ABSTRACT. We summarize the progress that has been made on the finitistic dimension conjecture for finite dimensional algebras since it's conception in 1960. Special emphasize is put on showing which classes of algebras are known to satisfy the conjecture.

SAMMENDRAG. Vi oppsummerer arbeidet gjort på finitistisk dimensjonsformodning for endeligdimensjonale algebraer, siden den først ble postulert i 1960. Vi fokuserer spesielt på å vise hvilke klasser av algebraer som det er kjent at tilfredsstiller formodningen.

Preface

This thesis was written as part of an integrated PhD position, supervised by Professor Øyvind Solberg. It marks the transition from my time as a Master student to PhD student.

I would especially like to thank Solberg for their excellent guidance and weekly meetings, which helped motivate me throughout the writing process.

I would like to thank the Research Council of Norway and NTNU for financing my position in the ARTaC project.

I would like to thank my partner and family for supporting me and being there for me all throughout my studies, allowing me to pursue this degree.

Lastly, none of the work of this thesis is original, but is a compilation of the work of many authors. All of these authors deserve gratitude for their excellent papers and books, which this thesis is based on.

Jacob Fjeld Grevstad
Trondheim, 2021

Contents

Notation

Throughout this thesis k will be a field, and Λ will be a finite dimensional algebra over k. We will use J to refer to the Jacobson radical of Λ.

We will use $\operatorname{mod}\Lambda$ to refer to the category of finite dimensional left Λ-modules, and $\operatorname{Mod}\Lambda$ to the category of all left Λ-modules. Any modules considered will be left modules if not specified otherwise. When there is ambiguity we may write ${}_{\Lambda}M$ to specify that we are considering M as a left Λ-module, and M_{Λ} to specify that we are considering M as a right Λ-module. Similarly ${}_{\Gamma}M_{\Lambda}$ means we are considering M as a Γ-Λ-bimodule.

Since right Λ-modules are the same as left Λ^{op}-modules we use these interchangeably. We use the symbol D to denote the duality functor $D\colon \operatorname{mod}\Lambda \leftrightarrow \operatorname{mod}\Lambda^{\mathrm{op}}$ where $DM = \operatorname{Hom}_k(M, k)$. Typically $D\Lambda$ will refer to the left module $D\Lambda_{\Lambda}$.

A quiver is a direct graph with a finite number of vertices. We write composition of paths right to left. I.e. for paths $\alpha\colon i \to j$ and $\beta\colon k \to l$ the composition $\alpha\beta$ is defined if and only if $l = i$. For a quiver Q, the path algebra kQ is the free vector space of all paths, including a trivial path for each vertex. Multiplication of paths is defined to be composition when it is defined, and 0 otherwise. The multiplication extends linearly to make kQ and algebra.

When working over a category $\mathcal{C}$ we will denote the set of morphisms either as $\operatorname{Hom}_{\mathcal{C}}(M, N)$ or as $\mathcal{C}(M, N)$. When the ambient category is clear we may also simply write $\operatorname{Hom}(M, N)$ or (M, N).

The categories we are considering are all k-linear and all functors are assumed to be k-linear as well.

For an exact category $\mathcal{A}$ we write:

- $\mathscr{D}(\mathcal{A})$ to refer to the derived category,
- $\mathscr{D}^b(\mathcal{A})$ to refer to the bounded derived category,

- $K^b(\mathcal{A})$ to refer to the bounded homotopy category,
- $K^{+,b}(\mathcal{A})$ (respectively $K^{-,b}(\mathcal{A})$) to refer to the homotopy category of complexes bounded below (respectively above) that are bounded in homology.

We also write $\mathscr{D}^b(\Lambda)$ instead of $\mathscr{D}^b(\operatorname{mod}\Lambda)$ and $\mathscr{D}(\Lambda)$ instead of $\mathscr{D}(\operatorname{Mod}\Lambda)$.

In all of these triangulated categories $X[i]$ will denote the complex X shifted i degrees down. That is, $(X[i])^n = X^{n+i}$. The hard truncation is the complex defined by $(X^{\geq n})^m = X^m$ when $m \geq n$ and $(X^{\geq n})^m = 0$ otherwise. We denote the hard truncation of X by $X^{\geq n}$. The other hard truncation, $X^{\leq n}$, is defined similarly.

For a module M we will write $I(M)$ for its injective envelope, and $P(M)$ for its projective cover. We may also write

$$\begin{tikzcd}
\cdots \arrow[r] & P_M^2 \arrow[r, "d_M^2"] & P_M^1 \arrow[r, "d_M^1"] & P_M^0 \arrow[r] \arrow[d, "d_M^0"] & 0 \\
& & & M &
\end{tikzcd}$$

for its minimal projective resolution. We let the nth syzygies of M be the kernel of d_M^{n-1}, denoted by $\Omega^n M$. We also define $\Omega^0 M$ to be M.

The projective dimension of M is i if P_M^i is the last non-zero module in the minimal projective resolution, and ∞ if there is no such module. We denote the projective dimension by $\operatorname{pd} M$.

Introduction

In representation theory of finite dimensional algebras, there are several related conjectures known as the “homological conejctures”. The strongest of these conjectures is the Finitistic Dimension Conjecture. It concerns the homological invariant called the finitistic dimension. For a noetherian ring we define

$$\begin{aligned}\operatorname{findim}(R) &:= \sup\{\operatorname{pd} M \mid M \in \operatorname{mod} R, \operatorname{pd} M < \infty\}\\ \operatorname{Findim}(R) &:= \sup\{\operatorname{pd} M \mid M \in \operatorname{Mod} R, \operatorname{pd} M < \infty\}\end{aligned}$$

The finitistic dimension conjecture states that $\operatorname{findim}(\Lambda) < \infty$, whenever Λ is a finite dimensional algebra. Note that $\operatorname{findim}(R) \leq \operatorname{Findim}(R)$, and so a stronger conjecture is whether $\operatorname{Findim}(\Lambda) < \infty$, but in this thesis we are mainly interested in the small finitistic dimension.

History

The finitistic dimension was introduced by Auslander–Buchsbaum in the late 1950s to study commutative noetherian rings. They proved that for a local noetherian commutative ring the finitistic dimension equals the depth [AB57]. Later it was shown by Bass and Gruson–Raynaud that for any commutative noetherian ring the (big) finitistic dimension equals the Krull dimension [Bas62, RG71].

The non-commutative case turned out to be more difficult. In 1960 Bass published to important questions about the finitistic dimension [Bas60], which they credit to Rosenberg and Zelinsky. Their first question asks whether the small finitistic dimension equals the big finitistic dimension. This was shown to be false even for monomial algebras by Huisgen-Zimmerman in 1992 [ZH92].

Their second question is what we here call the finitistic dimension conjecture. Much progress have been done on the problem over the last 60 years. Huisgen-Zimmerman has a great paper summarizing most of the results [ZH95]. Here we try to do something similar to said paper, with the focus on establishing which classes of algebras the conjecture is known to hold for. We try to keep the thesis self contained by writing out all the proofs, and in addition we include some results not covered in Huisgen-Zimmermann's paper.

Overview

The sections of this thesis are self contained, and can be read independently of one another, except for Section 5 which relies on results from Section 4. In Section 8 we summarize for which algebras the conjecture is known to hold. This relies only on Sections 3 to 7, and not on Sections 1 and 2.

In addition to the main sections of this thesis, there is an appendix, Section A, where we cover general theorems from homological algebra that would break the flow of the main text. These results are referenced when used.

In Section 1 we discuss the homological conjectures, and show the implications between them. All the conjectures concerns a specific property of an algebra that is conjectured to hold for all algebras. In Proposition 1.15 we give an overview of how the conjectures are related on the level of individual algebras.

In Section 2 we introduce a sort of “short exact sequence” of triangulated categories, known as a recollement. We show that if the derived category of Λ is a recollement of the derived categories of Λ' and Λ'', then finitistic dimension of Λ is finite if and only if the finitistic dimension of both Λ' and Λ'' are. The idea of using recollements to study the finitistic dimension is due to Happel, and most of the section is based on their paper [Hap93]. We also consider a related technique concerning triangular matrix rings, due to Fossum-Griffith-Reiten [FGR75], and discuss the similarities.

In Section 3 we show that if the subcategory of modules with finite projective dimension is contravariantly finite, then the algebra has finite finitistic dimension. This is a result due to Auslander–Reiten [AR91]. In Example 3.6, due to Igusa–Smalø–Todorov [IST90], we show that this subcategory can fail to be contravariantly finite even for monomial algebras with radical cubed equal to 0.

In Section 4 we introduce the Igusa–Todorov function, and use it to show that algebras with representation dimension less than or equal to 3 satisfies the finitistic dimension conjecture. We also give examples of two classes of algebras that are known to have representation dimension at most 3, due to Xi and Erdmann–Holm–Iyama–Schröer respectively [Xi02,EHIS04]. Preprints of Igusa–Todorov's paper [IT05] was circulated in the mid 90s, but it was not published until later, when several corollaries could be included.

In Section 5 we discuss restriction one can impose on the radical for the algebra to satisfy the finitistic dimension conjecture. Specifically we look at algebras $J^{2l+1} = 0$ and Λ/J^l is representation finite, and algebras where the composition factors of J^2 have finite projective dimension.

In Section 6 we show that the finitistic dimension of a monomial algebra is always finite. This proof is due to Green–Kirkman–Kuzmanovich [GKK91]. An alternate proof was given by Igusa–Zacharia [IZ90], but we don't discuss that here.

In Section 7 we discuss a more recent result, due to Rickard [Ric19]. In contrast to the rest of this thesis, instead of cinsidering the small finitistic dimension, we give a condition for when the big finitistic dimension is finite. Specifically we show that $\mathrm{Findim}(\Lambda) < \infty$ if the inejctives generate the unbounded derived category. Many of the algebras considered in previous sections also satisfies this more general condition. We state this more precisely in Theorem 8.2(g).

The intended reader

This thesis is written to be understandable to someone who has taken a course on representation theory of finite dimensional algebras and homological algebra. The reader should be familiar with:

- representation theory of quivers and path algebras,
- projective dimension and the Ext-functor,
- the long exact sequence in Ext and Tor,
- the basic definitions of category theory, including (co)limits and adjoint functors,
- the derived category and triangulated categories.

These subject are covered in the courses *MA3203 – Ring Theory* and *MA3204 – Homological Algebra* offered at NTNU, or in classical textbooks such as [ARS97] and [Wei94].

1 The homological conjectures

The finitistic dimension conjecture is part of a larger family of homological conjectures about finite dimensional algebras. In this section we outline some of these conjectures, and show how they are related.

All of the conjectures are formulated as a specific property conjectured to hold for all finite dimensional algebras. In Proposition 1.15 we summarize how these implications work on the level of individual algebras.

Finitistic Dimension Conjecture (FDC)

Definition 1.1 (Finitistic dimension)**.** For a finite dimensional algebra Λ the *finitistic dimension* of Λ, denoted $\operatorname{findim}(\Lambda)$ is defined by

$$\operatorname{findim}(\Lambda) = \{\operatorname{pd} M \mid M \in \operatorname{mod} \Lambda, \operatorname{pd} M < \infty\}.$$

There is also the analogous definition for $\mathrm{Mod}\,\Lambda$, which is sometimes called the *big finitistic dimension*, and is denoted $\mathrm{Findim}(\Lambda)$. A natural question to ask, which is sometimes also called the finitistic dimension conjecture is whether $\mathrm{findim}(\Lambda)$ always equals $\mathrm{Findim}(\Lambda)$. This was shown to be false by Huisgen-Zimmermann in 1992 [ZH92]. The conjecture we consider is due to Rosenberg and Zelinsky [Bas60], and asks about when the finitistic dimension is finite.

Conjecrture 1 (Finitistic dimension conjecture)**.** *For a finite dimensional algebra the finitistic dimension is always finite.*

$$\mathrm{findim}(\Lambda) < \infty$$

Wakamatsu Tilting Conjecture (WTC)

In 1988 Wakamatsu introduced a generalization of tilting modules, now known as Wakamatsu tilting modules [Wak88].

Definition 1.2 (Wakamatsu tilting)**.** Let T be a module in $\mathrm{mod}\,\Lambda$ for a finite dimensional algebra Λ. Then T is *Wakamatsu tilting* if

i) We have that $\mathrm{Ext}^n(T, T) = 0$ for all $n > 0$.

ii) There is an exact sequence

$$\eta\colon 0 \longrightarrow \Lambda \xrightarrow{d_{-1}} T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} \cdots$$

where T_i is in $\mathrm{add}\,T$.

iii) The sequence $\mathrm{Hom}(\eta, T)$ is exact. Which is equivalent to the condition that $\mathrm{Ext}^1(\ker d_i, T) = 0$ for every differential d_i in η.

The definition is distinct from the definition of a tilting module in two key ways: the projective dimension of T is not assumed to be finite, and η is not assumed to be bounded. The Wakamatsu tilting conjecture states that this last condition is unnecessary.

Conjecrture 2 (Wakamatsu tilting conjecture)**.** *If T is Wakamatsu tilting and has finite projective dimension, then T is a tilting module. In other words we can choose η to be bounded.*

Gorenstein Symmetry Conjecture (GSC)

Definition 1.3 (Gorenstein algebra)**.** A finite dimensional algebra is said to be *Gorenstein* if all projective modules have finite injective dimension and all injective modules have finite projective dimension.

The Gorenstein symmetry conjecture says that we only need one of the two conditions for our algebra to be Gorenstein.

Conjecrture 3 (Gorenstein symmetry conjecture)**.** *If Λ is a finite dimensional algebra the injective dimension of Λ as a left module is finite if and only if the projective dimension of $D\Lambda_\Lambda$ is finite.*

The conjecture describes a sort of symmetry between Λ and Λ^{op}. An equivalent formulation would be that Λ has finite injective dimension as a left module if and only if it has finite injective dimension as a right module.

Another noteworthy property of Gorenstein algebras is that a module has finite projective dimension if and only if it has finite injective dimension.

Proposition 1.4. *If Λ is Gorenstein and M is a Λ-module, then* $\operatorname{pd} M < \infty$ *if and only if* $\operatorname{id} M < \infty$*.*

Proof. From the projective and injective resolution of M we get short exact sequences:

$$0 \longrightarrow \Omega M \longrightarrow P \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow M \longrightarrow I \longrightarrow \mho M \longrightarrow 0.$$

From the long exact sequences in $\operatorname{Ext}(\Lambda/J, -)$ and $\operatorname{Ext}(-, \Lambda/J)$ it follows that $\operatorname{id} M \leq \max\{\operatorname{id} P, \operatorname{id} \Omega M\}$ and $\operatorname{pd} M \leq \max\{\operatorname{pd} I, \operatorname{pd} \mho M\}$. Iterating this construction it follows that for all n we have $\operatorname{id} M \leq \max\{\operatorname{id} \Lambda, \operatorname{id} \Omega^n M\}$ and $\operatorname{pd} \leq \max\{\operatorname{pd} D\Lambda, \operatorname{pd} \mho^n M\}$.

If M has finite projective dimension, then there is an n such that $\Omega^n M = 0$, which implies $\operatorname{id} M \leq \operatorname{id} \Lambda < \infty$. Conversely if M has finite injective dimension, then there is an n such that $\mho^n M = 0$, and so $\operatorname{pd} M \leq \operatorname{pd} D\Lambda < \infty$. □

Vanishing Conjecture (VC)

We remind the reader that when Λ is a finite dimensional algebra, we have an equivalence of categories between $K^{+,b}(\operatorname{inj} \Lambda)$ and the bounded derived category $\mathscr{D}^b(\Lambda)$ given by injective resolutions. This allows us to consider $K^b(\operatorname{inj} \Lambda)$ as a subcategory of $\mathscr{D}^b(\Lambda)$. Using this we define the perpendicular subcategory

$$K^b(\operatorname{inj} \Lambda)^{\perp} = \{X \in \mathscr{D}^b(\Lambda) \mid \operatorname{Hom}_{\mathscr{D}^b(\Lambda)}(I, X) = 0 \text{ for all } I \in K^b(\operatorname{inj} \Lambda)\}.$$

The vanishing conjecture then states that this subcategory is trivial.

Conjecrture 4 (Vanishing conjecture)**.** *If Λ is a finite dimensional algebra, then $K^b(\operatorname{inj} \Lambda)^{\perp} = 0$.*

In Section 7 we investigate an analog of this conjecture for the unbounded derived category.

Nunke Condition (NuC)

The Nunke condition is similar to the vanishing conjecture in that it considers modules which are “perpendicular” to the injective modules. Such a module is called a *Nunke module*, and an algebra is said to satisfy the Nunke condition if the only Nunke module is the zero module.

Conjecrture 5 (Nunke condition)**.** *If $X \neq 0$ is a module over a finite dimensional algebra Λ, then there is an $n \geq 0$ such that $\mathrm{Ext}^n(D\Lambda, X) \neq 0$.*

Strong Nakayama Conjecture (SNC)

The strong Nakayama conjecture is simply the dual of the Nunke condition. For the sake of completeness we include both in this summary.

Conjecrture 6 (Strong Nakayama conjecture)**.** *If $X \neq 0$ is a module over a finite dimensional algebra Λ, then there is an integer $n \geq 0$ such that $\mathrm{Ext}^n(X, \Lambda) \neq 0$.*

Generalized Nakayama Conjecture (GNC)

The generalized Nakayama conjecture is a slight weakening of the Strong Nakayama conjecture.

Conjecrture 7 (Generalized Nakayama conjecture)**.** *If S is a simple module over a finite dimensional algebra Λ, then there is an integer $n \geq 0$ such that $\mathrm{Ext}^n(S, \Lambda) \neq 0$.*

We can also formulate the conjecture as all indecomposable injectives appearing in the minimal injective resolution of Λ. We give a short proof that this is an equivalent formulation here.

Proposition 1.5. *A finite dimensional algebra Λ satisfies GNC if and only if every indecomposable injective appears in the minimal injective resolution of Λ.*

Proof. Let the minimal injective resolution of Λ be given by

$$0 \longrightarrow \Lambda \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

Since the resolution is minimal, we have that $\operatorname{Ext}^n(S,\Lambda) = \operatorname{Hom}(S, I_n)$ for any simple module S. This is non-zero if and only if S is in the socle of I_n. Thus $\operatorname{Ext}^n(S,\Lambda)$ is non-zero if and only if the injective envelope of S is a direct summand of I_n. Since every indecomposable injective module is the injective envelope of a simple module, we have that Λ satisfies GNC if and only if every indecomposable injective appears in the resolution. $\square$

Auslander–Reiten Conjecture (ARC)

Conjecrture 8 (Auslander–Reiten conjecture)**.** *Let Λ be finite dimensional algebra. If M is a generator in* $\operatorname{mod}\Lambda$ *such that* $\operatorname{Ext}^n(M,M) = 0$ *for all $n > 0$, then M is projective.*

Nakayama Conjecture (NC)

Definition 1.6 (Dominant dimension)**.** Let Λ be a finite dimensional algebra, and let

$$0 \longrightarrow \Lambda \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

be the minimal injective resolution of Λ. Then the *dominated dimension* of Λ is

$$\operatorname{domdim}(\Lambda) = \inf\{n \mid I^n \text{ is not projective}\}.$$

Conjecrture 9 (Nakayama conjecture)**.** *If Λ has infinite dominant dimension, then Λ is selfinjective.*

1.1 Implications

The homological conjectures are related in the way presented in the diagram below.

$$\begin{array}{ccccccccccccc}
\text{FDC} & \Longrightarrow & \text{WTC} & \Longrightarrow & \text{GSC} & & & & & & & & \\
\Downarrow & & & & & & & & & & & & \\
\text{VC} & \Longrightarrow & \text{NuC} & \Longleftrightarrow & \text{SNC} & \Longrightarrow & \text{GNC} & \Longleftrightarrow & \text{ARC} & \Longrightarrow & \text{NC}
\end{array}$$

The remainder of this section is used to prove these implications.

Theorem 1.7. *[MR04, Proposition 4.4] The finitistic dimension conjecture implies the Wakamatsu tilting conjecture.*

Proof. Assume Λ satisfies FDC, and let T be a Wakamatsu tilting module with $\operatorname{pd} T < \infty$. By definition we have an exact sequence

$$\eta\colon 0 \longrightarrow \Lambda \xrightarrow{d_{-1}} T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} \cdots$$

We want to show that η can be replaced by a bounded sequence of the same form.

Let K_i denote the kernel of d_i. First we prove by induction on i that $\operatorname{Ext}^{>0}(K_i, T) = 0$. For $i = 0$ we have $K_0 = \Lambda$, so we have $\operatorname{Ext}^{>0}(K_0, T) = 0$. Now assume that $\operatorname{Ext}^{>0}(K_i, T) = 0$ for some $i \geq 0$. We have a short exact sequence

$$0 \longrightarrow K_i \longrightarrow T_i \longrightarrow K_{i+1} \longrightarrow 0.$$

Applying the long exact sequence in $\operatorname{Ext}(-, T)$ we get

$$\operatorname{Ext}^n(T_i, T) \longrightarrow \operatorname{Ext}^n(K_i, T) \longrightarrow \operatorname{Ext}^{n+1}(K_{i+1}, T) \longrightarrow \operatorname{Ext}^{n+1}(T_i, T)$$

Since T_i is in $\operatorname{add} T$ we have that $\operatorname{Ext}^n(T_i, T) = 0$ for all $n > 0$. Then by exactness we have that $\operatorname{Ext}^{n+1}(K_{i+1}, T) \cong \operatorname{Ext}^n(K_i, T) = 0$ for all $n \geq 1$. Since T is Wakamatsu tilting we have that $\operatorname{Ext}^1(K_{i+1}, T) = 0$, so by induction $\operatorname{Ext}^{>0}(K_i, T) = 0$ for all $i \geq 0$.

By a similar argument we now wish to show that

$$\operatorname{Ext}^1(K_m, K_{m-1}) \cong \operatorname{Ext}^i(K_m, K_{m-i})$$

for all $i \leq m$. We proceed by induction on i. When $i = 1$ the statement is evident. Now assume that

$$\operatorname{Ext}^1(K_m, K_{m-1}) \cong \operatorname{Ext}^i(K_m, K_{m-i})$$

for some $i \geq 1$. Then it is sufficient to show that

$$\operatorname{Ext}^i(K_m, K_{m-i}) \cong \operatorname{Ext}^{i+1}(K_m, K_{m-i-1}).$$

We have a short exact sequence

$$0 \longrightarrow K_{m-i-1} \longrightarrow T_{m-i-1} \longrightarrow K_{m-i} \longrightarrow 0.$$

Taking the long exact sequence in $\operatorname{Ext}(K_m, -)$ we get the exact sequence

$$\begin{tikzcd}
\operatorname{Ext}^i(K_m, T_{m-i-1}) \arrow[r] & \operatorname{Ext}^i(K_m, K_{m-i}) \arrow[dll, rounded corners] \\
\operatorname{Ext}^{i+1}(K_m, K_{m-i-1}) \arrow[r] & \operatorname{Ext}^{i+1}(K_m, T_{m-i-1}).
\end{tikzcd}$$

Since we showed above that $\operatorname{Ext}^{>0}(K_m, T) = 0$ and T_{m-i-1} is in $\operatorname{add} T$ we get that $\operatorname{Ext}^{>0}(K_m, T_{m-i-1}) = 0$. Thus

$$\operatorname{Ext}^i(K_m, K_{m-i}) \cong \operatorname{Ext}^{i+1}(K_m, K_{m-i-1}),$$

and by induction we have that

$$\operatorname{Ext}^1(K_m, K_{m-1}) \cong \operatorname{Ext}^i(K_m, K_{m-i})$$

for all $i \leq m$.

Next we show that $\operatorname{pd} K_i < \infty$ for all $i \geq 0$. We again proceed by induction on i. The projective dimension of $K_0 = \Lambda$ is 0, which is finite. For $i > 0$ we have a short exact sequence

$$0 \longrightarrow K_{i-1} \longrightarrow T_{i-1} \longrightarrow K_i \longrightarrow 0.$$

Therefore $\operatorname{pd} K_i \leq \sup\{\operatorname{pd} T_{i-1}, \operatorname{pd} K_{i-1} + 1\} < \infty$.

Lastly, let $n = \operatorname{findim}(\Lambda) < \infty$. Then we have that

$$\operatorname{Ext}^1(K_{n+1}, K_n) \cong \operatorname{Ext}^{n+1}(K_{n+1}, K_0) = 0$$

where the last equality comes from $\operatorname{pd} K_{n+1} \leq n$. Now if we apply $\operatorname{Hom}(K_{n+1}, -)$ to the short exact sequence

$$0 \longrightarrow K_n \longrightarrow T_n \longrightarrow K_{n+1} \longrightarrow 0,$$

we get an exact sequence

$$\operatorname{Hom}(K_{n+1}, T_n) \longrightarrow \operatorname{Hom}(K_{n+1}, K_{n+1}) \longrightarrow \operatorname{Ext}^1(K_{n+1}, K_n) = 0.$$

This means that K_{n+1} is a direct summand of T_n, and thus is in $\operatorname{add} T$. Then we get a bounded version of η by

$$\eta' : 0 \longrightarrow \Lambda \xrightarrow{d_{-1}} T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} T_n \xrightarrow{d_n} K_{n+1} \longrightarrow 0.$$

Hence T is a tilting module, and thus Λ satisfies WTC. $\square$

Theorem 1.8. *The Wakamatsu tilting conjecture implies the Gorenstein symmetry conjecture.*

Proof. The left module $D(\Lambda_\Lambda)$ is Wakamatsu tilting. WTC then gives us that if $D(\Lambda_\Lambda)$ has finite projective dimension, then ${}_\Lambda\Lambda$ has a finite coresolution by modules in $\operatorname{add} D(\Lambda_\Lambda)$. In other words ${}_\Lambda\Lambda$ has finite injective dimension.

For the other direction assume ${}_\Lambda\Lambda$ has finite injective dimension. Then the right module $D({}_\Lambda\Lambda)$ has finite projective dimension, so WTC gives us that Λ_Λ has finite injective dimension. Which means $D(\Lambda_\Lambda)$ has finite projective dimension. □

Theorem 1.9. *[Hap93, 1.2] The finitistic dimension conjecture implies the vanishing conjecture.*

Proof. Assume Λ doesn't satisfy VC, and let $I^\bullet \in K^b(\operatorname{inj}\Lambda)^\perp$ be non-zero complex. Since $\mathscr{D}^b(\Lambda) \cong K^{+,b}(\operatorname{inj}\Lambda)$ we can consider $I^\bullet$ as a complex of injectives, and without loss of generality we may assume it is concentrated in degrees $i \geq 0$, and that $d^0 \colon I^0 \to I^1$ is not split mono. Since if it's concentrated in degrees $i \geq k$ we can just shift it, and if d^0 is split mono, then replacing I^0 by 0 and I^1 by I^1/I^0 gives a homotopic complex.

The module $\operatorname{Hom}(D\Lambda, I^i)$ is in $\operatorname{add}\operatorname{Hom}(D\Lambda, D\Lambda) = \operatorname{add}\Lambda$ so $\operatorname{Hom}(D\Lambda, I^\bullet)$ is a complex of projectives. We show that this complex is acyclic by considering the following diagram.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & D\Lambda \arrow[r] \arrow[d, "f"] \arrow[dl, dashed] & 0 \arrow[d] \\
I^{i-1} \arrow[r, "d^{i-1}"] & I^i \arrow[r, "d^i"] & I^{i+1}
\end{tikzcd}$$

Since $I^\bullet$ is in $K^b(\operatorname{inj}\Lambda)^\perp$ and $D\Lambda$ is in $K^b(\operatorname{inj}\Lambda)$, we have that whenever $d^i f = 0$, the morphism $f^\bullet$ is nullhomotopic. In other words, f factors through d^{i-1}. This means that $\operatorname{Hom}(D\Lambda, I^\bullet)$ is an acyclic complex. Further since $\operatorname{Hom}(D\Lambda, -)$ is an equivalence between $\operatorname{inj}\Lambda$ and $\operatorname{proj}\Lambda$ (c.f. Theorem A.5) and d^0 is not split mono, we have that $\operatorname{Hom}(D\Lambda, d^0)$ is not split mono.

The cokernel of $\operatorname{Hom}(D\Lambda, d^i)$ has a projective resolution of length i. This resolution is the direct sum of its minimal resolution and an acyclic bounded complex of projectives. Since bounded acyclic complexes of projectives are split and $\operatorname{Hom}(D\Lambda, d^0)$ is not, we must have that the minimal resolution has length i, and so $\operatorname{findim}(\Lambda) = \infty$. □

Theorem 1.10. *[Hap93, 1.2] The vanishing conjecture implies the Nunke condition.*

Proof. Assume Λ doesn't satisfy NuC. That is, there is an $X \neq 0$ with $\operatorname{Ext}^i(D\Lambda, X) = 0$ for all $i \geq 0$. Then we claim that X considered as a stalk complex is in $K^b(\operatorname{inj}\Lambda)^{\perp}$. To show this we proceed by induction on the width of $I^\bullet \in K^b(\operatorname{inj}\Lambda)$. If the width is 1, then $I^\bullet = I[-i] \in K^b(\operatorname{inj}\Lambda)$ is a stalk complex. Then $\mathscr{D}^b(I[-i], X) = \operatorname{Ext}^i(I, X)$, which is 0 because I is in $\operatorname{add} D\Lambda$ and $\operatorname{Ext}^i(D\Lambda, X) = 0$.

Let $I^\bullet \in K^b(\operatorname{inj}\Lambda)$ be a complex of width n. without loss of generality we may assume $I^\bullet$ is concentrated in degrees $0 \leq i < n$. Then

$$I^{>0} \longrightarrow I \longrightarrow I^0 \longrightarrow I^{>0}[1]$$

is a triangle with $I^{>0}$ of width $n-1$ and I^0 of width 1. Taking the long exact sequence in $\mathscr{D}^b(-, X)$ it follows that $\mathscr{D}^b(I, X) = 0$. So X is a non-zero complex in $K^b(\operatorname{inj}\Lambda)^{\perp}$, and hence Λ does not satisfy VC. $\square$

Before we can prove the equivalence between the generalized Nakayama conjecture and the Auslander–Reiten Conjecture we will need the following proposition.

Proposition 1.11. *Let M be a module and I an injective module. If the projective cover of the socle of I is in* $\operatorname{add} M$*, then (M, I) is an injective $\Gamma := \operatorname{End}(M)^{\mathrm{op}}$-module. In particular if M is a generator then $(M, -)$ preserves injectives.*

Proof. Let $J \leq \Gamma$ be a left ideal and let $\psi \colon J \to (M, I)$ be any Γ-linear map. By Lemma A.1 in the appendix it is enough to show that ψ factors through Γ to conclude that (M, I) is injective. Assume J is generated by $\{f_i\}$. If we can find $\gamma \colon M \to I$ such that $\gamma \circ f_i = \psi(f_i)$ then we would get our factorization of ψ by $J \hookrightarrow \Gamma \xrightarrow{\gamma \circ -} (M, I)$. To construct such a γ we consider the following diagram.

$$\begin{tikzcd} \bigoplus M \arrow[d, "\sum f_i"'] \arrow[dr, "\sum \psi(f_i)"] & \\ M \arrow[r, dashed, "\gamma"'] & I \end{tikzcd}$$

We want to show that the kernel of $\sum \psi(f_i)$ contains the kernel of $\sum f_i$, so that we can use the injective property of I. To see this let K be the kernel of $\sum f_i$ and let K' be the kernel of $\sum \psi(f_i)$. If K' does not contain K, then $Q := K/K' \cap K$ is a nonzero module that is mapped injectively into I. So the socle of Q is a summand of the socle of I. Then by assumption the projective cover of the socle of Q is in $\operatorname{add} M$, so there is a non-zero map $M \to Q$ that factors through a projective. By the lifting property of projectives we get a map $M \to K$ such that the composition with $\sum \psi(f_i)$ is non-zero.

Let a_i be the composition $M \longrightarrow K \hookrightarrow \bigoplus M \xrightarrow{\pi_i} M$. Then we get that $\sum f_i \circ a_i = 0$. Applying ψ we get $\sum \psi(f_i) \circ a_i = 0$, which gives a contradiction since a_i was explicitly constructed such that $\sum \psi(f_i) \circ a_i$ is non-zero. Thus K' contains K.

Using this we get the following commutative diagram:

$$\begin{tikzcd} \bigoplus M \arrow[d] \arrow[drr, "\sum \psi(f_i)"] \arrow[dd, bend right=60, "\sum f_i"'] & & \\ \left(\bigoplus M\right)/K \arrow[rr] \arrow[d, hook] & & I \\ M \arrow[urr, dashed, "\exists \gamma"'] & & \end{tikzcd}$$

Since I is injective it lifts monomorphisms so we know that γ exists. Thus (M, I) is an injective Γ-module. $\square$

Theorem 1.12. *The generalized Nakayama conjecture implies the Auslander–Reiten conjecture.*

Proof. The proof goes by contraposition. Assume Λ does not satisfy ARC. Then we have a nonprojective generator M such that $\mathrm{Ext}^n(M, M) = 0$ for all $n > 0$. We wish to show that $\Gamma := \mathrm{End}(M)^{\mathrm{op}}$ does not satisfy GNC. Let

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

be an injective resolution of M. Since $\mathrm{Ext}^n(M, M) = 0$, when we apply $(M, -) := \mathrm{Hom}(M, -)$ we get an exact sequence.

$$0 \longrightarrow \Gamma \longrightarrow (M, I_0) \longrightarrow (M, I_1) \longrightarrow \cdots$$

By Proposition 1.11 this is an injective resolution of Γ.

Since M is a non-projective generator it has every indecomposable projective as a summand and a nonprojective summand. So M has more indecomposable summands than Λ which means that Γ has more indecomposable projectives than Λ. It follows that Γ also has more injectives and thus has an injective not on the form (M, I). Since all modules that appear in the injective resolution of Γ are on the form (M, I), not all indecomposable injectives appear in the resolution. Therefore by Proposition 1.5 we have that Γ does not satisfy GNC. $\square$

Theorem 1.13. *[Yam96, Theorem 3.4.3] The Auslander–Reiten conjecture implies the generalized Nakayama conjecture.*

Proof. Assume that ARC holds, and let Γ be a finite dimensional algebra. We wish to show that Γ satisfies GNC. Let the minimal injective resolution of Γ be given by

$$0 \longrightarrow \Gamma \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

Let I be the minimal injective module such that each I_i is in $\mathrm{add}\, I$. If we can show that I is a cogenerator, then it will follow that Γ satisfies

GNC. Let $P = DI$ be the projective right Γ-module dual to I, and let $\Lambda = \mathrm{End}_\Gamma(P)$ be its endomorphism ring.

Using the Hom-Tensor adjunction we see that

$$\begin{aligned} D(P \otimes_\Gamma X) &= \mathrm{Hom}_k(P \otimes_\Gamma X, k) \\ &= \mathrm{Hom}_\Gamma(P, \mathrm{Hom}_k(X, k)) \\ &= \mathrm{Hom}_\Gamma(P, DX) \end{aligned}$$

In particular we have that $D(P \otimes_\Gamma I) = \mathrm{End}_\Gamma(P) = \Lambda$ as right Λ-modules, and so $P \otimes_\Gamma I = D\Lambda$.

Now let $\mathcal{S} \subseteq \mathrm{mod}\,\Gamma$ be the full subcategory of Γ-modules that have a copresentation in $\mathrm{add}\,I$. Then we claim there is an equivalence of categories

$$\begin{tikzcd} \mathcal{S} \arrow[r, bend left, "P\otimes_\Gamma -"] & \mathrm{mod}\,\Lambda \arrow[l, bend left, "\mathrm{Hom}_\Lambda(P,-)"] \end{tikzcd}$$

To see this we first note the following identities

$$\begin{aligned} \mathrm{Hom}_\Lambda(P, P \otimes_\Gamma I) &= \mathrm{Hom}_\Lambda(P, D\Lambda) \\ &= \mathrm{Hom}_k(\Lambda \otimes_\Lambda P, k) \\ &= DP = I \end{aligned}$$

$$\begin{aligned} P \otimes_\Gamma \mathrm{Hom}_\Lambda(P, D\Lambda) &= P \otimes_\Gamma DP \\ &= D\Lambda \end{aligned}$$

Since P_Γ is projective $P \otimes_\Gamma -$ is exact, so both functors are left exact. This means they induce equivalences between the subcategories with copresentations in $\mathrm{add}\,I$ and $\mathrm{add}\,D\Lambda$ respectively. Thus we get our wanted equivalence.

Now if we apply $P \otimes_\Gamma -$ to the injective resolution $I_\bullet$, we get an injective resolution of $P \otimes_\Gamma \Gamma = P$ as a Λ-module. Applying $\mathrm{Hom}_\Lambda(P, -)$ gives us back the complex $I_\bullet$ and thus we have that $\mathrm{Ext}^n_\Lambda(P, P) = 0$ for all $n > 0$.

Since $\mathrm{Hom}_\Lambda(P,-)$ is an equivalence, it is faithful. This says exactly that P is a generator in $\mathrm{mod}\,\Lambda$. Since by assumption ARC holds, we get that P is projective as a Λ-module. Thus $\mathrm{Hom}_\Lambda(P,-)$ is right exact. Since Γ is in $\mathcal{S}$, the equivalence give us $\mathrm{Hom}_\Lambda(P,P) = \mathrm{Hom}_\Lambda(P, P\otimes\Gamma) = \Gamma$. Combining these two facts we get that $\mathrm{Hom}_\Lambda(P,-)$ induces an equivalence between modules with a presentation in $\mathrm{add}\,P$ and modules with a presentation in $\mathrm{add}\,\Gamma$. We conclude that $\mathcal{S} = \mathrm{mod}\,\Gamma$, and thus that I is a cogenerator.

Since I is a cogenerator all indecomposable injective modules appear in the resolution of Γ, and thus Γ satisfies GNC. □

Proposition 1.14. *[AR75] The generalized Nakayama conjecture implies the Nakayama conjecture*

Proof. Assume Λ satisfies GNC and that the dominant dimension of Λ is ∞. As shown in Proposition 1.5 if $\mathrm{Ext}^\bullet(S,\Lambda)$ is nonzero that means the injective envelope $I(S)$ appears in the minimal injective resolution of Λ. If all injectives appear in the resolution and the dominant dimension is infinity then all injectives are projective. Thus Λ is self injective, and hence Λ satisfies NC. □

The proofs above do not necessarily work on the level of individual algebras. For example, for the proof that WTC implies GSC we need to assume that WTC holds for both Λ and Λ^{op} to prove that Λ satisfies GSC. Although it is implicit in the proofs, for the convenience of the reader, we list the relationships between the conjectures for individual algebras here.

Proposition 1.15. *The implications between the conjectures on the level of individual algebras can be described as follows:*

a) If Λ satisfies FDC, then Λ also satisfies WTC.

b) If both Λ and Λ^{op} satisfy WTC, then both Λ and Λ^{op} satisfy GSC.

c) The implications FDC $\Rightarrow$ VC $\Rightarrow$ NuC hold on the level of individual algebras.

d) An algebra Λ satisfies Nuc if and only if Λ^{op} satisfies SNC.

e) The implications SNC $\Rightarrow$ GNC $\Rightarrow$ NC hold on the level of individual algebras.

f) If Γ satisfies GNC whenever $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$ for a generator M in $\mathrm{mod}\,\Lambda$, then Λ satisfies ARC.

g) If $\mathrm{End}(I)^{\mathrm{op}}$ satisfies ARC, where I is an injective module such that $\mathrm{add}\,I$ contains every injective in the minimal resolution of Λ, then Λ satisfies GNC.

h) An algebra Λ satisfies NC if and only if Λ^{op} does [Mül68, Theorem 4].

read proof more fully

2 Recollement

In this section we will discuss a reduction technique known as recollement. The idea of reduction techniques is to reduce the work of proving an algebra has finite finitistic dimension to proving the same for "simpler" algebras. In Section 2.1 we will consider a reduction technique of triangular matrix algebras. The triangular matrix rings are closely related to recollements, and we discuss their relationship more closely in Section 2.2.

We begin by defining a recollement of triangulated categories.

Definition 2.1 (Recollement)**.** A *recollement* between triangulated categories $\mathcal{T}'$, $\mathcal{T}$ and $\mathcal{T}''$ is a collection of six functors satisfying:

$$
\begin{tikzcd}[column sep=huge]
\mathcal{T}' \arrow[r, "i_*=i_!"] & \mathcal{T} \arrow[l, bend right=40, "i^*"'] \arrow[l, bend left=40, "i^!"] \arrow[l, phantom, shift right=3ex, "\perp"] \arrow[l, phantom, shift left=3ex, "\perp"] \arrow[r, "j^!=j^*"] & \mathcal{T}'' \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*"] \arrow[l, phantom, shift right=3ex, "\perp"] \arrow[l, phantom, shift left=3ex, "\perp"]
\end{tikzcd}
$$

(i) All functors are exact, and we have adjoint pairs (i^*, i_*), $(i_!, i^!)$, $(j_!, j^!)$, (j^*, j_*).

(ii) The composition $j^*i_* = 0$ vanishes.

(iii) We have natural isomorphisms $i^*i_* \cong i^!i_! \cong \mathrm{id}_{\mathcal{T}'}$ induced by the units and counits of the adjunctions.

(iv) We have natural isomorphisms $j^!j_! \cong j^*j_* \cong \mathrm{id}_{\mathcal{T}''}$, also induced by the units and counits.

(v) For every $X \in \mathcal{T}$ we have the following distinguished triangles:

$$j_!j^!X \xrightarrow{\varepsilon} X \xrightarrow{\eta} i_*i^*X \longrightarrow j_!j^!X[1]$$

$$i_!i^!X \xrightarrow{\varepsilon} X \xrightarrow{\eta} j_*j^*X \longrightarrow i_!i^!X[1].$$

Note that (iii) and (iv) are equivalent to i_*, $j_!$, and j_* being fully faithful.

We are specifically interested in recollements where the triangulated categories in question are (bounded) derived categories of finite dimensional algebras.

We now give some properties of such functors, when they are defined over bounded derived categories.

Lemma 2.2. *Let* $\mathscr{D}^b(\Lambda') \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathscr{D}^b(\Lambda)$ *be exact functors with an adjoint pair* (i^*, i_*)*. Then* i^* *preserves bounded projective complexes and* i_* *preserves bounded injective complexes.*

Proof. The bounded projective complexes can be characterized up to isomorphism as the complexes P such that for any complex Y there is an integer t_Y with $\mathscr{D}^b(\Lambda)(P, Y[t]) = 0$ for $t \geq t_Y$. One can see this by using the equivalence $\mathscr{D}^b(\Lambda) \cong K^{-,b}(\operatorname{proj}\Lambda)$.

Let P be a bounded complex of projectives in $\mathscr{D}^b(\Lambda)$. Then we want to show that i^*P is as well. Let Y be any complex in $\mathscr{D}^b(\Lambda')$. Then

$\mathscr{D}^b(\Lambda')(i^*P, Y[t]) = \mathscr{D}^b(\Lambda)(P, i_*Y[t])$, so since P is a bounded complex of projectives there is t_Y such that this vanishes for $t \geq t_Y$.

The statement for injectives is exactly dual, and so we do not write it out here, but leave it to the reader. $\square$

The fact that these functors preserve bounded projective/injective complexes can be used to bound the homology of i_*X for modules X.

Lemma 2.3. *Let* $\begin{tikzcd} \mathscr{D}^b(\Lambda') \arrow[r, "i_*"] & \mathscr{D}^b(\Lambda) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \end{tikzcd}$ *be exact functors with adjoint pairs* (i^*, i_*) *and* $(i_*, i^!)$*. Then the homology of* i_*X *is uniformly bounded for* $X \in \operatorname{mod}\Lambda'$ *considered as a complex concentrated in degree 0. I.e. there is an* r*, independent of* X*, such that* $H^j(i_*X) = 0$ *for* $j \notin (-r, r)$*.*

Proof. We first prove that there is an r', independent of X, such that $H^j(i_*X) = 0$ for $j \geq r'$. Let P be $i^*\Lambda \in \mathscr{D}^b(\Lambda')$. Then by Lemma 2.2 P is a bounded complex of projectives.

Thus there is an r' such that $P^{-j} = 0$ for $j \geq r'$. Then

$$\mathscr{D}^b(\Lambda')(P, X[j]) = \mathscr{D}^b(\Lambda)(\Lambda, i_*X[j]) = H^j(i_*X) = 0$$

for $j \geq r'$ and any Λ'-module X, when considered as a complex concentrated in degree 0.

Next we prove that there is an r'' such that $H^{-j}(i_*X) = 0$ for $j \geq r''$. The argument is completely dual. Let I be $i^!D\Lambda \in \mathscr{D}^b(\Lambda') \cong K^{+,b}(\operatorname{inj}\Lambda')$. Then again by Lemma 2.2 I is a bounded complex of injectives.

Thus there is an r'' such that $I^j = 0$ for $j \geq r''$. Then

$$\mathscr{D}^b(\Lambda')(X, I[j]) = \mathscr{D}^b(\Lambda)(i_*X, D\Lambda[j]) = H^{-j}(i_*X) = 0$$

for $j \geq r''$ and any Λ'-module X, when considered as a complex concentrated in degree 0.

Letting r be the maximum of r' and r'' we get that $H^j(X)$ is zero outside of $(-r, r)$. $\square$

Now that we have a good understanding of how the functors in a recollement interact with homology, we can use this to say something about the projective dimension of modules, and thus about the finitistic dimension.

Theorem 2.4. *[Hap93, 3.3] Given a recollement between bounded derived categories*

$$\begin{tikzcd}
\mathscr{D}^b(\Lambda') \arrow[rr, "i_*=i_!"] & & \mathscr{D}^b(\Lambda) \arrow[ll, bend right, "i^*"'] \arrow[ll, bend left, "i^!"] \arrow[rr, "j^!=j^*"] & & \mathscr{D}^b(\Lambda''), \arrow[ll, bend right, "j_!"'] \arrow[ll, bend left, "j_*"]
\end{tikzcd}$$

then we have that $\operatorname{findim}(\Lambda) < \infty$ *if and only if we have that* $\operatorname{findim}(\Lambda') < \infty$ *and* $\operatorname{findim}(\Lambda'') < \infty$*.*

Proof. Assume $\operatorname{findim}(\Lambda) < \infty$. We begin by showing that $\operatorname{findim}(\Lambda') < \infty$.

Let $T = \Lambda' / \operatorname{rad} \Lambda'$ be the sum of all simple Λ'-modules. Then the projective dimension of X is the largest t for which $\operatorname{Ext}^t(X, T) \neq 0$. Let X be a module in $\operatorname{mod} \Lambda'$ with finite projective dimension. We consider X as a complex concentrated in degree 0. Then since X is isomorphic to its projective resolution, by Lemma 2.2 i_*X is a bounded complex of projectives. Say:

$$i_*X = 0 \to P^{-s} \to \cdots \to P^{s'} \to 0$$

By Lemma 2.3 we know there is an r independent of X such that $H^{-j}(i_*X) = 0$ for $j \geq r$. Truncating i_*X at $-r$ gives a projective resolution of $\ker d^{-r}_{i_*X}$. So $\ker d^{-r}_{i_*X}$ has projective dimension $-r - (-s) = s - r$. Since $\operatorname{findim}(\Lambda) < \infty$ this means that $s \leq r + \operatorname{findim}(\Lambda)$.

Since i_*T is in $\mathscr{D}^b(\Lambda)$ it is a bounded complex, in particular there is a t_0 such that $i_*T^t = 0$ for $t \geq t_0$. Then by the bounds above $\mathscr{D}^b(\Lambda)(i_*X, i_*T[t]) = 0$ for $t \geq t_0 + s \geq t_0 + r + \operatorname{findim}(\Lambda)$. Since i_* is fully faithful this equals $\mathscr{D}^b(\Lambda')(X, T[t])$, and so $\operatorname{findim}(\Lambda') \leq t_0 + r + \operatorname{findim}(\Lambda)$. In particular it is finite.

The proof for $\operatorname{findim}(\Lambda'')$ is the same, just replacing i_* with $j_!$. We leave writing out the details to the reader.

For the converse assume Λ' and Λ'' both have finite finitistic dimension. Let $T = \Lambda/\operatorname{rad}\Lambda$, and X be a Λ-module with finite projective dimension, and consider both modules as a complex concentrated in degree 0. By Definition 2.1(v) we have distinguished triangles:

$$j_!j^!X \longrightarrow X \longrightarrow i_*i^*X \longrightarrow j_!j^!X[1]$$

$$i_!i^!T \longrightarrow T \longrightarrow j_*j^*T \longrightarrow i_!i^!T[1].$$

We write $(-,-)_m$ instead of $\mathscr{D}^b(\Lambda)(-,-[m])$, and make the following abbreviation:

$$\begin{aligned} X_j &:= j_!j^!X & \qquad X_i &:= i_*i^*X \\ T_i &:= i_!i^!T & \qquad T_j &:= j_*j^*T. \end{aligned}$$

Taking the long exact sequence in homfuntors we get the long exact sequences:

$$\cdots \longrightarrow (X, T_i)_m \longrightarrow (X, T)_m \longrightarrow (X, T_j)_m \longrightarrow (X, T_i)_{m+1} \longrightarrow \cdots$$

$$\cdots \longrightarrow (X_i, T_i)_m \longrightarrow (X, T_i)_m \longrightarrow (X_j, T_i)_m \longrightarrow (X_i, T_i)_{m+1} \longrightarrow \cdots$$

$$\cdots \longrightarrow (X_i, T_j)_m \longrightarrow (X, T_j)_m \longrightarrow (X_j, T_j)_m \longrightarrow (X_i, T_j)_{m+1} \longrightarrow \cdots$$

Using the fact that $j^*i_* = j^!i_! = 0$ from Definition 2.1(ii) we deduce that

$$(X_i, T_j)_m = (i_*i^*X, j_*j^*T)_m = (j^*i_*i^*X, j^*T)_m = 0$$

and

$$(X_j, T_i)_m = (j_!j^!X, i_!i^!T)_m = (j^!X, j^!i_!i^!T)_m = 0.$$

Combining this with the long exact sequences gives us that

$$(X_i, T_i)_m = (X, T_i)_m \text{ and } (X_j, T_j)_m = (X, T_j)_m.$$

If we can show that $(X_i, T_i)_m$ and $(X_j, T_j)_m$ are bounded, then $(X, T_i)_m$ and $(X, T_j)_m$ would be bounded as well. Consequently we would have that $(X, T)_m$ is bounded. This would give us a bound on the projective dimension of X.

We start by bounding $(X, T_i)_m = (X_i, T_i)_m$. First note that since $i^*i_* \cong \mathrm{id}$ we have that

$$(X_i, T_i)_m = (i_*i^*X, i_!i^!T)_m = (i^*i_*i^*X, i^!T)_m = (i^*X, i^!T)_m$$

Since X has finite projective dimension we can think of it as a bounded complex of projectives. Then by Lemma 2.2 i^*X is as well. By the second half of Lemma 2.3 (using (i^*, i_*) instead of $(i_*, i^!)$) we have that there is an r such that $H^{-j}(i^*X) = 0$ for all $j \geq r$. This means that thinking of i^*X as a complex of projectives, it is 0 in degree $-t$ for all $t \geq r + \mathrm{pd}\ker d^{-r}_{i^*X}$, in particular it is 0 for all $t \geq r + \mathrm{findim}(\Lambda')$. Since $i^!T$ is a bounded complex, it has an upper bound, say t_0. Thus $(i^*X, i^!T)_m = 0$ for all $m \geq t_0 + r + \mathrm{findim}(\Lambda')$.

The bound on $(X, T_j)_m$ is similar, using the finitistic dimension of Λ''. Taking the maximum of these two bounds we get a bound on $(X, T)_m$, which gives a bound on the projective dimension independent of X, hence a bound on $\mathrm{findim}(\Lambda)$. □

2.1 Triangular matrix rings

In this section we will relate the finitistic dimension of the triangular matrix ring $\Lambda = \begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$ to the finitistic dimension of R and S. Specifically the finitistic dimension of Λ will be finite if the finitistic dimensions of both R and S are finite.

In Section 2.2 we give some further conditions on M for which we get a recollement between the bounded derived categories of S, R and Λ.

We will first define the concept of a comma category and describe some of its homological properties. In Theorem 2.12 we give a bound on the finitistic dimension of the comma category. Then in Proposition 2.15 we show that for Λ a triangular matrix ring as above, we have that $\operatorname{mod}\Lambda$ is isomorphic to the comma category of $M \otimes_R -\colon \operatorname{mod} R \to \operatorname{mod} S$, which means we get a bound on $\operatorname{findim}(\Lambda)$.

Definition 2.5 (Comma category)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor. Then the *comma category* $(F, \mathcal{B})$ has as objects triplets (A, B, f) with $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $f\colon FA \to B$ a morphism in $\mathcal{B}$. The morphisms are pairs $(\alpha, \beta)\colon (A, B, f) \to (A', B', f')$ with $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ such that the following diagram commutes:

$$\begin{tikzcd} FA \arrow[r, "f"] \arrow[d, "F\alpha"'] & B \arrow[d, "\beta"] \\ FA' \arrow[r, "f'"] & B'. \end{tikzcd}$$

The composition is what one would expect. Namely, $(\alpha, \beta) \circ (\alpha', \beta') = (\alpha \circ \alpha', \beta \circ \beta')$.

Proposition 2.6. *If $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and F is right exact, then the comma category $(F, \mathcal{B})$ is abelian. Further a sequence*

$$(A'', B'', f'') \xrightarrow{(\alpha', \beta')} (A, B, f) \xrightarrow{(\alpha, \beta)} (A', B', f')$$

is exact if and only if the two related sequences in $\mathcal{A}$ and $\mathcal{B}$ are exact.

$$A'' \xrightarrow{\alpha'} A \xrightarrow{\alpha} A'$$

$$B'' \xrightarrow{\beta'} B \xrightarrow{\beta} B'$$

Proof. We need to show that $(F, \mathcal{B})$ has kernels and cokernels, and that for any map the image equals the coimage. First we show that it contains kernels. Let $(\alpha, \beta)\colon (A, B, f) \to (C, D, g)$ be a morphism in the comma category. Then we have a diagram:

$$\begin{tikzcd}
 & F\ker\alpha \arrow[r, "F\iota_\alpha"] \arrow[d, dashed, "\theta"] & FA \arrow[r, "F\alpha"] \arrow[d, "f"] & FC \arrow[d, "g"] \\
0 \arrow[r] & \ker\beta \arrow[r, "\iota_\beta"] & B \arrow[r, "\beta"] & D
\end{tikzcd}$$

Since $\beta f F\iota_\alpha = f' F\alpha F\iota_\alpha = 0$ there is a unique θ making the diagram commute. I claim the kernel of (α, β) is $(\ker\alpha, \ker\beta, \theta)$. Indeed if $(\alpha', \beta')\colon (A', B', f') \to (A, B, f)$ is any map such that $(\alpha, \beta) \circ (\alpha', \beta') = 0$, then $\alpha\alpha' = 0$ and $\beta\beta' = 0$. This means both α' and β' factor uniquely through ι_α and ι_β. Let α'' and β'' be the morphisms such that $\alpha' = \iota_\alpha \circ \alpha''$ and $\beta' = \iota_\beta \circ \beta''$. Then we claim (α', β') factors through $(\iota_\alpha, \iota_\beta)$ as indicated in the diagram below.

$$\begin{tikzcd}
FA' \arrow[r, "F\alpha''"] \arrow[d, "f'"] & F\ker\alpha \arrow[r, "F\iota_\alpha"] \arrow[d, "\theta"] & FA \arrow[d, "f"] \\
B' \arrow[r, "\beta''"] & \ker\beta \arrow[r, "\iota_\beta"] & B
\end{tikzcd}$$

The only thing left to verify is that the left square commutes. This follows from the outer rectangle commuting, and that ι_β is a monomorphism.

Showing that cokernels exists is similar, but relies on F being right exact. The construction is completely dual, but to verify commutativity at the end instead of using that ι_β is mono we must use that $F\pi_\alpha\colon FA' \to F\operatorname{cok}\alpha$

is an epimorphism. This follows from F being right exact. We leave the details to the reader.

Since kernels and cokernels are directly induced by the kernels and cokernels in $\mathcal{A}$ and $\mathcal{B}$ it is clear that a sequence in $(F, \mathcal{B})$ is exact if and only if the two related sequences are exact. Similarly that the image equals the coimage follows from this being true in $\mathcal{A}$ and $\mathcal{B}$. □

For the rest of this section we assume F is a right exact functor between abelian catgeories so that the comma category is abelian. We also assume $\mathcal{A}$ and $\mathcal{B}$ has enough projectives. In particular we are interested in the case when $\mathcal{A}$ and $\mathcal{B}$ are module categories over finite dimensional algebras.

Definition 2.7. For $\mathcal{A}$ and $\mathcal{B}$ abelian categories and F right exact we define the following functors:

$$\begin{aligned} T\colon \mathcal{A} \times \mathcal{B} &\longrightarrow (F, \mathcal{B}) \\ (A, B) &\longmapsto (A, B \oplus FA, FA \hookrightarrow FA \oplus B) \\ (\alpha, \beta) &\longmapsto (\alpha, F\alpha \oplus \beta) \end{aligned}$$

$$\begin{aligned} U\colon (F, \mathcal{B}) &\longrightarrow \mathcal{A} \times \mathcal{B} \\ (A, B, f) &\longmapsto (A, B) \\ (\alpha, \beta) &\longmapsto (\alpha, \beta) \end{aligned} \qquad \begin{aligned} C\colon (F, \mathcal{B}) &\longrightarrow \mathcal{A} \times \mathcal{B} \\ (A, B, f) &\longmapsto (A, \operatorname{cok} f) \\ (\alpha, \beta) &\longmapsto (\alpha, \hat{\beta}) \end{aligned}$$

$$\begin{aligned} Z\colon \mathcal{A} \times \mathcal{B} &\longrightarrow (F, B) \\ (A, B) &\longmapsto (A, B, 0) \\ (\alpha, \beta) &\longmapsto (\alpha, \beta) \end{aligned}$$

Proposition 2.8. *With the definitions above U and Z become exact functors.*

Proof. Using the characterization of exact sequences shown in Proposition 2.6 a short exact sequence in $(F, \mathcal{B})$ is a commutative diagram

$$\begin{tikzcd}
 & FA'' \arrow[r, "F\alpha'"] \arrow[d, "f''"] & FA \arrow[r, "F\alpha"] \arrow[d, "f"] & FA' \arrow[r] \arrow[d, "f'"] & 0 \\
0 \arrow[r] & B'' \arrow[r, "\beta'"] & B \arrow[r, "\beta"] & B' \arrow[r] & 0
\end{tikzcd}$$

such that the sequences

$$0 \longrightarrow A'' \xrightarrow{\alpha'} A \xrightarrow{\alpha} A' \longrightarrow 0$$

$$0 \longrightarrow B'' \xrightarrow{\beta'} B \xrightarrow{\beta} B' \longrightarrow 0$$

are short exact. Since when we apply U we simply get the product of these two sequences, U is exact.

Similarly for Z since the two sequences we start with are assumed to be exact the resulting sequence will be exact by the characterization in Proposition 2.6. □

Proposition 2.9. *[FGR75, Proposition 1.3] The pairs of functors (T, U) and (C, Z) form adjoint pairs.*

Proof. We want to establish an isomorphism

$$\operatorname{Hom}(T(A,B),(A',B',f)) \cong \operatorname{Hom}((A,B),(A',B')).$$

A morphism $\left(\alpha, \begin{bmatrix}\beta & \gamma\end{bmatrix}\right) : T(A,B) \to (A',B',f)$ is given by a commutative diagram

$$\begin{tikzcd}
T(A,B)\colon \arrow[d, "{\left(\alpha,\begin{bmatrix}\beta & \gamma\end{bmatrix}\right)}"'] & FA \arrow[r, "\begin{bmatrix}0\\1\end{bmatrix}"] \arrow[d, "F\alpha"'] & B\oplus FA \arrow[d, "\begin{bmatrix}\beta & \gamma\end{bmatrix}"] \\
(A',B',f)\colon & FA' \arrow[r, "f"] & B'.
\end{tikzcd}$$

The isomorphism is then given by sending this to (α, β). This is clearly surjective.

For injectivity assume $(\alpha, \beta) = 0$, then $\gamma = \begin{bmatrix}\beta & \gamma\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = fF\alpha = 0$. So the map is injective, and (T, U) is an adjoint pair.

Next we consider (C, Z). We want an isomorphism

$$\begin{aligned}\operatorname{Hom}(C(A,B,f),(A',B')) &= \operatorname{Hom}((A,\operatorname{cok} f),(A',B'))\\ &\cong \operatorname{Hom}((A,B,f),(A',B',0)).\end{aligned}$$

A morphism in $\operatorname{Hom}((A,B,f),(A',B',0))$ is a commutative diagram

$$\begin{tikzcd}
FA \arrow[r, "f"] \arrow[d, "F\alpha"'] & B \arrow[d, "\beta"] \\
FA' \arrow[r, "0"] & B'
\end{tikzcd}$$

Since $\beta f = 0$, we have that β factors through the cokernel of f uniquely. Let the factorization be given by the map $\beta'\colon \operatorname{cok} f \to B'$. Then we send this diagram to (α, β'). Since the choice of β' was unique this is an isomorphism, so (C, Z) is an adjoint pair. □

Corollary 2.9.1. *The functors T and C preserve projective objects.*

Proof. What we need to check is that for projective objects P and Q in $(\mathcal{A}\times\mathcal{B})$ and $(F, \mathcal{B})$ respectively we have that $\operatorname{Hom}(TP, -)$ and $\operatorname{Hom}(CQ, -)$ are exact. By adjointness these are equal to $\operatorname{Hom}(P, U-)$ and $\operatorname{Hom}(Q, Z-)$

respectively. Since U and Z are exact this holds, and so T and C preserve projective objects. $\square$

We will now use these four functors to understand the structure of projective objects in the comma category, and consequently projective resolutions.

Proposition 2.10. *[FGR75, Corollary 1.6c] For a projective object P in $(F, \mathcal{B})$ we have that $T(C(P)) \cong P$, in particular all projectives are of the form $T(P')$ for a projective $P' \in \mathcal{A} \times \mathcal{B}$.*

Proof. Let P be given by $f \colon FA \to B$. Applying C we get $(A, \operatorname{cok} f)$. We have morphisms $P \to ZC(P)$ and $TC(P) \to ZC(P)$ given by the following diagram

$$\begin{tikzcd}
FA \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, two heads] \\
FA \arrow[r, "0"] \arrow[d, equal] & \operatorname{cok} f \\
FA \arrow[r, hook] & \operatorname{cok} f \oplus FA. \arrow[u, two heads]
\end{tikzcd}$$

By the projective property of P there is some morphism β factorizing the map $P \to ZC(P)$, which gives us the diagram:

$$\begin{tikzcd}
FA \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, "\beta"] \\
FA \arrow[r, hook] \arrow[d, equal] & \operatorname{cok} f \oplus FA \arrow[d, two heads] \\
FA \arrow[r, "0"] & \operatorname{cok} f.
\end{tikzcd}$$

Since $FA \hookrightarrow \operatorname{cok} f \oplus FA$ is split mono, f is split mono. This means that B splits as a direct sum of the image and cokernel of f, i.e. B is isomorphic to $\operatorname{cok} f \oplus \operatorname{Im} f \cong \operatorname{cok} f \oplus FA$. From the diagram we see that β induces an isomorphism on each component, and thus β is an isomorphism. So we have $P \cong TC(P)$. $\square$

Proposition 2.11. *[FGR75, Lemma 4.16] Let $X = (A, B, f)$ be an object in the comma category. Then $\operatorname{pd} X \geq \operatorname{pd} A$, and if $A = 0$ then $\operatorname{pd} X = \operatorname{pd} B$.*

Proof. We first show that $\operatorname{pd} X \geq \operatorname{pd} A$. Note that there is an equality $\operatorname{pd} C(X) = \max\{\operatorname{pd} A, \operatorname{pd} \operatorname{cok} f\}$ so we always have $\operatorname{pd} C(X) \geq \operatorname{pd} A$. If $\operatorname{pd} X = \infty$ then the statement holds so let us assume $\operatorname{pd} X = n < \infty$. We proceed by induction on n. If $n = 0$ then $C(X)$ is projective so $\operatorname{pd} X = \operatorname{pd} C(X) = \operatorname{pd} A = 0$. Next assume the statement holds whenever the projective dimension is less than n. Let $P \to A$ and $P' \to \operatorname{cok} f$ be epimorphisms from projectives. Then we have an epimorphism $T(P, P') \to X$. If we let ΩA be the kernel of $P \to A$ and $X' = (\Omega A, K, \theta)$ be the kernel of $T(P, P') \to X$ as shown in the following diagram

$$\begin{tikzcd}
 & F\Omega A \arrow[r] \arrow[d, "\theta"'] & FP \arrow[r] \arrow[d, hook] & FA \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K \arrow[r] & P' \oplus FP \arrow[r] & B \arrow[r] & 0,
\end{tikzcd}$$

then we have $\operatorname{pd} A \leq \operatorname{pd} \Omega A + 1$ and $\operatorname{pd} X = \operatorname{pd} X' + 1$. By induction we have that $\operatorname{pd} X' \geq \operatorname{pd} \Omega A$ and so $\operatorname{pd} X \geq \operatorname{pd} \Omega A + 1 \geq \operatorname{pd} A$.

If $A = 0$ then we can associate $C(X) = (0, B)$ with B. Any projective resolution $P_B^\bullet$ of B gives a resolution of X by $T(0, P_B^\bullet)$, and any resolution $P_X^\bullet$ of X gives a resolution of $(0, B)$ by $C(P_X^\bullet)$. Thus $\operatorname{pd} X = \operatorname{pd} B$. $\square$

Now we are ready for the main theorem of this section, where we give an upper bound on the finitistic dimension of the comma category.

Theorem 2.12. *[FGR75, Theorem 4.20] The finitistic dimension of the comma category $(F, \mathcal{B})$ is bounded above by $\operatorname{findim}(\mathcal{A}) + \operatorname{findim}(\mathcal{B}) + 1$.*

Proof. Let $X = (A, B, f)$ be an element of the comma category with finite projective dimension. Let $P_A^\bullet$ be a projective resolution of A shorter than $\operatorname{findim}(\mathcal{A})$. Similar to what we did in Proposition 2.11 define P_X^0 to be $T(P_A^0, P(\operatorname{cok} f))$ where $P(\operatorname{cok} f)$ is a projective module with an epimorphism onto $\operatorname{cok} f$. Then let the kernel of $P_X^0 \to X$ be $(\Omega A, K^0, \theta^0)$. We continue inductively, defining P_X^n to be $T(P_A^n, \operatorname{cok} \theta^{n-1})$. Then $\Omega^{\operatorname{findim}(\mathcal{A})+1} X = (0, K^{\operatorname{findim}(\mathcal{A})}, 0)$. Then by Proposition 2.11 we know that $\operatorname{pd} \Omega^{\operatorname{findim}(\mathcal{A})+1} X = \operatorname{pd} K^{\operatorname{findim}(\mathcal{A})} \leq \operatorname{findim}(\mathcal{B})$. So

$$\operatorname{pd} X \leq \operatorname{findim}(\mathcal{A}) + \operatorname{findim}(\mathcal{B}) + 1.$$

□

Before applying this to triangular matrix rings, let us have a look at a simple example.

Example 2.13. If k is a field, $\mathcal{A} = \mathcal{B} = \operatorname{mod} k$, and F is the identity, then the comma category $(F, \mathcal{B})$ is equivalent to the category of finite dimensional representations of $\mathbb{A}_2$ over k.

In this example $\mathcal{A}$ and $\mathcal{B}$ both have finitistic dimension 0, while $(F, \mathcal{B})$ has finitistic dimension 1. So the bound shown above is tight.

Definition 2.14 (Triangular matrix ring)**.** Let R and S be rings, and let M be an S-R-bimodule. Then the *triangular matrix ring* $\begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$ is the ring of all matricies $\begin{bmatrix} r & 0 \\ m & s \end{bmatrix}$ with $r \in R$, $s \in S$, and $m \in M$. The multplication is given by

$$\begin{bmatrix} r & 0 \\ m & s \end{bmatrix} \begin{bmatrix} r' & 0 \\ m' & s' \end{bmatrix} = \begin{bmatrix} rr' & 0 \\ mr' + sm' & ss' \end{bmatrix}.$$

We have already hinted at an example of this in Example 2.13. The algebra $k\mathbb{A}_2$ is isomorphic to the matrix ring $\begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$, and we saw how $\operatorname{mod} k\mathbb{A}_2$ becomes the comma category for a functor between $\operatorname{mod} k$ and $\operatorname{mod} k$. In fact whenever Λ is a triangular matrix ring, the module category $\operatorname{mod} \Lambda$ will be the comma category for a specific functor.

Proposition 2.15. *If* $\Lambda = \begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$ *is a triangular matrix ring and* M *is finitely generated as an* S*-module, then* $\operatorname{mod}\Lambda$ *is isomorphic to the comma category* $(M \otimes_R -, \operatorname{mod} S)$*. In particular this holds if* Λ *is also a finite dimensional algebra.*

Proof. Notice, if N is a Λ-module, then as an abelian group N splits as a direct sum into

$$N = N_R \oplus N_S := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} N \oplus \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} N.$$

By restriction of scalars we can think of N_R as an R-module and N_S as an S-module. Further multiplication by $\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$ is 0 on N_S and maps N_R into N_S. So N consists of an R-module N_R, an S-module N_S and a S-R-linear map $M \to \mathrm{Hom}_{\mathbb{Z}}(N_R, N_S)$, or equivalently an S-linear map $M \otimes_R N_R \to N_S$.

This gives us the equivalence between $\operatorname{mod}\Lambda$ and $(M \otimes_R -, \operatorname{mod} S)$. $\square$

Corollary 2.15.1. *When* Λ *is the triangular matrix algebra above, then*

$$\operatorname{findim}(\Lambda) \leq \operatorname{findim}(R) + \operatorname{findim}(S) + 1.$$

2.2 Recollements for triangular matrix rings

There is an analogues definition of recollement between abelian categories. If Λ is a triangulated matrix algebra as above then we do get a recollement of abelian categories

$$\begin{tikzcd}[column sep=huge]
\operatorname{mod} S \arrow[r, "\mathrm{inc}"] \arrow[r, phantom, shift left=4ex, "\perp"] \arrow[r, phantom, shift right=4ex, "\perp"] & \operatorname{mod}\Lambda \arrow[l, bend right=40, "\Lambda/\Lambda e_R\Lambda\otimes_\Lambda -"'] \arrow[l, bend left=40, "\mathrm{Hom}(\Lambda e_S,-)"'] \arrow[r, "\mathrm{Hom}(\Lambda e_R,-)=e_R\Lambda\otimes -"] \arrow[r, phantom, shift left=4ex, "\perp"] \arrow[r, phantom, shift right=4ex, "\perp"] & \operatorname{mod} R \arrow[l, bend right=40, "\Lambda e_R\otimes -"'] \arrow[l, bend left=40, "\mathrm{Hom}(e_R\Lambda,-)"']
\end{tikzcd}$$

In fact, by a result due to Psaroudakis–Vitória [PV14, Corollary 5.5], if Λ is semiprimary, then all recollements of module categories are of this form.

By taking derived functors we get a recollement of unbounded derived categories, which also restricts to a recollement between $D^{-}(S)$, $D^{-}(\Lambda)$ and $D^{-}(R)$, as shown by König [Kön91, Corollary 15].

This does not in general restrict to a recollement of bounded derived categories, but if M has finite projective dimension both as an R-module and an S-module then it does.

3 Contravariantly finite subcategories

In this section we will study the subcategory of modules with finite projective dimension, which we denote by $\mathcal{P}^{\infty}$. In Corollary 3.5.1 we show that an algebra has finite finitistic dimension when $\mathcal{P}^{\infty}$ is contravariantly finite. Example 3.6, discovered by Igusa–Smalø–Todorov, shows that $\mathcal{P}^{\infty}$ can fail to be contravariantly finite even for monomial algebras with radical cubed equal to 0.

It is known that $\mathcal{P}^{\infty}$ is contravariantly finite when the algebra is stably equivalent to a hereditary algebra. This was shown by Auslander–Reiten in their original paper [AR91]. We consider a generalization of this class in Section 4.2 through the perspective of the Igusa–Todorov-function.

Throughout this section we, as usual, assume Λ is a finite dimensional algebra, though it should be noted that all the results still hold if we let Λ be an artin algebra.

Definition 3.1 (Resolving)**.** A full subcategory of an abelian category is called *resolving* if

i) It is closed under extensions.

ii) It contains the projectives.

iii) It contains the kernel of any epimorphism between two of its objects.

Note that $\mathcal{P}^{\infty}$ is a resolving subcategory.

The main theorem of this section will hold for resolving subcategories in general. In the next few propositions we will consider a resolving subcategory $\mathcal{X}$, and its Ext-orthogonal complement

$$\mathcal{Y} := \ker \operatorname{Ext}^{\geq 1}(\mathcal{X}, -) = \{Y \in \mathcal{C} \mid \operatorname{Ext}^i(X, Y) = 0, \forall X \in \mathcal{X}, \forall i \geq 1\},$$

which we now show is equal to

$$\ker \operatorname{Ext}^1(\mathcal{X}, -) = \{Y \in \mathcal{C} \mid \operatorname{Ext}^1(X, Y) = 0, \forall X \in \mathcal{X}\}.$$

Lemma 3.2. *Let $\mathcal{X}$ be a resolving subcategory. Then $\operatorname{Ext}^1(\mathcal{X}, Y) = 0$ implies that $\operatorname{Ext}^i(\mathcal{X}, Y) = 0$ for all $i \geq 1$.*

Proof. Since $\mathcal{X}$ contains the projectives, ΩX is the kernel of an epimorphism between objects in $\mathcal{X}$. Thus $\mathcal{X}$ contains all syzygies, and we have $\operatorname{Ext}^i(X, Y) = \operatorname{Ext}^1(\Omega^{i-1} X, Y) = 0$. □

Proposition 3.3. *If $\mathcal{X}$ is resolving, then $\mathcal{Y} := \ker \operatorname{Ext}^{\geq 1}(\mathcal{X}, -) = \ker \operatorname{Ext}^1(\mathcal{X}, -)$ is closed under extensions.*

Proof. Let $0 \to Y \to E \to Y' \to 0$ be an extension of objects in $\mathcal{Y}$, and let X be an object of $\mathcal{X}$. Then we get an exact sequence

$$0 = \operatorname{Ext}^i(X, Y) \longrightarrow \operatorname{Ext}^i(X, E) \longrightarrow \operatorname{Ext}^i(X, Y') = 0$$

Thus $\operatorname{Ext}^i(X, E) = 0$ for all $i \geq 1$ and E is in $\mathcal{Y}$. □

Lemma 3.4. *Let $\mathcal{X}$ be a contravariantly finite, resolving subcategory of $\operatorname{mod} \Lambda$. Then for every object $C \in \operatorname{mod} \Lambda$ there is a short exact sequence*

$$0 \to Y \to X \to C \to 0$$

with $X \to C$ minimal $\mathcal{X}$-approximation and $\operatorname{Ext}^i(\mathcal{X}, Y) = 0$ for all $i \geq 1$.

Proof. Since $\mathcal{X}$ is contravariantly finite, C has a minimal $\mathcal{X}$-approximation $X \to C$. Since $\mathcal{X}$ contains the projective cover of C this approximation must be an epimorphism. So it is part of a short exact sequence

$$0 \to Y \to X \to C \to 0.$$

Let X' be an arbitrary object in $\mathcal{X}$. Taking the long exact sequence in $\mathrm{Ext}(X', -)$ gives us

$$\begin{tikzcd}
\mathrm{Hom}(X',Y) \arrow[r] & \mathrm{Hom}(X',X) \arrow[r] & \mathrm{Hom}(X',C) \arrow[dll, rounded corners] \\
\mathrm{Ext}^1(X',Y) \arrow[r] & \mathrm{Ext}(X',X)^1 \arrow[r] & \mathrm{Ext}^1(X',C)
\end{tikzcd}$$

Since $X \to C$ is an approximation, we know that $\mathrm{Hom}(X', X) \to \mathrm{Hom}(X', C)$ is epi. Thus if we can prove that $\mathrm{Ext}^1(X', X) \to \mathrm{Ext}^1(X', C)$ is mono we would have that $\mathrm{Ext}^1(X', Y) = 0$.

Assume we have an element of $\mathrm{Ext}^1(X', X)$ that is mapped to 0, i.e. we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] & C \oplus X' \arrow[r] & X' \arrow[r] & 0
\end{tikzcd}$$

Since $\mathcal{X}$ is closed under extensions E is in $\mathcal{X}$. By composing with projection $C \oplus X' \to C$ we get a commutative triangle

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & E \arrow[dl] \\
C &
\end{tikzcd}$$

since $X \to C$ is an approximation we get that $E \to C$ factors through X. The endomorphism $X \to E \to X$ leaves the approximation unchanged, so by minimality it must be an isomorphism. Hence

$$0 \to X \to E \to X' \to 0$$

is split and $\mathrm{Ext}^1(X', X) \to \mathrm{Ext}^1(X', C)$ is injective. Thus $\mathrm{Ext}^1(X', Y) = 0$, and by Lemma 3.2 we have $\mathrm{Ext}^i(X', Y) = 0$ for all $i \geq 1$. □

We now prove the main theorem of this section, about the structure of approximations for a resolving subcategory.

Theorem 3.5. *[AR91, 3.8] Let $\mathcal{X}$ be a contravariantly finite, resolving subcategory of* $\operatorname{mod} \Lambda$. *Let X_i be the minimal approximation of S_i. Then any $X \in \mathcal{X}$ is a direct summand of an X_i-filtered module.*

Proof. The first part of the proof is to show by induction on length that any module C is in an exact sequence $0 \to Y \to X \to C \to 0$ with X X_i-filtered and $\mathrm{Ext}^1(\mathcal{X}, Y) = 0$.

For the base case if $C = S_i$ is simple then by Lemma 3.4 we have an exact sequence $0 \to Y \to X_i \to C \to 0$ with the desired properties stated above.

For the induction step, assume it holds for all modules of length less than n, and let C be a module of length n. Then by Jordan-Hölder C is the extension of two modules of length less than n. Say

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

Applying the induction hypothesis we get a diagram on the form

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & Y' & & & & Y'' & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & X' & & & & X'' & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

Taking the pullback of $X'' \to C''$ we get a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C' & \longrightarrow & E & \longrightarrow & X'' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since Y' satisfies $\mathrm{Ext}^1(\mathcal{X}, Y') = 0$ by Lemma 3.2 it also satisfies $\mathrm{Ext}^2(\mathcal{X}, Y') = 0$. In particular from the long exact sequence

$$0 = \mathrm{Ext}^1(X'', Y) \to \mathrm{Ext}^1(X'', X') \to \mathrm{Ext}^1(X'', C) \to \mathrm{Ext}^2(X'', Y) = 0$$

we get that $X' \to C'$ induces an isomorphism $\mathrm{Ext}^1(X'', X') \to \mathrm{Ext}^1(X'', C)$. Thus the short exact sequence $0 \to C' \to E \to X'' \to 0$ must come from a sequence $0 \to X' \to X \to X'' \to 0$. This gives us a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & Y' \arrow[d] & & Y'' \arrow[d] & \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

Applying the Snake Lemma we can fill out the diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since X is an extension of X_i-filtered modules, it is also X_i-filtered. Since Y is the extension of Y'' and Y' it follows from Proposition 3.3 that $\mathrm{Ext}(\mathcal{X}, Y) = 0$.

Hence any C fits into a sequence $0 \to Y \to X \to C \to 0$ with X being X_i-filtered and $\mathrm{Ext}^{\geq 1}(\mathcal{X}, Y) = 0$.

Now suppose that C is in $\mathcal{X}$, and let $0 \to Y \to X \to C \to 0$ be as before. Then we get that

$$\operatorname{Hom}(C, X) \longrightarrow \operatorname{Hom}(C, C) \longrightarrow \operatorname{Ext}^1(C, Y) = 0$$

is exact, and thus C is a direct summand of X. So every object in $\mathcal{X}$ is a direct summand of an X_i-filtered module. $\square$

Applying this to $\mathcal{P}^\infty$ we get our wanted result about the finitistic dimension.

Corollary 3.5.1. *If $\mathcal{P}^\infty$ is contravariantly finite, then the finitistic dimension is the supremum of the projective dimension of X_i. In particular it is finite.*

To finish this section of we give two examples. The first example is due to Igusa–Smalø–Todorov, which shows that $\mathcal{P}^\infty$ need not be contravariantly finite even for monomial algebras with $J^3 = 0$.

Example 3.6. [IST90, Proposition 2.3] Let Λ be the path algebra of

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

with relations $\alpha\gamma$, $\beta\gamma$, and $\gamma\alpha$ over an algebraically closed field k. Then $\operatorname{findim}(\Lambda) = 1$, but $\mathcal{P}^\infty$ is not contravariantly finite.

Proof. The indecomposable projective Λ-modules are given by the following quivers

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 & & 2 \arrow[d, "\gamma"] \\ & & 1 \end{tikzcd} \qquad\qquad \begin{tikzcd} 2 \arrow[d, "\gamma"] \\ 1 \end{tikzcd}$$

Note that both the indecomposable projectives have even dimension, so any projective module has even dimension. Then if X is a module with finite projective dimension, since $\dim X = \sum (-1)^i \dim P_X^i$ the dimension of X is also even. In particular the two simple modules have infinite projective dimension.

The radical of P_1 is $P_2 \oplus S_2$ and the radical of P_2 is S_1, so the radical of an arbitrary projective looks like $P_2^n \oplus S_1^m \oplus S_2^n$. Let $P \to X$ be the projective cover of a module with finite projective dimension. Then ΩX is a submodule of $JP = P_2^n \oplus S_1^m \oplus S_2^n$. Let M be an indecomposable summand of ΩX, and consider the composition $M \to JP \to P_2$ for any possible projection to P_2. If this is epi then we must have $M = P_2$. If none of these are epi then M is contained in $JP_2^n \oplus S_1^m \oplus S_2^n = S_1^{m+n} \oplus S_2^n$. This would mean $M = S_1$ or $M = S_2$, but S_1 and S_2 both have infinite projective dimension. Thus we must have ΩX projective, and so $\operatorname{pd} X \leq 1$.

Next we want to show that S_1 has no minimal approximation by modules with finite projective dimension. Assume for the sake of contradiction that $X \to S_1$ is such a minimal approximation. Then we claim that P_2 is not a submodule of X. If X had P_2 as a submodule, then since $\operatorname{Hom}(P_2, S_1) = 0$ the approximation would factor through $X' = X/P_2$. From the short exact sequence $0 \to P_2 \to X \to X' \to 0$ it follows that

$$\operatorname{pd} X' \leq \max\{\operatorname{pd} P_2 + 1, \operatorname{pd} X\} < \infty$$

and so X' would give an approximation of shorter length, contradicting the minimality of X.

This means that $\gamma X = 0$, because if there was an element $x \in X$ with $\gamma x \neq 0$, then $(e_2 x)$ would be a submodule of X isomorphic to P_2. So X is a $\Lambda/(\gamma)$ module.

The algebra $\Lambda/(\gamma)$ is the path algebra of the 2-Kronecker quiver, whose representation theory is well understood (c.f. [ARS97, Chapter VIII.7] or [Rin84, Chapter 3.2]). Specifically $\Lambda/(\gamma)$ can be associated with the subquiver highlighted below.

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

The indecomposable modules are as given in the table below.

$\begin{tikzcd} k^n \arrow[r, bend left, "\begin{bmatrix} I_n \\ 0 \end{bmatrix}"] \arrow[r, swap, "\begin{bmatrix} 0 \\ I_n \end{bmatrix}"] & k^{n+1} \end{tikzcd}$	$\begin{tikzcd} k^n \arrow[r, bend left, "J(n{,}\lambda)"] \arrow[r, swap, "I_n"] & k^n \end{tikzcd}$ $\begin{tikzcd} k^n \arrow[r, bend left, "I_n"] \arrow[r, swap, "J(n{,}0)"] & k^n \end{tikzcd}$	$\begin{tikzcd} k^{n+1} \arrow[r, bend left, "\begin{bmatrix} I_n & 0 \end{bmatrix}"] \arrow[r, swap, "\begin{bmatrix} 0 & I_n \end{bmatrix}"] & k^n \end{tikzcd}$
preprojective	regular	preinjective

We see that the preprojective and preinjective modules both have odd dimension, so they will have infinite projective dimension as Λ-modules. We can easily verify that the $\Lambda/(\gamma)$-modules $\begin{tikzcd} k \arrow[r, bend left, "\lambda"] \arrow[r, swap, "1"] & k \end{tikzcd}$ all have finite projective dimension as Λ-modules and that they have a nonzero map onto S_1. So each of these modules would need to have a nonzero map to X. But it is easy to verify that there is a nonzero homomorphism between the regular modules only if they have the same value of λ. So for it to be possible for X to factorize all these maps we would need X to have infinitely many direct summands. Since we are working with finitely generated modules this is impossible, hence S_1 has no approximation, and the subcategory is not contravariantly finite. □

In the next example we look at the opposite algebra of Λ, for which $\mathcal{P}^\infty$ is contravariantly finite. This shows that there is no immediate relationship between $\mathcal{P}^\infty$ being contravariantly finite for Λ and for Λ^{op}.

Example 3.7. Let Γ be the opposite algebra of the one in Example 3.6. That is, Γ is the path algebra of

$$\begin{tikzcd} 2 \arrow[r, bend left=50, "\hat{\alpha}"] \arrow[r, "\hat{\beta}"] & 1 \arrow[l, bend left=50, "\hat{\gamma}"] \end{tikzcd}$$

with relations $\hat{\gamma}\hat{\alpha}$, $\hat{\gamma}\hat{\beta}$, and $\hat{\alpha}\hat{\gamma}$. Then $\mathcal{P}^{\infty}$ is contravariantly finite. In other words the subcategory of Λ-modules with finite injective dimension is covariantly finite.

Proof. The indecomposable projective Γ-modules are given by the following quivers

$$\begin{tikzcd} 1 \arrow[d, "\hat{\gamma}"] \\ 2 \arrow[d, "\hat{\beta}"] \\ 1 \end{tikzcd} \qquad \begin{tikzcd} & 2 \arrow[dl, "\hat{\alpha}"'] \arrow[dr, "\hat{\beta}"] & \\ 1 & & 1 \end{tikzcd}$$

Similar to before, notice that the indecomposable projective modules are 3-dimensional and thus every module with finite projective dimension will have a k-dimension that is a multiple of 3. So in particular the simple modules have infinite projective dimension.

Let X be a module with finite projective dimension, and let P be its projective cover. We have that ΩX is a submodule of JP. Notice that $\hat{\alpha}J = \hat{\gamma}J = 0$, so ΩX is a $\Gamma/(\hat{\alpha}, \hat{\gamma})$-module. But $\Gamma/(\hat{\alpha}, \hat{\gamma})$ is simply isomorphic to the path algebra of $2 \longrightarrow 1$, over which there are just 3 indecomposable modules. We already know that the simple modules cannot be summands of ΩX, because they have infinite projective dimension. The non-simple module $k \xrightarrow{1} k$ is 2-dimensional and thus also has infinite projective dimension over Γ. So we conclude that $\Omega X = 0$, so X is projective.

So the only modules with finite projective dimension are the projectives themselves. In particular there are only a finite number of indecomposable modules with finite projective dimension. So the subcategory is contravariantly finite. □

4 The Igusa–Todorov functions

In this section we introduce the Igusa–Todorov functions, which are important tools for bounding the projective dimensions of modules in $\operatorname{mod}\Lambda$. The main theorem is Theorem 4.3 in which we give a bound for the projective dimension of modules in a short exact sequence. In Section 4.1 we use this to show that algebras with representation dimension at most 3, has finite finitistic dimension, and in Section 4.2 we give an example of a class of algebras which are known to have representation dimension 3.

From this point forward we let K_0 be the abelian group generated by isomorphism classes of modules in $\operatorname{mod}\Lambda$, with relations given by $[A \oplus B] - [A] - [B] = 0$ for any modules A and B, and $[P] = 0$ when P is projective. We define the linear map $L\colon K_0 \to K_0$ by $L[A] = [\Omega A]$. For any module X, we let $[\operatorname{add} X]$ be the finitely generated subgroup of K_0 generated by modules in $\operatorname{add} X$.

Fitting's lemma (Theorem A.6) tells us that there is an integer η_X such that $L\colon L^m[\operatorname{add} X] \to L^{m+1}[\operatorname{add} X]$ is an isomorphism for every $m \geq \eta_X$. We use this to define two important functions from $\operatorname{mod}\Lambda$ to $\mathbb{N}$.

Definition 4.1 (The Igusa–Todorov functions)**.** We define two functions ϕ and ψ from $\operatorname{mod}\Lambda$ to $\mathbb{N}$. For a module $M \in \operatorname{mod}\Lambda$ we define $\phi(M)$ to be the integer η_M coming from Fitting's lemma, as explained above. In other words, $\phi(M)$ is the smallest integer such that

$$L\colon L^m[\operatorname{add} M] \to L^{m+1}[\operatorname{add} M]$$

is an isomorphism for every $m \geq \phi(M)$. We define $\psi(M)$ in a similar way, but adding on an extra term to account for the structure of $\Omega^{\phi(M)}M$.

$$\psi(M) = \phi(M) + \sup\left\{\operatorname{pd} Z \;\middle|\; \operatorname{pd} Z < \infty, Z \in \operatorname{add} \Omega^{\phi(M)} M\right\}$$

We now list the properties needed to prove our main theorem.

Lemma 4.2. *[IT05, Lemma 3]*

i) $\psi(M) = \operatorname{pd} M$, *when* $\operatorname{pd} M < \infty$.

ii) $\psi(M^k) = \psi(M)$.

iii) $\psi(M) \leq \psi(M \oplus N)$.

iv) If Z *is a direct summand of* $\Omega^n(M)$ *where* $n \leq \phi(M)$ *and* $\operatorname{pd} Z < \infty$, *then* $\operatorname{pd} Z + n \leq \psi(M)$.

Proof.

i) If $\operatorname{pd} M < \infty$, then $L^m[\operatorname{add} M] \neq 0$ for $m < \operatorname{pd} M$, and $L^m[\operatorname{add} M] = 0$ for $m \geq \operatorname{pd} M$. So $\psi(M) = \phi(M) = \operatorname{pd} M$.

ii) The subcategory $\operatorname{add} M^k = \operatorname{add} M$, and ψ is defined only in terms of the additive subcategory $\operatorname{add} M$, so $\psi(M^k) = \psi(M)$.

iii) The subcategory $\operatorname{add} M$ is contained in $\operatorname{add} M \oplus N$, so if L is injective when restricted to $L^m(\operatorname{add} M \oplus N)$ then L is injective when restricted to $L^m(\operatorname{add} M)$. Thus we have $\phi(M) \leq \phi(M \oplus N)$. Further

$$\Omega^{\phi(M\oplus N)-\phi(M)}\left(\operatorname{add} \Omega^{\phi(M)} M\right) \subseteq \operatorname{add} \Omega^{\phi(M\oplus N)} M \oplus N,$$

so $\psi(M) \leq \psi(M \oplus N)$.

iv) Let $p = \operatorname{pd} Z$ and $k = \phi(M) - n$. Then $\Omega^k Z$ is in $\operatorname{add} \Omega^{\phi(M)} M$, so $\operatorname{pd} \Omega^k Z + \phi(M) \leq \psi(M)$. Thus

$$\operatorname{pd} Z + n = p + n = (p - k) + \phi(M) \leq \operatorname{pd} \Omega^k Z + \phi(M) \leq \psi(M).$$

□

We will now apply these properties to get a bound on the projective dimension of modules in a short exact sequence, in terms of the ψ-function.

Theorem 4.3. *[IT05, Theorem 4] Let $0 \to A \to B \to C \to 0$ be a short exact sequence of modules with $\operatorname{pd} C < \infty$. Then $\operatorname{pd} C \leq \psi(A \oplus B) + 1$.*

Proof. Let $P_A^\bullet$ and $P_C^\bullet$ be the minimal projective resolutions of A and C. Then we get a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & P_A^0 \arrow[r] \arrow[d] & P_A^0 \oplus P_C^0 \arrow[r] \arrow[d] & P_C^0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Applying the Snake Lemma we get $0 \to \Omega A \to \Omega B \oplus P \to \Omega C \to 0$ for some projective module P. Thus for some $n \leq \operatorname{pd} C$ we have $L^n[A] = L^n[B]$, and let n be the minimal such number. Clearly $n \leq \phi(A \oplus B)$. Let $X = \Omega^n A$, then our sequence of n-syzygies looks like

$$0 \longrightarrow X \longrightarrow X \oplus P \longrightarrow \Omega^n C \longrightarrow 0.$$

Let f be the composition $X \longrightarrow X \oplus P \xrightarrow{\pi_X} X$. Then by Fitting's Lemma (Corollary A.6.1) X decomposes as a direct sum into two summands $X = Z \oplus Y$ such that $f = f_Z \oplus f_Y$ with f_Y an isomorphism and f_Z nilpotent. In other words the sequence above can be written as

$$0 \longrightarrow Z \oplus Y \longrightarrow Z \oplus Y \oplus P \longrightarrow \Omega^n C \longrightarrow 0.$$

with the left map being

$$\begin{bmatrix} f_Z & 0 \\ 0 & f_Y \\ * & * \end{bmatrix} \sim \begin{bmatrix} f_Z & 0 \\ 0 & 1_Y \\ * & 0 \end{bmatrix}$$

So by changing basis this restricts to another short exact sequence

$$0 \longrightarrow Z \longrightarrow Z \oplus P \longrightarrow \Omega^n C \longrightarrow 0.$$

Let $T = \Lambda/J$ and apply the long exact sequence in $\mathrm{Ext}(-,T)$. Then we get an exact sequence

$$\mathrm{Ext}^k(Z,T) \longrightarrow \mathrm{Ext}^k(Z \oplus P,T) \longrightarrow \mathrm{Ext}^{k+1}(\Omega^n C,T)$$

where the left map is induced by f_Z since $\mathrm{Ext}^k(Z \oplus P,T) \cong \mathrm{Ext}^k(Z,T)$. Now since f_Z is nilpotent the induced map is surjective if and only if $\mathrm{Ext}^k(Z,T) = 0$. We know that, since $\Omega^n C$ has finite projective dimension, $\mathrm{Ext}^{k+1}(\Omega^n C,T)$ is 0 for k large enough. Then we must have that $\mathrm{Ext}^k(Z,T) = 0$, and thus Z has finite projective dimension. Specifically we have bounds given by $\operatorname{pd}\Omega^n C - 1 \leq \operatorname{pd} Z \leq \operatorname{pd}\Omega^n C$.

Since Z is a direct summand of $\Omega^n(A \oplus B)$, by Lemma 4.2 we have that $\operatorname{pd} Z + n \leq \psi(A \oplus B)$, and thus $\operatorname{pd}\Omega^n C - 1 + n = \operatorname{pd} C - 1 \leq \psi(A \oplus B)$. $\square$

With a bit of diagram chasing we can extend this theorem to get a bound for $\operatorname{pd} A$ and $\operatorname{pd} B$ as well.

Corollary 4.3.1. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of modules.*

i) If $\operatorname{pd} A < \infty$, then $\operatorname{pd} A \leq \psi(\Omega B \oplus \Omega C) + 1$.

ii) If $\operatorname{pd} B < \infty$ then $\operatorname{pd} B \leq \psi(\Omega A \oplus \Omega^2 C) + 2$.

Proof. Let $P_B \to B$ be a projective cover of B. Then we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & P_B \arrow[r, equal] \arrow[d] & P_B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Applying the Snake Lemma we get a short exact sequence

$$0 \longrightarrow \Omega B \longrightarrow \Omega C \oplus P \longrightarrow A \longrightarrow 0$$

for some projective module P. Then using the theorem we have that if $\operatorname{pd} A \leq \infty$, then $\operatorname{pd} A \leq \psi(\Omega B \oplus \Omega C \oplus P) + 1 = \psi(\Omega B \oplus \Omega C) + 1$.

Applying the same reasoning to $0 \to \Omega B \to \Omega C \oplus P \to A \to 0$ gives us that if $\operatorname{pd} B \leq \infty$, then $\operatorname{pd} \Omega B \leq \psi(\Omega A \oplus \Omega^2 C) + 1$. Hence we get that $\operatorname{pd} B \leq \psi(\Omega A \oplus \Omega^2 C) + 2$. □

These are all the results we need about the Igusa–Todorov functions. We will now use them to find families of algebras with $\operatorname{findim}(\Lambda) < \infty$.

4.1 Representation dimension

In this section we look at the representation dimension of an algebra. This is another useful homological invariant of the representation theory for a finite dimensional algebra. The representation dimension is less than or equal to 2 if and only if Λ is representation finite, so it is natural to think that the representation dimension in some sense measures the *complexity* of $\operatorname{mod} \Lambda$. In Corollary 4.9.1 we show that $\operatorname{findim}(\Lambda) < \infty$ when $\operatorname{repdim}(\Lambda) \leq 3$, and in Section 4.2 we give an example of a family of algebras that satisfy this.

Definition 4.4 (Representation dimension)**.** Let Λ be a finite dimensional algebra. The *representation dimension* of Λ, denoted $\operatorname{repdim}(\Lambda)$, is the minimal global dimension of $\operatorname{End}(M)^{\mathrm{op}}$ for M a generator-cogenerator in $\operatorname{mod} \Lambda$. We call a generator-cogenerator that achieves this minimum an *Auslander-generator.*

The representation dimension can also be defined using $\mathcal{M}$-resolutions, which we define here.

Definition 4.5 ($\mathcal{M}$-resolutions)**.** Let X be an object in $\operatorname{mod}\Lambda$ and $\mathcal{M}$ a contravariantly finite subcategory. We consider a diagram as the one below.

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr, two heads] & M_2 \arrow[r] \arrow[dr, two heads] & M_1 \arrow[r] \arrow[dr, two heads] & M_0 \arrow[dr, two heads] & \\
& \Omega^3_M X \arrow[u, hook] & \Omega^2_M X \arrow[u, hook] & \Omega_M X \arrow[u, hook] & X
\end{tikzcd}$$

If the maps $M_n \twoheadrightarrow \Omega^n_M X$ are minimal right $\mathcal{M}$-approximations for $n \geq 0$ (they need not be surjective), and $\Omega^{n+1}_M \hookrightarrow M_n$ are their kernels, then this is a minimal $\mathcal{M}$*-resolution* of X. The $\mathcal{M}$*-res-dimension* of X is the length of this sequence of (nonzero) M_i's, and the $\mathcal{M}$-res-dimension of Λ is the supremum of the dimension on its objects.

An $\mathcal{M}$-resolution of X should be thought of as a projective resolution of $\operatorname{Hom}(-, X)|_{\mathcal{M}}$ in the category of coherent functors on $\mathcal{M}$. When $\mathcal{M} = \operatorname{add} M$ the category of coherent functors is isomorphic to $\operatorname{mod}\operatorname{End}(M)^{\mathrm{op}}$, where $\operatorname{Hom}(-, X)|_{\mathcal{M}}$ corresponds to $\operatorname{Hom}(M, X)$. In the proof of the next proposition we use this correspondence, and we write M-res-dim instead of $(\operatorname{add} M)$-res-dim.

Proposition 4.6. *If the representation dimension of Λ is at least 2, then* $\operatorname{repdim}(\Lambda) - 2$ *equals the minimum of* M-res-dim$(\operatorname{mod}\Lambda)$ *for M a generator-cogenerator. In fact, for any generator-cogenerator,* M-res-dim$(\operatorname{mod}\Lambda)$ *is two less than the global dimension of* $\operatorname{End}(M)^{\mathrm{op}}$*.*

Proof. Let M be a generator-cogenerator. We first show that the global dimension of $\operatorname{End}(M)^{\mathrm{op}}$ is less than or equal to M-res-dim$(\operatorname{mod}\Lambda) + 2$.

The functor $\operatorname{Hom}(M, -)$ is an equivalence from $\operatorname{add} M$ to $\operatorname{proj}\operatorname{End}(M)^{\mathrm{op}}$, which maps minimal M-approximations to projective covers. Let X be any module in $\operatorname{mod}\operatorname{End}(M)^{\mathrm{op}}$ with projective dimension at least 2. Then it has a projective presentation

$$\Omega^2 X \longrightarrow (M, M_1) \longrightarrow (M, M_0) \longrightarrow X.$$

Because of the equivalence this is induced by a map $f\colon M_1 \to M_0$. Since $\operatorname{Hom}(M,-)$ is left exact we have that $\Omega^2 X \cong \operatorname{Hom}(M, \ker f)$, and so the projective dimension of X is 2 more than the resolution dimension of $\ker f$ with respect to M. Hence we have that

$$\operatorname{gl.dim} \operatorname{End}(M)^{\mathrm{op}} \leq M\text{-res-dim}(\operatorname{mod} \Lambda) + 2.$$

Next we prove the other inequality.

Since M is a cogenerator, any module Y in $\operatorname{mod} \Lambda$ has a copresentation

$$0 \longrightarrow Y \longrightarrow M_0 \xrightarrow{f} M_1.$$

Applying $(M,-) := \operatorname{Hom}(M,-)$ we get

$$0 \longrightarrow (M,Y) \longrightarrow (M,M_0) \xrightarrow{(M,f)} (M,M_1) \longrightarrow \operatorname{cok}(M,f) \longrightarrow 0.$$

If the projective dimension of $\operatorname{cok}(M,f)$ is less than 2, then (M,Y) is a direct summand of (M,M_0). This means that $(M,Y) \cong (M,M')$, so the minimal M-approximation of Y is M', and $(M, \Omega_M Y) = 0$. Since M is a generator this means $\Omega_M Y = 0$ and thus $M\text{-res-dim}(Y) = 0$.

So provided the projective dimension of $\operatorname{cok}(M,f)$ is larger than or equal to 2, it equals $M\text{-res-dim}(Y) + 2$. In particular the global dimension of $\operatorname{End}(M)^{\mathrm{op}}$ is larger than or equal to M-res-dim$(\operatorname{mod} \Lambda) + 2$. Hence they are equal. □

The next two results paint an important picture of the representation dimension as an invariant, but are not relevant for the other results in this thesis.

Theorem 4.7. *The representation dimension of an artin algebra is always finite.*

Proof. The proof is omitted here, but can be found in [Iya02]. . □

I should go over proof

Proposition 4.8. *The representation dimension of Λ is less than or equal to 2 if and only if Λ is representation finite.*

Proof. Assume Λ is representation finite and let M be the direct sum of all indecomposable modules up to isomorphism. Then M is a generator-cogenerator. Let X be an $\mathrm{End}(M)^{\mathrm{op}}$-module with projective presentation

$$(M, M_1) \to (M, M_0) \to X \to 0.$$

Let M_2 be the kernel of $M_1 \to M_0$. Since M is the sum of all indecomposables M_2 is in $\operatorname{add} M$, so

$$0 \to (M, M_2) \to (M, M_1) \to (M, M_0) \to X \to 0$$

is a projective resolution of X. So Λ has representation dimension at most 2.

Assume Λ has representation dimension at most 2, and let M be an Auslander-generator. We want to show that $\operatorname{add} M = \operatorname{mod} \Lambda$. Let X be any Λ-module, and let

$$0 \to X \to I_0 \to I_1$$

be a minimal injective presentation. If $I_0 \to I_1$ is split then X is injective and thus in $\operatorname{add} M$. Let M_X be a minimal M-approximation of X, let $\Omega_M X$ be the kernel of the approximation, and let Y be the cokernel of $(M, I_0) \to (M, I_1)$. Then

$$(M, \Omega_M X) \to (M, M_X) \to (M, I_0) \to (M, I_1) \to Y \to 0$$

is a minimal exact sequence. Since the global dimension of $\mathrm{End}(M)^{\mathrm{op}}$ is at most 2 this means that $(M, \Omega_M X) = 0$. Consequently we have that $\Omega_M X = 0$ and that $X = M_X$, so X is in $\operatorname{add} M$. Thus Λ is representation finite. □

We conclude this subsection by proving that $\operatorname{findim}(\Lambda)$ is finite when Λ has representation dimension at most 3. To do this we first prove a slight generalization of this.

Theorem 4.9. *[IT05, Corollary 8] If $\Lambda = \mathrm{End}_\Gamma(P)^{\mathrm{op}}$ for an algebra Γ with global dimension at most 3, and P projective, then* $\mathrm{findim}(\Lambda) < \infty$.

Proof. Let X be any Λ-module with finite projective dimension. Then it has a projective presentation $(P, P_1) \to (P, P_0) \to X \to 0$ where $(P, P_i) = \mathrm{Hom}_\Gamma(P, P_i)$ with $P_i \in \mathrm{add}\, P$. Since $(P, -)$ is an equivalence from $\mathrm{add}\, P$ to $\mathrm{proj}\, \Lambda$ this corresponds to a map $P_1 \to P_0$ which we can extend to a projective resolution in Γ:

$$0 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0.$$

Applying the exact functor $(P, -)$, we get an exact sequence

$$0 \longrightarrow (P, P_3) \longrightarrow (P, P_2) \longrightarrow (P, P_1) \longrightarrow (P, P_0) \longrightarrow X \longrightarrow 0.$$

Truncating this we get a short exact sequence

$$0 \longrightarrow (P, P_3) \longrightarrow (P, P_2) \longrightarrow \Omega^2 X \longrightarrow 0.$$

Then by Theorem 4.3 the projective dimension of $\Omega^2 X$ is bounded by $\psi((P, P_3) \oplus (P, P_2)) + 1$. Which means

$$\mathrm{pd}\, X \leq \psi((P, P_3) \oplus (P, P_2)) + 3 \leq \psi((P, \Gamma)) + 3$$

Since this bound doesn't depend on X, Λ has finite finitistic dimension. □

Corollary 4.9.1. *If* $\mathrm{repdim}(\Lambda) \leq 3$ *then* $\mathrm{findim}(\Lambda) < \infty$.

Proof. If Λ has rep-dimension less than or equal to 3, then there is a generator-cogenerator M in $\mathrm{mod}\, \Lambda$ such that $\Gamma := \mathrm{End}_\Lambda(M)^{\mathrm{op}}$ has global dimension 3 or less. Then since M is a generator Λ is in $\operatorname{add} M$ and so $\operatorname{Hom}_\Lambda(M, \Lambda)$ is a projective Γ-module with

$$\operatorname{End}_\Gamma(\operatorname{Hom}_\Lambda(M, \Lambda))^{\mathrm{op}} = \operatorname{End}_\Lambda(\Lambda)^{\mathrm{op}} = \Lambda.$$

□

4.2 Stably hereditary algebras

In this section we introduce the class of stably hereditary algebras, and show that they have representation dimension at most 3. Then from what we showed earlier in this section it follows that they have finite finitistic dimension.

Hereditary algebras are those where all torsionfree modules are projective. This corresponds exactly to the algebra having global dimension 1 or less. Stably hereditary algebras are a generalization of these where we also allow simple modules to be torsionfree without being projective. This turns out to include the class of algebras that are stably equivalent to a hereditary algebra, hence the name. We now remind the reader of the definition of torsionfree.

Definition 4.10 ((co)torsionfree)**.** A module is called *torsionfree* if it is a submodule of a projective module. Dually, a module is called *cotorsionfree* if it is a factor module of an injective.

Defining hereditary algebras to be those where cotorsionfree modules are injective would give an equivalent definition. When we generalize to stably hereditary algebras, the dual condition is no longer equivalent, so we include both.

Definition 4.11 (Stably hereditary algebra)**.** An algebra is called *stably hereditary* if any indecomposable torsionfree module is projective or simple, and any indecomposable cotorsionfree moule is injective or simple.

Like we said above, the archetypal example of a stably hereditary algebra is one whose stably equivalent to a hereditary algebra. Two algebras being stably equivalent means they have the same stable category. We now remind the reader of the definition.

Definition 4.12 (The stable category)**.** For an algebra Λ, *the stable category* $\underline{\mathrm{mod}}\Lambda$ has the same objects as $\mathrm{mod}\,\Lambda$, but the sets of homomorphisms are given by

$$\mathrm{Hom}_{\underline{\mathrm{mod}}\Lambda}(M,N) = \mathrm{Hom}_{\Lambda}(M,N)/\mathcal{P}(M,N)$$

where $\mathcal{P}(M,N)$ is the ideal of all morphisms factoring through a projective.

Proposition 4.13. *If for an algebra Λ there is a hereditary algebra H such that $\underline{\mathrm{mod}}\Lambda \cong \underline{\mathrm{mod}}H$ then Λ is stably hereditary.*

Proof. The proof is omitted here, but can be found in [AR73, Chapter IV, Theorem 1.5]. □

There exists stably hereditary algebras that are not stably equivalent to a hereditary algebra, but the simple defining property of stably hereditary algebras together with the Igusa–Todorov function is all we need to prove our main theorem.

Theorem 4.14. *[Xi02, Theorem 3.5] If Λ is stably hereditary, then it has representation dimension at most 3.*

Proof. By Proposition 4.6 it is enough to find a generator-cogenerator V such that $V\text{-res-dim}(\Lambda) \leq 1$.

Let V be the direct sum of all the indecomposable projective, all the indecomposable injective, and all the simple modules. Then V is a generator-cogenerator. So we just need to show that $V\text{-res-dim}(\Lambda) \leq 1$.

In other words we need to show that for any Λ-module M there is a short exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow M \longrightarrow 0$$

with V_i in $\operatorname{add} V$, and such that

$$0 \longrightarrow (V, V_1) \longrightarrow (V, V_0) \longrightarrow (V, M) \longrightarrow 0$$

is exact.

To construct V_1 and V_0 let M' be the sum of the maximal injective summand of M and the socle of M. Then let P be the projective cover of M/M'. Taking the pullback of $M \to M/M' \leftarrow P$ gives us the diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & M' \oplus P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] \arrow[d] & M/M' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We claim that $0 \to K \to M' \oplus P \to M \to 0$ is the desired sequence. Firstly $M' \oplus P$ is in $\operatorname{add} V$ since it is the sum of an injective, a semisimple, and a projective module. Further K is a submodule of P, hence torsionfree. So since Λ is stably hereditary K is the sum of a projective and a semisimple module, so K is also in $\operatorname{add} V$.

Next we need to show that

$$0 \longrightarrow (V, K) \longrightarrow (V, M' \oplus P) \longrightarrow (V, M) \longrightarrow 0$$

is exact. The only thing needed to show here is that the map $(V, M' \oplus P) \to (V, M)$ is surjective. We do this by showing that $(W, M' \oplus P) \to (W, M)$ is surjective for any indecomposable summand W of V. If W is projective, this holds by definition. If W is simple, then any map from W to M factors through the socle and hence through M', so it's surjective. Lastly if W is injective, then the image of W in M is a cotorsionfree module, so it is the sum of simple modules and an injective module. Hence the map from W to M factors through M'.

This shows that V-res-dim$(\Lambda) \leq 1$ and thus that repdim$(\Lambda) \leq 3$. $\square$

Now because of Corollary 4.9.1 this means that findim$(\Lambda) < \infty$ whenever Λ is stably hereditary.

4.3 Special biserial algebras

In this section we shall consider two finite dimensional algebras, with a homomorphism between them. We denote these by Λ and Γ, and we denote their radicals by J_Λ and J_Γ respectively.

The goal of the section is to show that special biserial algebras have representation dimension less than or equal to 3, and consequently that they have finite finitistic dimension. We do this in several parts. In Theorem 4.19 we show that an algebra that has a radical embedding into a representation finite algebra has representation dimension at most 3. In Theorem 4.20 and Proposition 4.22 we show that for every special biserial algebra there is a string algebra with larger representation dimension. Lastly in Theorem 4.23 we construct a radical embedding of any string algebra into a representation finite algebra.

First we discuss some general properties of homomorphisms of algebras.

Definition 4.15 (Coinduced module)**.** Given a homomorphism of algebras $\psi\colon \Lambda \to \Gamma$ we can consider every Γ-module as a Λ-module, where multiplication by λ is given by multiplication with $\psi(\lambda)$. This defines a functor $\operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$ known as *restriction of scalars*. The right adjoint to this functor is called the *coinduction functor*. For a Λ-module M the coinduced module is the Γ-module defined as

$$M' := \mathrm{Hom}_\Lambda(\Gamma, M)$$

where we consider Γ as a Λ-Γ-bimodule through restriction of scalars. If we identify M with $\mathrm{Hom}_\Lambda(\Lambda, M)$ then the counit of the adjunction is given by precomposing with ψ. Specifically we get the map

$$M' \xrightarrow{\ \varepsilon_M\ } M$$

$$f \longmapsto f(\psi(1)) = f(1).$$

Proposition 4.16. *[EHIS04, Lemma 2.2] The coinduced functor as defined above is the right adjoint to restriction of scalars, and ε is the counit.*

Proof. Let M be a Λ-module and let N be a Γ-module. Then we get an isomoprhism from the Hom-Tensor adjunction

$$\mathrm{Hom}_\Gamma(N, \mathrm{Hom}_\Lambda(\Gamma, M)) \cong \mathrm{Hom}_\Lambda(\Gamma \otimes_\Gamma N, M).$$

Notice that ${}_\Lambda\Gamma \otimes_\Gamma N \cong {}_\Lambda N$ is exactly restriction of scalars. Further the counit $\Gamma \otimes_\Gamma M' = M' \to M$ is given by $f \mapsto f(1)$, which is exactly how we defined ε above. $\square$

Next, in preperation for Theorem 4.19, we restrict to the case where ψ is the inclusion of a radical embbeding.

Definition 4.17 (Radical embedding)**.** A subalgebra $\Lambda \subseteq \Gamma$ is called a *radical embedding* if the two radicals coincide, $J_\Lambda = J_\Gamma$.

Lemma 4.18. *[EHIS04, Lemma 2.3] If $\Lambda \subseteq \Gamma$ is a radical embedding, then $\ker \varepsilon_M$ and $\operatorname{cok} \varepsilon_M$ are both semisimple for any Λ-module M.*

Proof. If we apply $\mathrm{Hom}_\Lambda(-, M)$ to the short exact sequence of Λ-modules
$0 \longrightarrow \Lambda \xrightarrow{\ \psi\ } \Gamma \longrightarrow \Gamma/\Lambda \longrightarrow 0,$ we get

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\Gamma/\Lambda, M) \arrow[r] & M' \arrow[r, "\varepsilon_M"] & M \arrow[r] \arrow[dr, two heads] & \mathrm{Ext}^1(\Gamma/\Lambda, M) \\
 & & & & \operatorname{cok} \varepsilon_M \arrow[u, hook]
\end{tikzcd}$$

Thus $\mathrm{Hom}(\Gamma/\Lambda, M)$ is the kernel of ε_M and the cokernel is a submodule of $\mathrm{Ext}^1(\Gamma/\Lambda, M)$. Since $J_\Gamma = J_\Lambda \subseteq \Lambda$ we have that $(\Gamma/\Lambda)J_\Lambda = 0$. Thus $J_\Lambda \mathrm{Hom}(\Gamma/\Lambda, M)$ and $J_\Lambda \mathrm{Ext}^1(\Gamma/\Lambda, M)$ are both 0, which means they are both semisimple. Since $\operatorname{cok} \varepsilon_M$ is a submodule of $\mathrm{Ext}^1(\Gamma/\Lambda, M)$, it is also semisimple. $\square$

We now use the radical embedding to say somethign about the representation dimension of Λ.

INTERESTINGQUESTION: what happens if we replace the hypothesis Γ-repfinite with repdim of $\Gamma = n - 1$. Same proof should give repdim $\Lambda = n$ except there might be problem with the exactness of the sequence since there are more Λ-linear maps. Find counterexample

Theorem 4.19. *[EHIS04, Theorem 1.1] If Γ is representation finite and $\Lambda \subseteq \Gamma$ is a radical embedding, then the representation dimension of Λ is at most 3.*

Proof. Since Γ is representation finite there is a finite set of indecomposable Γ-modules up to isomorphism. Let X be the direct sum of all of these. Since Λ is a subalgebra of Γ we can consider X as a Λ-module. Now define V to be $\Lambda \oplus D\Lambda \oplus X$, i.e. V is the sum of all projective Λ-modules, all injective Λ-modules, and all Γ-modules. We claim that V-res-dim$(\Lambda) \leq 1$, which by Proposition 4.6 would imply that $\mathrm{repdim}(\Lambda) \leq 3$.

As in Theorem 4.14 we do this by showing that for any Λ-module M there is a short exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow M \longrightarrow 0$$

with V_i in $\operatorname{add} V$, such that

$$0 \longrightarrow (V, V_1) \longrightarrow (V, V_0) \longrightarrow (V, M) \longrightarrow 0$$

is exact.

Now let M be any Λ-module. If M is injective, then M is in $\operatorname{add} V$, and so we may simply choose $V_2 = M$ and $V_1 = 0$. From here on out assume that M has no injective summands.

Let M' be the coinduced module of M, and $\varepsilon_M \colon M' \to M$ be the counit map. Now if we let P be the projective cover of $\operatorname{cok}\varepsilon_M$, then by lifting the map $P \to \operatorname{cok}\varepsilon_M$ we get a surjective map $M' \oplus P \to M$. Since M' is a Γ-module and P is projective $M' \oplus P$ is in $\operatorname{add} V$. We let this be our V_0.

Next, we let V_1 be the kernel of the map $V_0 \to M$. Then we wish to show that this is in $\operatorname{add} V$. Since $M \to \operatorname{cok}\varepsilon_M$ is an epimorphism and $P \to \operatorname{cok}\varepsilon_M$ is a projective cover, we can lift this to a morphism $P \to M$. Taking the pullback along $\operatorname{Im}\varepsilon_M \to M$ we get a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r] \arrow[d] & \operatorname{cok}\varepsilon_M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Im}\varepsilon_M \arrow[r] & M \arrow[r] & \operatorname{cok}\varepsilon_M \arrow[r] & 0
\end{tikzcd}$$

By Lemma 4.18 we have that $\operatorname{cok}\varepsilon_M$ is semisimple, and thus $K = J_\Lambda P$. Since $J_\Lambda = J_\Gamma$ this means that $J_\Lambda P$ is a Γ-module, and thus is in $\operatorname{add} V$. Next we take the pullback again, this time along $M' \to \operatorname{Im}\varepsilon_M$.

$$\begin{tikzcd}
0 \arrow[r] & \ker\varepsilon_M \arrow[r] \arrow[d, equal] & M' \prod_M J_\Lambda P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & J_\Lambda P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\varepsilon_M \arrow[r] & M' \arrow[r] & \operatorname{Im}\varepsilon_M \arrow[r] & 0
\end{tikzcd}$$

Notice that $M' \prod_M J_\Lambda P = M' \prod_M P$, which is the kernel of $V_0 \to M$. In other words it is equal to V_1.

Since $J_\Lambda P$ is a Γ-module we get a map of abelian groups by postcomposing with ε_M:

$$\begin{aligned} \mathrm{Hom}_\Gamma(J_\Lambda P, M') &\xrightarrow{\varepsilon_M \circ -} \mathrm{Hom}_\Lambda(J_\Lambda P, M) \\ f &\longmapsto (p \mapsto f(p)(1)) \end{aligned}$$

This is excatly the isomoprhism of the adjuntion between restriction of scalars and the coinduction functor in Proposition 4.16.

In other words the map $J_\Lambda P \to \operatorname{Im} \varepsilon_M$ factorizes through M'. Then using the pullback property, we get that the map $V_1 \to J_\Lambda P$ splits, and so $V_1 = \ker \varepsilon_M \oplus J_\Lambda P$.

We have already established that $J_\Lambda P$ is a Γ-module. By Lemma 4.18 we have that $\ker \varepsilon_M$ is semisimple. We now show that $\ker \varepsilon_M$ is in $\operatorname{add} V$, by showing that all simple modules are.

Let S be a simple Λ-module, and let e be an idempotent such that $S \cong \Lambda e / J_\Lambda e$. We have a semisimple Γ-module $\hat{S} := \Gamma e / J_\Gamma e$ that contains S. Since $J_\Gamma = J_\Lambda$ we have that $\hat{S}$ is also semisimple as a Λ-module. Thus S is a direct summand of $\hat{S}$. Since $\hat{S}$ is in $\operatorname{add} V$, we get that S is as well. Thus V_1 is in $\operatorname{add} V$.

Lastly we show that we get an exact sequence

$$0 \longrightarrow (V, V_1) \longrightarrow (V, V_2) \longrightarrow (V, M) \longrightarrow 0.$$

The only thing we need to show is that the last map is surjective. We do this by verifying the three cases for an indecomposable summand of V. Firstly let W be a Γ-module. Then $\mathrm{Hom}_\Lambda(W, V_2)$ breaks up as a direct sum into $\mathrm{Hom}_\Lambda(W, M') \oplus \mathrm{Hom}_\Lambda(W, P)$. We saw in Proposition 4.16 that the composition $\mathrm{Hom}_\Gamma(W, M') \overset{\subseteq}{\to} \mathrm{Hom}_\Lambda(W, M') \to \mathrm{Hom}_\Lambda(W, M)$

is an isomorphism. Thus the map $\mathrm{Hom}_\Gamma(W, M') \to \mathrm{Hom}_\Lambda(W, M)$ is surjective.

If W is projective, then $\mathrm{Hom}_\Lambda(W, -)$ is exact, and there is nothing we need to show.

If W is an indecomposable injective, since we assumed M had no injective summands, a map $W \to M$ cannot be injective. This means that it factors through $W/\operatorname{soc}(W)$. Since $D(W/\operatorname{soc}(W)) = (DW)J_\Lambda = (DW)J_\Gamma$ this means that $W/\operatorname{soc}(W)$ is a Γ-module. Then from the argument above it follows that the map is surjective.

This shows that V-res-dim$(\Lambda) \leq 1$, and thus the representation dimension of Λ is at most 3. □

Now we move away from the case where ψ is a radical embedding, and instead look at a specific quotient map.

Theorem 4.20. *[EHIS04, Proposition 1.2] Let Λ be a basic finite dimensional algebra and let P be a basic projective-injective Λ-module. Then the socle of P is a two-sided ideal, which allows us to define the ring $\Gamma := \Lambda/\operatorname{soc} P$. Then we have that* $\operatorname{repdim}(\Lambda) \leq \max\{2, \operatorname{repdim}(\Gamma)\}$.

Proof. First we show that the socle of P is a two-sided ideal. Multiplication on the right defines a homomorphism $- \cdot \lambda \colon \Lambda \to \Lambda$. Any homomorphism maps the socle to the socle, so $(\operatorname{soc} P) \cdot \lambda \subseteq \operatorname{soc} \Lambda$. Now let $s \in \operatorname{soc} P$ be some element such that $s\lambda$ is non-zero. Then the injective envelope $I(s)$ is a direct summand of P and thus projective-injective. Further since $- \cdot \lambda \colon (s) \to (s\lambda)$ is an injective map, $I(s)$ is mapped injectively into Λ by $- \cdot \lambda$, which means $- \cdot \lambda \colon I(s) \to \Lambda$ splits. Since Λ is basic this means that $I(s)\lambda \subseteq P$, and thus $s\lambda \in \operatorname{soc} P$, so the socle of P is a two-sided ideal.

Next we note that any indecomposable Λ-module is either a Γ-module, or a direct summand of P. To see this, let M be any indecomposable Λ-module and consider $(\operatorname{soc} P)M$. If this is zero, then M is a Γ-module. If on the other hand there is some $s \in \operatorname{soc} P$ and $m \in M$ such that $sm \neq 0$, then let $I(s)$ be the injective envelope of s and let e be the idempotent such that $I(s) = \Lambda e$. Then we get a map $I(s) \to M$ which maps λe to λem. Since $sm \neq 0$ this maps the socle of $I(s)$ injectively. Now, since $I(s)$ is injective this mean that $I(s)$ is a direct summand of M. Since M is indecomposable we have that $M \cong I(s)$, and thus M is a direct summand of P.

Now we show that $\operatorname{repdim}(\Lambda) \leq \max\{2, \operatorname{repdim}(\Gamma)\}$. By Proposition 4.6 it suffices to find a generator-cogenerator V such that V-res-dim$(\operatorname{mod}\Lambda) \leq \max\{0, \operatorname{repdim}(\Gamma) - 2\}$. Let N be the generator-cogenerator in $\operatorname{mod}\Gamma$ that achieves the minimal resolution dimension. Then we claim $V = N \oplus P$ is our desired generator-cogenerator. This is a generator-cogenerator because any indecomposable projective or injective module that is not a summand of P will be a summand of N, since all Λ-modules that are not summands of P are Γ-modules.

To show that V-res-dim$(\operatorname{mod}\Lambda) \leq \max\{0, \operatorname{repdim}(\Gamma) - 2\}$ we explicitly construct the resolutions. Let M be an indecomposable Λ-module. Then we wish to construct an exact sequence

$$0 \longrightarrow V_n \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow M \longrightarrow 0$$

such that V_i is in $\operatorname{add} V$, $n \leq \max\{0, \operatorname{repdim}(\Gamma) - 2\}$, and $\operatorname{Hom}(V, -)$ is exact on the sequence. If M is a summand of P we may choose $V_0 = M$ and $V_i = 0$ for $i > 0$.

If M is not a summand of P then M is a Γ-module. Then we already have an exact sequence

$$0 \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 \longrightarrow M \longrightarrow 0$$

with $N_i \in \operatorname{add} N$. Since $\Lambda \to \Gamma$ is surjective we get that $\operatorname{Hom}_\Lambda(N, -) = \operatorname{Hom}_\Gamma(N, -)$ on Γ-modules. So if we apply $\operatorname{Hom}_\Lambda(N, -)$ to the sequence it remains exact. Lastly since $\operatorname{Hom}(V, -) = \operatorname{Hom}(N, -) \oplus \operatorname{Hom}(P, -)$ and $\operatorname{Hom}(P, -)$ is an exact functor, if we apply $\operatorname{Hom}(V, -)$ to the sequences it still remains exact. Thus V-res-dim(mod Λ) $\leq \max\{0, \operatorname{repdim}(\Gamma) - 2\}$ and $\operatorname{repdim}(\Lambda) \leq \max\{2, \operatorname{repdim}(\Gamma)\}$. $\square$

We now give the definition of special biserial algebras, and string algebras.

Definition 4.21 (Special biserial algebra)**.** A finite dimensional algebra Λ is called *special biserial* if it is isomorphic to a path algebra kQ/I such that

i) Each vertex in Q is the initial vertex for at most two arrows, and the terminal vertex for at most two arrows.

ii) For any arrow β in Q there is at most on arrow α such that $\alpha\beta \notin I$ and at most one arrow γ such that $\beta\gamma \notin I$.

A special biserial algebra is called a *string algebra* if it is also monomial. I.e. I is generated by paths.

We now show that given a special biserial algebra we can always construct a string algebra, by moding out socles of projective injective modules like we do in Theorem 4.20.

Proposition 4.22. *If $\Lambda = kQ/I$ is special biserial, then I is generated by monomial and binomial relations. Further if $\gamma + t\gamma'$ is a binomial relation such that $\gamma \notin I$, then (γ) is the socle of a projective-injective module.*

Proof. Let ρ be a relation. Then we may assume ρ is some linear combinations of paths which start in the same vertex and end in the same vertex. Assume by induction that ρ is a combination of n distinct paths for some $n \geq 3$, and let γ^1, γ^2, and γ^3 be three of those paths. Write each path as a composition of arrows $\gamma^1 = \alpha^1_{t_1} \cdots \alpha^1_1\alpha^1_0$, $\gamma^2 = \alpha^2_{t_2} \cdots \alpha^2_1\alpha^2_0$, and $\gamma^3 = \alpha^3_{t_3} \cdots \alpha^3_1\alpha^3_0$.

Since there can be at most two arrows out of any vertex, it cannot be the case that α^1_0, α^2_0, and α^3_0 are all distinct. Let us assume $\alpha^1_0 = \alpha^2_0$. Since we assume γ^1 and γ^2 are distinct there must be a smallest k such that

$\alpha_k^1 \neq \alpha_k^2$. But then it must be the case that either $\alpha_k^1\alpha_{k-1}^1$ or $\alpha_k^2\alpha_{k-1}^1$ is a relation. That means that either γ^1 or γ^2 is a relation. Thus ρ is the sum of a monomial relation and a relation that is the linear combination of $(n-1)$ paths. Then by induction each relation in I is the sum of binomial relations.

Now let $\gamma + t\gamma'$ be a binomial relation such that $\gamma \notin I$. Let i be the origin vertex of γ, let j be the terminal vertex, and let e_i and e_j be the corresponding idempotents. Then we claim that Λe_i is projective-injective, and that (γ) is its socle.

As above decompose the two paths into a product of arrows $\gamma = \alpha_t \cdots \alpha_1\alpha_0$ and $\gamma' = \alpha'_{t'} \cdots \alpha_1\alpha_0$, and let k be the smallest integer such that $\alpha_k \neq \alpha'_k$. If k is bigger than 0, then as before we get that either $\alpha_k\alpha_{k-1}$ or $\alpha'_k\alpha_{k-1}$ is a relation. Consequently both γ and γ' would be relations contradicting our assumption. Similarly if we let k be the smallest integer such that $\alpha_{t-k} \neq \alpha'_{t'-k}$ we get that k cannot be bigger than 0, by exactly the same argument. This means that $\alpha_0 \neq \alpha'_0$ and that $\alpha_t \neq \alpha'_{t'}$, which will be important later.

We show that (γ) is simple, by showing that $\alpha\gamma$ is a relation for every arrow α. We have that $\alpha(\gamma + t\gamma')$ is a relation. Since $\alpha_t \neq \alpha'_{t'}$ we have that either $\alpha\alpha_t = 0$ or $\alpha\alpha'_{t'} = 0$. If $\alpha\alpha_t = 0$, then $\alpha\gamma = 0$ and we are done. If $\alpha\alpha'_{t'} = 0$, then $\alpha\gamma' = 0$ which means that $\alpha\gamma = \alpha(\gamma + t\gamma') - t\alpha\gamma'$ is as well. So (γ) is simple and hence in the socle of Λe_i.

By exactly the same argument as above, any path in Λe_i is an initial subpath of either γ or γ'. This gives us that $\operatorname{soc} \Lambda e_i = (\gamma)$.

Lastly we need to show that Λe_i is injective. We can do this by constructing an isomorphism $\varphi\colon \Lambda e_i \to D(e_j\Lambda)$. We define the map by $\varphi(e_i) = \gamma^*$. By the same argument as before (γ) is the socle of $e_j\Lambda$ as right modules. Thus γ^* generates the top of $D(e_j\Lambda)$, and φ is surjective. Since $\varphi(\gamma) = e_j^*$ and (γ) is the socle of Λe_i we have that φ is injective, and so it is an isomorphism.

Hence Λe_i is projective-injective, and so (γ) is the socle of a projective-

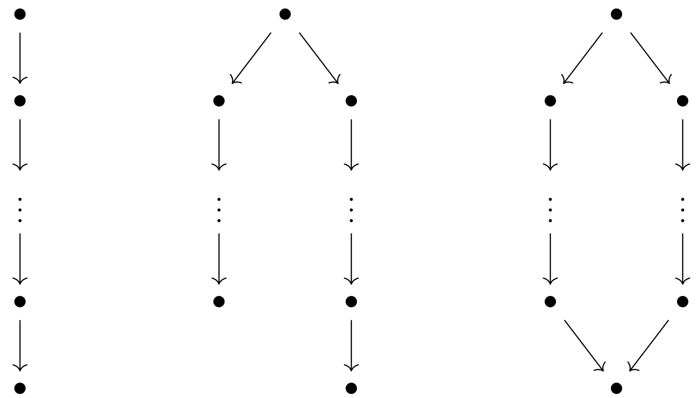

Figure 1: The possible shapes for an indecomposable projective module.

injective module. $\square$

This explains where the name *special biserial* comes from; the radical of each indecomposable projective of a special biserial algebra is biserial. I.e. it is the sum of two uniserial modules. In fact for an indecomposable projective P, either P is uniserial or $JP/\operatorname{soc} P$ is the direct sum of two uniserial modules, as visualized in Fig. 1.

Combining Theorem 4.20 and Proposition 4.22 we can reduce the problem of computing the representation dimension of a special biserial algebra to string algebras, by modding out all binomial relations. We now do this to show that special biserial algebras have representation dimension at most 3.

Theorem 4.23. *[EHIS04, Corollary 1.3] If $\Lambda = kQ/I$ is a special biserial algebra, then $\operatorname{repdim}(\Lambda) \leq 3$, and thus $\operatorname{findim}(\Lambda) < \infty$.*

Proof. By Theorem 4.20 we may assume Λ is a string algebra. If we can construct a radical embedding of Λ into a representation finite algebra, then by Theorem 4.19 our result would follow.

For any vertex $l \in Q$ define $E(l)$ to be the number of arrows ending in l

and $S(l)$ the number of arrows starting in l. Define $c(\Lambda)$ to be the sum of the number of vertices l with $E(l) \geq 2$ and the number of vertices k with $S(k) \geq 2$. The proof goes by induction on $c(\Lambda)$.

If $c(\Lambda) = 0$, then Q is the disjoint union of linearly oriented quivers of type $\mathbb{A}$ and cyclically oriented quivers of type $\tilde{\mathbb{A}}$. Finite dimensional algebras arising from such quivers are well known to be representation finite (c.f. [ARS97, Chapter VI.2] or [ASS06, Chapter V.3]), and so the identity map on Λ is a radical embedding into an algebra of finite representation type.

$$1 \longrightarrow \;\; ------ \;\; \longrightarrow n$$

Linearly oriented
quiver of type $\mathbb{A}_n$

$$\begin{tikzcd}
& 0 \arrow[r] & 1 \arrow[dr] & \\
n \arrow[ur] & & & 2 \arrow[dl] \\
& {} \arrow[ul] & {---} &
\end{tikzcd}$$

Cyclically oriented
quiver of type $\tilde{\mathbb{A}}_n$

If $c(\Lambda) = n \geq 1$, then there is a vertex l with either $E(l) = 2$ or $S(l) = 2$. We now construct a new string algebra Γ and a radical embedding $\Lambda \to \Gamma$ such that $c(\Gamma) \leq n - 1$.

The two cases are completely symmetric, so we only show the case $E(l) = 2$ here. Let α_1 and α_2 be the two arrows ending in l. Define the quiver Q' to have the same vertices as Q, except we replace l by two vertices l_1 and l_2. The arrows of Q' are exactly the same, except now α_1 ends in l_1 and α_2 ends in l_2. For any arrow $\beta \in Q$ that starts in l, the corresponding arrow in Q' starts in l_1 if and only if $\beta\alpha_1$ is not a relation.

We may consider I as an ideal in kQ' simply by setting paths to 0 if they are no longer defined in Q'. Then $\Gamma := kQ'/I$ is a string algebra, and the map $\Lambda \to \Gamma$ that sends e_l to $e_{l_1} + e_{l_2}$ and all other paths to themselves is a radical embedding.

For each vertex $k \neq l$, we have $E_\Lambda(k) = E_\Gamma(k)$ and $S_\Lambda(k) = S_\Gamma(k)$. We also have $E_\Lambda(l) = 2$, $E_\Gamma(l_1) = E_\Gamma(l_2) = 1$, and $S_\Lambda(l) = S_\Gamma(l_1) + S_\Gamma(l_2)$. Since $S_\Lambda(l) \leq 2$ it follows that $c(\Gamma) \leq n - 1$.

By induction there is a radical embedding of Λ into an algebra Γ with $c(\Gamma) = 0$, which is representation finite. Then by Theorem 4.19 we get that $\operatorname{repdim}(\Lambda) \leq 3$, and by Corollary 4.9.1 we have $\operatorname{findim}(\Lambda) < \infty$. $\square$

5 Vanishing radical powers

We remind the reader that throughout this section Λ is a finite dimensional algebra, and J is its radical. The Loewy length of an algebra is the smallest integer n such that $J^n = 0$. In this section show that algebras with short Loewy length have finite finitistic dimension.

Historically the two important conditions for showing that $\operatorname{findim}(\Lambda) < \infty$ has been that $J^2 = 0$ and $J^3 = 0$. Note that both of these are special case of this sections main theorem, Theorem 5.3, where we show that $\operatorname{findim}(\Lambda) < \infty$ for "half representation finite" algebras. This proof is due to Wang [Wan94].

We first give an alternate proof for the case $J^2 = 0$.

Theorem 5.1. *If* $J^2 = 0$ *then* $\operatorname{findim}(\Lambda) < \infty$.

Proof. Let $d = \max\{\operatorname{pd} S_i \mid \operatorname{pd} S_i < \infty\}$ where S_i ranges over the simple Λ-modules. Let M be a module with $\operatorname{pd} M < \infty$. Let $P \to M$ be a projective cover. Then ΩM is contained in JP and since $J^2P = 0$, ΩM is annihilated by J and is thus semisimple. This means $\operatorname{pd} \Omega M \leq d$, and thus $\operatorname{pd} M \leq d + 1$. So $\operatorname{findim}(\Lambda) \leq d + 1 < \infty$. $\square$

The proof of the above theorem is relatively elementary, but it first appeared as a corollary to a more general result by Mochizuki [Moc65]. We outline the proof of this as well.

Theorem 5.2. *[Moc65, Theorem 3.1] Let Λ be a finite dimensional algebra such that J^i/J^{i+1} has finite projective dimension for all $i \geq 2$. Then* $\operatorname{findim}(\Lambda) < \infty$.

Proof. As before let $d = \max\{\operatorname{pd} S_i \mid \operatorname{pd} S_i < \infty\}$ where S_i ranges over the simple Λ-modules. We want to show that $\operatorname{findim}(\Lambda) \leq d + 1$.

First note that since J^i/J^{i+1} is semisimple and of finite projective dimension, we have $\operatorname{pd} J^i/J^{i+1} \leq d$. Now let M be a Λ-module with finite projective dimension. We see that $J^iM/J^{i+1}M$ is in $\operatorname{add} J^i/J^{i+1}$, because it is semisimple and for each nonzero simple summand (λm), we have that $(\lambda m) \cong (\lambda) \subseteq J^i/J^{i+1}$.

So $\operatorname{pd} J^iM/J^{i+1}M \leq d$ for all $i \geq 2$. For each i we have a short exact sequence

$$0 \longrightarrow J^{i+1}M \longrightarrow J^iM \longrightarrow J^iM/J^{i+1}M \longrightarrow 0,$$

which gives us that $\operatorname{pd} J^iM \leq \max\{\operatorname{pd} J^{i+1}M, J^iM/J^{i+1}M\}$. Since there is an n such that $J^nM = 0$ it follows by induction that $\operatorname{pd} J^2M \leq d$.

If we consider the short exact sequence

$$0 \longrightarrow J^2M \longrightarrow M \longrightarrow M/J^2M \longrightarrow 0,$$

we get that $\operatorname{pd} M/J^2M \leq \max\{\operatorname{pd} J^2M + 1, \operatorname{pd} M\}$. In particular if $M = \Lambda$, then we get $\operatorname{pd} \Lambda/J^2 \leq d + 1$. If we let $P \to M$ be the projective cover of M, we get a short exact sequence

$$0 \longrightarrow K \longrightarrow P/J^2P \longrightarrow M/J^2M \longrightarrow 0$$

for some module $K \subseteq JP/J^2P$. Since we assumed M had finite projective dimension, and $\operatorname{pd} M/J^2M \leq \max\{\operatorname{pd} J^2M + 1, \operatorname{pd} M\}$, both M/J^2M and P/J^2P has finite projective dimension. Thus K is a semisimple module with finite projective dimension, and we have $\operatorname{pd} K \leq d$. Thus $\operatorname{pd} M/J^2M \leq \max\{\operatorname{pd} K + 1, \operatorname{pd} P/J^2P\} \leq d + 1$.

Lastly since $\operatorname{pd} M \leq \max\{\operatorname{pd} J^2M, M/J^2M\}$ we get that $\operatorname{pd} M \leq d + 1$, and consequently $\operatorname{findim}(\Lambda) \leq d + 1 < \infty$. $\square$

The case for $J^3 = 0$ was first proved by Green–Huisgen-Zimmerman [GZH91, Theorem 16]. Simplified proofs where given by Fuller–Saorin [FS92], and Igusa–Todorov [IT05, Corollary 6]. Igusa–Todorov's proof was then generalized by Wang to so called "half representation finite" algebras [Wan94]. We give this proof here.

Theorem 5.3. *[Wan94] If $J^{2l+1} = 0$ and Λ/J^l is representation-finite, then $\operatorname{findim}(\Lambda) < \infty$.*

Proof. Let M be a module with $\operatorname{pd} M < \infty$. We have a short exact sequence

$$0 \longrightarrow J^l\Omega M \longrightarrow \Omega M \longrightarrow \Omega M/J^l\Omega M \longrightarrow 0.$$

Since $\Omega M \subseteq JP_M^0$ we have $J^{2l}\Omega M = 0$. This means that $J^l\Omega M$ and $\Omega M/J^l\Omega M$ are Λ/J^l-modules. We use this, the fact that Λ/J^l is representation finite, and the Igusa–Todorov function to create a bound for $\operatorname{pd} M$.

Applying Corollary 4.3.1 (ii) we have that:

$$\operatorname{pd} \Omega M \leq \psi(\Omega(J^l\Omega M) \oplus \Omega^2(\Omega M/J^l\Omega M)) + 2.$$

Since Λ/J^l is representation finite, there are only finitely many indecomposable Λ/J^l-modules, up to isomorphism. Let V be the sum of all of them. Then since $J^l\Omega M$ and $\Omega M/J^l\Omega M$ are in $\operatorname{add} V$, using Lemma 4.2 we have that

$$\psi(\Omega(J^l\Omega M) \oplus \Omega^2(\Omega M/J^l\Omega M)) \leq \psi(\Omega V \oplus \Omega^2 V).$$

So $\operatorname{pd} M \leq \psi(\Omega V \oplus \Omega^2 V) + 3$, and thus $\operatorname{findim}(\Lambda) < \infty$. $\square$

6 Monomial algebras

In this section we show a particularly nice way to construct a minimal projective resolution of the right module Λ/J for a monomial algebra Λ. We use this to compute $\mathrm{Tor}_i(\Lambda/J, M)$ and/or $\mathrm{Ext}^i(M, D(\Lambda/J))$ to get a bound on the projective dimension of all modules M.

In Proposition 6.4 we define the projective resolution. Then in Theorem 6.8 we use this to get a bound ion the finitistic dimension, giving us that monomial algebras satisfies the finitistic dimension conjecture.

Definition 6.1 (Monomial algebra)**.** A *monomial algebra* is a path algebra with admissible relations that are generated by monomials. That is, we do not allow the generators for the relations to consist of nontrivial linear combinations of paths.

From now on we will assume that the relations of our algebra are contained in J^2. If our relations includes an arrow or a vertex, we may simply replace our quiver by one where said vertex or arrow is removed. Thus we do not lose any generality by assuming this.

We will now define the set of m-chains, which will serve as a basis for our projective resolution.

Definition 6.2 (m-chains)**.** [GKK91] Let $\Lambda = k\Gamma/(\rho)$ be a monomial algebra, with ρ a minimal generating set of paths. As usual we define Γ_0 to be the vertices of Γ, and Γ_1 to be the arrows. Recursively define the set of $(m-1)$-chains, Γ_m, as the paths γ with the following criteria:

i) $\gamma = \beta\delta\tau$ with $\beta \in \Gamma_{m-2}$, $\beta\delta \in \Gamma_{m-1}$, and τ a non-zero path of length at least 1.

ii) $\delta\tau$ is 0 in Λ, i.e. it is in the ideal of relations.

iii) γ is left-minimal in the sense that if $\gamma = \gamma'\sigma$ such that γ' satisfies the above conditions, then $\gamma = \gamma'$.

Before we can construct our projective resolution we will need a key property of m-chains.

Lemma 6.3. *Any $\gamma \in \Gamma_m$ for $m \geq 1$ can be factored uniquely as $\gamma_1\gamma_0$ with $\gamma_1 \in \Gamma_{m-1}$, and γ_0 a non-zero path of length at least 1.*

Proof. When $m = 1$ this should be clear, since Γ_1 is the set of arrows, and Γ_0 is the set of vertices, so if $\gamma \in \Gamma_1$ is an arrow $i \to j$ then $\gamma = e_j\gamma$.

When $m > 1$ we know from the definition of Γ_m that γ can be written as $\gamma_1\gamma_0$. Assume there is another decomposition $\gamma = \gamma_1'\gamma_0'$. Then without loss of generality we may assume that γ_1' is shorter than γ_1. Then there is a σ such that $\gamma_1'\sigma = \gamma_1$. By minimality this means that $\gamma_1' = \gamma_1$, and so the decomposition is unique. $\square$

From now on we write R for the ring Λ/J, which we identify with the subring of Λ generated by the paths of length 0. Let $k\Gamma_m$ be the free vector space generated by Γ_m. Notice that $k\Gamma_m$ has a canonical structure as an R-R-bimodule. This means we can construct projective right Λ-modules by $P^m := k\Gamma_m \otimes_R \Lambda$.

Proposition 6.4. *Define the map $\delta_m \colon P^m \to P^{m-1}$ by $\delta_m(\gamma \otimes \alpha) = \gamma_1 \otimes \gamma_0\alpha$ where $\gamma_1\gamma_0$ is the unique decomposition of γ, and define $\delta_0 \colon P^0 \to \Lambda/J$ by $\delta_0(e_i \otimes \alpha) = e_i\alpha + J$. Then we have a minimal projective resolution of the right Λ-module Λ/J by*

$$\begin{tikzcd}
\cdots \arrow[r] & P^3 \arrow[r, "\delta_3"] & P^2 \arrow[r, "\delta_2"] & P^1 \arrow[r, "\delta_1"] & P^0 \arrow[r] \arrow[d, two heads, "\delta_0"] & 0 \\
& & & & \Lambda/J &
\end{tikzcd}$$

Before proving this proposition we require the following lemma.

Lemma 6.5. *[GKK91, Lemma 2.1] Let M be a Λ-module, and x an element in the kernel of of $\delta_m \otimes M \colon k\Gamma_m \otimes_R M \to k\Gamma_{m-1} \otimes_R M$. Write*

x on the form

$$x = \sum_j \sum_{k=0}^{n_j} \gamma_j \gamma_j^k \otimes m_j^k$$

with $\gamma_i \in \Gamma_{m-1}$ and $\gamma_i \neq \gamma_j$ when $i \neq j$ and $\gamma_j^k \neq \gamma_j^l$ when $k \neq l$. Then

$$\sum_{k=0}^{n_j} \gamma_j \gamma_j^k \otimes m_j^k$$

is also in the kernel for each j.

Proof. Let x be as given above. Applying $\delta_m \otimes M$ we get that

$$\sum_j \gamma_j \otimes \sum_{k=0}^{n_j} \gamma_j^k m_j^k = 0.$$

Since the γ_js are distinct we can deduce that

$$\sum_{k=0}^{n_j} \gamma_j^k m_j^k = 0.$$

From this it follows that

$$\sum_{k=0}^{n_j} \gamma_j \gamma_j^k \otimes m_j^k$$

is also in the kernel of $\delta_m \otimes M$. □

Using this lemma we can now prove the proposition.

Proof of Proposition 6.4. For all i the module P^i is projective as a right Λ-module and the image of δ_m is clearly contained in $P^{m-1}J$, so the only thing left to show is exactness. First we show that $\delta_m \delta_{m-1} = 0$. Let $\gamma \otimes \alpha$ be in P^m for $m \geq 2$. Then we can decompose γ uniquely as $\gamma_2\gamma_1\gamma_0$ and $\delta_m\delta_{m-1}(\gamma \otimes \alpha) = \gamma_2 \otimes \gamma_1\gamma_0\alpha$. By the way we defined Γ_m, $\gamma_1\gamma_0$ is 0 in Λ, and so $\gamma_2 \otimes \gamma_1\gamma_0\alpha = 0$.

Next we want to show that $\ker \delta_{m-1} \subseteq \operatorname{Im} \delta_m$. Let x be in the kernel of δ_{m-1}. By Lemma 6.5 it is sufficient to assume x is of the form

$$\sum_k \gamma\gamma_k \otimes \alpha_k$$

with $\gamma \in \Gamma_{m-2}$ and the γ_ks all distinct. Then $\sum_k \gamma_k \alpha_k = 0$. By the minimality conditions in the way we define m-chains we have that none of the γ_ks divide each other. Since Λ only has monomial relations, this gives us that $\gamma_k \alpha_k = 0$.

Because of this we have that $\gamma\gamma_k\alpha_k = \zeta_k\sigma_k$ for some m-chain ζ_k and some path σ_k (possibly of length 0). This gives us that x is the image of

$$\sum_k \zeta_k \otimes \sigma_k$$

by δ_m. Hence $\ker \delta_{m-1} \subseteq \operatorname{Im} \delta_m$, and the sequence is exact. So this gives a minimal projective resolution of Λ/J as a right Λ-module. □

The next thing we will do is find a repeating pattern in this resolution to aid us in bounding projective dimensions. To do this we introduce the concept of a special segment.

Definition 6.6 (Special segments)**.** We call a path τ in Γ a *special segment* for $\Lambda = k\Gamma/(\rho)$ if there is a path γ such that $\gamma\tau$ is a minimal relation.

Note that when we decompose an m-chain γ in Lemma 6.3 into $\gamma_1\gamma_0$, then γ_0 is a special segment, and that the set of special segments is finite.

Lemma 6.7. *[GKK91, Theorem 2.2] Let d be the number of special segments for Λ. If $s \geq d+3$ and γ is in Γ_s, then for any integer N there is an $n \geq N$ and a $\hat{\gamma} \in \Gamma_n$ such that for any path τ and any integer $r \geq 1$ we have $\gamma\tau \in \Gamma_{s+r}$ if and only if $\hat{\gamma}\tau \in \Gamma_{n+r}$.*

Proof. Applying Lemma 6.3 recursively we get that γ can be written as $\gamma = \tau_0\tau_1 \cdots \tau_{s-1}$ where $\tau_0\tau_1 \cdots \tau_{i-1} \in \Gamma_i$. In particular each τ_i is a special segment.

Since $s \geq d+3$ we must have that there exists i and j, $1 \leq i < j \leq s-1$ such that $\tau_i = \tau_j$. Let $\beta = \tau_{i+1}\tau_{i+2}\cdots\tau_j$. Then

$$\gamma_k := \tau_0\tau_1\cdots\tau_{j-1}\tau_j\beta^k\tau_{j+1}\cdots\tau_{s-1} \in \Gamma_{s+k(j-i)}$$

where β^k means β repeated k times. If we now choose k large enough such that $s+k(j-i) \geq N$ we can choose $n = s+k(j-i)$ and $\hat{\gamma} = \gamma_k$. Then we see that for any path τ, the composition $\gamma\tau$ is in Γ_{s+r} if and only if $\hat{\gamma}\tau$ is in Γ_{n+r}. $\square$

This gives us a pattern in the projective resolution that we now use to bound the finitistic dimension of our algebra.

Theorem 6.8. *[GKK91, Corollary 2.4] Let $\Lambda = k\Gamma/(\rho)$ be a monomial relation algebra. Then* $\operatorname{findim}(\Lambda) \leq d+3$ *where d is the number of special segments for Λ.*

Proof. Let M be a module of finite projective dimension and let N be $\operatorname{pd} M$. The projective dimension of M can be characterized as the largest integer c such that $\operatorname{Tor}_c(\Lambda/J, M) \neq 0$. We show that this is at most $d+3$. Let $s \geq d+3$ be an integer. Then we want to show that $\operatorname{Tor}_{s+1}(\Lambda/J, M) = 0$. We compute this by taking the projective resolution of Λ/J found in Proposition 6.4 and tensoring with M.

$$\cdots \longrightarrow k\Gamma_{s+2} \otimes M \xrightarrow{\delta_{s+2}\otimes M} k\Gamma_{s+1} \otimes M \xrightarrow{\delta_{s+1}\otimes M} k\Gamma_s \otimes M \longrightarrow \cdots$$

Let x be in the kernel of $\delta_{s+1} \otimes M$. Then by Lemma 6.5 we may assume x is on the form

$$x = \sum_j \gamma\gamma_j \otimes m_j$$

with γ in Γ_s and all the γ_js distinct. Then Lemma 6.7 gives us that there is an $n \geq N$ and a $\hat{\gamma} \in \Gamma_n$ such that $\gamma\tau$ is in Γ_{s+r} if and only if $\hat{\gamma}\tau$ is in Γ_{n+r}.

Then $\hat{x} := \sum \hat{\gamma}\gamma_j \otimes m_j$ is in the kernel of $\delta_{n+1} \otimes M$. Since $n+1 > N = \operatorname{pd} M$ the complex is exact at $n+1$. This means that there are elements γ_j^k and m_j^k such that

$$\hat{x} = \delta_{n+2}\left(\sum_j \sum_{k=0}^{n_j} \hat{\gamma}\gamma_j\gamma_j^k \otimes m_j^k\right) = \sum_j \sum_{k=0}^{n_j} \hat{\gamma}\gamma_j \otimes \gamma_j^k m_j^k$$

Since $\hat{\gamma}\gamma_j\gamma_j^k$ is in Γ_{n+2} if and only if $\gamma\gamma_j\gamma_j^k$ is in Γ_{s+2} we have that

$$x = \delta_{s+2}\left(\sum_j \sum_{k=0}^{n_j} \gamma\gamma_j\gamma_j^k \otimes m_j^k\right)$$

and thus $\operatorname{Tor}_{s+1}(\Lambda/J, M) = 0$ so $\operatorname{pd} M \leq d+3$. Since M was arbitrary this means that $\operatorname{findim}(\Lambda) \leq d+3$. $\square$

7 Unbounded derived category

So far we have been focused on the finite dimensional version of the finitistic dimension, known as the little finitistic dimension. Namely

$$\operatorname{findim}(\Lambda) = \sup\{\operatorname{pd} M \mid M \in \operatorname{mod} \Lambda, \operatorname{pd} M < \infty\}.$$

In this section we will consider infinite dimensional modules, and thus it is natural for us to look at the infinite dimensional version of the finitistic dimension, known as the big finitistic dimension. It is defined, as you would expect, by considering not just finite dimensional modules, but all Λ-modules:

$$\operatorname{Findim}(\Lambda) = \sup\{\operatorname{pd} M \mid M \in \operatorname{Mod} \Lambda, \operatorname{pd} M < \infty\}.$$

Note that $\operatorname{findim}(\Lambda) \leq \operatorname{Findim}(\Lambda)$ and so if we can show that $\operatorname{Findim}(\Lambda) < \infty$ we have also shown that $\operatorname{findim}(\Lambda) < \infty$.

In Theorem 1.9 we showed that if $\operatorname{findim}(\Lambda) < \infty$, then $D\Lambda$ becomes a generator in $\mathscr{D}^b(\Lambda)$. In this section we show that if we instead consider the unbounded derived category of all Λ-modules, then we get an analogous converse result.

Definition 7.1 (Localizing subcategory)**.** A full subcategory of a triangulated category $\mathcal{T}$ is called *localizing* if

i) It is triangulated. I.e. it is closed under shifts and cones.

ii) It is closed under arbitrary coproducts.

For a class of objects $\mathcal{S} \subset \mathcal{T}$ we call the smallest localizing subcategory that contains $\mathcal{S}$ the localizing category generated by $\mathcal{S}$, and we write $\langle \mathcal{S} \rangle$.

It's a well known fact that Λ generates the derived category as a localizing subcategory. We also have a dual notion, a colocalizing subcategory. Similarly it is true that $D\Lambda$ generates the derived category as a colocalizing subcategory. In the below theorem we do something a bit unexpected, we ask whether the derived category also is generated by $D\Lambda$ as a localizing subcategory.

Theorem 7.2. *[Ric19, Theorem 4.3] If the localizing subcategory generated by $D\Lambda$ is the entire unbounded derived category, then* $\operatorname{Findim}(\Lambda) < \infty$.

Proof. Assume $\operatorname{Findim}(\Lambda) = \infty$. Then there are modules M_i with projective dimension i for every $i \geq 0$. Let P_i be the minimal projective resolution of M_i, and consider $\bigoplus P_i[-i]$ and $\prod P_i[-i]$. Both of these have homology M_i in degree i, and are concentrated in non-negative degrees.

The inclusion from the sum to the product is clearly a quasi-isomorphism. We want to show that it is not a homotopy equivalence. Assume for the sake of contradiction that it was. Then tensoring with Λ/J would give us another homotopy equivalence. Since Λ/J is finitely presented tensoring preserves both products and coproducts. Because all the resolutions were minimal, tensoring with Λ/J gives us a complex with differentials equal to 0. In degree 0 we get

$$\bigoplus \mathrm{Tor}_i(\Lambda/J, M_i) \to \prod \mathrm{Tor}_i(\Lambda/J, M_i).$$

Since $\mathrm{Tor}_i(\Lambda/J, M_i)$ is nonzero for every M_i this map is not an isomorphism, and so we don't have a homotopy equivalence.

Let C be the cone of $\bigoplus P_i[-i] \to \prod P_i[-i]$. Then C is 0 in the derived category, but non-zero in the homotopy category. Since Λ is artinian, the product of projectives is projective [Cha60, Theorem 3.3], so $\prod P_i[-i]$ is a complex of projectives, which means that C is a complex of projectives.

In other words C is an acyclic lower bounded complex of projectives that is not contractible. Tensoring with $D\Lambda$ is an equivalence from projectives to injectives with inverse $\mathrm{Hom}(D\Lambda, -)$ (c.f. Theorem A.5 in the appendix), so $D\Lambda \otimes C$ is a lower bounded complex of injectives that is not contractible. Such a complex cannot be acyclic so $D\Lambda \otimes C$ has homology, and is thus non-zero in $\mathscr{D}(\Lambda)$.

The homology of C is 0, so $K(\Lambda)(\Lambda, C[i]) = 0$. Applying the equivalence $D\Lambda \otimes -$ we get

$$0 = K(\Lambda)(D\Lambda, D\Lambda \otimes C[i]) = \mathscr{D}(\Lambda)(D\Lambda, D\Lambda \otimes C[i]).$$

The full subcategory of objects X with $\mathscr{D}(\Lambda)(X, D\Lambda \otimes C[i]) = 0$ is localizing and contains $D\Lambda$, so it contains $\langle D\Lambda \rangle$.

This means that $D\Lambda \otimes C$ is not in $\langle D\Lambda \rangle$, and so that can not be the entire derived category. □

Theorem 7.3. *[Ric19, Theorem 4.4] For a finite dimensional algebra Λ we have* $\mathrm{Findim}(\Lambda) < \infty$ *if and only if* $D\Lambda^{\perp} \cap \mathscr{D}^{+}(\Lambda) = 0$.

Proof. In the theorem above we proved that when the finitistic dimension is infinite, then there is a non-zero complex in $\mathscr{D}^{+}(\Lambda)$ perpendicular to $D\Lambda$.

The proof of the converse is the same as for Theorem 1.9. If we have a non-zero object $X \in D\Lambda^{\perp} \cap \mathscr{D}^{+}(\Lambda)$, then by replaccing X by its minimal injective resolution we see that $\mathscr{D}(\Lambda)(D\Lambda, X)$ is an acyclic minimal complex of projectives that continue arbitrarily to the right. So the cokernels have arbitrarily large projective dimension. $\square$

8 Summary

8.1 Classes of algebras

We conclude the thesis by summarizing for which families of algebras the finitistic dimension conjecture has been shown to hold.

Theorem 8.1. *The following classes of algebras satisfies the finitistic dimension conjecture:*

a) Representation-finite algebras

b) Monomial algebras

c) Gorenstein algebras

d) Algebras with finite global dimension

e) Self-injective algebras

f) Algebras where the radical squares to 0

g) Local artin algebras

h) Stably hereditary algebras

i) Special biserial algebras

j) "Half representation-finite" algebras, i.e. algebras such that Λ/J^l is representation-finite and $J^{2l+1} = 0$.

Proof.

(a) The supremum over a finite set is finite so $\operatorname{findim}(\Lambda) < \infty$ for a representation finite algebra.

(b) This is the content of Section 6.

(c) Over a Gorenstein algebra, if a module has finite projective dimension, then its projective dimension is less than $\operatorname{pd} D\Lambda$. Thus $\operatorname{findim}(\Lambda) = \operatorname{pd} D\Lambda$. The proof of this is implicit in Proposition 1.4.

(d) If an algebra Λ has finite global dimension, then $\operatorname{findim}(\Lambda) = \operatorname{gl.dim}(\Lambda)$.

(e) If Λ is self-injective, then all projective modules are injective. Thus any monomorphism from a projective module is split. It follows that the only modules with finite projective dimension are the projectives themselves, and so $\operatorname{findim}(\Lambda) = 0$.

(f) This was shown in Theorem 5.1.

(g) Local artin algebras have finitistic dimension 0. A proof of this is included in the appendix, Theorem A.4.

(h) Stably hereditary algebras are considered in Section 4.2.

(i) Special biseral algebras are considered in Section 4.3.

(j) Half representation-finite algebras are considered Section 5.

□

In this thesis our main focus has been on the small finitistic dimension. We now summarize for which algebras it is known that the big finitistic dimension is finite.

Theorem 8.2. *The following classes of algebras satisfy* $\operatorname{Findim}(\Lambda) < \infty$.

a) Representation-finite algebras

b) Monomial algebras

c) Gorenstein algebras

d) Algebras with finite global dimension

e) Self-injective algebras

f) Algebras with $J^2 = 0$

g) Any algebra derived equivalent to any of the above

h) Local artin algebras

Proof.

(a) It was shown by Auslander and Ringel–Tachikawa that if an artin ring is representation finite, then any module is the direct sum of finitely generated modules [Aus74, II Proposition 4.3(c)] [RT74, Corollary 4.4]. This implies that $\mathrm{Findim}(\Lambda) = \mathrm{findim}(\Lambda) < \infty$ for a representation finite algebra Λ.

(b) Although Section 6 is formulated in terms of finitely generated modules, all the same arguments hold if we consider infinitely generated modules.

(c) By the same argument as above, we have that $\mathrm{Findim}(\Lambda) = \mathrm{pd}\, D\Lambda$ for a Gorenstein algebra.

(d) Any infinitely generated module is the direct limit of its finitely generated submodules. Since all finitely generated submodules has projective dimension less than the global dimension and $\mathrm{Tor}_\bullet(\Lambda/J, -)$ commutes with direct limits, it follows that $\mathrm{Findim}(\Lambda) = \mathrm{gl.dim}(\Lambda)$.

(e) By the same argument as above we have that $\mathrm{Findim}(\Lambda) = 0$ for a self-injective algebra.

(f) Theorem 5.1 does not depend on the module being finitely generated, so the same proof works equally well to prove that $\mathrm{Findim}(\Lambda) < \infty$ when $J^2 = 0$.

(g) Rickard showed that injectives generates the derived category for all the classes of algebras above [Ric19, Theoreom 3.2, Corrolary 7.4-7.6]. This also gives an alternate proof that all the algebras above satisfies $\text{Findim}(\Lambda) < \infty$. We can combine this with the fact that whether injectives generate is preserved under derived equivalence [Ric19, Theorem 3.4]. Then we get that any algebra derived equivalent to any of the above satisfies $\text{Findim}(\Lambda) < \infty$.

(h) Like above, Theorem A.4 gives us that $\text{Findim}(\Lambda) = 0$ for a local artin algebra.

$\square$

As far as the author is aware it is not known whether stably hereditary algebras, special biserial algebras or half representation-finite algebras are known to satisfy $\text{Findim}(\Lambda) < \infty$ in general.

Appendices

A Homological algebra

In this section we collect relevant theorems from homological algebra that would be distracting within the text itself.

Lemma A.1. *[CE99, Chapter I, theorem 3.2] Let R be a noetherian ring. Then an R-module Q is injective if and only if it has the injective lifting property for inclusions of ideals into R.*

Proof. If Q is injective then Q has the lifting property for all monomorphisms, so one direction is clear. Assume we have a diagram

$$\begin{tikzcd} Q & \\ M \arrow[u, "f"] \arrow[r, hook] & N \arrow[ul, dashed] \end{tikzcd}$$

We want to show that the dashed arrow exists. Let S be the partially ordered set $\{(M', f') : M \leq M', f'|_M = f\}$. By Zorn's lemma this has a maximal element (M', f'). Assume $M' \neq N$, then there is an element $x \in N - M'$. The set of r such that $rx \in M'$ forms an ideal I. Define the map $g : I \to Q$ by $I(r) = f'(rx)$. By hypothesis g lifts to a map $\tilde{g} : R \to Q$. Let q be $\tilde{g}(1)$. Then $\tilde{f} : M' + Rx \to Q$ defined by $\tilde{f}(m + rx) = f'(m) + rq$ gives us a bigger element of S, contradicting maximality. Thus $M' = N$ and Q is injective. □

Theorem A.2. *Let R be a noetherian ring. Then an arbitrary coproduct of injectives is injective.*

Proof. By the lemma above it is enough to show the lifting property on ideals of R. Let I be an ideal and $f : I \to \bigoplus_i Q_i$ be a map to a coproduct of injectives. Since R is notherian I is finitely generated so f factors through a finite sum $I \to \bigoplus_{i=0}^n Q_i \to \bigoplus Q_i$. Since finite coproducts of injectives are injective we are done.

$$\begin{tikzcd} \bigoplus Q_i & \\ \bigoplus\limits_{i=0}^{n} Q_i \arrow[u] & \\ I \arrow[u] \arrow[r, hook] & R \arrow[ul, dashed] \end{tikzcd}$$

□

Theorem A.3. *[CE99, Chapter I, Exercise 8] Let R be a noetherian ring. Then direct limits of injectives is injective.*

Proof. By the lemma above it is enough to show the lifting property on ideals of R. Let I be an ideal and let $Q = \varinjlim Q_i$ be a direct limit of injectives.

Since R is noetherian I is finitely presented, say $R^n \to R^m \to I \to 0$. Applying $\mathrm{Hom}(-, Q)$ we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(I, Q) \longrightarrow \mathrm{Hom}(R^m, Q) \longrightarrow \mathrm{Hom}(R^n, Q)$$

Since direct limits are exact we also have an exact sequence

$$0 \longrightarrow \varinjlim \mathrm{Hom}(I, Q_i) \longrightarrow \varinjlim \mathrm{Hom}(R^m, Q_i) \longrightarrow \varinjlim \mathrm{Hom}(R^n, Q_i)$$

We also have a natural map $\varinjlim \mathrm{Hom}(-, Q_i) \to \mathrm{Hom}(-, Q)$. $\mathrm{Hom}(R^n, Q_i)$ just equals Q_i^n, so this map is an isomorphism at R^n. Then by the five lemma applied to the two sequences above we get that $\mathrm{Hom}(I, Q) \cong \varinjlim \mathrm{Hom}(I, Q_i)$ for all ideals I. So since

$$\varinjlim \mathrm{Hom}(R, Q_i) \longrightarrow \varinjlim \mathrm{Hom}(I, Q_i) \longrightarrow 0$$

is exact, we get that

$$\mathrm{Hom}(R, Q) \longrightarrow \mathrm{Hom}(I, Q) \longrightarrow 0$$

is exact. Hence Q is injective. □

Theorem A.4. *If R is a local artinian ring, then all modules with finite projective dimensions are projective. In other words we have that* $\mathrm{Findim}(R) = 0$.

Proof. Assume there is a non-projective module with finite projective dimension. Then in particular we have one with projective dimension equal to 1. Since all projective modules are free this means we have a short exact sequence

$$0 \longrightarrow R^{(I')} \longrightarrow R^{(I)} \longrightarrow M \longrightarrow 0$$

where $R^{(I')}$ maps into $JR^{(I)}$. Let k be the minimal integer such that $J^k = 0$. Let a be a generator in $R^{(I')}$ and let r be a non-zero element of J^{k-1}. Then ra is non-zero, but is mapped to something in $J^{k-1}JR^m = 0$, thus the map is not injective which gives a contradiction. □

Theorem A.5. *Let Λ be an artin algebra. Then we have an equivalence of categories*

$$\begin{tikzcd} \operatorname{Proj}\Lambda \arrow[r, bend left, "D\Lambda\otimes -"] & \operatorname{Inj}\Lambda \arrow[l, bend left, "\operatorname{Hom}(D\Lambda,-)"] \end{tikzcd}$$

where the tensor product is over Λ, and $\operatorname{Hom}(D\Lambda, X)$ *is considered as a Λ-module by considering $D\Lambda$ as a bimodule.*

Proof. First we note the following isomorphisms of Λ-modules when evaluating the functors at Λ and $D\Lambda$

$$\begin{aligned} \operatorname{Hom}(D\Lambda, D\Lambda \otimes \Lambda) &\cong \operatorname{End}(D\Lambda) \\ &\cong \operatorname{End}(\Lambda_\Lambda) \\ &\cong \Lambda \end{aligned}$$

and

$$\begin{aligned} D\Lambda \otimes \operatorname{Hom}(D\Lambda, D\Lambda) &\cong D\Lambda \otimes \Lambda \\ &\cong D\Lambda. \end{aligned}$$

Since $D\Lambda$ is finitely presented $D\Lambda \otimes -$ and $\mathrm{Hom}(D\Lambda, -)$ preserve both products and coproducts. Then since $\mathrm{Proj}\,\Lambda = \mathrm{Add}\,\Lambda$ and $\mathrm{Inj}\,\Lambda = \mathrm{Prod}\,D\Lambda$ it follows from the equations above that $\mathrm{Hom}(D\Lambda, D\Lambda \otimes -)$ and $D\Lambda \otimes \mathrm{Hom}(D\Lambda, -)$ are isomorphic to the identity on $\mathrm{Proj}\,\Lambda$ and $\mathrm{Inj}\,\Lambda$ respectively.

Lastly we verify that the maps are well defined. Since Λ is an artin algebra each injective module is the injective envelope of its socle. Since the socle is semisimple it is the direct sum of simple modules. Thus each injective is the sum of indecomposable injective modules, and hence we have that $\mathrm{Add}\,D\Lambda = \mathrm{Inj}\,\Lambda$. It is true for any ring that $\mathrm{Add}\,\Lambda = \mathrm{Proj}\,\Lambda$, and so we have the following:

$$D\Lambda \otimes (\mathrm{Proj}\,\Lambda) = D\Lambda \otimes (\mathrm{Add}\,\Lambda) = \mathrm{Add}\,D\Lambda = \mathrm{Inj}\,\Lambda,$$

and

$$\mathrm{Hom}(D\Lambda, \mathrm{Inj}\,\Lambda) = \mathrm{Hom}(D\Lambda, \mathrm{Add}\,D\Lambda) = \mathrm{Add}\,\Lambda = \mathrm{Proj}\,\Lambda.$$

So the maps induce an equivalence of categories. □

Theorem A.6 (Fitting's Lemma)**.** *Let R be a ring, M an R-module, and $L\colon M \to M$ an endomorphism. If X is a noetherian submodule of M, then there exists a positive integer η_X such that $L|_{L^n(X)}\colon L^n(X) \to M$ is injective for all $n \geq \eta_X$.*

Proof. We have an increasing sequence of submodules of X given by:

$$\ker L \cap X \subseteq \ker L^2 \cap X \subseteq \ker L^3 \cap X \subseteq \cdots$$

Since X is noetherian this sequence stabilizes, i.e. there is an integer η_X such that $\ker L^n \cap X = \ker L^{n+1} \cap X$ for all $n \geq \eta_X$. We know that $L^n(X) \cong X/\ker L^n \cap X$, and that through this isomorphism the map $L\colon L^n(X) \to M$ is induced by $L^{n+1}\colon X/\ker L^n \cap X \to L^{n+1}(X) \subseteq M$. Since for $n \geq \eta_X$ we have that $\ker L^n \cap X = \ker L^{n+1} \cap X$ this map is injective, and so the theorem holds. □

Interesting examples of Fitting's Lemma comes from R being a noetherian ring and X being a finitely generated modules. In particular the case when $R = \mathbb{Z}$ appears in Section 4.

An important special case of Fitting's Lemma that comes up when working with artinian rings is when $X = M$ and X has finite length. Remember that over an artin ring all finitely generated modules have finite length.

Corollary A.6.1. *Let X be a module of finite length, and let $L\colon X \to X$ be an endomorphism. Then L splits as a direct sum $L_1 \oplus L_2\colon X_1 \oplus X_2 \to X_1 \oplus X_2$ such that L_1 is nilpotent and L_2 is an isomorphism.*

Proof. Since X has finite length it is noetherian, thus we can apply Fitting's Lemma. Let n be the positive integer we get from Fitting's Lemma, and let K be $\ker L^n$. We wish to show that X is the direct sum of K and $L^n(X)$. Note that since L is inejctive when restricted to $L^n(X)$ we have that $K \cap L^n(X) = 0$, so all we have to show is that $X = K + L^n(X)$.

We have a short exact sequence

$$0 \longrightarrow K \longrightarrow X \longrightarrow L^n(X) \longrightarrow 0.$$

From this we conclude that the length of $L^n(X)$ is equal to the length of X minus the length of K. Since $\ker L^n = \ker L^{2n}$ we also have that the length of $L^n(X)$ and $L^{2n}(X)$ are equal. Since $L^{2n}(X)$ is a submodule of $L^n(X)$ this means that $L^n(X) = L^{2n}(X)$. Thus L restricts to an automorphism on $L^n(X)$. Let ψ be its inverse. Then for any $x \in X$ we have $x = \psi L^n(x) + x - \psi L^n(x)$. Clearly $\psi L^n(x)$ is in $L^n(X)$. Applying L^n to $x - \psi L^n(x)$ we get

$$\begin{aligned} L^n(x - \psi L^n(x)) &= L^n(x) - L^n \psi L^n(x) \\ &= L^n(x) - L^n(x) \\ &= 0 \end{aligned}$$

Thus $x - \psi L^n(x)$ is in the kernel and so $X = K \oplus L^n(X)$. Then we see that L breaks down as a direct sum $L = L_1 \oplus L_2$ with $L_1\colon K \to K$ nilpotent and $L_2\colon L^n(X) \to L^n(X)$ an isomorphism. $\square$

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