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Extragalactic Homework I

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1. PROBLEM 1

Given the Planck function, $B_\nu(T)$,

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT)-1} \tag{1}$$

show that the following alternative form for the Planck function, $B_\lambda(T)$, is given by:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{\exp(hc/\lambda kT)-1} \tag{2}$$

To begin with, let's rewrite the Planck function to represent all of its variables, $B_\nu(\nu, T)$. If we set $\nu(\lambda) = {}^{c}/_{\lambda}$, then the following steps can be taken:

$$\begin{aligned} -B_\nu(\nu(\lambda), T)d\nu &= B_\lambda(\lambda, T)d\lambda \\ -B_\nu(\nu(\lambda), T)\frac{d\nu}{d\lambda} &= B_\lambda(\lambda, T) \end{aligned} \tag{3}$$

Taking into account that $\nu(\lambda) = {}^{c}/_{\lambda}$,

$$\frac{d\nu}{d\lambda} = -{}^{c}/_{\lambda^2} \tag{4}$$

Substituting these into the original $B_\nu(\nu(\lambda), T)$ equation (1) yields:

$$\begin{aligned} B_\nu({}^{c}/_{\lambda}, T)\,({}^{c}/_{\lambda^2}) &= \frac{2h({}^{c}/_{\lambda})^3}{c^2}\frac{1}{\exp(h({}^{c}/_{\lambda})/kT)-1}\,({}^{c}/_{\lambda^2}) \\ B_\nu({}^{c}/_{\lambda}, T)\,({}^{c}/_{\lambda^2}) &= \frac{2hc^2}{\lambda^5}\frac{1}{\exp(hc/\lambda kT)-1} \end{aligned} \tag{5}$$

$$\boxed{B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{\exp(hc/\lambda kT)-1}}$$

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Figure 1 shows a series of Planck functions (also known as a 'blackbody' curves), representing a range of temperatures. The curves have been plotted using both version of the Planck function, $B_\nu(T)$ and $B_\lambda(T)$, represented by the solid and dotted lines, respectively.

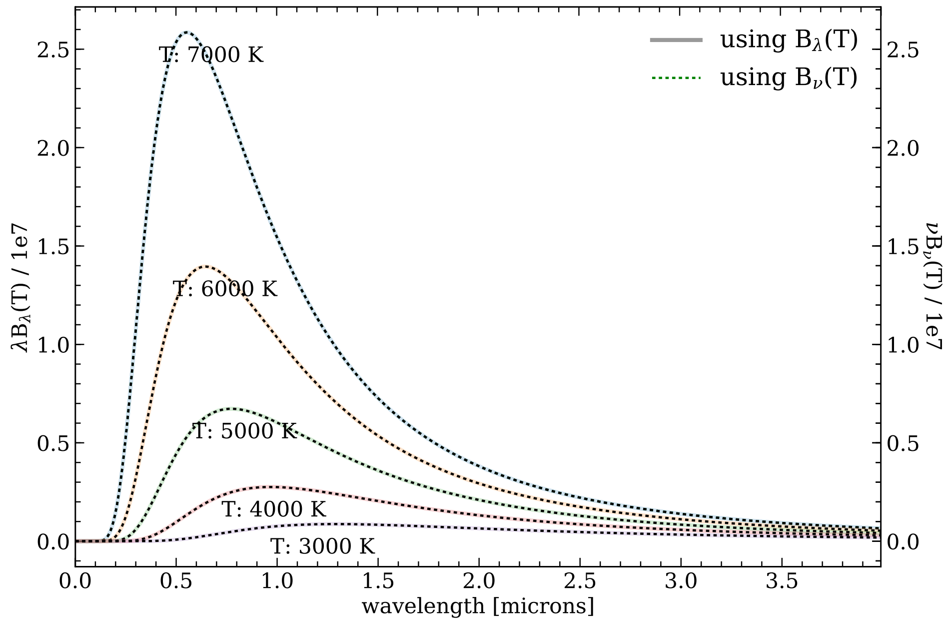


Figure 1. Visualization of integrating over a bandpass.

2. PROBLEM 2

Compare the $U-B$ and $B-V$ colors of stars with those of (perfect) black bodies. The transmission curves for the three filters are provided, along with the Kurucz stellar model spectra.

Given a transmission curve for a bandpass 'x' ($T_x\nu$) and a spectrum in units of [erg/s/cm^2/Hz] (S_ν), we can calculate the apparent magnitude using the following equation:

$$f_{tot,x} = \frac{\int T_x(\nu) S_\nu/\nu \; d\nu}{\int T_x(\nu)/\nu \; d\nu}$$
$$m_x = -2.5\log_{10}(f_{tot,x}) + \text{constant} \tag{6}$$
$$\text{(note: this assumes that } T_x(\nu) \text{ is } \textbf{not} \text{ in units of 1/eV)}$$

This can be shown for a series of bandpasses (U, B, V) in Figure 2, where the transmission curves are shown as the shaded regions, the spectrum (a blackbody with T=10^4 K) is the black curve, and the points represent $f_{tot,x}$ for each bandpass.

When applied for a series of bandpasses (ie. U, B, V), we can calculate the color of the spectrum by subtracting the apparent magnitudes. However we need to ensure that the color-order is correct, so $U-B, B-V$.

(a) Calculate the $U-B$ and $B-V$ colors on the Absolute Bolometric (AB) system for blackbodies with a temperatures of 3000, 6000, 10,000, and 15,000 K.

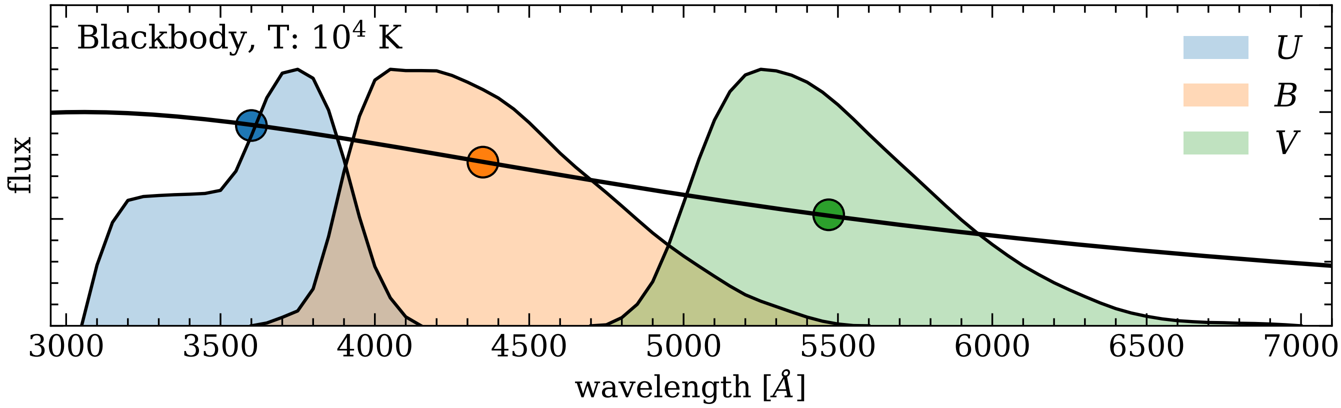


Figure 2. Visualization of integrating over a bandpass.

⟶ See Table 1 for blackbody colors.

(b) Calculate the $U - B$ and $B - V$ colors on the AB magnitude system for stars of spectral type M0V, K5V, K0V, G5V, G0V, F5V, F0V, A5V, A0V, B8V, and B3V.

⟶ See Table 1 for Kurucz model colors.

Table 1. Colors for Blackbody & Kurucz Spectra.

Spectrum	Temp.	$U-B$	$B-V$	Spectrum	Type	$U-B$	$B-V$
Kurucz	B3V	-0.318	0.074	Blackbody	25000 K	-0.308	-0.218
Kurucz	B8V	-0.216	0.502	Blackbody	15000 K	-0.15	-0.045
Kurucz	A0V	-0.118	0.817	Blackbody	10000 K	0.07	0.196
Kurucz	A5V	0.016	0.92	Blackbody	8000 K	0.244	0.387
Kurucz	F0V	0.179	0.853	Blackbody	6000 K	0.542	0.712
Kurucz	F5V	0.332	0.838	Blackbody	5000 K	0.779	0.975
Kurucz	G0V	0.482	0.892	Blackbody	4000 K	1.13	1.372
Kurucz	G5V	0.561	0.984	Blackbody	3000 K	1.691	2.043
Kurucz	K0V	0.732	1.256	...	...	...	...
Kurucz	K5V	1.071	1.909	...	...	...	...
Kurucz	M0V	1.12	1.888	...	...	...	...

Note—The Kurucz stellar model values are from the eCampus data, not from my personal Kurucz models (see caption of Figure 3 for clarification).

(c) Make a plot of $U - B$ vs $B - V$ for your answers in parts (a) and (b), and compare it to fig 2.8 in your textbook. Discuss the differences between the colors of blackbodies and stellar spectra.

Figure 3 shows the $U - B$ vs $B - V$ for the blackbody spectra and the Kurucz model spectra. I've included two separate scenarios for the Kurucz models – originally I had been using my own Kurucz models (pulled from the STSci website), but on a whim I checked my results against the models provided on eCampus. The left panel in Figure 3 shows the Kurucz model data provided on eCampus; the right panel shows the Kurucz models from my computer. Thankfully they do still follow the same path (so no physics has changed!) but it is interesting that there are differences at all.

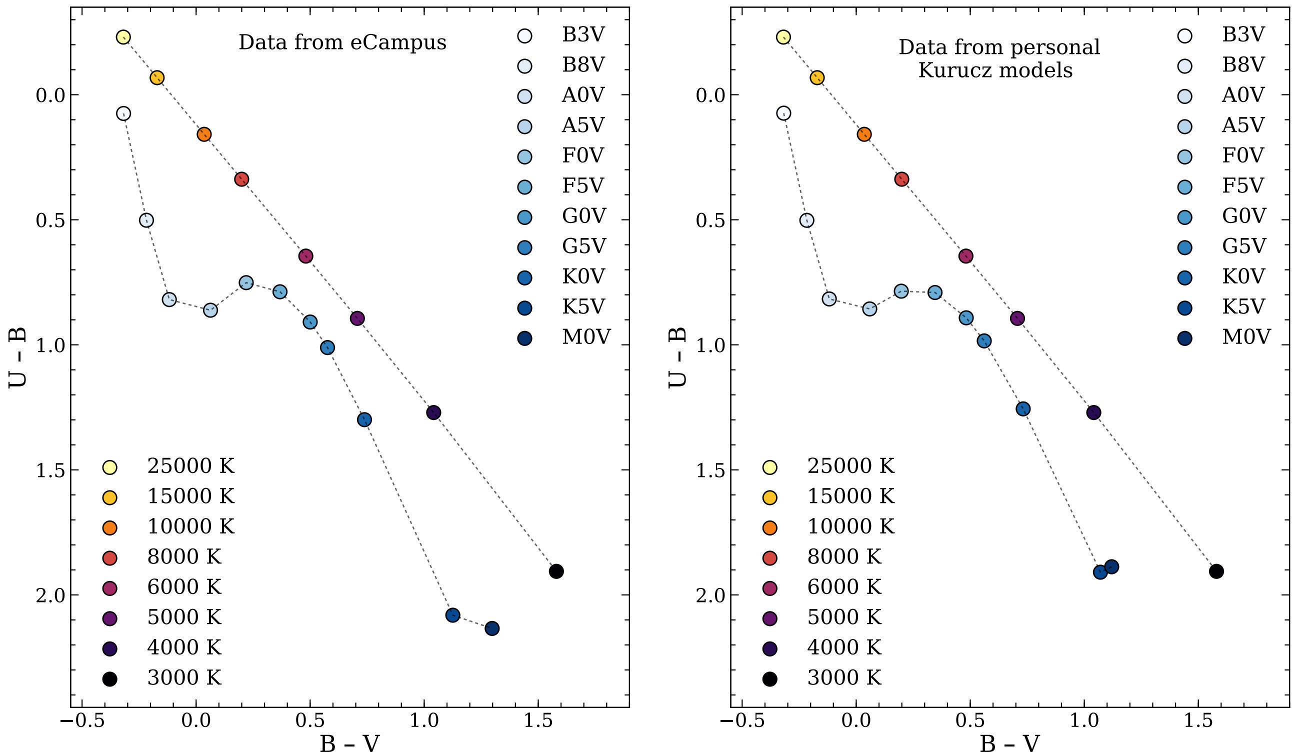


Figure 3. (left) Correct $U - B$ vs $B - V$ plot using Kurucz models pulled from eCampus. (right) $U - B$ vs $B - V$ plot using my personal Castelli & Kurucz models, pulled from the STSci website.

Aside from that, when comparing Figure 3 to fig. 2.8 in the textbook, the first thing to note is that I used AB magnitudes and the fig. 2.8 used Vega magnitudes. A simple calculation can tell you the conversion to Vega for all of the points – let's assume we have a source that measures

$$m_U = 1 \text{ [AB]}, \quad m_B = 1 \text{ [AB]}, \quad m_V = 1 \text{ [AB]} \tag{7}$$

The color would then be

$$U - B = 0, \quad B - V = 0 \tag{8}$$

Table 2. AB to Vega Mag.

Band	λ_{eff}	$m_{AB} - m_{Vega}$
U	3571	0.79
B	4344	-0.09
V	5456	0.02

Now the conversion from AB magnitudes to Vega magnitudes, $m_{AB} - m_{Vega}$[1], using Table 2 provides

$$m_U = 0.21 \text{ [Vega]}, \quad m_B = 1.09 \text{ [Vega]}, \quad m_V = 0.98 \text{ [Vega]}$$
$$(U-B)_{\text{Vega}} = -0.88, \ \ (B-V)_{\text{Vega}} = 0.11 \tag{9}$$

So $B - V$ would not shift much, but imagine every point shifting ~1 magnitude up (so more negative) for $U - B$ (this isn't important for the next paragraph but is useful if you want to compare the two figures by values).

Now, when looking at Figure 3, the blackbody points seem to follow a nice slope upwards with the higher temperature blackbodies appearing at bluer and bluer colors. In a similar way, the Kurucz model spectra seem to track upward and to the left for the bluer stars with higher T_{eff}, however they lie distinctly below the space occupied by the blackbodies.

This is expected for the stellar spectra, as they have absorption lines of varying strength depending upon the stellar type (one example, Balmer lines affecting F5V-A0V points). The blackbodies can then be seen as the perfect result (think spherical space cow)[2], while the model stellar spectra are the more realistic results.

3. PROBLEM 3

Studies have provided the distance measures for two Galactic Globular Clusters, NGC 6397 & NGC 6934, in the table below (see 3).

(a) What is the extinction coefficient, A_V, and color excess $E(B-V)$ towards each globular cluster?

The values in the final two columns, A_V and $E(B-V)$, of Table 3 were calculated as follows.

Table 3. Measured Values for Galactic GCs.

Name	$V - M_V$ (mag)	D (kpc)	A_V	$E(B-V)$
NGC 6397	12.37	2.3	0.561	0.181
NGC 6934	16.28	15.6	0.314	0.101

[1] Values pulled from http://www.astronomy.ohio-state.edu/~martini/usefuldata.html
[2] referencing the joke about a 'perfect physics world'

$$V - A_V - M_V = 5\log(D/\text{pc}) - 5 \tag{10}$$

$$\boxed{A_V = -5\log(D/\text{pc}) + 5 + (V - M_V)}$$

Taking A_V and using R_V, the factor of proportionality between the color excess ($E(B-V)$) and A_V, we can calculate the color excess by

$$\boxed{E(B-V) = \frac{A_V}{R_V}}$$

(b) Figure 1 (in prompt) shows observed color-magnitude diagrams using raw data (ie. uncorrected for extinction) for the globular clusters.

i. Estimate the location of the main-sequence turnoff for each cluster both from the raw data and corrected for dust extinction using your answer from part (a).

The original location of both clusters' main-sequence turnoff were estimated by eye and are shown in Table 4. The new locations were derived by the following equations.

Let's think about how the color excess $E(B-V)$ is defined:

$$E(B-V) = (B-V) - (B_0 - V_0) \tag{11}$$

Given that the x-axis values in the color-magnitude diagram (CMD) are the raw (ie. not corrected for extinction) $B-V$ colors, we can rewrite (11) to solve for the corrected color $(B_0 - V_0)$.

$$\boxed{(B_0 - V_0) = (B-V) - E(B-V)}$$

For the correction applied to the magnitude, let's revisit the distance modulus equation (10) used to calculate A_V, shown below. Another way to write the distance modulus is also shown.

$$\begin{aligned} V - A_V - M_V &= 5\log(D/\text{pc}) - 5 \\ m - M_V &= 5\log(D/\text{pc}) - 5 \end{aligned} \tag{12}$$

This would lead to the notion that the corrected apparent magnitude can be defined as

$$\boxed{m = V - A_V}$$

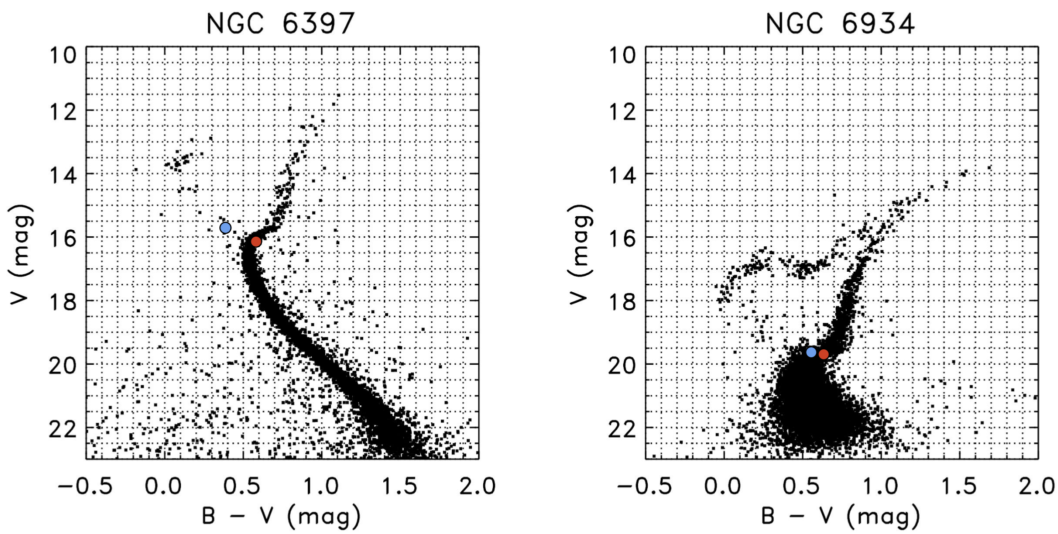


Figure 4. CMD showing the estimated points for the main-sequence turnoff (red points) and the de-reddened locations for the points (blue points).

Table 4. Estimated Values for Galactic GCs.

Name	V (mag)	$B - V$ (mag)	V_0 (mag)	$B_0 - V_0$ (mag)
NGC 6397	16.0	0.55	15.44	0.37
NGC 6934	19.5	0.65	19.19	0.55

ii. Estimate the age of each globular cluster from the main-sequence turnoff. Discuss how much your answer would change if you neglect dust extinction.

Using fig. 2.5 in our textbook, which shows both absolute magnitude and age (Myrs) as a function of color index. From this figure, and $M_V = 3.63(3.22)$ for NGC 6397 (NGC 6934), I would estimate that the age for each likely falls in the range of 5,000 to 7,100 Myrs. The uncertainty on this range is large, which would help explain why these seemingly very different stellar populations (one each cluster) seem to imply very different ages.

4. PROBLEM 4

The mass of the Milky Way Galaxy is calculated to be $M = 5.4 \times 10^{11} M_\odot$ within $r = 50$ kpc, and $M = 1.9 \times 10^{12} M_\odot$ within $r = 230$ kpc.

(a) Estimate the values for the constants ρ_0 and a for the NFW matter density profile,

$$\rho(r) = \frac{\rho_0}{(r/a)(1 + r/a)^2} \tag{13}$$

The mass given for each radius incorporates all of the mass out to that point. Therefore, in order to derive the constants we need to set up an integral to reflect this:

$$M = \int_0^{R_{max}} 4\pi r^2 \rho(r)\ dr \tag{14}$$

From Wolfram this integral can be approximated as

$$M = 4\pi\rho_0 a^3 \left[\ln\left(\frac{z + r_{max}}{a}\right) - \left(\frac{r_{max}}{a + r_{max}}\right)\right] \tag{15}$$

In this equation, r_{max} represents the radius that corresponds to the integrated M. By using both radii and M measurements and solving both equations for ρ_0, this yields

$$\frac{M_1}{\ln\left(\frac{z+r_1}{a}\right) - \left(\frac{r_1}{a+r_1}\right)} = \frac{M_2}{\ln\left(\frac{z+r_2}{a}\right) - \left(\frac{r_2}{a+r_2}\right)} \tag{16}$$

Which, when re-ordered to work with `scipy.optimize`'s `fsolve` equation

$$\frac{\ln\left(\frac{z+r_1}{a}\right) - \left(\frac{r_1}{a+r_1}\right)}{\ln\left(\frac{z+r_2}{a}\right) - \left(\frac{r_2}{a+r_2}\right)} = \frac{M_1}{M_2} \tag{17}$$

This yeilds an $a = 28.89$ kpc. Plugging that into equation (15) to solve for ρ_0 resulted in $\rho_0 = 4.81 \times 10^6\ M_\odot/\text{kpc}^3$.

$$\boxed{a = 28.89 \text{ kpc } \&\ \rho_0 = 4.81 \times 10^6\ M_\odot/\text{kpc}^3}$$

(b) Assume that all starlight from the Milky Way comes from the disk with a luminosity density that follows Eqn. 2.35 in your textbook,

$$l(R, z) = l_0 \frac{\exp(-R/h_R)}{\cosh^2(z/h_z)} \tag{18}$$

with $h_R = 3.5$ kpc and $h_z = 0.65$ kpc. Given that the total blue-light luminosity of the disk integrated over all R and z is $L_B = 1.8 \times 10^{10} L_\odot$, derive l_0 in units of $L_\odot/\text{pc}^3$ in the equation above.

In a similar way to part (a), first we need to set up the integral

$$L_B = \int_{-\infty}^{\infty}\int_0^{\infty} l(R, z) R\ dR\ dz \tag{19}$$

We can break this down to two equations, as l's z and R dependence are separable.

$$\begin{aligned}
&\text{for R: } \int_0^\infty \exp(-R/h_R) R \; dR \\
&\text{for z: } \int_{-\infty}^\infty \cosh^{-2}(z/h_z) \; dz
\end{aligned} \tag{20}$$

The resulting solutions being

$$\begin{aligned}
&\text{for R: } -h_R \exp(-R/h_R)(h_R + R)\Big|_0^\infty = h_R^2 \\
&\text{for z: } h_z \tanh(z/h_z)\Big|_{-\infty}^\infty = 2h_z
\end{aligned} \tag{21}$$

Plugging the solutions back into equation (19) and solving for l_0 (using $h_R = 3.5$ kpc, $h_z = 0.65$ kpc, and $L_B = 1.8 \times 10^{10} L_\odot$) yields

$$\boxed{l_0 = 0.36 L_\odot/\text{pc}^3}$$

(c) Using your answers to parts (a) and (b), and assume that the stars have a constant mass-to-light ratio of $M/L_B = 3M_\odot/L_\odot$, plot the total Galaxy M/L_B as a function of radius, R. What is the total Galaxy M/L_B interior to a radius of 3, 10, 30, 100, and 300 kpc?

For the total Galaxy M/L_B, we first need to account for both the dark matter (using the NFW profile) and the stellar matter (assuming a constant $M/L_B = 3M_\odot/L_\odot$ for the stars).

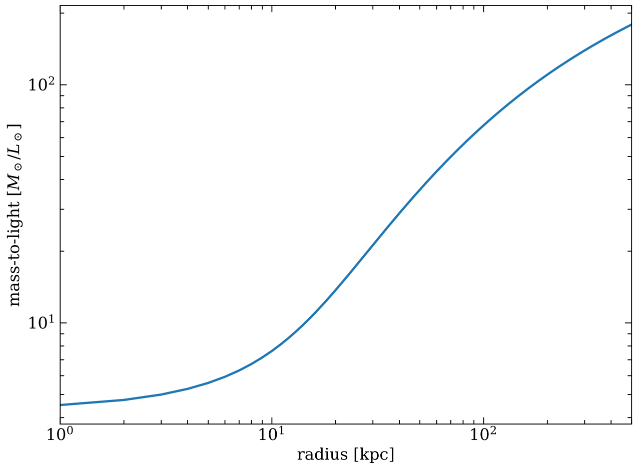
$$\begin{aligned}
M(R) &= \int_0^{R_{max}} 4\pi r^2 \rho(r) \; dr \; + \; M_* \\
\text{where } \; M_* &= 3 \int_0^{R_{max}} 2\pi h_z l(r) r \; dr \\
\text{and } \int_0^{R_{max}} 2\pi h_z l(r) r \; dr &= \int_z \int_0^{R_{max}} l(r,z) r \; dr \; dz \\
&\text{(note: marginalizing over } z \text{ in equation (19).)}
\end{aligned} \tag{22}$$

The luminosity can then be calculated as

$$L_B(R) = \int_0^{R_{max}} 2\pi h_z l(r) r \; dr \tag{23}$$

The mass-to-light ratio, as a function of galactocentric distance, is shown in Figure 5. I've set the plot in log-space in order to make it easier to see the gentle change in the shape at both ends (in radius) of the curve.

As a note, I had some trouble getting this shape to how it looks currently (my functions that I was integrating over were confusing `python` for a time) – I am fairly certain that this curve is correct (rather, I seem to be in agreement with other students).



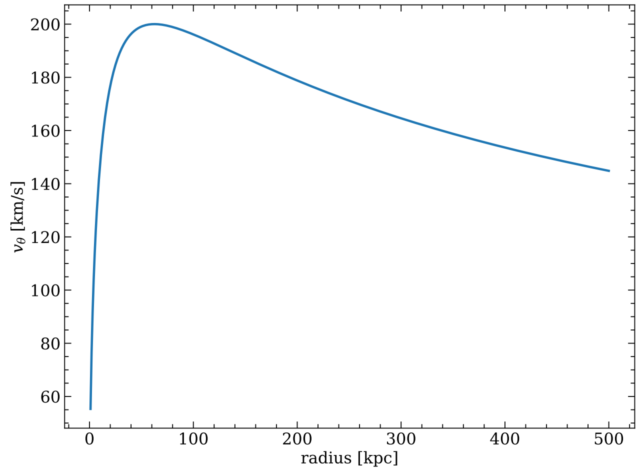


Figure 5. (left) Mass-to-light as a function of radius, or galactocentric distance, and (right) the circular velocity, v_θ, as a function of radius.

(d) Using your result from part (c) and Eqn. 2.35, derive an expression for the circular velocity, v_θ, for a star orbiting the center of the Milky Way Galaxy as a function of the galactocentric distance R. Compare your result to the rotation curve data and models in Kafle et al. (2012).

To find the equation for the circular velocity, we can use the conservation of energy

$$\frac{1}{2}v_\theta^2 = \frac{-GM}{R} \tag{24}$$

where the M term is equal to the integrated mass up to a given galactocentric distance, R.

The circular velocity, as a function of galactocentric distance, is shown in Figure 5 (right panel). When comparing my curve to fig. 10 in Kafle et al. (2012), it is apparent that I only modeled a disk component and the dark matter halo (using the NFW profile) of the overall circular velocity curve. By adding in a bulge component, my curve would flatten out and better match their 'total' curve (shown in red).

5. PROBLEM 5

Calculate the Free Fall Time for a spherical gas cloud of mass M, inital radius R, and constant density $\rho_0 = \frac{M}{\frac{4}{3}\pi R^3}$.

(a) In class we worked out that the equation of motion $((d^2r/dt^2) = -GM/r^2)$ for a spherical shell with enclosed mass, $M(< r)$, starting with radius R and 'falling' to radius r can be written as

$$u^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right) \tag{25}$$

Starting there, and recalling that $u = dr/dt$, show that the 'free-fall' time, t_{ff}, can be given by

$$t_{ff} = \left(\frac{3\pi}{32G\rho_0}\right) \tag{26}$$

To being, let's rewrite equation (24) slightly, setting $M = \frac{4}{3}\pi\rho_0 R^3$

$$u^2 = \left(\frac{dr}{dt}\right)^2 = \frac{8}{3}G\pi\rho_0 R^3\left(\frac{R}{r} - 1\right) \tag{27}$$

Next, to ease in the integration, we'll substitute $r = R\,\cos^2(u)$. Therefore,

$$\begin{aligned} \frac{dr}{dt} &= \frac{d}{dt}\left(R\,\cos^2(u)\right) = -2R\,\cos(u)\,\sin(u)\,\frac{du}{dt} \\ \frac{du}{dt} &= (2\cos(u)\,\sin(u))^{-1}\left(\frac{8}{3}G\pi\rho_0\left(\frac{1}{\cos^2(u)} - 1\right)\right)^{\frac{1}{2}} \end{aligned} \tag{28}$$

This can now be reordered in preparation for integration

$$\begin{aligned} \frac{du}{dt} &= (2\cos^2(u)\,\sin(u))^{-1}\left(\frac{8}{3}G\pi\rho_0\left(1 - \cos^2(u)\right)\right)^{\frac{1}{2}} \\ \text{where}\quad 1 - \cos^2(u) = \sin^2(u),\quad \frac{du}{dt} &= (2\cos^2(u))^{-1}\left(\frac{8}{3}G\pi\rho_0\right)^{\frac{1}{2}} \end{aligned} \tag{29}$$

Now, to integrate the function while pulling out the constants:

$$\int_{u_0}^{u}\cos^2(u')\,du' = \frac{1}{2}\left(\frac{8}{3}G\pi\rho_0\right)^{\frac{1}{2}}\int_0^{t_{eff}} dt = \frac{t_{eff}}{2}\left(\frac{8}{3}G\pi\rho_0\right)^{\frac{1}{2}} \tag{30}$$

(note: $u = 0$ when $t = 0$; $u = \frac{\pi}{s}$ when $t = t_{eff}$ $\because$ r=0=R $\cos^2(u)$)

$$\int_0^{\frac{\pi}{2}}\cos^2(u)\,du = \frac{u}{2} - \frac{\sin(2u)}{4}\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4} \tag{31}$$

Therefore, when setting $\frac{\pi}{4}$ equal to the far right hand side of the equation (29), we derive

$$\boxed{t_{eff} = \left(\frac{3\pi}{32G\rho_0}\right)^{\frac{1}{2}}}$$

(b) Calculate the free fall time for a spherical gas cloud with $M = 5\times10^{11}M_\odot$ for the case that it has initial radius $R = 50$ kpc and compare it to the case where the initial radius is $R = 200$ kpc.

I'm not 100% sure this is correct, but I got $t_{eff} = 261.4$ Myr for $R = 50$ kpc and $t_{eff} = 2.09$ Gyr for $R = 200$ kpc. However, it is entirely possible that I incorrectly converted one of my constants to cgs units.

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