of a prismatic joint, then two rotational and two translation degrees of freedom perpendicular on the prismatic axis of the first joint. The mobility of the three legs can be written respectively as:

$$E(\mathbf{x}_1) = \text{Tz, Rx, Ry, Tx, Ty}$$

$$E(\mathbf{x}_2) = \text{Tx, Ry, Rz, Ty, Tz}$$

$$E(\mathbf{x}_3) = \text{Ty, Rx, Ry, Tx, Tz.}$$

Thus

$$n_{dof} = \dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2) \cap E(\mathbf{x}_3)) = 3$$

$$E(\mathbf{x}_p) = \text{Tx, Ty, Tz}$$

$$N_{dof} = L - \sum_{j=1}^{3} \dim(E(\mathbf{x}_j)) + \dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2) \cap E(\mathbf{x}_3)) = 3.$$

We selected the three prismatic joints as actuators.

A.4.4 The Tripteron

This *PKM* is shown in Fig. 9.3. It is composed of three legs, each of them having a serial architecture with 4 joints. The first joint is prismatic and the other joints are revolute. All the *P* and *R* joints of the same leg have parallel axes.

We can easily deduce that:

$$E(\mathbf{x}_1) = \text{Tx, Ty, Tz, Rx}$$

$$E(\mathbf{x}_2) = \text{Tx, Ty, Tz, Ry}$$

$$E(\mathbf{x}_3) = \text{Tx, Ty, Tz, Rz.}$$

Thus

$$n_{dof} = \dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2) \cap E(\mathbf{x}_3)) = 3$$

$$E(\mathbf{x}_p) = \text{Tx, Ty, Tz}$$

$$N_{dof} = L - \sum_{j=1}^{3} \dim(E(\mathbf{x}_j)) + \dim(E(\mathbf{x}_1) \cap E(\mathbf{x}_2) \cap E(\mathbf{x}_3)) = 3.$$

We selected the three prismatic joints as actuators.

Remarks

Note that the simple relation of (A.2) can give the correct result for the first two examples (planar five-bar mechanism, and 3–$R\underline{P}R$), but it does not give the correct result for the Orthoglide nor for the Tripteron.

Appendix B
Lagrange Equations with Multipliers

Let us consider a mechanical system whose Lagrangian L can computed by the knowledge of the generalized coordinates $\mathbf{q}$ (and $\dot{\mathbf{q}}$). Let us assume that $\mathbf{q}$ groups two sets of variables $\mathbf{q}_a$ and $\mathbf{q}_d$ ($\mathbf{q}^T = [\mathbf{q}_a^T \ \mathbf{q}_d^T]$ and $\dot{\mathbf{q}}^T = [\dot{\mathbf{q}}_a^T \ \dot{\mathbf{q}}_d^T]$) which are not independent and are related through the expressions:

$$\mathbf{h}(\mathbf{q}_a, \mathbf{q}_d) = \mathbf{0} \tag{B.1}$$

and:

$$\mathbf{A}(\mathbf{q}_a, \mathbf{q}_d)\dot{\mathbf{q}}_d + \mathbf{B}(\mathbf{q}_a, \mathbf{q}_d)\dot{\mathbf{q}}_a = \mathbf{0} \tag{B.2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are two matrices depending on $\mathbf{q}_a$ and $\mathbf{q}_d$:

$$\mathbf{A}(\mathbf{q}_a, \mathbf{q}_d) = \left[\frac{\partial \mathbf{h}(\mathbf{q}_a, \mathbf{q}_d)}{\partial \mathbf{q}_d}\right] \tag{B.3}$$

and

$$\mathbf{B}(\mathbf{q}_a, \mathbf{q}_d) = \left[\frac{\partial \mathbf{h}(\mathbf{q}_a, \mathbf{q}_d)}{\partial \mathbf{q}_a}\right]. \tag{B.4}$$

Moreover, we consider that the variables $\mathbf{q}_a$ are some variables corresponding (in the frame of this book) to motor coordinates, motors which are exerting some input efforts $\boldsymbol{\tau}$ on the system. This is not the case for the coordinates $\mathbf{q}_d$.

The usual Lagrange equations (6.1) cannot be derived because all coordinates in $\mathbf{q}$ are not independent. In order to modify the Lagrange equations (6.1) to take into account the constraints (B.1) and (B.2), we must include some generalized constraint forces $\boldsymbol{\gamma}_a$ and $\boldsymbol{\gamma}_d$ such that:

S. Briot and W. Khalil, *Dynamics of Parallel Robots*,
Mechanisms and Machine Science 35, DOI 10.1007/978-3-319-19788-3

$$\begin{aligned}\boldsymbol{\tau} + \boldsymbol{\gamma}_a &= \boldsymbol{\tau}_a, \text{ where } \boldsymbol{\tau}_a = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_a}\right)^T - \left(\frac{\partial L}{\partial \mathbf{q}_a}\right)^T \\ \boldsymbol{\gamma}_d &= \boldsymbol{\tau}_d, \text{ where } \boldsymbol{\tau}_d = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_d}\right)^T - \left(\frac{\partial L}{\partial \mathbf{q}_d}\right)^T.\end{aligned} \tag{B.5}$$

Of course, these generalized constraint forces $\boldsymbol{\gamma}_a$ and $\boldsymbol{\gamma}_d$ are internal to the system and produce no work, i.e. the *PVP* states that we have, for any arbitrary velocities $\dot{\mathbf{q}}_a^*$ and $\dot{\mathbf{q}}_d^*$,

$$\dot{\mathbf{q}}_d^{*T}\boldsymbol{\gamma}_d + \dot{\mathbf{q}}_a^{*T}\boldsymbol{\gamma}_a = 0. \tag{B.6}$$

Now, taking the transposed expression of (B.2), we obtain

$$\dot{\mathbf{q}}_d^T\mathbf{A}^T + \dot{\mathbf{q}}_a^T\mathbf{B}^T = \mathbf{0}. \tag{B.7}$$

This expression is also true if we right-multiply it by any arbitrary vector $\boldsymbol{\lambda}$:

$$\dot{\mathbf{q}}_d^T\mathbf{A}^T\boldsymbol{\lambda} + \dot{\mathbf{q}}_a^T\mathbf{B}^T\boldsymbol{\lambda} = 0. \tag{B.8}$$

By identification between (B.6) and (B.8), we can see that we have:

$$\boldsymbol{\gamma}_a = \mathbf{B}^T\boldsymbol{\lambda} \tag{B.9}$$

and

$$\boldsymbol{\gamma}_d = \mathbf{A}^T\boldsymbol{\lambda} \tag{B.10}$$

from which we obtain the new set of Lagrange equations, in which $\boldsymbol{\lambda}$ is called the vector of Lagrange multipliers:

$$\begin{aligned}\boldsymbol{\tau} + \mathbf{B}^T\boldsymbol{\lambda} &= \boldsymbol{\tau}_a, \text{ where } \boldsymbol{\tau}_a = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_a}\right)^T - \left(\frac{\partial L}{\partial \mathbf{q}_a}\right)^T \\ \mathbf{A}^T\boldsymbol{\lambda} &= \boldsymbol{\tau}_d, \text{ where } \boldsymbol{\tau}_d = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_d}\right)^T - \left(\frac{\partial L}{\partial \mathbf{q}_d}\right)^T.\end{aligned} \tag{B.11}$$

Appendix C
Computation of Wrenches Reciprocal to a System of Twists

In this Appendix, we compute the actuation and constraint wrenches associated to some common *PKM* legs.

C.1 Definitions

The twist $\mathbf{t}$ of a body is parameterized by two vectors, the translational velocity $\mathbf{v}$ and the rotational velocity $\boldsymbol{\omega}$, such that we can define a vector of dimension 6:

$$\mathbf{t} = \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix}. \tag{C.1}$$

The twist $\mathbf{t}$ is also called the velocity screw. $\boldsymbol{\omega}$ is the resultant of the screw and $\mathbf{v}$ is its moment.

A wrench $\mathbf{w}$ is parameterized by two vectors, the force $\mathbf{f}$ and the moment $\mathbf{m}$, such that we can define a vector of dimension 6:

$$\mathbf{w} = \begin{bmatrix} \mathbf{f} \\ \mathbf{m} \end{bmatrix}. \tag{C.2}$$

The wrench $\mathbf{w}$ is a screw in which $\mathbf{f}$ is the resultant and $\mathbf{m}$ is the moment of the screw.

Let us define the screws:

- $\$$ which describes a unit twist; as a result, if
  - $\$$ characterizes a pure translation, it can be written as

$$\$ = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{0} \end{bmatrix} \tag{C.3}$$

in which $\mathbf{u}_1$ represents the direction of the translation and $\|\mathbf{u}_1\| = 1$.

S. Briot and W. Khalil, *Dynamics of Parallel Robots*,
Mechanisms and Machine Science 35, DOI 10.1007/978-3-319-19788-3

– $\$$ characterizes a pure rotation, it can be written as

$$\$ = \begin{bmatrix} \mathbf{u}_1 \times \mathbf{r} \\ \mathbf{u}_1 \end{bmatrix} \tag{C.4}$$

in which $\mathbf{u}_1$ represents the axis of the rotation ($\|\mathbf{u}_1\| = 1$) and $\mathbf{r}$ is the vector defining the distance between the axis of rotation and the point at which $\$$ is expressed.

- $\boldsymbol{\zeta}$ which describes a unit wrench.

  – $\boldsymbol{\zeta}$ characterizes a pure moment, it can be written as

$$\boldsymbol{\zeta} = \begin{bmatrix} \mathbf{0} \\ \mathbf{u}_2 \end{bmatrix} \tag{C.5}$$

in which $\mathbf{u}_2$ represents the direction around which the moment is applied and $\|\mathbf{u}_2\| = 1$.

  – $\boldsymbol{\zeta}$ characterizes a pure force, it can be written as

$$\boldsymbol{\zeta} = \begin{bmatrix} \mathbf{u}_2 \\ \mathbf{u}_2 \times \mathbf{r} \end{bmatrix} \tag{C.6}$$

in which $\mathbf{u}_2$ represents the direction along which the force is applied ($\|\mathbf{u}_2\| = 1$) and $\mathbf{r}$ is the vector defining the distance between the point on which the force is applied and the point at which $\boldsymbol{\zeta}$ is expressed.

C.2 Condition of Reciprocity

A twist $\$$ is reciprocal to a wrench $\boldsymbol{\zeta}$ if their product is null, i.e. $\$^T \boldsymbol{\zeta} = \boldsymbol{\zeta}^T \$ = 0$. This means that the power developed by the wrench $\boldsymbol{\zeta}$ along the motion defined by $\$$ is null.

From the definition of the unit twist and wrench $\$$ and $\boldsymbol{\zeta}$, we can define the following rules (Zhao et al. 2009):

- for a revolute joint with axis along the direction $\boldsymbol{u}$, the reciprocal wrenches are:
  – forces coplanar to $\boldsymbol{u}$, i.e. forces directed along an axis either parallel to $\boldsymbol{u}$ or intersecting $\boldsymbol{u}$ at a point,
  – moments whose axes are orthogonal to $\boldsymbol{u}$.
- for a prismatic joint with axis aligned along $\boldsymbol{u}$, the reciprocal wrenches are:
  – forces whose directions are orthogonal to $\boldsymbol{u}$,
  – any moment.

These rules can be used to find the $(6 - n)$ reciprocal wrenches for a system of n twists.

Similarly, we can define the twists reciprocal to a given wrench or system of wrenches. The interested reader could find more details in (Zhao et al. 2009).

C.3 Computation of Wrenches Reciprocal to a System of Twists Constrained in a Plane

This case where a system of twists is constrained in a plane appear for the *PPM* (planar parallel manipulators). These mechanisms are made of joints whose displacements are all constrained in the same plane. The most common legs of *PPM* are presented in Fig. C.1. They are all made of three joints, one of them being actuated (the joints in gray).

Let us consider that the plane of motion is the $(O_0, \boldsymbol{x}_0, \boldsymbol{y}_0)$ plane. As a result, any twist $\$_i$ associated to the motion of a joint i has the following form:

$$\$_i = \left[v_{xi}\ v_{yi}\ 0\ 0\ 0\ \omega_{zi}\right]^T . \tag{C.7}$$

This means that, for any twist system of dimension n (representing a planar leg composed of n active or passive joints –$n \in [1, +\infty[$) defined by $\$ = \left[\$_1 \ldots \$_n\right]$, three constraint wrenches $\boldsymbol{\zeta}_{c1}$, $\boldsymbol{\zeta}_{c2}$ and $\boldsymbol{\zeta}_{c3}$ (i.e. the wrenches reciprocal to all twists representing the passive and active joint motions) can be easily defined as

$$\boldsymbol{\zeta}_{c1} = \left[0\ 0\ 1\ 0\ 0\ 0\right]^T \tag{C.8}$$

which represents a force along z_0,

$$\boldsymbol{\zeta}_{c2} = \left[0\ 0\ 0\ 1\ 0\ 0\right]^T \tag{C.9}$$

which represents a moment around $\boldsymbol{x}_0$, and

$$\boldsymbol{\zeta}_{c3} = \left[0\ 0\ 0\ 0\ 1\ 0\right]^T \tag{C.10}$$

which represents a moment around $\boldsymbol{y}_0$.

These three constraint wrenches prevent the translation along z_0 and the rotations around $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$ of the body located at the leg extremity. All legs presented in the Fig. C.1 impose these three constraint wrenches to the platform.

Now, let us consider that one joint of the legs depicted in Fig. C.1 is actuated (the joints in gray). This is the case for the most usual *PPM*. Starting from this consideration, only two cases can appear:

1. the passive system is made of two *R* joints (Fig. C.2a),
2. the passive system is made of one *P* joint and one *R* joint (Fig. C.2b).

Fig. C.1 Usual legs for *PPM*. **a** $\underline{R}RR$ leg. **b** $R\underline{R}R$ leg. **c** $\underline{R}PR$ leg. **d** $R\underline{P}R$ leg. **e** $\underline{P}PR$ leg. **f** $P\underline{P}R$ leg. **g** $\underline{P}RR$. **h** $\underline{P}RR$ leg. **i** $\underline{P}RP$ leg. **j** $\underline{R}RR$ leg

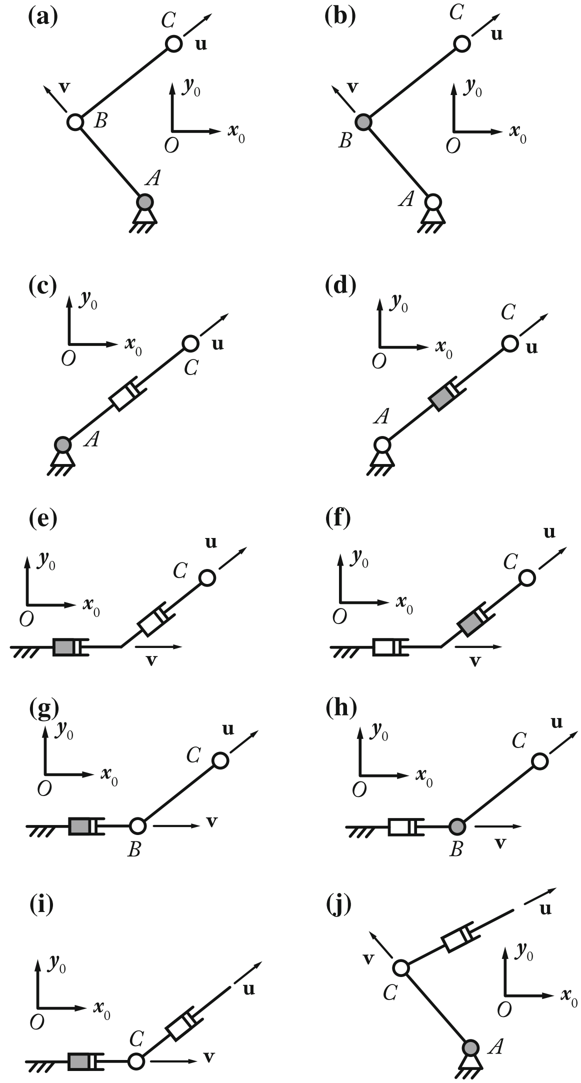
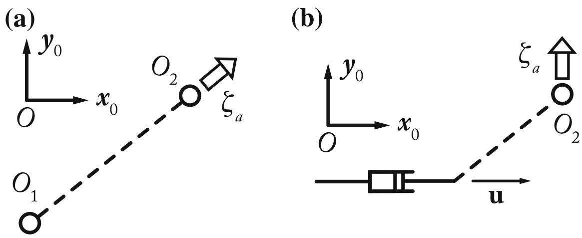


Fig. C.2 General passive systems for the usual legs of *PPM*. **a** *RR* passive system ($\boldsymbol{\zeta}_a$is aligned with $(O_1 O_2)$). **b** *PR* passive system ($\boldsymbol{\zeta}_a \perp \mathbf{u}$)



C.3.1 Computation of Wrenches Reciprocal to a Twist System Representing the Motion of Two Passive **R** *Joints*

For a system composed of two R joints (one located at point O_1 and the second one at point O_2, point O_i having the coordinates (x_i, y_i) in the plane $(O_0, \boldsymbol{x}_0, \boldsymbol{y}_0)$– Fig. C.2a, it is possible to define two twists $\$_{R1}$ and $\$_{R2}$ parameterized by (when expressed at point O_2):

$$^0\$_{R1} = \left[-(y_2 - y_1)\ x_2 - x_1\ 0\ 0\ 0\ 1\right]^T \tag{C.11}$$

$$^0\$_{R2} = \left[0\ 0\ 0\ 0\ 0\ 1\right]^T . \tag{C.12}$$

The constraint wrenches $\boldsymbol{\zeta}_{ci}$ $(i = 1, 2, 3)$ are reciprocal to $\$_{R1}$ and $\$_{R2}$ but one additional wrench, denoted as the actuation wrench $\boldsymbol{\zeta}_a$ is reciprocal to these twists. It can be easily seen from (Zhao et al. 2009) that

$$^0\boldsymbol{\zeta}_a = \left[x_2 - x_1\ y_2 - y_1\ 0\ 0\ 0\ 0\right]^T \tag{C.13}$$

i.e. it is a force passing through the centers of the two R joints (see Fig. C.2a). It should be noted that $\boldsymbol{\zeta}_a$ must not be reciprocal to the twist of the third (actuated) joint

C.3.2 Computation of Wrenches Reciprocal to a Twist System Representing the Motion of One **P** *Joint and one* **R** *Joint*

For a system composed of P joint of direction $\mathbf{u}_1 = [u_{1x}\ u_{1y}\ 0]^T$ and one R joint located at point O_2 (Fig. C.2b), it is possible to define two twists $\$_{P1}$ and $\$_{R2}$ parameterized by (when expressed at point O_2):

$$^0\$_{P1} = \left[u_{1x}\ u_{1y}\ 0\ 0\ 0\ 0\right]^T \tag{C.14}$$

$$^0\$_{R2} = \left[0\ 0\ 0\ 0\ 0\ 1\right]^T . \tag{C.15}$$

The constraint wrenches $\boldsymbol{\zeta}_{ci}$ $(i = 1, 2, 3)$ are reciprocal to $\$_{P1}$ and $\$_{R2}$ but one additional wrench, denoted as the actuation wrench $\boldsymbol{\zeta}_a$ is reciprocal to these twists ($\boldsymbol{\zeta}_a$ must not be reciprocal to the twist of the third (actuated) joint). It can be easily seen from (Zhao et al. 2009) that

$$^0\boldsymbol{\zeta}_a = \left[-u_{1y}\ u_{1x}\ 0\ 0\ 0\ 0\right]^T \tag{C.16}$$

i.e. it is a force lying in the plane $(O_0, \boldsymbol{x}_0, \boldsymbol{y}_0)$ passing through O_2 and orthogonal to the prismatic joint direction (see Fig. C.2b).

C.4 Computation of Wrenches Reciprocal to Other Types of Twist Systems

For *SPM*, other types of twist systems appear. Due to the large number of possible leg architectures for the existing *SPM*, it is impossible to deal with all possible twist systems in this Appendix. However, we will compute the wrenches reciprocal to the twist systems corresponding to:

- *U<u>P</u>S* legs (legs of the Gough-Stewart platform—see Sect. 7.2.2.5)
- *U<u>P</u>U* legs (legs of the Tsai mechanism (Tsai and Joshi 2000)),
- *<u>R</u>US* legs (legs of Hexa-like robots, and most of Delta-like robots (Clavel 1989; Company et al. 2002; Pashkevich et al. 2006)).

C.4.1 Computation of Wrenches Reciprocal to a Twist System Representing the Motion of a U<u>P</u>S *Leg*

Let us consider a *U<u>P</u>S* leg composed of an actuated *<u>P</u>* joint of direction ${}^i\mathbf{u} = [u_x\ 0\ u_z]^T$ in the leg frame $\mathscr{F}_i$ (Fig. C.3), one passive *U* joint which can be represented as an assembly of two *R* joints whose axes $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal to the direction of the *<u>P</u>* joint, i.e. $\mathbf{a}_1 = [1\ 0\ 0]^T$ and $\mathbf{a}_2 = [0\ 1\ 0]^T$ in the leg frame $\mathscr{F}_i$, and one passive *S* joint allowing three independent rotations around three axes $\mathbf{a}_1 = [1\ 0\ 0]^T$ $\mathbf{a}_2 = [0\ 1\ 0]^T$ and $\mathbf{a}_3 = [0\ 0\ 1]^T$ without loss of generality in the leg frame $\mathscr{F}_i$ (Fig. C.3).

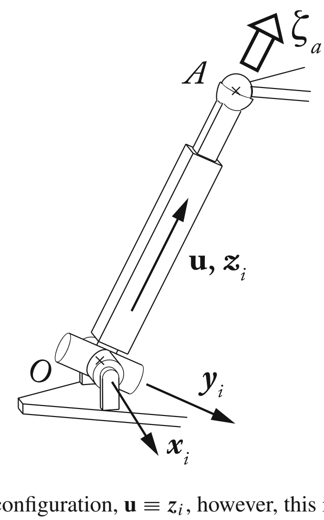


Fig. C.3 A *U<u>P</u>S* leg (in this configuration, $\mathbf{u} \equiv z_i$, however, this is not the general case)

As a result, the twist system representing the motion of the leg in the frame $\mathscr{F}_i$ and expressed at the center A of the S joint is given by:

$$ {}^i\$_1 = \begin{bmatrix} 0 & -z_A & 0 & 1 & 0 & 0 \end{bmatrix}^T \tag{C.17}$$

$$ {}^i\$_2 = \begin{bmatrix} z_A & 0 & -x_A & 0 & 1 & 0 \end{bmatrix}^T \tag{C.18}$$

$$ {}^i\$_3 = \begin{bmatrix} u_x & 0 & u_z & 0 & 0 & 0 \end{bmatrix}^T \tag{C.19}$$

$$ {}^i\$_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \tag{C.20}$$

$$ {}^i\$_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \tag{C.21}$$

$$ {}^i\$_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \tag{C.22}$$

where x_A and z_A are the coordinates of the point A along the axes $\boldsymbol{x}_i$ and z_i of the frame $\mathscr{F}_i$.

The total twist system $\$ = \begin{bmatrix} \$_1 & \dots & \$_6 \end{bmatrix}$ is of rank 6. Therefore, there are no constraint wrenches.

Now, let us compute the actuation wrench when the $\underline{P}$ joint is considered actuated. We must thus consider the twist system $\$_d = \begin{bmatrix} \$_1 & \$_2 & \$_4 & \$_5 & \$_6 \end{bmatrix}$. It automatically gives that the actuation wrench $\boldsymbol{\zeta}_a$ is equal to, from (Zhao et al. 2009):

$$ {}^i\boldsymbol{\zeta}_a = \begin{bmatrix} u_x & 0 & u_z & 0 & 0 & 0 \end{bmatrix}^T \tag{C.23}$$

i.e. it is a force directed along the direction connecting the centres of the U and S joints. We note that this force is not reciprocal to the actuated prismatic joint axis.

C.4.2 Computation of Wrenches Reciprocal to a Twist System Representing the Motion of a U$\underline{\text{P}}$U Leg

Let us consider a $U\underline{P}U$ leg composed of an actuated $\underline{P}$ joint of direction $\mathbf{u} = [u_x \; 0 \; u_z^T$ in the leg frame $\mathscr{F}_i$, and two passive U joints. Each passive U joint can be represented as an assembly of two R joints whose axes $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal to the direction of the $\underline{P}$ joint, i.e. $\mathbf{a}_1 = [1 \; 0 \; 0]^T$ and $\mathbf{a}_2 = [0 \; 1 \; 0]^T$ in the leg frame $\mathscr{F}_i$ (Fig. C.4).

As a result, the twist system representing the motion of the leg in the frame $\mathscr{F}_i$ and expressed at the center A of the second U joint is given by:

$$ {}^i\$_1 = \begin{bmatrix} 0 & -z_A & y_A & 1 & 0 & 0 \end{bmatrix}^T \tag{C.24}$$

$$ {}^i\$_2 = \begin{bmatrix} z_A & 0 & -x_A & 0 & 1 & 0 \end{bmatrix}^T \tag{C.25}$$

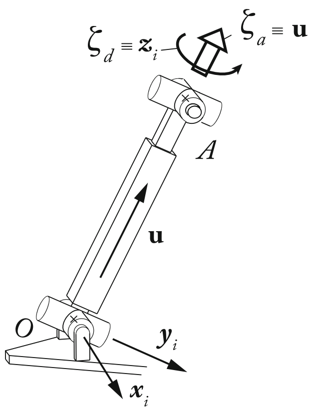


Fig. C.4 A $U\underline{P}U$ leg (in this configuration, $\mathbf{u} \equiv z_i$, however, this is not the general case)

$$^{i}\$_3 = \begin{bmatrix} u_x & 0 & u_z & 0 & 0 & 0 \end{bmatrix}^T \tag{C.26}$$

$$^{i}\$_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \tag{C.27}$$

$$^{i}\$_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T . \tag{C.28}$$

The total twist system $\$ = \begin{bmatrix} \$_1 & \ldots & \$_5 \end{bmatrix}$ is of rank 5. As a result, there is a constraint wrench given from (Zhao et al. 2009) by

$$^{i}\boldsymbol{\zeta}_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \tag{C.29}$$

i.e. it is a moment around the z_i axis (Fig. C.4).

Now, let us compute the actuation wrench when the $\underline{P}$ joint is considered actuated. We must thus consider the twist system $\$_d = \begin{bmatrix} \$_1 & \$_2 & \$_4 & \$_5 \end{bmatrix}$. It automatically gives that the actuation wrench $\boldsymbol{\zeta}_a$ is equal to, from [Zhao *et al.*., 2009]:

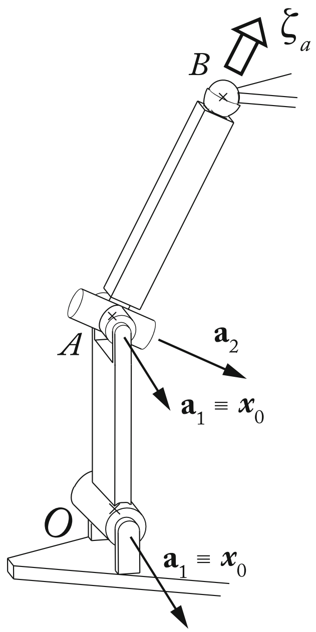
$$^{i}\boldsymbol{\zeta}_a = \begin{bmatrix} u_x & 0 & u_z & 0 & 0 & 0 \end{bmatrix}^T \tag{C.30}$$

i.e. it is a force directed along the direction of the prismatic joint (Fig. C.4).

C.4.3 Computation of Wrenches Reciprocal to a Twist System Representing the Motion of a **R**US *Leg*

Let us consider a $\underline{R}US$ leg composed of an actuated $\underline{R}$ joint of direction $^{0}\mathbf{a}_1 = [1\ 0\ 0]^T$ in the base frame $\mathscr{F}_0$, one passive U joint which can be represented as an assembly of two R joints whose axes are $\mathbf{a}_1$ and $\mathbf{a}_2$, with $\mathbf{a}_1 = [1\ 0\ 0]^T$ and $\mathbf{a}_2 = [0\ a_y\ a_z]^T$ in the leg frame $\mathscr{F}_i$, and one passive S joint allowing three independent rotations around three axes $\mathbf{a}_1 = [1\ 0\ 0]^T$ $\mathbf{a}_3 = [0\ 1\ 0]^T$ and $\mathbf{a}_4 = [0\ 0\ 1]^T$ without loss of generality in the leg frame $\mathscr{F}_i$ (Fig. C.5).

Fig. C.5 A $\underline{R}US$ leg



As a result, the twist system representing the motion of the leg in the frame $\mathscr{F}_0$ and expressed at the center B of the S joint (with coordinates in the base frame x_B, y_B and z_B) is given by:

$$^{i}\$_1 = \begin{bmatrix} 0 & -z_B & y_B & 1 & 0 & 0 \end{bmatrix}^T \tag{C.31}$$

$$^{i}\$_2 = \begin{bmatrix} 0 & -z_{AB} & y_{AB} & 1 & 0 & 0 \end{bmatrix}^T \tag{C.32}$$

$$^{i}\$_3 = \begin{bmatrix} a_y z_{AB} - a_z y_{AB} & a_z x_{AB} & -a_y x_{AB} & 0 & a_y & a_z \end{bmatrix}^T \tag{C.33}$$

$$^{i}\$_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \tag{C.34}$$

$$^{i}\$_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \tag{C.35}$$

$$^{i}\$_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \tag{C.36}$$

where x_{AB}, y_{AB} and z_{AB} are the coordinates of the vector $\overrightarrow{AB}$ along the axes of the frame $\mathscr{F}_0$. Note that $\overrightarrow{AB} \perp \mathbf{a}_2$, i.e. that $y_{AB} a_y + z_{AB} a_z = 0$.

The total twist system $\$ = \begin{bmatrix} \$_1 & \dots & \$_6 \end{bmatrix}$ is of rank 6. Therefore, there are no constraint wrenches.

Now, let us compute the actuation wrench when the first $\underline{R}$ joint is considered actuated. We must thus consider the twist system $\$_d = \left[\$_2 \ \ldots \ \$_6\right]$. It automatically gives that the actuation wrench $\boldsymbol{\zeta}_a$ is equal to, from (Zhao et al. 2009):

$$^{i}\boldsymbol{\zeta}_a = \left[x_{AB} \ y_{AB} \ z_{AB} \ 0 \ 0 \ 0\right]^T / \sqrt{x_{AB}^2 + y_{AB}^2 + z_{AB}^2} \tag{C.37}$$

i.e. it is a force directed along the direction given by the vector $\overrightarrow{AB}$ (Fig. C.5). We can verify that $\boldsymbol{\zeta}_a$ is not orthogonal to the unit twist of the actuated joint.

Appendix D
Point-to-Point Trajectory Generation

Let us consider a robot displacement between an initial configuration A_0 parameterized by the Cartesian variables $\mathbf{x}_0$ and a final configuration A_f parameterized by the Cartesian variables $\mathbf{x}_f$. The trajectory between these two configurations can be defined by the functions

$$\mathbf{x}(t) = s(t)\left(\mathbf{x}_f - \mathbf{x}_0\right) + \mathbf{x}_0 \tag{D.1}$$

$$\dot{\mathbf{x}}(t) = \dot{s}(t)\left(\mathbf{x}_f - \mathbf{x}_0\right) \tag{D.2}$$

$$\ddot{\mathbf{x}}(t) = \ddot{s}(t)\left(\mathbf{x}_f - \mathbf{x}_0\right) \tag{D.3}$$

where:

- $t \in [0,\ t_f]$, where $t = 0$ s is the time at which the robot starts to move from the initial configuration A_0 and t_f is the time at which the robot arrives at the final configuration A_f
- $\mathbf{x}(t)$ denotes the robot Cartesian variables at the time t,
- $\dot{\mathbf{x}}(t)$ denotes the first derivative w.r.t. time of the robot Cartesian variables at the time t,
- $\ddot{\mathbf{x}}(t)$ denotes the second derivative w.r.t. time of the robot Cartesian variables at the time t,
- $s(t)$ is an interpolation function.

We can deduce from (D.1) that the path in the Cartesian space will be defined by a straight line. The boundary conditions for $s(t)$ are deduced as:

$$s(t = 0\ \text{s}) = 0 \tag{D.4}$$

$$s(t = t_f) = 1. \tag{D.5}$$

S. Briot and W. Khalil, *Dynamics of Parallel Robots*,
Mechanisms and Machine Science 35, DOI 10.1007/978-3-319-19788-3

Moreover, if at the initial and final configurations, the velocities and accelerations are null, we have

$$\dot{s}(t = 0 \text{ s}) = 0 \tag{D.6}$$
$$\dot{s}(t = t_f) = 0 \tag{D.7}$$
$$\ddot{s}(t = 0 \text{ s}) = 0 \tag{D.8}$$
$$\ddot{s}(t = t_f) = 0. \tag{D.9}$$

From these boundary conditions, and assuming that the interpolation function $s(t)$ is a polynomial of the form:

$$s(t) = \sum_{k=0}^{n} a_k \; t^k \tag{D.10}$$

we can find the coefficients a_k.

For high speed robots or when a robot is handling heavy or delicate loads, it is worth ensuring the continuity of the position, velocity, and accelerations as well, in order to avoid exciting resonances in the mechanics. The trajectory is said to be of class C^2. Thus, from (D.2) and (D.3), we must define the functions $\dot{s}(t)$ and $\ddot{s}(t)$, given by

$$\dot{s}(t) = \sum_{k=1}^{n} k \; a_k \; t^{k-1} \tag{D.11}$$

and

$$\ddot{s}(t) = \sum_{k=2}^{n} k \, (k-1) \; a_k \; t^{k-2}. \tag{D.12}$$

Since six constraints (D.4)–(D.9) have to be satisfied, the interpolation requires a polynomial of at least fifth degree (Binford et al. 1977).

Solving the six constraints yields the following interpolation function:

$$s(t) = 10 \left(\frac{t}{t_f}\right)^3 - 6 \left(\frac{t}{t_f}\right)^4 + 15 \left(\frac{t}{t_f}\right)^5. \tag{D.13}$$

Obviously, if we increase the number of boundary conditions to take into account, the order of the polynomial will increase. For example, if n_c constraints have to be satisfied, the interpolation requires a polynomial of at least $n_c - 1$ degree.

Appendix E
Calculation of the Terms $\mathbf{f}_{acc_1}$, $\mathbf{f}_{acc_2}$ and $\mathbf{f}_{acc_3}$ in Chapter 10

E.1 Calculation of the Term $\mathbf{f}_{acc_1}$

From (10.12) and (10.15), we get that $\mathbf{f}_{acc_1}$ is given by

$$\begin{aligned}
\mathbf{f}_{acc_1} &= \int_{\mathscr{B}_j} \dot{\mathbf{v}}_{M_j} \mathrm{d}m \\
&= \int_{\mathscr{B}_j} \left(\dot{\mathbf{v}}_j + \boldsymbol{\Phi}_{d_j}(M_{0j})\, \ddot{\mathbf{q}}_{e_j} + 2\boldsymbol{\omega}_j \times \boldsymbol{\Phi}_{d_j}(M_{0j})\dot{\mathbf{q}}_{e_j}\right) \mathrm{d}m \\
&+ \int_{\mathscr{B}_j} \left(\boldsymbol{\omega}_j \times \left(\boldsymbol{\omega}_j \times \mathbf{r}_{O_j M_j}\right) + \dot{\boldsymbol{\omega}}_j \times \mathbf{r}_{O_j M_j}\right) \mathrm{d}m
\end{aligned} \tag{E.1}$$

which can be expanded to be rewritten as

$$\begin{aligned}
\mathbf{f}_{acc_1} = &\int_{\mathscr{B}_j} \dot{\mathbf{v}}_j \mathrm{d}m + \int_{\mathscr{B}_j} \boldsymbol{\Phi}_{d_j}(M_{0j}) \mathrm{d}m \ddot{\mathbf{q}}_{e_j} \\
&+ \dot{\boldsymbol{\omega}}_j \times \left(\int_{\mathscr{B}_j} \mathbf{r}_{O_j M_j} \mathrm{d}m\right) \\
&+ 2\boldsymbol{\omega}_j \times \left(\int_{\mathscr{B}_j} \boldsymbol{\Phi}_{d_j}(M_{0j}) \mathrm{d}m\ \dot{\mathbf{q}}_{e_j}\right) \\
&+ \boldsymbol{\omega}_j \times \left(\boldsymbol{\omega}_j \times \left(\int_{\mathscr{B}_j} \mathbf{r}_{O_j M_j} \mathrm{d}m\right)\right).
\end{aligned} \tag{E.2}$$

E.2 Calculation of the Term $\mathbf{f}_{acc_2}$

From (10.12) and (10.15), we get that $\mathbf{f}_{acc_2}$ is given by

S. Briot and W. Khalil, *Dynamics of Parallel Robots*,
Mechanisms and Machine Science 35, DOI 10.1007/978-3-319-19788-3

$$\begin{aligned}
\mathbf{f}_{acc_2} &= \int_{\mathscr{B}_j} \mathbf{r}_{O_jM_j} \times \dot{\mathbf{v}}_{M_j} \mathrm{d}m \\
&= \int_{\mathscr{B}_j} \mathbf{r}_{O_jM_j} \times \left(\dot{\mathbf{v}}_j + \boldsymbol{\Phi}_{d_j}(M_{0j})\ddot{\mathbf{q}}_{e_j} + 2\boldsymbol{\omega}_j \times \boldsymbol{\Phi}_{d_j}(M_{0j})\dot{\mathbf{q}}_{e_j}\right) \mathrm{d}m \\
&\quad + \int_{\mathscr{B}_j} \mathbf{r}_{O_jM_j} \times \left(\boldsymbol{\omega}_j \times \left(\boldsymbol{\omega}_j \times \mathbf{r}_{O_jM_j}\right) + \dot{\boldsymbol{\omega}}_j \times \mathbf{r}_{O_jM_j}\right) \mathrm{d}m
\end{aligned} \tag{E.3}$$

which can be expanded to be rewritten under a sum of five terms:

$$\mathbf{f}_{acc_2} = \sum_{k=1}^{5} \mathbf{a}_k \tag{E.4}$$

where:

$$\begin{aligned}
\mathbf{a}_1 &= \int_{\mathscr{B}_j} \mathbf{r}_{O_jM_j} \times \dot{\mathbf{v}}_j \mathrm{d}m \\
&= \left(\int_{\mathscr{B}_j} \mathbf{r}_{O_jM_j} \mathrm{d}m\right) \times \dot{\mathbf{v}}_j
\end{aligned} \tag{E.5}$$

$$\begin{aligned}
\mathbf{a}_2 &= \int_{\mathscr{B}_j} \mathbf{r}_{O_jM_j} \times \left(\boldsymbol{\Phi}_{d_j}(M_{0j})\ddot{\mathbf{q}}_{e_j}\right) \mathrm{d}m \\
&= \int_{\mathscr{B}_j} \hat{\mathbf{r}}_{O_jM_j} \boldsymbol{\Phi}_{d_j}(M_{0j})\ddot{\mathbf{q}}_{e_j} \mathrm{d}m \\
&= \left(\int_{\mathscr{B}_j} \hat{\mathbf{r}}_{O_jM_j} \boldsymbol{\Phi}_{d_j}(M_{0j}) \mathrm{d}m\right) \ddot{\mathbf{q}}_{e_j}
\end{aligned} \tag{E.6}$$

$$\begin{aligned}
\mathbf{a}_3 &= \int_{\mathscr{B}_j} \mathbf{r}_{O_jM_j} \times \left(\dot{\boldsymbol{\omega}}_j \times \mathbf{r}_{O_jM_j}\right) \mathrm{d}m \\
&= -\int_{\mathscr{B}_j} \mathbf{r}_{O_jM_j} \times \left(\mathbf{r}_{O_jM_j} \times \dot{\boldsymbol{\omega}}_j\right) \mathrm{d}m \\
&= -\int_{\mathscr{B}_j} \hat{\mathbf{r}}_{O_jM_j} \left(\hat{\mathbf{r}}_{O_jM_j} \dot{\boldsymbol{\omega}}_j\right) \mathrm{d}m \\
&= \int_{\mathscr{B}_j} \hat{\mathbf{r}}^T_{O_jM_j} \hat{\mathbf{r}}_{O_jM_j} \dot{\boldsymbol{\omega}}_j \mathrm{d}m \\
&= \int_{\mathscr{B}_j} \hat{\mathbf{r}}^T_{O_jM_j} \hat{\mathbf{r}}_{O_jM_j} \mathrm{d}m \; \dot{\boldsymbol{\omega}}_j
\end{aligned} \tag{E.7}$$

$$\mathbf{a}_4 = \int_{\mathscr{B}_j} \mathbf{r}_{O_jM_j} \times \left(\boldsymbol{\omega}_j \times \left(\boldsymbol{\omega}_j \times \mathbf{r}_{O_jM_j}\right)\right) \mathrm{d}m \tag{E.8}$$

$$\mathbf{a}_5 = 2\int_{\mathscr{B}_j} \mathbf{r}_{O_jM_j} \times \left(\boldsymbol{\omega}_j \times \boldsymbol{\Phi}_{d_j}(M_{0j})\dot{\mathbf{q}}_{e_j}\right) \mathrm{d}m. \tag{E.9}$$

To simplify the term $\mathbf{a}_4$, let us recall the well-known identity for the double cross-product of three arbitrary vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) = \mathbf{0}. \tag{E.10}$$

If $\mathbf{u} = \mathbf{r}_{O_jM_j}$, $\mathbf{v} = \boldsymbol{\omega}_j$, $\mathbf{w} = \boldsymbol{\omega}_j \times \mathbf{r}_{O_jM_j}$, so $\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$ which leads to

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = -\mathbf{v} \times (\mathbf{w} \times \mathbf{u}) \tag{E.11}$$

or also, replacing the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ by their corresponding values

$$\mathbf{r}_{O_jM_j} \times \left(\boldsymbol{\omega}_j \times \left(\boldsymbol{\omega}_j \times \mathbf{r}_{O_jM_j}\right)\right) = -\boldsymbol{\omega}_j \times ((\boldsymbol{\omega}_j \times \mathbf{r}_{O_jM_j}) \times \mathbf{r}_{O_jM_j}). \tag{E.12}$$

Thus, $\mathbf{a}_4$ becomes

$$\begin{aligned}
\mathbf{a}_4 &= -\int_{\mathscr{B}_j} \boldsymbol{\omega}_j \times \left(\left(\boldsymbol{\omega}_j \times \mathbf{r}_{O_jM_j}\right) \times \mathbf{r}_{O_jM_j}\right) \mathrm{d}m \\
&= -\int_{\mathscr{B}_j} \boldsymbol{\omega}_j \times \left(\mathbf{r}_{O_jM_j} \times \left(\mathbf{r}_{O_jM_j} \times \boldsymbol{\omega}_j\right)\right) \mathrm{d}m \\
&= -\int_{\mathscr{B}_j} \boldsymbol{\omega}_j \times \left(\hat{\mathbf{r}}_{O_jM_j}\hat{\mathbf{r}}_{O_jM_j}\boldsymbol{\omega}_j\right) \mathrm{d}m \\
&= \boldsymbol{\omega}_j \times \left(\left(\int_{\mathscr{B}_j} \hat{\mathbf{r}}^T_{O_jM_j}\hat{\mathbf{r}}_{O_jM_j} \mathrm{d}m\right)\boldsymbol{\omega}_j\right).
\end{aligned} \tag{E.13}$$

Now, introducing (10.4) into (E.9), the expression of $\mathbf{a}_5$ becomes

$$\mathbf{a}_5 = 2(\mathbf{a}_{51} + \mathbf{a}_{52}) \tag{E.14}$$

where

$$\begin{aligned}
\mathbf{a}_{51} &= \int_{\mathscr{B}_j} \mathbf{r}_{O_jM_{0j}} \times \left(\boldsymbol{\omega}_j \times \boldsymbol{\Phi}_{d_j}(M_{0j})\dot{\mathbf{q}}_{e_j}\right) \mathrm{d}m \\
&= -\int_{\mathscr{B}_j} \mathbf{r}_{O_jM_{0j}} \times \left(\left(\boldsymbol{\Phi}_{d_j}(M_{0j})\dot{\mathbf{q}}_{e_j}\right) \times \boldsymbol{\omega}_j\right) \mathrm{d}m
\end{aligned} \tag{E.15}$$

$$\begin{aligned}
\mathbf{a}_{52} &= \int_{\mathscr{B}_j} \left(\boldsymbol{\Phi}_{d_j}(M_{0j})\mathbf{q}_{e_j}\right) \times \left(\boldsymbol{\omega}_j \times \boldsymbol{\Phi}_{d_j}(M_{0j})\dot{\mathbf{q}}_{e_j}\right) \mathrm{d}m \\
&= -\int_{\mathscr{B}_j} \left(\boldsymbol{\Phi}_{d_j}(M_{0j})\mathbf{q}_{e_j}\right) \times \left(\left(\boldsymbol{\Phi}_{d_j}(M_{0j})\dot{\mathbf{q}}_{e_j}\right) \times \boldsymbol{\omega}_j\right) \mathrm{d}m.
\end{aligned} \tag{E.16}$$

In order to simplify these two expressions, let us consider the fact that:

$$\boldsymbol{\Phi}_{d_j}(M_{0j})\mathbf{q}_{e_j} = \sum_{k=1}^{N_j} \boldsymbol{\Phi}_{dk_j}(M_{0j})q_{ek_j} \tag{E.17}$$

$$\boldsymbol{\Phi}_{d_j}(M_{0j})\dot{\mathbf{q}}_{e_j} = \sum_{k=1}^{N_j} \boldsymbol{\Phi}_{dk_j}(M_{0j})\dot{q}_{ek_j}. \tag{E.18}$$

Introducing (E.18) into (E.15) leads to:

$$\begin{aligned}
\mathbf{a}_{51} &= -\int_{\mathscr{B}_j} \mathbf{r}_{O_jM_{0j}} \times \left(\left(\sum_{k=1}^{N_j} \boldsymbol{\Phi}_{dk_j}(M_{0j})\dot{q}_{ek_j}\right) \times \boldsymbol{\omega}_j\right) \mathrm{d}m \\
&= -\sum_{k=1}^{N_j}\left(\int_{\mathscr{B}_j} \mathbf{r}_{O_jM_{0j}} \times \left(\left(\boldsymbol{\Phi}_{dk_j}(M_{0j})\dot{q}_{ek_j}\right) \times \boldsymbol{\omega}_j\right) \mathrm{d}m\right) \\
&= -\sum_{k=1}^{N_j}\left(\int_{\mathscr{B}_j} \mathbf{r}_{O_jM_{0j}} \times \left(\boldsymbol{\Phi}_{dk_j}(M_{0j}) \times \boldsymbol{\omega}_j\right) \mathrm{d}m\; \dot{q}_{ek_j}\right) \\
&= \sum_{k=1}^{N_j}\left(\int_{\mathscr{B}_j} \hat{\mathbf{r}}^T_{O_jM_{0j}} \hat{\boldsymbol{\Phi}}_{dk_j}(M_{0j})\mathrm{d}m\right) \boldsymbol{\omega}_j\dot{q}_{ek_j}.
\end{aligned} \tag{E.19}$$

Then, introducing (E.17) and (E.18) into (E.16) leads to:

$$\begin{aligned}
\mathbf{a}_{52} &= -\int_{\mathscr{B}_j} \left(\sum_{i=1}^{N_j} \boldsymbol{\Phi}_{di_j}(M_{0j})q_{ei_j}\right) \times \left(\left(\sum_{k=1}^{N_j} \boldsymbol{\Phi}_{dk_j}(M_{0j})\dot{q}_{ek_j}\right) \times \boldsymbol{\omega}_j\right) \mathrm{d}m \\
&= -\sum_{i=1}^{N_j}\sum_{k=1}^{N_j}\left(\int_{\mathscr{B}_j} \left(\boldsymbol{\Phi}_{di_j}(M_{0j})q_{ei_j}\right) \times \left(\left(\boldsymbol{\Phi}_{dk_j}(M_{0j})\dot{q}_{ek_j}\right) \times \boldsymbol{\omega}_j\right) \mathrm{d}m\right) \\
&= -\sum_{i=1}^{N_j}\sum_{k=1}^{N_j}\left(\int_{\mathscr{B}_j} \boldsymbol{\Phi}_{di_j}(M_{0j}) \times \left(\boldsymbol{\Phi}_{dk_j}(M_{0j}) \times \boldsymbol{\omega}_j\right) \mathrm{d}m\; q_{ei_j}\dot{q}_{ek_j}\right) \\
&= -\sum_{i=1}^{N_j}\sum_{k=1}^{N_j}\left(\int_{\mathscr{B}_j} \hat{\boldsymbol{\Phi}}_{di_j}(M_{0j})\hat{\boldsymbol{\Phi}}_{dk_j}(M_{0j})\boldsymbol{\omega}_j\mathrm{d}m\; q_{ei_j}\dot{q}_{ek_j}\right) \\
&= \sum_{i=1}^{N_j}\sum_{k=1}^{N_j}\left(\int_{\mathscr{B}_j} \hat{\boldsymbol{\Phi}}^T_{di_j}(M_{0j})\hat{\boldsymbol{\Phi}}_{dk_j}(M_{0j})\mathrm{d}m\right) \boldsymbol{\omega}_j q_{ei_j}\dot{q}_{ek_j}.
\end{aligned} \tag{E.20}$$

Finally, introducing expressions (E.5), (E.6), (E.7), (E.13), (E.14), (E.19), and (E.20) into (E.4), the expression (10.18) can be obtained.

E.3 Calculation of the Term $\mathbf{f}_{acc_3}$

From (10.12) and (10.15), we get that $\mathbf{f}_{acc_3}$ is given by

$$\begin{aligned}\mathbf{f}_{acc_3} &= \int_{\mathscr{B}_j} \mathbf{\Phi}_{d_j}^T(M_{0j})\dot{\mathbf{v}}_{M_j}\mathrm{d}m \\ &= \int_{\mathscr{B}_j} \mathbf{\Phi}_{d_j}^T(M_{0j})\left(\dot{\mathbf{v}}_j + \mathbf{\Phi}_{d_j}(M_{0j})\ddot{\mathbf{q}}_{e_j} + 2\boldsymbol{\omega}_j \times \mathbf{\Phi}_{d_j}(M_{0j})\dot{\mathbf{q}}_{e_j}\right)\mathrm{d}m \\ &\quad + \int_{\mathscr{B}_j} \mathbf{\Phi}_{d_j}^T(M_{0j})\left(\boldsymbol{\omega}_j \times \left(\boldsymbol{\omega}_j \times \mathbf{r}_{O_jM_j}\right) + \dot{\boldsymbol{\omega}}_j \times \mathbf{r}_{O_jM_j}\right)\mathrm{d}m \end{aligned} \tag{E.21}$$

which can be expanded to be rewritten under a sum of five terms:

$$\mathbf{f}_{acc_3} = \sum_{k=1}^{5} \mathbf{b}_k \tag{E.22}$$

where:

$$\begin{aligned}\mathbf{b}_1 &= \int_{\mathscr{B}_j} \mathbf{\Phi}_{d_j}(M_{0j})^T\dot{\mathbf{v}}_j\mathrm{d}m \\ &= \left(\int_{\mathscr{B}_j} \mathbf{\Phi}_{d_j}(M_{0j})\mathrm{d}m\right)^T \dot{\mathbf{v}}_j\end{aligned} \tag{E.23}$$

$$\begin{aligned}\mathbf{b}_2 &= \int_{\mathscr{B}_j} \mathbf{\Phi}_{d_j}^T(M_{0j})\left(\mathbf{\Phi}_{d_j}(M_{0j})\ddot{\mathbf{q}}_{e_j}\right)\mathrm{d}m \\ &= \left(\int_{\mathscr{B}_j} \mathbf{\Phi}_{d_j}^T(M_{0j})\mathbf{\Phi}_{d_j}(M_{0j})\mathrm{d}m\right)\ddot{\mathbf{q}}_{e_j}\end{aligned} \tag{E.24}$$

$$\begin{aligned}\mathbf{b}_3 &= \int_{\mathscr{B}_j} \mathbf{\Phi}_{d_j}^T(M_{0j})\left(\dot{\boldsymbol{\omega}}_j \times \mathbf{r}_{O_jM_j}\right)\mathrm{d}m \\ &= -\int_{\mathscr{B}_j} \mathbf{\Phi}_{d_j}^T(M_{0j})\left(\mathbf{r}_{O_jM_j} \times \dot{\boldsymbol{\omega}}_j\right)\mathrm{d}m \\ &= -\int_{\mathscr{B}_j} \mathbf{\Phi}_{d_j}^T(M_{0j})\hat{\mathbf{r}}_{O_jM_j}\dot{\boldsymbol{\omega}}_j\mathrm{d}m \\ &= \left(\int_{\mathscr{B}_j} \mathbf{\Phi}_{d_j}^T(M_{0j})\hat{\mathbf{r}}_{O_jM_j}^T\mathrm{d}m\right)\dot{\boldsymbol{\omega}}_j \\ &= \left(\int_{\mathscr{B}_j} \hat{\mathbf{r}}_{O_jM_j}\mathbf{\Phi}_{d_j}(M_{0j})\mathrm{d}m\right)^T \dot{\boldsymbol{\omega}}_j\end{aligned} \tag{E.25}$$

$$\mathbf{b}_4 = \int_{\mathscr{B}_j} \boldsymbol{\Phi}_{d_j}^T(M_{0j}) \left(\boldsymbol{\omega}_j \times \left(\boldsymbol{\omega}_j \times \mathbf{r}_{O_j M_j}\right)\right) \mathrm{d}m \tag{E.26}$$

$$\mathbf{b}_5 = 2 \int_{\mathscr{B}_j} \boldsymbol{\Phi}_{d_j}^T(M_{0j}) \left(\boldsymbol{\omega}_j \times \boldsymbol{\Phi}_{d_j}(M_{0j})\dot{\mathbf{q}}_{e_j}\right) \mathrm{d}m. \tag{E.27}$$

Considering the ith component $\mathbf{b}_4|_i$ of the vector $\mathbf{b}_4$, we can rewrite (E.26) as

$$\mathbf{b}_4|_i = \int_{\mathscr{B}_j} \boldsymbol{\Phi}_{di_j}^T(M_{0j}) \left(\boldsymbol{\omega}_j \times \left(\boldsymbol{\omega}_j \times \mathbf{r}_{O_j M_j}\right)\right) \mathrm{d}m \tag{E.28}$$

in which $\boldsymbol{\Phi}_{di_j}(M_{0j})$ is the ith column of the matrix $\boldsymbol{\Phi}_{d_j}(M_{0j})$.

Now, introducing (10.4) into (E.28), the expression of $\mathbf{b}_4|_i$ becomes

$$\mathbf{b}_4|_i = b_{4i1} + b_{4i2} \tag{E.29}$$

where

$$\begin{aligned} b_{4i1} &= \int_{\mathscr{B}_j} \boldsymbol{\Phi}_{di_j}^T(M_{0j}) \left(\boldsymbol{\omega}_j \times \left(\boldsymbol{\omega}_j \times \mathbf{r}_{O_j M_{0j}}\right)\right) \mathrm{d}m \\ &= \int_{\mathscr{B}_j} \boldsymbol{\Phi}_{di_j}^T(M_{0j}) \left(\left(\mathbf{r}_{O_j M_{0j}} \times \boldsymbol{\omega}_j\right) \times \boldsymbol{\omega}_j\right) \mathrm{d}m \end{aligned} \tag{E.30}$$

$$b_{4i2} = \int_{\mathscr{B}_j} \boldsymbol{\Phi}_{di_j}^T(M_{0j}) \left(\boldsymbol{\omega}_j \times \left(\boldsymbol{\omega}_j \times \left(\boldsymbol{\Phi}_{d_j}\mathbf{q}_{e_j}\right)\right)\right) \mathrm{d}m. \tag{E.31}$$

To simplify the term b_{4i1}, let us recall the well-known identity for the triple product of three arbitrary vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{u}^T(\mathbf{v} \times \mathbf{w}) = \mathbf{w}^T(\mathbf{u} \times \mathbf{v}). \tag{E.32}$$

Now, replacing $\mathbf{u}$ by $\boldsymbol{\Phi}_{di_j}(M_{0j})$, $\mathbf{v}$ by $\boldsymbol{\omega}_j \times \mathbf{r}_{O_j M_{0j}}$, and $\mathbf{w}$ by $\boldsymbol{\omega}_j$, we get

$$\begin{aligned} b_{4i1} &= \int_{\mathscr{B}_j} \boldsymbol{\omega}_j^T \left(\boldsymbol{\Phi}_{di_j}(M_{0j}) \times \left(\mathbf{r}_{O_j M_{0j}} \times \boldsymbol{\omega}_j\right)\right) \mathrm{d}m \\ &= \boldsymbol{\omega}_j^T \left(\int_{\mathscr{B}_j} \hat{\boldsymbol{\Phi}}_{di_j}(M_{0j})\hat{\mathbf{r}}_{O_j M_{0j}} \mathrm{d}m\right) \boldsymbol{\omega}_j \\ &= -\boldsymbol{\omega}_j^T \left(\int_{\mathscr{B}_j} \hat{\boldsymbol{\Phi}}_{di_j}^T(M_{0j})\hat{\mathbf{r}}_{O_j M_{0j}} \mathrm{d}m\right) \boldsymbol{\omega}_j \\ &= -\boldsymbol{\omega}_j^T \left(\int_{\mathscr{B}_j} \hat{\mathbf{r}}_{O_j M_{0j}}^T \hat{\boldsymbol{\Phi}}_{di_j}(M_{0j}) \mathrm{d}m\right)^T \boldsymbol{\omega}_j. \end{aligned} \tag{E.33}$$

Now, in order to simplify the term b_{4i2}, let us consider the fact that:

$$\boldsymbol{\Phi}_{d_j}(M_{0j})\mathbf{q}_{e_j} = \sum_{k=1}^{N_j} \boldsymbol{\Phi}_{dk_j}(M_{0j})q_{ek_j}. \tag{E.34}$$

Thus, b_{4i2} can be rewritten as

$$\begin{aligned} b_{4i2} &= \int_{\mathscr{B}_j} \boldsymbol{\Phi}^T_{di_j}(M_{0j}) \left(\boldsymbol{\omega}_j \times \left(\boldsymbol{\omega}_j \times \left(\sum_{k=1}^{N_j} \boldsymbol{\Phi}_{dk_j}(M_{0j})q_{ek_j}\right)\right)\right) \mathrm{d}m \\ &= \sum_{k=1}^{N_j} \int_{\mathscr{B}_j} \boldsymbol{\Phi}^T_{di_j}(M_{0j}) \left(\boldsymbol{\omega}_j \times \left(\boldsymbol{\omega}_j \times \boldsymbol{\Phi}_{dk_j}(M_{0j})q_{ek_j}\right)\right) \mathrm{d}m \\ &= \sum_{k=1}^{N_j} \int_{\mathscr{B}_j} \boldsymbol{\Phi}^T_{di_j}(M_{0j}) \left(\boldsymbol{\omega}_j \times \left(\boldsymbol{\omega}_j \times \boldsymbol{\Phi}_{dk_j}(M_{0j})\right)\right) \mathrm{d}m \; q_{ek_j}. \end{aligned} \tag{E.35}$$

Once again, using the identity (E.32) by replacing $\mathbf{u}$ by $\boldsymbol{\omega}_j$, $\mathbf{v}$ by $\boldsymbol{\omega}_j \times \boldsymbol{\Phi}_{dk_j}(M_{0j})$, and $\mathbf{w}$ by $\boldsymbol{\Phi}_{di_j}(M_{0j})$, we can obtain

$$\begin{aligned} b_{4i2} &= \sum_{k=1}^{N_j} \int_{\mathscr{B}_j} \boldsymbol{\omega}^T_j \left(\left(\boldsymbol{\omega}_j \times \boldsymbol{\Phi}_{dk_j}(M_{0j})\right) \times \boldsymbol{\Phi}_{di_j}(M_{0j})\right) \mathrm{d}m \; q_{ek_j} \\ &= \sum_{k=1}^{N_j} \boldsymbol{\omega}^T_j \left(\int_{\mathscr{B}_j} \boldsymbol{\Phi}_{di_j}(M_{0j}) \times \left(\boldsymbol{\Phi}_{dk_j}(M_{0j}) \times \boldsymbol{\omega}_j\right) \mathrm{d}m\right) q_{ek_j} \\ &= \sum_{k=1}^{N_j} \boldsymbol{\omega}^T_j \left(\int_{\mathscr{B}_j} \hat{\boldsymbol{\Phi}}_{di_j}(M_{0j})\hat{\boldsymbol{\Phi}}_{dk_j}(M_{0j})\boldsymbol{\omega}_j \mathrm{d}m\right) q_{ek_j} \\ &= -\sum_{k=1}^{N_j} \boldsymbol{\omega}^T_j \left(\int_{\mathscr{B}_j} \hat{\boldsymbol{\Phi}}^T_{di_j}(M_{0j})\hat{\boldsymbol{\Phi}}_{dk_j}(M_{0j})\mathrm{d}m\right) \boldsymbol{\omega}_j q_{ek_j}. \end{aligned} \tag{E.36}$$

Now, considering the ith component $\mathbf{b}_5|_i$ of the vector $\mathbf{b}_5$, we can rewrite (E.27) as

$$\mathbf{b}_5|_i = 2\int_{\mathscr{B}_j} \boldsymbol{\Phi}^T_{di_j}(M_{0j}) \left(\boldsymbol{\omega}_j \times \boldsymbol{\Phi}_{d_j}(M_{0j})\dot{\mathbf{q}}_{e_j}\right) \mathrm{d}m. \tag{E.37}$$

In order to simplify the term $\mathbf{b}_5|_i$, let us consider the fact that:

$$\boldsymbol{\Phi}_{d_j}(M_{0j})\dot{\mathbf{q}}_{e_j} = \sum_{k=1}^{N_j} \boldsymbol{\Phi}_{dk_j}(M_{0j})\dot{q}_{ek_j}. \tag{E.38}$$

Thus, $\mathbf{b}_5|_i$ can be rewritten as

$$\begin{aligned}
\mathbf{b}_5|_i &= 2\int_{\mathscr{B}_j} \boldsymbol{\Phi}^T_{di_j}(M_{0j}) \left(\boldsymbol{\omega}_j \times \left(\sum_{k=1}^{N_j} \boldsymbol{\Phi}_{dk_j}(M_{0j})\dot{q}_{ek_j}\right)\right) \mathrm{d}m \\
&= 2\sum_{k=1}^{N_j}\left(\int_{\mathscr{B}_j} \boldsymbol{\Phi}^T_{di_j}(M_{0j}) \left(\boldsymbol{\omega}_j \times \boldsymbol{\Phi}_{dk_j}(M_{0j})\right) \mathrm{d}m\ \dot{q}_{ek_j}\right) \qquad \text{(E.39)} \\
&= -2\sum_{k=1}^{N_j}\left(\int_{\mathscr{B}_j} \boldsymbol{\Phi}^T_{di_j}(M_{0j}) \left(\boldsymbol{\Phi}_{dk_j}(M_{0j}) \times \boldsymbol{\omega}_j\right) \mathrm{d}m\ \dot{q}_{ek_j}\right).
\end{aligned}$$

Using the identity (E.32) by replacing $\mathbf{u}$ by $\boldsymbol{\Phi}_{di_j}(M_{0j})$, $\mathbf{v}$ by $\boldsymbol{\omega}_j$, and $\mathbf{w}$ by $\boldsymbol{\Phi}_{dk_j}(M_{0j})$, we can obtain

$$\begin{aligned}
\mathbf{b}_5|_i &= -2\sum_{k=1}^{N_j}\left(\int_{\mathscr{B}_j} \boldsymbol{\Phi}^T_{di_j}(M_{0j}) \left(\boldsymbol{\Phi}_{dk_j}(M_{0j}) \times \boldsymbol{\omega}_j\right) \mathrm{d}m\ \dot{q}_{ek_j}\right) \\
&= -2\sum_{k=1}^{N_j}\left(\int_{\mathscr{B}_j} \boldsymbol{\omega}^T_j \left(\boldsymbol{\Phi}^T_{di_j}(M_{0j}) \times \boldsymbol{\Phi}_{dk_j}(M_{0j})\right) \mathrm{d}m\ \dot{q}_{ek_j}\right) \\
&= 2\sum_{k=1}^{N_j}\left(\int_{\mathscr{B}_j} \boldsymbol{\omega}^T_j \left(\boldsymbol{\Phi}_{dk_j}(M_{0j}) \times \boldsymbol{\Phi}^T_{di_j}(M_{0j})\right) \mathrm{d}m\ \dot{q}_{ek_j}\right) \qquad \text{(E.40)} \\
&= 2\sum_{k=1}^{N_j}\left(\int_{\mathscr{B}_j} \boldsymbol{\Phi}_{dk_j}(M_{0j}) \times \boldsymbol{\Phi}^T_{di_j}(M_{0j})\mathrm{d}m\right)^T \boldsymbol{\omega}_j\dot{q}_{ek_j}.
\end{aligned}$$

Finally, taking the ith rows of expressions (E.23), (E.24) and (E.25) and summing them with the expressions (E.33), (E.36) and (E.40), the expression (10.19) can be obtained.

Appendix F
Dynamics Equations for a Clamped-Free Flexible Beam

F.1 Shape Functions for a Free Flexible Beam

Computation of the mass and stiffness matrices of 3D beams is useful for elastodynamic modeling of parallel manipulators.

The Bernoulli model describes beam deformation under the assumption that the shear effect is negligible, that the cross-sections remain perpendicular to the neutral axis and that the rotational inertia of sections is assumed to be zero Blevins (2001). With such a model, the 3D beam deformation $\mathbf{u}_{e_j}(M_{0j})$ (see Sect. 10.2.1) can be characterized with the six shape functions $\mathbf{\Phi}_{dx_j}$, $\mathbf{\Phi}_{dy_j}$, $\mathbf{\Phi}_{dz_j}$, $\mathbf{\Phi}_{rx_j}$, $\mathbf{\Phi}_{ry_j}$ and $\mathbf{\Phi}_{rz_j}$, i.e. $N_j = 6$, defined as:

$$\mathbf{\Phi}_{dx_j} = \left[\xi \; 0 \; 0 \; 0 \; 0 \; 0\right] \tag{F.1a}$$

$$\mathbf{\Phi}_{dy_j} = \left[0 \; 3\xi^2 - 2\xi^3 \; 0 \; 0 \; 0 \; l_j\left(\xi^3 - \xi^2\right)\right] \tag{F.1b}$$

$$\mathbf{\Phi}_{dz_j} = \left[0 \; 0 \; 3\xi^2 - 2\xi^3 \; 0 \; -l_j\left(\xi^3 - \xi^2\right) \; 0\right] \tag{F.1c}$$

$$\mathbf{\Phi}_{rx_j} = \left[0 \; 0 \; 0 \; \xi \; 0 \; 0\right] \tag{F.1d}$$

$$\mathbf{\Phi}_{ry_j} = \left[0 \; 0 \; -6\left(\xi - \xi^2\right)/l_j \; 0 \; 3\xi^2 - 2\xi \; 0\right] \tag{F.1e}$$

$$\mathbf{\Phi}_{rz_j} = \left[0 \; 6\left(\xi - \xi^2\right)/l_j \; 0 \; 0 \; 0 \; 3\xi^2 - 2\xi\right] \tag{F.1f}$$

where $\xi = x/l_j$ and l_j is the beam length.

x, y and z denote the Cartesian coordinates of point M_{0j} expressed in the local frame $\mathscr{F}_j$ and $\mathbf{\Phi}_{d_j}(M_{0j})$ defined at (10.3) is a (3×6) matrix that takes the form:

$$\mathbf{\Phi}_{d_j}(M_{0j}) = \begin{bmatrix} \mathbf{\Phi}_{dx_j} - y\mathbf{\Phi}_{rz_j} + z\mathbf{\Phi}_{ry_j} \\ \mathbf{\Phi}_{dy_j} - z\mathbf{\Phi}_{rx_j} \\ \mathbf{\Phi}_{dz_j} + y\mathbf{\Phi}_{rx_j} \end{bmatrix} \tag{F.2}$$

S. Briot and W. Khalil, *Dynamics of Parallel Robots*,
Mechanisms and Machine Science 35, DOI 10.1007/978-3-319-19788-3

while $\boldsymbol{\Phi}_{r_j}(M_{0j})$ defined at (10.5) is a (3×6) matrix equal to:

$$\boldsymbol{\Phi}_{r_j}(M_{0j}) = \begin{bmatrix} \boldsymbol{\Phi}_{rx_j} \\ \boldsymbol{\Phi}_{ry_j} \\ \boldsymbol{\Phi}_{rz_j} \end{bmatrix}. \tag{F.3}$$

F.2 Stiffness Matrix for a Free Flexible Beam

In the beam model, it is assumed that (Shabana 2005)

$$\sigma_{j22} = \sigma_{j33} = \sigma_{j23} = 0 \tag{F.4}$$

$$\varepsilon_{j22} = \varepsilon_{j33} = \varepsilon_{j23} = 0 \tag{F.5}$$

$$\sigma_{j11} = E_j \varepsilon_{j11} \tag{F.6}$$

$$\sigma_{j12} = G_j \varepsilon_{j12} \tag{F.7}$$

$$\sigma_{j13} = G_j \varepsilon_{j13} \tag{F.8}$$

where E_j is the Young modulus of body j and $G_j = E_j/(2(1+\nu_j))$ is its shear modulus, ν_j being the Poisson's coefficient.

Introducing (F.1a) to (F.8) into (10.35), the stiffness matrix of body $\mathscr{B}_j$ takes the form:

$$\mathbf{K}_{ee_j} = \frac{1}{l_j^3} \begin{bmatrix} E_j A_j l_j^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 E_j I_{z_j} & 0 & 0 & 0 & -6 E_j I_{z_j} l_j \\ 0 & 0 & 12 E_j I_{y_j} & 0 & 6 E_j I_{y_j} l_j & 0 \\ 0 & 0 & 0 & I_{0_j} G_j l_j^2 & 0 & 0 \\ 0 & 0 & 6 E_j I_{y_j} l_j & 0 & 4 E_j I_{y_j} l_j^2 & 0 \\ 0 & -6 E_j I_{z_j} l_j & 0 & 0 & 0 & 4 E_j I_{z_j} l_j^2 \end{bmatrix} \tag{F.9}$$

where A_j is the beam cross-section area, I_{y_j} and I_{z_j} are the second moments of area around axes y and z of the local frame, I_{0_j} is the torsion constant.

F.3 Evaluation of the Inertia Matrix of a Free Flexible 3D Bernoulli Beam for $\mathbf{q}_{e_j} = \mathbf{0}$

For $\mathbf{q}_{e_j} = \mathbf{0}$, the inertia matrix of the flexible 3D Bernoulli beam becomes, from Sect. 10.2.24:

$$\mathbf{M}_j = \begin{bmatrix} m_j \mathbf{1}_3 & \widehat{\mathbf{ms}}_{r_j}^T & \mathbf{MS}_{de_j} \\ \widehat{\mathbf{ms}}_{r_j} & \mathbf{I}_{rr_j} & \mathbf{MS}_{re_j}^r \\ \mathbf{MS}_{de_j}^T & \mathbf{MS}_{re_j}^{r\,T} & \mathbf{M}_{ee_j} \end{bmatrix} \tag{F.10}$$

where

$$\mathbf{MS}^{r}_{re_j} = \left[\boldsymbol{\beta}_{1_j} \;\cdots\; \boldsymbol{\beta}_{N_{j,j}}\right]. \tag{F.11}$$

After simplifications, we get that

$$\mathbf{MS}_{de_j} = \begin{bmatrix} \frac{m_j}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{m_j}{2} & 0 & 0 & 0 & \frac{m_j l_j}{12} \\ 0 & 0 & \frac{m_j}{2} & 0 & \frac{m_j l_j}{12} & 0 \end{bmatrix} \tag{F.12}$$

$$\mathbf{MS}^{r}_{re_j} = \begin{bmatrix} 0 & 0 & 0 & \frac{\rho_j l_j I_{p_j}}{2} & 0 & 0 \\ 0 & 0 & -\rho_j I_{y_j} - \frac{7 m_j l_j}{20} & 0 & -\frac{m_j l_j^2}{20} & 0 \\ 0 & \rho_j I_{z_j} + \frac{7 m_j l_j}{20} & 0 & 0 & 0 & -\frac{m_j l_j^2}{20} \end{bmatrix} \tag{F.13}$$

and

$$\mathbf{M}_{ee_j} = \begin{bmatrix} \frac{m_j}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{13 m_j}{35} + \frac{6 \rho_j I_{z_j}}{5 l_j} & 0 & 0 & 0 & -\frac{11 m_j l_j + 21 \rho_j I_{z_j}}{210} \\ 0 & 0 & \frac{13 m_j}{35} + \frac{6 \rho_j I_{y_j}}{5 l_j} & 0 & \frac{11 m_j l_j + 21 \rho_j I_{y_j}}{210} & 0 \\ 0 & 0 & 0 & \frac{\rho_j l_j I_{p_j}}{3} & 0 & 0 \\ 0 & 0 & \frac{11 m_j l_j + 21 \rho_j I_{y_j}}{210} & 0 & \frac{m_j l_j^2 + 14 \rho_j I_{y_j} l_j}{105} & 0 \\ 0 & -\frac{11 m_j l_j + 21 \rho_j I_{z_j}}{210} & 0 & 0 & 0 & \frac{m_j l_j^2 + 14 \rho_j I_{z_j} l_j}{105} \end{bmatrix} \tag{F.14}$$

where $I_{p_j} = I_{y_j} + I_{z_j}$ is the polar moment of inertia.

References

Abdellatif, H., & Heimann, B. (2009). Computational efficient inverse dynamics of 6-dof fully parallel manipulators by using the lagrangian formalism. *Mechanism and Machine Theory*, *44*(1), 192–207.

Afroun, M., Dequidt, A., & Vermeiren, L. (2012). Revisiting the inverse dynamics of the Gough-Stewart platform manipulator with special emphasis on universal-prismatic-spherical leg and internal singularity. *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, *226*(10), 2422–2439.

Ait-Ahmed, M. (1993). Contribution à la modélisation géométrique et dynamique des robots parallèles. Ph.D. thesis, LAAS, Toulouse.

Alizadeh, D. (2009). On the home posture of the McGill schoenflies motion generator. Master's thesis, McGill University, Montreal, QC, Canada.

Amine, S., Tale-Masouleh, M., Caro, S., Wenger, P., & Gosselin, C. (2011). Singularity analysis of the 4-RUU parallel manipulator using Grassmann-Cayley algebra. *Transactions of the Canadian Society for Mechanical Engineering*, *35*(5), 515–528.

Amine, S., Tale Masouleh, M., Caro, S., Wenger, P., & Gosselin, C. (2012a). Singularity analysis of 3T2R parallel mechanisms using Grassmann-Cayley algebra and Grassmann geometry. *Mechanism and Machine Theory*, *52*, 326–340.

Amine, S., Tale-Masouleh, M., Caro, S., Wenger, P., & Gosselin, C. (2012b). Singularity analysis of 3T2R parallel mechanisms using Grassmann-Cayley algebra and Grassmann line geometry. *Mechanism and Machine Theory*, *52*, 326–340.

Andreff, N., & Martinet, P. (2006). Vision-based kinematic modelling of some parallel manipulators for control purposes. In *Proceedings of EuCoMeS, the First European Conference on Mechanism Science*. Obergurgl, Austria.

Andreff, N., Marchadier, A., & Martinet, P. (2005). Vision-based control of a Gough-Stewart parallel mechanism using legs observation. In *Proceedings of the IEEE International Conference on Robotics and Automation, ICRA'05* (pp. 2546–2551). Barcelona, Spain.

Andreff, N., Dallej, T., & Martinet, P. (2007). Image-based visual servoing of gough-stewart parallel manipulators using legs observation. *International Journal of Robotics Research*, *26*(7), 677–687.

Angeles, J. (2003). *Fundamentals of robotic mechanical systems—theory, methods, and algorithms* (2nd ed.). New York: Springer.

Arai, T., & Cleary, K. et al. (1990). Design, analysis and construction of a prototype parallel link manipulator. In *Proceedings of the IEEE/RJS Conference on Intelligent Robots and Systems (IROS 1990)* (pp. 205–212). Ibaraki, Japan.

S. Briot and W. Khalil, *Dynamics of Parallel Robots*,
Mechanisms and Machine Science 35, DOI 10.1007/978-3-319-19788-3

Arakelian, V., Briot, S., & Glazunov, V. (2008). Increase of singularity-free zones in the workspace of parallel manipulators using mechanisms of variable structure. *Mechanism and Machine Theory*, *43*(9), 1129–1140.

Armstrong, B. (1988). *Dynamics for robot control: friction modeling and ensuring excitation during parameter identification*. Ph.D. thesis, Department of Electrical Engineering, Stanford University.

Armstrong, B. (1991). *Control of machines with frictions*. Dordrecht: Kluwer Academic.

Ayusawa, K., Venture, G., & Nakamura, Y. (2008). Identification of humanoid robots dynamics using floating-base motion dynamics. In *Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS 2008)*.

Baillieul, J. (1986). Avoiding obstacles and resolving kinematic redundancy. In *Proceedings of the 1986 IEEE International Conference on Robotics and Automation (ICRA 1986)* (pp. 1698–1704). San Francisco, CA, USA.

Baillieul, J., Hollerbach, J., & Brockett, R. (1984). Programming and control of kinematically redundant manipulators. In: *Proceedings of the 23rd IEEE Conference on Decision and Control (CDC 1984)* (pp. 768–774). Las Vegas.

Bauchau, O. (2011). *Flexible multibody dynamics*. New York: Springer.

Ben-Horin, P., & Shoham, M. (2006). Singularity analysis of a class of parallel robots based on Grassmann-Cayley algebra. *Mechanism and Machine Theory*, *41*(8), 958–970.

Ben-Horin, P., & Shoham, M. (2009). Application of Grassmann-Cayley algebra to geometrical interpretation of parallel robot singularities. *The International Journal of Robotics Research*, *28*(1), 127–141.

Ben-Horin, P., Shoham, M., Caro, S., Chablat, D., & Wenger, P. (2008). SINGULAB—a graphical user interface for the singularity analysis of parallel robots based on Grassmann-Cayley algebra. Advances in Robot Kinematics.

Bézout, E. (1764). *Recherches sur le degré des équations résultantes de l'éva-nouissement des inconnues*. Histoire de l'Académie Royale des Sciences.

Bhattacharya, S., Hatwal, H., & Ghosh, A. (1997). An on-line estimation scheme for generalized Stewart platform type parallel manipulators. *Mechanism and Machine Theory*, *32*(1), 79–89.

Bhattacharya, S., Nenchev, D., & Uchiyama, M. (1998). A recursive formula for the inverse of the inertia matrix of a parallel manipulator. *Mechanism and Machine Theory*, *33*(7), 957–964.

Binford, T., et al. (1977). Exploratory study of computer integrated assembly system—discussion of trajectory calculation methods. *Progress Report Memo AIM-285.4*. Stanford: Stanford Artificial Intelligence Laboratory.

Blevins, R. (2001). *Formulas for natural frequency and mode shape*. New York: Krieger Publishing Company.

Bonev, I. (2002). Geometric analysis of parallel mechanisms. Ph.D. thesis, Université Laval, QC, Canada.

Bonev, I. (2008). Direct kinematics of zero-torsion parallel mechanisms. In *Proceedings of the 2008 IEEE International Conference on Robotics and Automation (ICRA 2008)*.

Bonev, I., & Gosselin, C. (2005a). Singularity loci of spherical parallel mechanisms. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA2005)* (pp. 2968–2973). Barcelona, Spain.

Bonev, I., & Gosselin, C. (2005b). Singularity loci of spherical parallel mechanisms. In *Proceedings of the 2005 IEEE International Conference on Robotics and Automation (ICRA 2005)* (pp. 2968–2973). Barcelona, Spain.

Bonev, I., & Gosselin, C. (2006). Analytical determination of the workspace of symmetrical spherical parallel mechanisms. *IEEE Transactions on Robotics*, *22*(5), 1011–1017.

Bonev, I., & Ryu, J. (1999). Orientation workspace analysis of 6-dof parallel manipulators. In *Proceedings of the ASME 1999 Design Engineering Technical Conferences (IDETC99)*. Las Vegas, NV, USA.

Bonev, I., Zlatanov, D., & Gosselin, C. (2002a). Advantages of the modified Euler angles in the design and control of PKMs. In *Proceedings of the Parallel Kinematic Machines International Conference* (pp. 171–188). Chemnitz, Germany.

Bonev, I., Zlatanov, D., Gosselin, C. (2002b). Advantages of the modified euler angles in the design and control of PKMs. In *Proceedings of the 2002 Parallel Kinematic Machines International Conference (PKS 2002)* (pp. 171–188). Chemnitz, Germany.

Bonev, I., Zlatanov, D., & Gosselin, C. (2003). Singularity analysis of 3-dof planar parallel mechanisms via screw theory. *ASME Journal of Mechanical Design*, *125*(3), 573–581.

Bonev, I., Chablat, D., & Wenger, P. (2006). Working and assembly modes of the agile eye. In *Proceedings of the IEEE International Conference On Robotics And Automation (ICRA 1996)*. Orlando, FL, USA.

Borrel, P. (1986). *Contribution à la modélisation géométrique des robots-manipulateurs: Application à la conception assistée par ordinateur*. USTL, Montpellier, France: Thèse d'etat.

Bourbonais, F., Bigras, P., & Bonev, I. (2014). Minimum-time trajectory planning and control of a reconfigurable pick-and-place parallel robot. *IEEE/ASME Transactions on Mechatronics*.

Bouzgarrou, B. C., Ray, P., & Gogu, G. (2005). New approach for dynamic modelling of flexible manipulators. *Part K: Journal of Multi-body Dynamics*, *219*(3), 285–298.

Boyer, F., & Coiffet, P. (1996). Symbolic modelling of a flexible manipulator via assembling of its generalized Newton-Euler model. *Mechanism and Machine Theory*, *31*, 45–56.

Boyer, F., & Khalil, W. (1998). An efficient calculation of the flexible manipulator inverse dynamics. *International Journal of Robotics Research*, *17*(3), 282–293.

Boyer, F., Khalil, W., Benosman, M., & LeVey, G. (2007). Robot manipulators. Modeling, performance, analysis and control: Modeling and control of flexible robots. *Control Systems, Robotics and Manufacturing Series* (pp. 337–394). ISTE

Briot, S., & Arakelian, V. (2008). Optimal force generation of parallel manipulators for passing through the singular positions. *International Journal of Robotics Research*, *27*(8), 967–983.

Briot, S., & Arakelian, V. (2009). Complete shaking force and shaking moment balancing of the position-orientation decoupled PAMINSA manipulator. In *Proceedings of the IEEE/ASME International Conference on Advanced Intelligent Mechatronics (AIM2009)*. Singapore.

Briot, S., & Arakelian, V. (2010). On the dynamic properties of rigid-link flexible-joint parallel manipulators in the presence of type 2 singularities. *ASME Journal of Mechanisms and Robotics*, *2*(2).

Briot, S., & Arakelian, V. (2011). On the dynamic properties of flexible parallel manipulators in the presence of Type 2 singularities. *ASME Journal of Mechanisms and Robotics*, *3*(3).

Briot, S., & Bonev, I. (2008). Singularity loci of zero-torsion parallel mechanisms. In *Proceedings of the IEEE/RJS 11th International Conference on Intelligent Robots and Systems* (IROS 2008). Nice, France.

Briot, S., & Bonev, I. (2009a). Pantopteron: A new fully-decoupled 3-dof translational parallel robot for pick-and-place applications. *ASME Journal of Mechanisms and Robotics*, *1*(2).

Briot, S., & Bonev, I. (2009b). Self motions of the pantopteron. In *Proceedings of the ASME 2009 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference (IDETC/CIE 2009)*. San Diego, California, USA.

Briot, S., & Bonev, I. (2010). Pantopteron-4: A new 3T1R decoupled parallel manipulator for pick-and-place applications. *Mechanism and Machine Theory*, *45*(5), 707–721.

Briot, S., & Gautier, M. (2012). Global identification of drive gains and dynamic parameters of parallel robots—part 1: Theory. In *Proceedings of the 19th CISM-IFToMM Symposium on Robot Design, Dynamics, and Control (RoManSy)*.

Briot, S., & Gautier, M. (2014). Global identification of joint drive gains and dynamic parameters of parallel robots. *Multibody System Dynamics* (In press).

Briot, S., & Khalil, W. (2014a). Recursive and symbolic calculation of the elastodynamic model of flexible parallel robots. *The International Journal of Robotics Research*, *33*(3), 469–483.

Briot, S. & Khalil, W. (2014b). Recursive and symbolic calculation of the stiffness and mass matrices of parallel robots. In *Proceedings of the 20th CISM-IFToMM Symposium on Theory and Practice of Robots and Manipulators* (RoManSy 2014). Moscow, Russia.

Briot, S. & Martinet, P. (2013). Minimal representation for the control of Gough-Stewart platforms via leg observation considering a hidden robot model. In *Proceedings of the 2013 IEEE International Conference on Robotics and Automation (ICRA 2013)*. Karlsruhe, Germany.

Briot, S., Bonev, I., Chablat, D., Wenger, P., & Arakelian, V. (2008). Self motions of general 3-rpr planar parallel robots. *International Journal of Robotics Research*, *27*(7), 855–866.

Briot, S., Pashkevich, A., & Chablat, D. (2009). On the optimal design of parallel robots taking into account their deformations and natural frequencies. In *Proceedings of the ASME 2009 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference IDETC/CIE*. San Diego, California.

Briot, S., Pashkevich, A., & Chablat, D. (2010). Optimal technology-oriented design of parallel robots for high-speed machining applications. In *Proceedings of the 2010 IEEE International Conference on Robotics and Automation (ICRA 2010)*. Anchorage, Alaska, USA.

Briot, S., Pashkevich, A. & Chablat, D. (2011). Reduced elastodynamic modelling of parallel robots for the computation of their natural frequencies. In *World Congress in Mechanism and Machine Science*, 13.

Briot, S., Arakelian, V., & Le Baron, J. (2012a). Shaking force minimization of high-speed robots via centre of mass acceleration control. *Mechanism and Machine Theory*, *57*, 1–12.

Briot, S., Caro, S., & Germain, C. (2012b). Robot Parallèle à Deux Degrés de Liberté Présentant Deux Chaînes Cinématiques dont la Raideur en Flexion est Maximisée. FR, FR2967603 - JP 2013–540316 - US 2014–0020500 - PCT/EP2011/070598.

Bruyninckx, H. (1998). Closed-form position kinematics of a (3-1-1-1) fully parallel manipulator. *IEEE Transactions on Robotics and Automation*, *14*(2), 326–328.

Cammarata, A., Condorelli, D., & Sinatra, R. (2013). An algorithm to study the elastodynamics of parallel kinematic machines with lower kinematic pairs. *ASME Transactions Journal of Mechanisms and Robotics*, *5*(1).

Campos, L., Bourbonais, F., Bonev, I. & Bigras, P. (2010). Development of a five-bar parallel robot with large workspace. In *Proceedings of the ASME 2010 International Design Engineering Technical Conferences*. Montréal, QC, Canada.

Canudas de Wit, C., & Aubin, A. (1990). Parameters identification of robots manipulators via sequential hybrid estimation algorithms. In *Proceedings IFAC Congress* (pp. 178–183).

Canudas de Wit, C., Noël, P., Aubin, A., Brogliato, B., & Drevet, P. (1989). Adaptive friction compensation in robot manipulators: Low-velocities. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA 1989)* (pp. 1352–1357). Scottsdale.

Carricato, M., & Gosselin, C. (2009). On the modeling of leg constraints in the dynamic analysis of Gough/Stewart-type platforms. *ASME Journal of Computational and Nonlinear Dynamics*, *4*(1), 1–8.

Carricato, M., & Parenti-Castelli, V. (2002). Singularity-free fully-isotropic translational parallel manipulators. *International Journal of Robotics Research*, *21*(2), 161–174.

Chablat, D., & Wenger, P. (2003). Architecture optimization of a 3-dof parallel mechanism for machining applications, the Orthoglide. *IEEE Transactions on Robotics and Automation*, *19*(3), 403–410.

Chablat, D., & Wenger, P. (2005). Design of a spherical wrist with parallel architecture: Application to vertebrae of an eel robot. In *Proceedings of the 2005 IEEE International Conference on Robotics and Automation (ICRA 2005)*. Barcelona, Spain.

Chablat, D., Moroz, G., & Wenger, P. (2011). Uniqueness domains and non singular assembly mode changing trajectories. In *Proceedings of the 2011 IEEE International Conference on Robotics and Automation (ICRA 2011)*. Shanghai, China.

Charentus, S., & Renaud, M. (1989). Modeling and control of a modular redundant robot manipulator. In *Proceedings of the First International Symposium on Experimental Robotics*.

Chedmail, P., & Gautier, M. (1990). Optimum choice of robot actuators. *ASME Journal of Engineering for Industry*, *112*(4), 361–367.

Chedmail, P., Gautier, M., & Khalil, W. (1986). *Automatic modelling of robots including parameters of links and actuators* In *Proceedings of the IFAC Symposium on Theory of Robots* (pp. 295–299). Vienna, Austria

Clavel, R. (1989). Une nouvelle structure de manipulateur parallèle pour la robotique légère. *Automatique-Productique Informatique Industrielle*, *23*(6), 501–519.

Clavel, R. (1990). *Device for the movement and positioning of an element in space*. US, 4976582.

Codourey, A., & Burdet, E. (1997). A body oriented method for finding a linear form of the dynamic equatiojns of fully parallel robot. In *Proceedings of the 1997 IEEE International Conferenace on Robotics and Automation (ICRA 1997)* (pp. 1612–1619). Albuquerque, New Mexico, USA.

Company, O. & Pierrot, F. (1999). H4: A new family of 4-dof parallel robots. In *Proceedings of the IEEE/ASME International Conference on Advanced Intelligent Mechatronics (AIM1999)* (pp. 508–513). Atlanta, Georgia, USA.

Company, O., Pierrot, F., & Krut, S. (2002). Modelling and preliminary design issues of a four-axis parallel machine for heavy parts handling. *Proceedings of the Institution of Mechanical Engineers, Part K: Journal of Multi-body Dynamics*, *216*(1), 1–12.

Conconi, M., & Carricato, M. (2009). A new assessment of singularities of parallel kinematic chains. *IEEE Transactions on Robotics*, *25*(4), 757–770.

Craig, J. (1986). *Introduction to robotics: Mechanics and control*. Reading: Addison Wesley Publishing Company.

Craig, R. (1981). *Structural dynamics*. London: Wiley.

Craig, R., & Bampton, M. (1968). Coupling of substructures for dynamic analysis. *AIAA Journal*, *6*(7).

Crawford, N., Yamaguchi, G., & Dickman, C. (1999). A new technique for determining 3-D joint angles: The tilt/twist method. *Clinical Biomechanics*, *14*(3), 153–165.

Dahl, P. (1977). Measurements of solid friction parameters of ball bearings. In *Proceedings of the 6th Annual Symposium on Incremental Motion Control Systems and Devices*. University of Illinois, USA.

Dasgupta, B., & Choudhury, P. (1999). A general strategy based on the Newton-Euler approach for the dynamic formulation of parallel manipulators. *Mechanism and Machine Theory*, *34*(6), 801–824.

Dasgupta, B., & Mruthyunjaya, T. (1998a). Closed-form dynamic equations of the general Stewart platform through the Newton-Euler approach. *Mechanism and Machine Theory*, *33*(7), 993–1012.

Dasgupta, B., & Mruthyunjaya, T. (1998b). A Newton-Euler formulation for the inverse dynamics of the Stewart platform manipulator. *Mechanism and Machine Theory*, *33*(8), 1135–1152.

de Casteljau, P. (1987). *Les quaternions*. Paris, France: Hermès.

de Jalon, J. & Bayo, E. (1994). *Kinematic and dynamic simulations of multibody systems*. New York: Springer.

Dedieu, J., & Norton, G. (1990). Stewart varieties : A direct algebraic model for stewart platform. *SIGSAM Bulletin*, *24*(4), 42–59.

Denavit, J., & Hartenberg, R. (1955). A kinematic notation for lower pair mechanism based on matrices. *ASME Journal of Applied Mechanics*, *22*, 215–221.

Di Gregorio, R. (2003). Kinematics of the 3-UPU wrist. *Mechanism and Machine Theory*, *38*(3), 253–263.

Di Gregorio, R. (2004a). The 3-RRS wrist: A new, simple and non-overconstrained spherical parallel manipulator. *ASME Journal of Mechanical Design*, *126*(5), 850–855.

Di Gregorio, R. (2004b). Determination of singularities in Delta-like manipulators. *International Journal of Robotics Research*, *23*(1), 89–96.

Di Gregorio, R. (2005). Analytic form solution of the direct position analysis of a wide family of three-legged parallel manipulators. *ASME Journal of Mechanical Design*, *128*(1), 264–271.

Dwivedy, S. K. & Eberhard, P. (2006a). Dynamic analysis of flexible manipulators, a literature review. *Mechanism and Machine Theory*, 41 (7), 749–777. http://www.sciencedirect.com/science/article/B6V46-4JHMRPP-1/2/b2f90925df5e89d47b0aba9b6d67d06c.

Dwivedy, S., & Eberhard, P. (2006b). Dynamic analysis of flexible manipulators, a litterature review. *Mechanism and Machine Theory*, *41*(7), 749–777.

Ebrahimi, I., Carretero, J., & Boudreau, R. (2008). Kinematic analysis and path planning of a new kinematically redundant planar parallel manipulator. *Robotica*, *26*(3).

El Omri, J. (1996). Analyse géométrique et cinématique des mécanismes de type manipulateur. Ph.D. thesis, Université de Nantes et Ecole Centrale de Nantes, Nantes, France.

Espiau, B., Chaumette, F., & Rives, P. (1992). A new approach to visual servoing in robotics. *IEEE Transactions on Robotics and Automation*, *8*(3).

Faugère, J., & Lazard, D. (1995). Combinatorial classes of parallel manipulators. *Mechanism and Machine Theory*, *30*(6), 765–776.

Fichter, E. (1986). A Stewart platform based manipulator: General theory and practical construction. *The International Journal of Robotics Research*, *5*(2), 157–181.

Firmani, F., Zibil, A., Nokleby, S., & Podhorodeski, R. (2007). Force-moment capabilities of revolute-jointed planar parallel manipulators with additional actuated branches. *Transactions of the Canadian Society for Mechanical Engineering*, *31*(4), 469–481.

Forsythe, G., Malcom, M., & Moler, C. (1977). *Computer methods for mathematical computations*. Englewood Cliffs: Prentice-Hall.

Foucault, S., & Gosselin, C. (2004). Synthesis, design, and prototyping of a planar three degree-of-freedom re-actionless parallel mechanism. *ASME Journal of Mechanical Design*, *126*, 992–999.

Fournier, A. (1980). *Génération de mouvements en robotique : application des inverses généralisées et des pseudo-inverses*. USTL, Montpellier, France: Thèse d'etat.

Fu, S., Yao, Y., & Wu, Y. (2007). Comments on a Newton-Euler formulation for the inverse dynamics of the Stewart platform manipulator. *Mechanism and Machine Theory*, *42*(12), 1668–1671.

Gallardo, J., Rico, J., Frisoli, A., Checcacci, D., & Bergamasco, M. (2003). Dynamics of parallel manipulators by means of screw theory. *Mechanism and Machine Theory*, *38*(11), 1113–1131.

Gautier, M. (1986). Identification of robots dynamics. In *Proceedings IFAC Symposium on Theory of Robots* (pp. 351–356). Vienne, Austria

Gautier, M. (1991). Numerical calculation of the base inertial parameters. *Journal of Robotics Systems*, *8*(4), 485–506.

Gautier, M. (1997). Dynamic identification of robots with power model. *In Proceedings IEEE ICRA* (pp. 1922–1927). Albuquerque, USA

Gautier, M., & Briot, S. (2011a). New method for global identification of the joint drive gains of robots using a known inertial payload. In *Proceedings IEEE ECC CDC* (pp. 526–531). Orlando, FL, USA.

Gautier, M., & Briot, S. (2011b). New method for global identification of the joint drive gains of robots using a known payload mass. In *Proceedings IEEE IROS*. San Francisco, CA, USA.

Gautier, M. & Briot, S. (2012). Global identification of drive gains parameters of robots using a known payload. In *Proceedings of the 2012 International Conference on Robotics and Automation (ICRA 2012)*.

Gautier, M. & Briot, S. (2013). Dynamic parameter identification of a 6 dof industrial robot using power model. In: *Proceedings of the 2013 IEEE International Conference on Robotics and Automation (ICRA 2013)*. Karlsruhe, Germany.

Gautier, M., & Khalil, W. (1990). Direct calculation of minimum set of inertial parameters of serial robots. *IEEE Transactions on Robotics and Automation*, *6*(3), 368–373.

Gautier, M., Vandanjon, P., & Presse, C. (1994). Identification of inertial and drive gain parameters of robots. In *Proceedings IEEE CDC* (pp. 3764–3769). Lake Buena Vista, FL, USA

Gautier, M., Briot, S. & Venture, G. (2013). Identification of consistent standard dynamic parameters of industrial robots. In *Proceedings of the 2013 IEEE/ASME International Conference on Advanced Intelligent Mechatronics (AIM 2013)*. Wollongong, Australia.

Geng, Z., Haynes, S., Lee, J., & Carrol, R. (1992). On the dynamic model and kinematic analysis of a class of Stewart platforms. *Robotics and Autonomous Systems*, *9*, 237–254.

Germain, C., Caro, S., Briot, S. & Wenger, P. (2013). Optimal design of the IRSBot-2 based on an optimized test trajectory. In *Proceedings of the ASME 2011 International Design Engineering*

Technical Conferences and Computers and Information in Engineering Conference (IDETC/CIE 2013). Portland, Oregon, USA.

Gogu, G. (2004). Structural synthesis of fully-isotropic translational parallel robots via theory of linear transformations. *European Journal of Mechanics. A/Solids*, *23*(6), 1021–1039.

Gogu, G. (2008). *Structural synthesis of parallel robots*. New York: Springer.

Gogu, G. (2009). *Structural synthesis of parallel robots—part 2: Translational topologies with two and three degrees of freedom*. New York: Springer.

Gogu, G. (2010). *Structural synthesis of parallel robots—part 3: Topologies with planar motion of the moving platform*. New York: Springer.

Gogu, G. (2012). Structural synthesis of parallel robots—part 4: Other topologies with two and three degrees of freedom. New York: Springer.

Gogu, G. (2014). *Structural synthesis of parallel robots—part 5: Basic overconstrained topologies with schönflies motions*. New York: Springer.

Golub, G. & van Loan, C. (1983). Matrix computation (2nd ed.). *J. Hopkins*.

Gosselin, C. (1993). Parallel computational algorithms for the kinematics and dynamics of parallel manipulators. In *Proceedings of the 1993 IEEE International Conference on Robotics and Automation* (pp. 883–889). NY, USA.

Gosselin, C. (2009). Compact dynamic models for the tripteron and quadrupteron parallel manipulators. *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, *223*(1), 1–11.

Gosselin, C., & Angeles, J. (1990). Singularity analysis of closed-loop kinematic chains. *IEEE Transactions on Robotics and Automation*, *6*(3), 281–290.

Gosselin, C., & Sefrioui, J. (1992). Determination of the singular loci of spherical 3 dof parallel manipulators. In *22nd Biennial Mechanisms Conference* (Vol. DE-45, pp. 329–336). Scottsdale.

Gosselin, C., Sefrioui, J., & Richard, M. (1992). Solution polynomiale au problème de la cinématique directe des manipulateurs parallèles plans à 3 degrés de liberté. *Mechanism and Machine Theory*, *27*(2), 107–119.

Gosselin, C., Saint-Pierre, E., & Gagné, M. (1996). On the development of the Agile Eye: Mechanical design, control issues and experimentation. *IEEE Robotics and Automation Society Magazine*, *3*(4), 29–37.

Gugliemetti, P., & Longchamp, R. (1994). A closed form inverse dynamic model ofthe Delta parallel robot. In *Proceedings of the 4th IFAC Symposium on Robot Control (SyRoCo 1994)* (pp. 51–56). Capri.

Hamon, P., Gautier, M., Garrec, P., & Janot, A. (2010). Dynamic identification of robot with a load-dependent joint friction model. In *Proceedings of the International Conference on Robotics, Automation and Mechatronics* (pp. 129–135). Singapore.

Hervé, J. (1978). Analyse structurelle des mécanismes par groupe de déplacement. *Mechanism and Machine Theory*, *13*(4), 437–450.

Hesselbach, J., Helm, M., & Soetebier, S. (2002). Connecting assembly modes for workspace enlargement. In *Advances in Robot Kinematics (ARK 2002)* (pp. 347–356).

Hesselbach, J., Wrege, J., Raatz, A., & Becker, O. (2004). Aspects on the design of high precision parallel robots. *Assembly Automation*, *24*(1), 49–57.

Hoffman, R., & Hoffman, M. (1979). Vibrational modes of an aircraft simulator motion system. In *Proceedings of the 5th World Congress on Theory of Machines and Mechanisms* (pp. 603–606). Montreal.

Hollerbach, J., Khalil, W., & Gautier, M. (2008). Handbook of robotics. *Model Identification* (pp. 321–344). New York: Springer.

Hollerbach, J., & Suh, K. (1985). Redundancy resolution of manipulators through torque optimization. In *Proceedings of the 1985 IEEE International Conference on Robotics and Automation (ICRA 1985)* (pp. 1016–1021). St Louis.

Horaud, R., Dornaika, F., & Espiau, B. (1998). Visually guided object grasping. *IEEE Transactions on Robotics and Automation*, *14*(4), 525–532.

Huang, T., Wang, J., & Whitehouse, D. (1999). Closed form solution to workspace of hexapod-based virtual axis machine tools. *ASME Journal of Mechanical Design*, *121*(1), 26–31.

Hunt, K., & Primrose, E. (1993). Assembly configurations of some in-parallel actuated manipulators. *Mechanism and Machine Theory*, *28*(1), 31–42.

Husain, M., & Waldron, K. (1994). Direct position kinematics of the 3-1-1-1 Stewart platform. *ASME Journal of Mechanical Design*, *116*(4), 1102–1108.

Husty, M., & Karger, A. (2000). Architecture singular parallel manipulators and their self-motions. In *Advances in Robot Kinematics (ARK 2000)*. Piran.

Husty, M. (1996). An algorithm for solving the direct kinematics of general Stewart-Gough platforms. *Mechanism and Machine Theory*, *31*(4), 365–379.

Ibrahim, O., & Khalil, W. (2010). Inverse and direct dynamic models of hybrid robots. *Mechanism and Machine Theory*, *45*, 627–640.

Ider, S. K. (2005). Inverse dynamics of parallel manipulators in the presence of drive singularities. *Mechanism and Machine Theory*, *40*, 33–44.

Innocenti, C. (1995). Direct kinematics in analytical form of the 6–4 fully parallel mechanism. *ASME Journal of Mechanical Design*, *117*(1), 89–95.

Innocenti, C., & Parenti-Castelli, V. (1991a). Direct kinematics of the 6–4 fully parallel manipulator with position and orientation uncoupled. In *Proceedings of the European Robotics and Intelligent Systems Conference* (pp. 23–28). Corfou, Italy.

Innocenti, C., & Parenti-Castelli, V. (1991b). Direct kinematics of the reverse Stewart platform mechanism. In *Proceedings of the3rd IFAC/IFIP/IMACS Symposium on Robot Control (SyRoCo 91)* (pp. 75–80). Vienna, Austria.

Innocenti, C., & Parenti-Castelli, V. (1992). Analytical form solution of the direct kinematics of a 4–4 fully in-parallel actuated six degree-of-freedom mechanism. In *Proceedings of the 9th CISM-IFToMM SYMPOSIUM on Theory and Practice of Robots and Manipulators (RoManSy)* (pp. 41–50). Udine, Italy.

Innocenti, C., & Parenti-Castelli, V. (1993). Direct kinematics in analytical form of a general 5–4 fully-parallel manipulators. In *Computational Kinematics* (pp. 141–152).

Innocenti, C., & Parenti-Castelli, V. (1992). Reduction singularities in kinematics solution of the general geometry 6–6 Stewart platform. In *Proceedings of the IMACS/SICE International Symposium on Robotics, Mechatronics, and Manufacturing Systems* (pp. 451–458). Kobe, Japan.

Inoue, H., Tsusaka, Y., & Fukuizumi, T. (1985). Parallel manipulator. In *Proceedings of the 3rd ISRR* (pp. 321–327). Gouvieux, France.

Ji, Z. (1993). Study of the effect of leg inertia in Stewart platform. In *Proceedings of the 1993 IEEE International Conference on Robotics and Automation (ICRA 1993)* (pp. 121–126). Atlanta.

Joubair, A., Slamani, M., & Bonev, I. (2012). A novel XY-Theta precision table and a geometric procedure for its kinematic calibration. *Robotics and Computer-Integrated Manufacturing*, *28*(1), 57–65.

Kanaan, D. (2008). Contribution à l'étude cinématique et dynamique des machines parallèles. Ph.D. thesis, École Centrale de Nantes.

Kanaan, D., Wenger, P., & Chablat, D. (2007). Kinematics analysis of the parallel module of the VERNE machine. In *Proceedings of the 12th World Congress in Mechanism and Machine Science*. Besançon, France.

Kanaan, D., Wenger, P., Caro, S., & Chablat, D. (2009). Singularity analysis of lower mobility parallel manipulators using Grassmann-Cayley algebra. *IEEE Transactions on Robotics*, *25*(5), 995–1004.

Khalil, W., & Bennis, F. (1995). Symbolic calculation of the base inertial parameters of closed-loop robots. *The International Journal of Robotics Research*, *14*(2), 112–128.

Khalil, W., & Creusot, D. (1997). Symoro+: A system for the symbolic modelling of robots. *Robotica*, *15*, 153–161.

Khalil, W., & Dombre, E. (2002). *Modeling, Identification and Control of Robots*. London: Hermes Penton

Khalil, W. & Gautier, M. (2000). Modeling of mechanical systems with lumped elasticity. In *Proceedings of the IEEE International Conference on Robotics and Automation* (pp. 3965–3970). San Francisco, CA, USA.

Khalil, W., & Guégan, S. (2002). A novel solution for the dynamic modeling of gough-stewart manipulators. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA 2002)*. Washington, DC.

Khalil, W., & Guégan, S. (2004). Inverse and direct dynamic modeling of Gough-Stewart robots. *IEEE Transactions on Robotics and Automation*, *20*(4), 754–762.

Khalil, W., & Ibrahim, O. (2007). General solution for the dynamic modeling of parallel robots. *Journal of Intelligent and Robotic Systems*, *49*(1), 19–37.

Khalil, W., & Kleinfinger, J. (1986). A new geometric notation for open and closed-loop robots. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA1986)* (pp. 1174–1180). San Francisco, CA, USA.

Khalil, W., & Kleinfinger, J. (1987). Minimum operations and minimum parameters of the dynamic model of tree structure robots. *IEEE Journal of Robotics and Automation*, *3*(6), 517–526.

Khalil, W., Kleinfinger, J., & Gautier, M. (1986). Reducing the computational burden of the dynamic model of robots. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA 1986)* (pp. 525–531). San Francisco, CA, USA.

Khalil, W., Gallot, G. & Boyer, F. (2007a). Dynamic modeling and simulation of a 3D serial eel-like robot. *IEEE Transactions on Systems, Man and Cybernetics. Part C: Application and Reviews*, *37*(6).

Khalil, W., Gautier, M., & Lemoine, P. (2007b). Identification of the payload inertial parameters of industrial manipulators. In *Proceedings IEEE ICRA* (pp. 4943–4948). Roma, Italy.

Khalil, W., Vijayalingam, A., Khomutenko, B., Mukhanov, I., Lemoine, P. and Ecorchard, G. (2014). OpenSYMORO: An open-source software package for symbolic modelling of robots. In *Proceedings of the 2014 IEEE/ASME International Conference on Advanced Intelligent Mechatronics (AIM 2014)* (pp. 1206–1211). Besançon, France.

Klein, C. (1984). Use of redundancy in the design of robotic systems. In *Proceedings of the 2nd International Symposium of Robotic Research* (pp. 58–65). Kyoto.

Kong, X., & Gosselin, C. (2002). A class of 3-dof translational parallel manipulators with linear input-output equations. In *Proceedings of the Workshop on Fundamental Issues and Future Research Directions for Parallel Mechanisms and Manipulators* (pp. 3–4). Québec City, QC, Canada.

Kong, X., & Gosselin, C. (2007). *Type synthesis of parallel mechanisms*. New York: Springer.

Kong, X., & Gosselin, C. (2011a). Forward displacement analysis of a quadratic 4-dof 3T1R parallel manipulator: The Quadrupteron. *Meccanica*, *46*(1), 147–154.

Kong, X., & Gosselin, C. (2011b). Forward displacement analysis of a quadratic 4-dof 3T1R parallel manipulator: The Quadrupteron. *Meccanica*, *46*(1), 147–154.

Korein, J. (1984). *A geometric investigation of reach*. Cambridge: MIT Press.

Kotlarski, J., Trung, D., Heimann, B., & Ortmaier, T. (2010). Optimization strategies for additional actuators of kinematically redundant parallel kinematic machines. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA 2010)* (pp. 656–661).

Kruszewski, J., Gawronski, W., Wittbrodt, E., Najbar, F., & Grabowski, S. (1975). *The rigid finite element method*. Warszawa: Arkady.

Kurtz, R., & Hayward, V. (1992). Multiple-goal kinematic optimization of a parallel spherical mechanism with actuator redundancy. *IEEE Transactions on Robotics and Automation*, *8*(5), 644–651.

Lazard, D. (1993). On the representation of rigid-body motions and its application to generalized platform manipulators. In J. Angeles, P. Kovacs & G. Hommel (Eds.), *Computational Kinematics* (pp. 175–182). Kluwer.

Le Borzec, R., & Lotterie, J. (1975). Principes de la théorie des mécanismes. Paris: Dunod.

le Rond d'Alembert, J. (1743). *Traité de dynamique*. Paris: David l'aine

Lebret, G., Liu, G., & Lewis, F. (1993). Dynamic analysis and control of a Stewart platform manipulator. *Journal of Robotic Systems*, *10*(5), 629–655.

Lee, K., & Shah, D. (1988). Dynamic analysis of a three-degrees-of-freedom in-parallel actuated manipulator. *IEEE Transactions on Robotics and Automation*, *4*(3), 361–368.

Leinonen, T. (1991). Terminology for the theory of machines and mechanisms. *Mechanism and Machine Theory*, 26.

Leonesio, M., & Bianchi, G. (2009). Self-locking analysis in closed kinematic chains. *Mechanism and Machine Theory*, *44*(11), 2038–2052.

Lin, W., Duffy, J., & Griffis, M. (1992). Forward displacement analysis of the 4–4 Stewart platform. *ASME Journal of Mechanical Design*, *114*(3), 444–450.

Lin, W., Crane, C., & Duffy, J. (1994). Closed-form forward displacement analysis of the 4–5 in-parallel platforms. *ASME Journal of Mechanical Design*, *116*(1), 47–53.

Liu M. J., Li, C. X., & Li, C. (2000). Dynamics analysis of the Gough-Stewart platform manipulator. *IEEE Transaction on Robotics and Automation*, *16*(1), 94–98.

Liu, X. J., Wang, J., & Pritschow, G. (2006). Performance atlases and optimum design of planar 5r symmetrical parallel mechanisms. *Mechanism and Machine Theory*, *41*(2), 119–144.

Llibre, M., Mampey, R., & Chrétien, J. (1983). Simulation de la dynamique des robots manipulateurs motorisés (pp. 197–207). Actes du Congrès AFCET: Productique et Robotique Intelligente. Besançon, France.

Long, P., Khalil, W., & Martinet, P. (2014). Dynamic modeling of parallel robots with flexible platforms. *Mechanism and Machine Theory*, *81*, 21–35.

Luh, J., Walker, M., & Paul, R. (1980). On-line computational scheme for mechanical manipulators. *ASME Journal of Dynamic Systems, Measurement and Control*, *102*(2), 69–76.

Maciejewski, A., & Klein, C. (1985). Obstacle avoidance for kinematically redundant manipulators in dynamically varying environments. *The International Journal of Robotics Research*, *4*(3), 109–117.

Martinet, P., Gallice, J., & Khadraoui, D. (1996). Vision based control law using 3D visual features. In *Proceedings of the World Automation Congress, WAC96, Robotics and Manufacturing Systems* (Vol. 3, pp. 497–502). Montpellier, France.

Mayeda, H., Yoshida, K., & Osuka, K. (1990). Base parameters of manipulator dynamic models. *IEEE Transactions on Robotics and Automation*, *6*(3), 312–321.

Merlet, J. (1989). Singular configurations of parallel manipulators and Grassmann geometry. *The International Journal of Robotics Research*, *8*(5), 45–56.

Merlet, J. (1997). Direct kinematics of planar parallel manipulators. In *Proceedings of the 1997 IEEE International Conference on Robotics and Automation (ICRA 1997)* (pp. 3744–3749). Minneapolis, USA

Merlet, J. (2004). Solving the forward kinematics of a Gough-type parallel manipulator with interval analysis. *The International Journal of Robotics Research*, *23*(3), 221–236.

Merlet, J. (2006a). Jacobian, manipulability, condition number, and accuracy of parallel robots. *ASME Transactions Journal of Mechanical Design*, *128*(1), 199–206.

Merlet, J. (2006b). *Parallel robots* (2nd ed.). New York: Springer

Merlet, J. (2014). www-sop.inria.fr/members/Jean-Pierre.Merlet/merlet.html.

Miller, K. (2004). Optimal design and modeling of spatial parallel manipulators. *The International Journal of Robotics Research*, *23*(2), 127–140.

Moon, F. (2007). Applied dynamics. J. Wiley and Sons.

Mourrain, B. (1993). The 40 generic positions of a parallel robot. In M. Bronstein (Ed.), *ISSAC'93* (pp. 173–182). Kiev, Ukraine: ACM Press.

Müller, A. (2005). Internal preload control of redundantly actuated parallel manipulators—its application to backlash avoiding control. *IEEE Transactions on Robotics*, *21*(4), 668–677.

Murphy, S., & Ting-Yung Wen, J. (1993). Analysis of active manipulator elements in space manipulation. *IEEE Transactions on Robotics and Automation*, *9*(5), 544–552.

Nabat, V. (2007). Parallel robot with articulated platforms: from concept to industrial solution for pick-and-place operations. Ph.D. thesis, Université de Montpellier.

Nahon, M., & Angeles, J. (1989). Force optimization in redundantly-actuated closed kinematic chains. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA 1989)*.

Nair, P. (1994). On the forward kinematics of parallel manipulators. *The International Journal of Robotics Research*, *13*(2), 171–188.

Nanua, P., & Waldron, K. (1991). Direct kinematic solution of a Stewart platform. *IEEE Transactions on Robotics and Automation*, *6*(4), 438–444.

Newman, W., & Sproull, R. (1979). *Principles of interactive computer graphics*. New York: McGraw Hill.

Notash, L., & Kamalzadeh, A. (2007). Inverse dynamics of wire-actuated parallel manipulators with a constraining linkage. *Mechanism and Machine Theory*, *42*(9), 1103–1118.

Ozgur, E., Andreff, N., & Martinet, P. (2011). Dynamic control of the quattro robot by the leg edgels. In *Proceedings of the IEEE International Conference on Robotics and Automation, ICRA11*. Shanghai, China.

Özgür, E., Andreff, N., & Martinet, P. (2013). Linear dynamic modeling of parallel kinematic manipulators from observable kinematic elements. *Mechanism and Machine Theory*.

Pagis, G., Bouton, N., Briot, S., & Martinet, P. (2014). Design of a controller for enlarging parallel robots workspace through type 2 singularity crossing. *Proceedings of 2014 IEEE International Conference on Robotics and Automation (ICRA 2014)*. Hong Kong, China.

Pagis, G., Bouton, N., Briot, S., & Martinet, P. (2015). Enlarging parallel robot workspace through type-2 singularity crossing. *Control Engineering Practice*, *39*, 1–11.

Park, F., Choi, J., & Ploen, S. (1999). Symbolic formulation of closed chain dynamics in independent coordinates. *Mechanism and Machine Theory*, *34*(5), 731–751.

Pashkevich, A., Chablat, D., & Wenger, P. (2006). Kinematics and workspace analysis of a three-axis parallel manipulator: The Orthoglide. *Robotica*, *24*(1), 39–49.

Paul, R. (1981). *Robot manipulators: Mathematics, programming and control*. Cambridge: MIT Press.

Pelaez, G., Pelaez, G., Perez, J., Vizan, A., & Bautista, E. (2005). Input shaping reference commands for trajectory following cartesian machines. *Control Engineering Practice*, *13*(8), 941–958. Fault Detection, Supervision and Safety of Technical Processes (Safeprocess 2003).

Pfurner, M., & Husty, M. (2010). New trends in mechanism science. In *Implementation of a new and efficient algorithm for the inverse kinematics of serial 6R chains* (pp. 91–98). New York: Springer.

Pieper, D. (1968). The kinematics of manipulators under computer control. Ph.D. thesis, Stanford University.

Pierrot, F., Uchiyama, M., Dauchez, P., & Fournier, A. (1990). A new design of a 6-dof parallel robot. In *Proceedings of the 23rd International Symposium on Industrial Robots. Jounrla of Robotics and Mechatronics* (pp. 308–315).

Pierrot, F., Baradat, C., Nabat, V., Company, O., & Gouttefarde, S. K. K. (2009). Above 40g acceleration for pick-and-place with a new 2-dof pkm. In *Proceedings of the 2009 IEEE international Conference on Robotics and Automation (ICRA 2009)*. Japan: Kobe International Conference Center.

Potkonjak, V. (1986). Thermal criterion for the selection of DC drives for industrial robots. In *Proceedings of the 16th International Symposium on Industrial Robots. Bruxelles* (pp. 129–140). Belgium.

Raghavan, M. (1993). The Stewart platform of general geometry has 40 configurations. *ASME Journal of Mechanical Design*, *115*(2), 277–282.

Rakotomanga, N., Chablat, D., & Caro, S. (2008). Kinetostatic performance of a planar parallel mechanism with variable actuation. *Advances in Robot Kinematics*.

Raucent, B., Bastin, G., Campion, G., Samin, J., & Willems, P. (1992). Identification of barycentric parameters of robotic manipulators from external measurements. *Automatica*, *28*(5), 1011–1016.

Reboulet, C. and Berthomieu, T. (1991). Dynamic models of a six degree of freedom parallel manipulators. In *Proceedings of the International Conference on Advanced Robotics (ICAR 1991)* (pp. 1153–1157). Pisa, Italy.

Rizk, R., Munteanu, M., Fauroux, J., & Gogu, G. (2007). A semi-analytical stiffness model of parallel robots from the Isoglide family via the sub-structuring principle. In *Proceedings of the 12th IFToMM World Congress*. Besançon, France.

Roberts, L. (1965). Homogeneous matrix representation and manipulation of N dimensional constructs. Master's thesis, MIT Lincoln Lab.

Rognant, M., Courteille, E., & Maurine, P. (2010). A systematic procedure for the elastodynamic modeling and identification of robot manipulators. *IEEE Transactions on Robotics*, *26*(6), 1085–1093.

Ronga, F., & Vust, T. (1992). Stewart platforms without computer? In *Proceedings of the Conference on Real Analytic and Algebraic Geometry* (pp. 197–212). Trento, Italy.

Sciavicco, L., Siciliano, B., & Villani, L. (1994). On dynamic modelling of gear-driven rigid robot manipulators. In *Proceedings of the 4th IFAC Symposium on Robot Control (SYROCO 1994)* (pp. 543–549). Capri, Italy.

Seward, N., & Bonev, I. (2014). A new 6-dof parallel robot with simple kinematic model. In *Proceedings of the 2014 IEEE International Conference on Robotics and Automation (ICRA 2014)*. Hong Kong, China.

Shabana, A. (2005). *Dynamics of multibody systems*. Cambridge: Cambridge University Press.

Shabana, A. (1990). Dynamics of flexible bodies using generalized Newton-Euler equations. *Journal of Dynamic Systems, Measurement, and Control*, *112*, 496–503.

Shah, S., Saha, S., & Dutt, J. (2013). *Dynamics of tree-type robotic systems*. New York: Springer.

Sharf, I., & Damaren, C. (1992). Simulation of flexible-link manipulators: basis functions and nonlinear terms in the motion equations. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA 1992)* (pp. 1956–1962). Nice, France.

Sheth, P., & Uicker, J. (1971). A generalized symbolic notation for mechanism. *ASME Journal of Engineering for Industry*, *93*, 102–112.

Singer, N. C., & Seering, W. P. (1988). Preshaping command inputs to reduce system vibration. Massachusetts Institute of Technology Artificial Intelligence Laboratory. A.I. Memo No. 1027.

Singh, T., & Singhose, W. E. (2002). Tutorial on input shaping/time delay control of maneuvering flexible structures. In *American Control Conference*.

Spong, M. (1987). Modeling and control of elastic joint robots. *ASME Journal of Dynamic Systems, Measurement, and Control*, *109*, 310–319.

Stachera, K., & Schumacher, W. (2008). Derivation and calculation of the dynamics of elastic parallel manipulators. *Automation and robotics*. I-Tech Education and Publishing.

Sugimoto, K. (1989). Computational scheme for dynamic analysis of parallel manipulators. *ASME Journal of Mechanics, Transmission and Automation in Design*, *111*, 29–33.

Tale-Masouleh, M., Gosselin, C., Husty, M., & Walter, D. (2011). Forward kinematic problem of 5-RPUR parallel mechanisms (3T2R) with identical limb structures. *Mechanism and Machine Theory*, *46*(7), 945–959.

Tsai, L. (2000). Solving the inverse dynamics of a Stewart-Gough manipulator by the principle of virtual work. *ASME Journal of Mechanical Design*, *122*, 3–9.

Tsai, L., & Joshi, S. (2000). Kinematics and optimization of a spatial 3-u Pu parallel manipulator. *ASME Journal of Mechanical Design*, *122*(4), 439–446.

Ur-Rehman, R., Caro, S., Chablat, D., & Wenger, P. (2009). Path placement optimization of manipulators based on energy consumption: Application to the Orthoglide 3-axis. *Transactions of the Canadian Society for Mechanical Engineering*, *33*(3).

Vakil, M., Pendar, H., & Zohoor, H. (2008). Comments on closed-form dynamic equations of the general Stewart platform through the Newton-Euler approach and a Newton-Euler formulation for the inverse dynamics of the Stewart platform manipulator. *Mechanism and Machine Theory*, *43*(10), 1349–1351.

van der Wijk, V., Krut, S., Pierrot, F., & Herder, J. (2011). Generic method for deriving the general shaking force balance conditions of parallel manipulators with application to a redundant planar 4-rrr parallel manipulator. In *Proceedings of the 13th World Congress in Mechanism and Machine Science*. Guanajuato, Mexico.

Voglewede, P., & Ebert-Uphoff, I. (2005). Overatching framework for measuring the closeness to singularities of parallel manipulators. *IEEE Transactions on Robotics*, *21*(6), 1037–1045.

Walker, M., & Orin, D. (1982). Efficient dynamic computer simulation of robotics mechanism. *ASME Journal of Dynamic Systems, Measurement, and Control*, *104*, 205–211.

Wampler, C. (1996). Forward displacement analysis of general six-in-parallel SPS (Stewart) platform manipulators using soma coordinates. *Mechanism and Machine Theory*, *31*(3), 331–337.

Wang, X. and Mills, J.K. (2006). Dynamic modeling of a flexible-link planar parallel platform using a substructuring approach. *Mechanism and Machine Theory*, *41*(6), 671–687. http://www.sciencedirect.com/science/article/B6V46-4HNSG0Y-1/2/81cd2e245e4875189413080286cf4c4c.

Wang, Y. (1999). Workspace analysis of a novel closed-chain manipulator. Ph.D. thesis, Case Western Reserve University, Cleveland, OH, USA.

Wei, W., & Simaan, N. (2010). Design of planar parallel robots with preloaded flexures for guaranteed backlash prevention. *ASME Journal of Mechanisms and Robotics*, *2*(1).

Wenger, P. (1992). A new general formalism for the kinematic analysis of all non redundant manipulators. In *Proceedings of the 1992 IEEE International Conference on Robotics and Automation (ICRA 1992)* (pp. 442–447). Nice, France.

Wenger, P., Chablat, D., & Zein, M. (2007). Degeneracy study of the forward kinematics of planar 3-r Pr parallel manipulators. *ASME Journal of Mechanical Design*, *129*(12), 1265–1268.

White, N. (2008). Grassmann-Cayley algebra and robotics applications. In *Handbook of geometric computing* (Vol. 8, pp. 629–656). New York: Springer

Whitney, D. (1969). Resolved motion rate control of manipulators and human prostheses. *IEEE Transactions on Man Machine Systems*, *10*(2), 47–53.

Wittbrodt, E., Adamiec-Wójcik, I., & Wojciech, S. (2006). *Dynamics of flexible multibody systems*. Berlin: Springer.

Yi, B., Freeman, R., & Tesar, D. (1994). Force and stiffness transmision in redundantly actuated mechanisms: The case for a spherical shoulder mechanism. *Robotics, Spatial Mechanisms, Mechanical Systems*, *45*, 163–172.

Yin, J., & Liang, C. (1994). The forward displacement analysis of a kind of special platform manipulator. *Mechanism and Machine Theory*, *29*(1), 1–9.

Yoshikawa, T. (1984). Analysis and control of robot manipulators with redundancy. In *The 1st International Symposium of Robotics Research* (pp. 735–748). Cambridge: MIT press.

Zein, M., Wenger, P., & Chablat, D. (2008). Non-singular assembly-mode changing motions for 3-RPR parallel manipulators. *Mechanism and Machine Theory*, *43*(4), 480–490.

Zhang, C., & Song, S. (1991). Forward kinematics of a class of parallel (Stewart) platforms with closed-form solutions. In *Proceedings of the 1991 IEEE International Conference on Robotics and Automation (ICRA 1991)* (pp. 2676–2681). Sacramento.

Zhao, J., Li, B., Yang, X., & Yu, H. (2009). Geometrical method to determine the reciprocal screws and applications to parallel manipulators. *Robotica*, *27*, 929–940.

Zlatanov, D., Fenton, R., & Benhabib, B. (1994a). Singularity analysis of mechanisms and robots via a motion-space model of the instantaneous kinematics. In *Proceedings of the 1994 IEEE International Conference on Robotics and Automation*. Ontario, Canada: Toronto University.

Zlatanov, D., Fenton, R., & Benhabib, B. (1994b). Singularity analysis of mechanisms and robots via a velocity-equation model of the instantaneous kinematics. In *Proceedings of the 1994 IEEE International Conference on Robotics and Automation*. Ontario, Canada: Toronto University.

Zlatanov, D., Bonev, I., & Gosselin, C. (2002). Constraint singularities of parallel mechanisms. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA 2002)*.

Index

A
Acceleration
  angular acceleration, 36
  linear acceleration, 36
  of a body, 36
  of flexible *PKM*, 264
  of flexible tree structure robot, 257
  of joints of open kinematic chains, 58
  of the active joints of *PKM*, 122
  of the passive joints of a *PKM*, 125
  of the platform of a *PKM*, 122
Active joint, 46
Architectures, 9
Aspect, 57
Atan2 function, 82

B
Base, 3
Base dynamic parameters, 189
Bernoulli beams, 248, 329

C
Cable-driven parallel robot, 15
Cartesian
  coordinate, 8, 19, 76
Center of mass, 63
Centrifugal effects, 62, 68, 149, 152, 153, 159, 175, 254, 261, 266
Clamped-free flexible body, 240
Closed loop, 44
Composite link, 150
Coriolis effects, 62, 68, 149, 152, 153, 159, 175, 254, 261, 266
Cross-product matrix, 35
Cuspidal robot, 57
Customized symbolic algorithm, 48, 152, 267, 288

D
Decoupled robot, 9, 103, 202
Degeneracy of the dynamic model, 201
Degree of freedom (*DOF*), 6
Denavit-Hartenberg, 40
DualEMPS, 269
  elastodynamicmodel, 273
  natural frequencies, 289
DualV, 14
  geometric and kinematic models, 179
  inverse dynamic model, 182
Dynamic model
  direct dynamic model (*DDM*), 140, 159, 175
  ground reaction model (*GRM*), 140, 184
  inverse dynamic model (*IDM*), 140, 144, 155, 172
  of parallel robots with actuation redundancy, 172
  of parallel robots with kinematic redundancy, 142
  of parallel robots with no redundancy, 153

E
Eig function, 232
Eigenvalue, 232, 279, 286
Eigenvector, 232, 286
Elastic deformation, 241
Elastic joint, 266, 287
Elastic modulus, 248
End-effector, 3

S. Briot and W. Khalil, *Dynamics of Parallel Robots*,
Mechanisms and Machine Science 35, DOI 10.1007/978-3-319-19788-3

acceleration, 121
frame, 76
twist, 105
Energy
elastic potential energy, 246, 280
energy model, 140, 187
kinetic energy, 62
potential energy, 64
Euler
angles, 27, 109, 124
parameters, 26

F
First moments of inertia
for a flexible body, 249
for a rigid body, 63
Five-bar mechanism, 9
base dynamic parameters, 198
direct dynamic model, 164
energy model, 188
first-order kinematic model, 114
forward geometric model, 94
ground reaction model, 185
inverse dynamic model, 162
inverse geometric model, 80
second-order kinematic model, 127
singularity, 137
trajectory generation through singularity, 206, 211
Flexible body, 240
Flexible joint, 266, 287
Flexible robot, 237, 279
Force, 34
Four-bar linkage, 97
Friction, 147

G
Gear transmission ratio, 147
Generalized accelerations, 264
Generalized coordinates, 40, 61
active generalized coordinates, 61, 65
elastic generalized coordinates, 241
passive generalized coordinates, 65
Generalized forces, 61
Generalized inertia matrix
of a flexible body, 253
of a flexible robot, 266
of a rigid body, 63
of a rigid robot, 62
Generalized Newton-Euler principle, 240, 252
Generalized velocities, 61, 261
elastic generalized velocities, 242
Geometric description, 40
Geometric model
forward geometric model (*FGM*), 92
inverse geometric model (*IGM*), 76
Geometric parameter, 40
Gough-Stewart platform, 13
forward geometric model, 100
inverse geometric model, 89
kinematic model, 120
singularity, 138
Grassmann geometric, 134
Grassmann-Cayley algebra, 135
Gravity, 62, 64, 149, 152, 153, 159, 175, 254, 261, 266
Grouped parameters, 196

H
Half-angle formula, 82
Homogeneous coordinate, 19
Homogeneous transformation matrix, 20
Hooke's law, 248
Hybrid parallel robot, 14

I
Inertia matrix
of a flexible body, 250
of a rigid body, 63
Inertia of actuators, 147, 260, 284
Inertial parameters, 67, 190

J
Jacobian
inverse Jacobian matrix, 108, 111
inverse kinematic Jacobian matrix, 108
Jacobian matrix, 108, 111
Jacobian matrix of a flexible parallel robot, 264
Jacobian matrix of a flexible tree-structure robot, 255
kinematic Jacobian matrix, 53, 108
Joint
active, 8
passive, 8
prismatic, 8
revolute, 8
spherical, 8
universal, 8

K
Kinematic constraint equation, 44, 65, 105, 121, 155, 172
Kinematic model
  forward kinematic model (*FKM*), 52, 108
  inverse kinematic model (*IKM*), 55, 108

L
Lagrange
  equation, 61, 143
  formulation, 61
  multiplier, 65, 156, 173
Lagrangian, 62, 143
Leg passive joint twist system (*LPJTS*), 131
Loop closure equation, 44, 76
Lower mobility robot, 6

M
Mathematica, 153, 268, 289
Matlab, 136, 232, 273, 279, 295
Mobility, 6
Modified Denavit-Hartenberg parameters, 40
Moment, 34
Moving platform, 3

N
Newton-Euler
  algorithm, 144
  equations, 67, 70
Newton-Raphson, 104
Null space, 56, 129
Number of degrees of freedom
  of the platform, 9
  of the robot, 9

O
Olinde-Rodrigues parameters, 26
OpenSYMORO, 58, 59
Orthoglide, 11
  base dynamic parameters, 198
  direct dynamic model, 167
  first-order kinematic model, 119
  forward geometric model, 98
  inverse dynamic model, 167
  inverse geometric model, 86
  second-order kinematic model, 128
  singularity, 138
Over-constrained robot, 262
Overconstraint, 66, 172, 173

P
Parallel kinematic machine (*PKM*), 3
Parallel robot, 3
Passive joint, 46
Payload, 3, 178
Plücker coordinates, 134
Planar 3–$R\underline{P}R$ parallel manipulator
  forward geometric model, 95
  forward kinematic model, 117
  inverse geometric model, 83
Planar parallel manipulators (*PPM*), 9
  forward geometric model, 98
  inverse geometric model, 86
  singularity, 138
Platform, 3
Platform coordinates, 76
Poisson coefficient, 248
Principle of virtual powers (*PVP*), 68, 156, 254
Pseudo-inverse, 56, 66, 108, 109, 123, 124, 173
Python, 153

Q
QR decomposition, 195
Quaternions, 26, 109

R
Recursive computation
  of the accelerations, 257
  of the dynamic models, 144
  of the kinematic Jacobian matrices, 58, 255
  of velocities, 57, 255
Redundancy
  actuation redundancy, 14
  kinematic redundancy, 13
Robot inertia matrix, 62
Roll-Pitch-Yaw angles, 29
Rotation matrix, 20
Rotor inertia, 147

S
Screw, 33
  transformation matrix, 35
Serial chain, 3
Sextic curve, 98
Shape functions, 241

of Bernoulli beams, 329
Shear modulus, 330
Sign function, 26
Simple open chain, 41
Singular value decomposition (*SVD*), 196
Singularities
LPJTS singularities, 131, 204
constraint singularities, 132
input-output singularities, 128
of open chains, 56
of Type 1, 128
of Type 2, 129, 204
of Type 3, 130
serial singularities, 130
Singularity analysis, 133
Spatial parallel manipulators (*SPM*), 10
forward geometric model, 101
singularity, 138
Static moments, 63
Stiffness matrix
of a flexible body, 248
of a flexible parallel robot, 285
of a flexible tree structure robot, 284
Strain vector, 246
Stress vector, 246

T
Tilt-and-Torsion angles, 31
Trajectory generation, 214, 319
Transformation matrix
between twists, 35
between wrenches, 36
homogeneous transformation matrix, 20
Translational parallel manipulators (*TPM*), 11
Tree-structure chain, 3
Tripteron, 11
geometric and kinematic models, 218
inverse dynamic model, 223
singularity, 204
Twist, 33

V
Velocity
angular velocity, 33
linear velocity, 33
of a body, 33
of flexible tree structure robot, 255
of joints of open kinematic chains, 52
of the active joints of a *PKM*, 108
of the passive joints of a *PKM*, 111
of the platform of *PKM*, 108, 109

W
Wrench, 34
actuation wrench, 106, 134
constraint wrench, 133