Finitistic dimension conjecture

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Abstract

FDC yo! This is abstract!

Contents

Notation

Throughout this thesis k will be a field, and Λ will be a finite dimensional algebra over k. We will use J to refer to the Jacobson radical of Λ.

We will use $\operatorname{mod}\Lambda$ to refer to the category of finite dimensional left Λ-modules, and $\operatorname{Mod}\Lambda$ to the category of all left Λ-modules. Any modules considered will be left modules if not specified otherwise. When there is ambiguity we may write ${}_\Lambda M$ to specify that we are considering M as a left Λ-module, and M_Λ to specify that we are considering M as a right Λ-module. Similarly ${}_\Gamma M_\Lambda$ means we are considering M as a Γ-Λ-bimodule.

Since right Λ-modules are the same as left Λ^{op}-modules we use these interchangeably. We use the symbol D to denote the duality functor $D\colon \operatorname{mod}\Lambda \leftrightarrow \operatorname{mod}\Lambda^{\mathrm{op}}$ where $DM = \operatorname{Hom}_k(M, k)$. Typically $D\Lambda$ will refer to the left module $D\Lambda_\Lambda$.

A quiver is a direct graph with a finite number of vertices. We write composition of paths right to left. That is, for paths $\alpha\colon i \to j$ and $\beta\colon k \to l$ the composition $\alpha\beta$ is defined if and only if $l = i$. For a quiver Q, the path algebra kQ is the free vector space of all paths, including a trivial path for each vertex. Multiplication of paths is defined to be composition when it is defined and 0 otherwise. The multiplication extends linearly to make kQ and algebra.

When working over a category $\mathcal{C}$ we will denote the set of morphisms either as $\operatorname{Hom}_{\mathcal{C}}(M, N)$ or as $\mathcal{C}(M, N)$. When the ambient category is clear we may also simply write $\operatorname{Hom}(M, N)$ or (M, N).

The categories we are considering are all k-linear and all functors are assumed to be k-linear as well.

For an exact category $\mathcal{A}$ we write:

- $\mathscr{D}(\mathcal{A})$ to refer to the derived category,
- $\mathscr{D}^b(\mathcal{A})$ to refer to the bounded derived category,
- $K^b(\mathcal{A})$ to refer to the bounded homotopy category,
- $K^{+,b}(\mathcal{A})$ (respectively $K^{-,b}(\mathcal{A})$) to refer to the homotopy category of complexes bounded below (respectively above) that are bounded in homology.

We also write $\mathscr{D}^b(\Lambda)$ instead of $\mathscr{D}^b(\operatorname{mod}\Lambda)$ and $\mathscr{D}(\Lambda)$ instead of $\mathscr{D}(\operatorname{Mod}\Lambda)$.

In all of these triangulated categories $X[i]$ will denote the complex X shifted i degrees down. That is, $(X[i])^n = X^{n+i}$. The hard truncation is the complex defined by $(X^{\geq n})^m$ equals X^m when $m \geq n$ and 0 otherwise. We denote the hard truncation of X by $X^{\geq n}$. The other hard truncation, $X^{\leq n}$, is defined similarly.

For a module M we will write $I(M)$ for its injective envelope, and $P(M)$ for its projective cover. We may also write

$$\begin{tikzcd} \cdots \arrow[r] & P_M^2 \arrow[r, "d_M^2"] & P_M^1 \arrow[r, "d_M^1"] & P_M^0 \arrow[r] \arrow[d, "d_M^0"] & 0 \\ & & & M & \end{tikzcd}$$

for its minimal projective resolution. We let the nth syzygies of M be the kernel of d_M^{n-1}, denoted by $\Omega^n M$. We also define $\Omega^0 M$ to be M.

The projective dimension of M is i if P_M^i is the last non-zero module in the minimal projective resolution, and ∞ if there is no such module. We denote the projective dimension by $\operatorname{pd} M$.

Introduction

This is an introduction

1 The homological conjectures

The finitistic dimension conjecture is part of a larger family of homological conjectures about finite dimensional algebras. In this section we outline some of these conjectures, and show which conjectures imply each other. Most of these implications can be proven only using the basic tools of representation theory and homological algebra, apart from the implications involving the Auslander–Reiten conjecture. For these we need some theorems from the theory of Wedderburn projectives, which is outlined in **??**.

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All of the conjectures are formulated as a specific property conjectured to hold for all finite dimensional algebras. In Proposition 1.14 we summarize how these implications work on the level of individual algebras.

Finitistic Dimension Conjecture (FDC)

Definition 1.1 (Finitistic dimension)**.** For a finite dimensional algebra Λ the *finitistic dimension* of Λ, denoted $\operatorname{findim}(\Lambda)$ is defined by

$$\operatorname{findim}(\Lambda) = \{\operatorname{pd} M \mid M \in \operatorname{mod}\Lambda, \operatorname{pd} M < \infty\}.$$

There is also the analogous definition for $\operatorname{Mod}\Lambda$, which is sometimes called the *big finitistic dimension*, and is denoted $\operatorname{Findim}(\Lambda)$. A natural question to ask, which is sometimes also called the finitistic dimension conjecture is whether $\operatorname{findim}(\Lambda)$ always equals $\operatorname{Findim}(\Lambda)$. This was shown to be false by Zimmermann-Huisgen in 1992 [ZH92]. The conjecture we consider is due to Rosenberg and Zelinsky [Bas60], and asks about when the finitistic dimension is finite.

Conjecrture 1 (Finitistic dimension conjecture)**.** *For a finite dimensional algebra the finitistic dimension is always finite.*

$$\operatorname{findim}(\Lambda) < \infty$$

Wakamatsu Tilting Conjecture (WTC)

In 1988 Wakamatsu introduced a generalization of tilting modules, now known as Wakamatsu tilting modules [Wak88].

Definition 1.2 (Wakamatsu tilting). Let T be a module in $\operatorname{mod}\Lambda$ for a finite dimensional algebra Λ. Then T is *Wakamatsu tilting* if

i) We have that $\operatorname{Ext}^n(T,T) = 0$ for all $n > 0$.

ii) There is an exact sequence

$$\eta\colon 0 \longrightarrow \Lambda \xrightarrow{d_{-1}} T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} \cdots$$

where T_i is in $\operatorname{add} T$.

iii) The sequence $\operatorname{Hom}(\eta, T)$ is exact. Which is equivalent to the condition that $\operatorname{Ext}^1(\operatorname{Ker} d_i, T) = 0$ for every differential d_i in η.

The definition is distinct from the definition of a tilting module in two key ways: the projective dimension of T is not assumed to be finite, and η is not assumed to be bounded. The Wakamatsu tilting conjecture states that this last condition is unnecessary.

Conjecrture 2 (Wakamatsu tilting conjecture). *If T is Wakamatsu tilting and has finite projective dimension, then T is a tilting module. In other words we can choose η to be bounded.*

Gorenstein Symmetry Conjecture (GSC)

Definition 1.3 (Gorenstein algebra). A finite dimensional algebra is said to be Gorenstein if all projective modules have finite injective dimension and all injective modules have finite projective dimension.

The Gorenstein symmetry conjecture says that we only need one of the two conditions for our algebra to be Gorenstein.

Conjecrture 3 (Gorenstein symmetry conjecture). *If Λ is a finite dimensional algebra the injective dimension of Λ as a left module is finite if and only if the projective dimension of $D\Lambda_\Lambda$ is finite.*

The conjecture describes a sort of symmetry between Λ and Λ^{op}. An equivalent formulation would be that Λ has finite injective dimension as a left module if and only if it has finite injective dimension as a right module.

Vanishing Conjecture (VC)

We remind the reader that when Λ is a finite dimensional algebra, we have an equivalence of categories between $K^{+,b}(\operatorname{inj}\Lambda)$ and the bounded derived category $\mathscr{D}^b(\Lambda)$ given by injective resolutions. This allows us to consider $K^b(\operatorname{inj}\Lambda)$ as a subcategory of $\mathscr{D}^b(\Lambda)$. Using this we define the perpendicular subcategory

$$K^b(\operatorname{inj}\Lambda)^\perp = \{X \in \mathscr{D}^b(\Lambda) \mid \operatorname{Hom}(I, X) = 0 \text{ for all } I \in K^b(\operatorname{inj}\Lambda)\}.$$

The vanishing conjecture then states that this subcategory is 0.

Conjecrture 4 (Vanishing conjecture). *If Λ is a finite dimensional algebra, then $K^b(\operatorname{inj}\Lambda)^\perp = 0$.*

In Section 7 we investigate an analog of this conjecture for the unbounded derived category.

Nunke Condition (NuC)

The Nunke condition is similar to the vanishing conjecture in that it considers modules which are “perpendicular” to the injective modules. Such a module is called a *Nunke module*, and an algebra is said to satisfy the Nunke condition if the only Nunke module is the zero module.

Conjecrture 5 (Nunke condition). *If $X \neq 0$ is a non-zero module over a finite dimensional algebra Λ, then there is an $n \geq 0$ such that $\operatorname{Ext}^n(D\Lambda, X) \neq 0$.*

Strong Nakayama Conjecture (SNC)

The strong Nakayama conjecture is simply the dual of the Nunke condition. We include both in this summary for completeness sake.

Conjecrture 6 (strong Nakayama Conjecture). *If $X \neq 0$ is a non-zero module over a finite dimensional algebra Λ, then there is an $n \geq 0$ such that $\operatorname{Ext}^n(X, \Lambda) \neq 0$.*

Generalized Nakayama Conjecture (GNC)

The generalized Nakayama conjecture is a slight weakening of the Strong Nakayama conjecture.

Conjecrture 7 (generalized Nakayama conjecture)**.** *If S is a simple module over a finite dimensional algebra Λ, then there is an $n \geq 0$ such that* $\operatorname{Ext}^n(S, \Lambda) \neq 0$.

We can also formulate the conjecture as all indecomposable injectives appearing in the minimal injective resolution of Λ. We give a short proof that this is an equivalent formulation here.

Proposition 1.4. *A finite dimensional algebra Λ satisfies GNC if and only if every indecomposable injective appears in the minimal injective resolution of Λ.*

Proof. Let the minimal injective resolution of Λ is given by

$$0 \longrightarrow \Lambda \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

Since the resolution is minimal, we have that $\operatorname{Ext}^n(S, \Lambda) = \operatorname{Hom}(S, I_n)$. This is non-zero if and only if the socle of S is a summand of I_n. Thus $\operatorname{Ext}^n(S, \Lambda)$ is non-zero for some n if the injective envelope of S appears in the minimal resolution, and I is a summand of I_n if $\operatorname{Ext}^n(\operatorname{soc} I, \Lambda) \neq 0$. □

Auslander–Reiten Conjecture (ARC)

Conjecrture 8 (Auslander–Reiten conjecture)**.** *Let Λ be finite dimensional algebra. If M is a generator in* $\operatorname{mod} \Lambda$ *such that* $\operatorname{Ext}^n(M, M) = 0$ *for all $n > 0$, then M is projective.*

Nakayama Conjecture (NC)

Definition 1.5 (Dominant dimension)**.** Let Λ be a finite dimensional algebra, and let

$$0 \longrightarrow \Lambda \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

be the minimal injective resolution of Λ. Then the *dominated dimension* of Λ is

$$\operatorname{domdim}(\Lambda) = \inf\{n \mid I^n \text{ is not projective}\}.$$

Conjecrture 9 (Nakayama conjecture)**.** *If Λ has infinite dominant dimension, then Λ is selfinjective.*

1.1 Implications

The homological conjectures are related in the way presented in the diagram below.

$$\begin{array}{ccccccccccc}
\text{FDC} & \Longrightarrow & \text{WTC} & \Longrightarrow & \text{GSC} & & & & & & \\
\Big\Downarrow & & & & & & & & & & \\
\text{VC} & \Longrightarrow & \text{NuC} & \Longleftrightarrow & \text{SNC} & \Longrightarrow & \text{GNC} & \Longleftrightarrow & \text{ARC} & \Longrightarrow & \text{NC}
\end{array}$$

The remainder of this section is used to prove these implications.

Theorem 1.6. *[MR04, Proposition 4.4] The finitistic dimension conjecture implies the Wakamatsu tilting conjecture.*

Proof. Assume Λ satisfies FDC, and let T be a Wakamatsu tilting module with $\operatorname{pd} T < \infty$. By definition we have an exact sequence

$$\eta \colon 0 \longrightarrow \Lambda \xrightarrow{d_{-1}} T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} \cdots$$

We want to show that η can be replaced by a bounded sequence of the same form.

Let K_i denote the kernel of d_i. First we prove by induction on i that $\operatorname{Ext}^{>0}(K_i, T) = 0$. For $i = 0$ we have $K_0 = \Lambda$, so we have $\operatorname{Ext}^{>0}(K_0, T) = 0$. Now assume that $\operatorname{Ext}^{>0}(K_i, T) = 0$ for some $i \geq 0$. We have a short exact sequence

$$0 \longrightarrow K_i \longrightarrow T_i \longrightarrow K_{i+1} \longrightarrow 0.$$

Applying the long exact sequence in $\operatorname{Ext}(-, T)$ we get

$$\operatorname{Ext}^n(T_i, T) \longrightarrow \operatorname{Ext}^n(K_i, T) \longrightarrow \operatorname{Ext}^{n+1}(K_{i+1}, T) \longrightarrow \operatorname{Ext}^{n+1}(T_i, T)$$

Since T_i is in $\operatorname{add} T$ we have that $\operatorname{Ext}^n(T_i, T) = 0$ for all $n > 0$. Then by exactness we have that $\operatorname{Ext}^{n+1}(K_{i+1}, T) \cong \operatorname{Ext}^n(K_i, T) = 0$ for all $n \geq 1$. Since T is Wakamatsu tilting we have that $\operatorname{Ext}^1(K_{i+1}, T) = 0$, so by induction $\operatorname{Ext}^{>0}(K_i, T) = 0$ for all $i \geq 0$.

By a similar argument we now wish to show that

$$\operatorname{Ext}^1(K_m, K_{m-1}) \cong \operatorname{Ext}^i(K_m, K_{m-i})$$

for all $i \leq m$. We proceed by induction on i. When $i = 1$ the statement is evident. Now assume that

$$\operatorname{Ext}^1(K_m, K_{m-1}) \cong \operatorname{Ext}^i(K_m, K_{m-i})$$

for some $i \geq 1$. Then it is sufficient to show that

$$\operatorname{Ext}^i(K_m, K_{m-i}) \cong \operatorname{Ext}^{i+1}(K_m, K_{m-i-1}).$$

We have a short exact sequence

$$0 \longrightarrow K_{m-i-1} \longrightarrow T_{m-i-1} \longrightarrow K_{m-i} \longrightarrow 0.$$

Taking the long exact sequence in $\operatorname{Ext}(K_m, -)$ we get the exact sequence

$$\begin{array}{c} \operatorname{Ext}^i(K_m, T_{m-i-1}) \longrightarrow \operatorname{Ext}^i(K_m, K_{m-i}) \\ \longrightarrow \operatorname{Ext}^{i+1}(K_m, K_{m-i-1}) \longrightarrow \operatorname{Ext}^{i+1}(K_m, T_{m-i-1}). \end{array}$$

Since we showed above that $\operatorname{Ext}^{>0}(K_m, T) = 0$ and T_{m-i-1} is in $\operatorname{add} T$ we get that $\operatorname{Ext}^{>0}(K_m, T_{m-i-1}) = 0$. Thus $\operatorname{Ext}^i(K_m, K_{m-i}) \cong \operatorname{Ext}^{i+1}(K_m, K_{m-i-1})$, and by induction we have that

$$\operatorname{Ext}^1(K_m, K_{m-1}) \cong \operatorname{Ext}^i(K_m, K_{m-i})$$

for all $i \leq m$.

Next we show that $\operatorname{pd} K_i < \infty$ for all $i \geq 0$. We proceed by induction on i. The projective dimension of $K_0 = \Lambda$ is 0, which is finite. For $i > 0$ we have a short exact sequence

$$0 \longrightarrow K_{i-1} \longrightarrow T_{i-1} \longrightarrow K_i \longrightarrow 0.$$

Therefore $\operatorname{pd} K_i \leq \sup\{\operatorname{pd} T_{i-1}, \operatorname{pd} K_{i-1} - 1\} < \infty$.

Lastly, let $n = \operatorname{findim}(\Lambda) < \infty$. Then we have that

$$\operatorname{Ext}^1(K_{n+1}, K_n) \cong \operatorname{Ext}^{n+1}(K_{n+1}, K_0) = 0$$

where the last equality comes from $\operatorname{pd} K_{n+1} \leq n$. Now if we apply $\operatorname{Hom}(K_{n+1}, -)$ to the short exact sequence

$$0 \longrightarrow K_n \longrightarrow T_n \longrightarrow K_{n+1} \longrightarrow 0,$$

we get an exact sequence

$$\operatorname{Hom}(K_{n+1}, T_n) \longrightarrow \operatorname{Hom}(K_{n+1}, K_{n+1}) \longrightarrow \operatorname{Ext}^1(K_{n+1}, K_n) = 0.$$

This means that K_{n+1} is a direct summand of T_n, and thus is in $\operatorname{add} T$. Then we get abounded version of η by

$$\eta' \colon 0 \longrightarrow \Lambda \xrightarrow{d_{-1}} T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} T_n \xrightarrow{d_n} K_{n+1} \longrightarrow 0.$$

Hence T is a tilting module, and thus Λ satisfies WTC. $\square$

Theorem 1.7. *The Wakamatsu tilting conjecture implies the Gorenstein symmetry conjecture.*

Proof. The left module $D(\Lambda_\Lambda)$ is Wakamatsu tilting. WTC then gives us that if $D(\Lambda_\Lambda)$ has finite projective dimension, then ${}_\Lambda\Lambda$ has a finite coresolution by modules in $\operatorname{add} D(\Lambda_\Lambda)$. In other words ${}_\Lambda\Lambda$ has finite injective dimension.

For the other direction assume ${}_\Lambda\Lambda$ has finite injective dimension. Then the right module $D({}_\Lambda\Lambda)$ has finite projective dimension, so WTC gives us that Λ_Λ has finite injective dimension. Which means $D(\Lambda_\Lambda)$ has finite projective dimension. $\square$

Theorem 1.8. *[Hap93, 1.2] The finitistic dimension conjecture implies the vanishing conjecture.*

Proof. Assume Λ doesn't satisfy VC, and let $I^\bullet \in K^b(\operatorname{inj}\Lambda)^\perp$ be non-zero complex. Since $\mathscr{D}^b(\Lambda) \cong K^{+,b}(\operatorname{inj}\Lambda)$ we may assume $I^\bullet$ is a complex of injectives, and without loss of generality we may assume it is concentrated in degrees $i \geq 0$, and that $d^0 \colon I^0 \to I^1$ is not split mono. Since if it's concentrated in degrees $i \geq k$ we can just shift it, and if d^0 is split mono, then replacing I^0 by 0 and I^1 by I^1/I^0 gives a homotopic complex.

The module $\operatorname{Hom}(D\Lambda, I^i)$ is in $\operatorname{add}\operatorname{Hom}(D\Lambda, D\Lambda) = \operatorname{add}\Lambda$ so $\operatorname{Hom}(D\Lambda, I^\bullet)$ is a complex of projectives. We show that this complex is acyclic by considering the following diagram.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & D\Lambda \arrow[r] \arrow[d, "f"] \arrow[dl, dashed] & 0 \arrow[d] \\
I^{i-1} \arrow[r, "d^{i-1}"] & I^i \arrow[r, "d^i"] & I^{i+1}
\end{tikzcd}$$

Since $I^\bullet$ is in $K^b(\operatorname{inj}\Lambda)^\perp$ and $D\Lambda$ is in $K^b(\operatorname{inj}\Lambda)$, we have that whenever $d^i f = 0$, $f^\bullet$ is homotopic to 0. Meaning f factors through d^{i-1}. This means that $\operatorname{Hom}(D\Lambda, I^\bullet)$ is an acyclic complex. Further since $\operatorname{Hom}(D\Lambda, -)$ is an equivalence between $\operatorname{inj}\Lambda$ and $\operatorname{proj}\Lambda$ and d^0 is not split mono, we have that $\operatorname{Hom}(D\Lambda, d^0)$ is not split mono.

The cokernel of $\operatorname{Hom}(D\Lambda, d^i)$ has a projective resolution of length i. This resolution is the direct sum of its minimal resolution and an acyclic bounded complex of projectives. Since bounded acyclic complexes of projectives are split and $\operatorname{Hom}(D\Lambda, d^0)$ is not, we must have that the minimal resolution has length i, and so $\operatorname{findim}(\Lambda) = \infty$. $\square$

Theorem 1.9. *[Hap93, 1.2] The vanishing conjecture implies the Nunke condition.*

Proof. Assume Λ doesn't satisfy NuC. That is, there is an $X \neq 0$ with $\operatorname{Ext}^i(D\Lambda, X) = 0$ for all $i \geq 0$. Then we claim that X considered as a stalk complex is in $K^b(\operatorname{inj}\Lambda)^\perp$. To show this we proceed by induction on the width of $I^\bullet \in K^b(\operatorname{inj}\Lambda)$. If the width is 1, then $I^\bullet = I[-i] \in K^b(\operatorname{inj}\Lambda)$ is a stalk complex. Then $\mathscr{D}^b(I[-i], X) = \operatorname{Ext}^i(I, X)$, which is 0 because I is in $\operatorname{add} D\Lambda$ and $\operatorname{Ext}^i(D\Lambda, X) = 0$.

Let $I^\bullet \in K^b(\operatorname{inj}\Lambda)$ be a complex of width n. without loss of generality we may assume $I^\bullet$ is concentrated in degrees $0 \leq i < n$. Then

$$I^{>0} \to I \to I^0 \to I^{>0}[1]$$

is a triangle with $I^{>0}$ of width $n-1$ and I^0 of width 1. Taking the long exact sequence in $\mathscr{D}^b(-, X)$ it follows that $\mathscr{D}^b(I, X) = 0$. So X is a non-zero complex in $K^b(\operatorname{inj}\Lambda)^\perp$, and hence Λ does not satisfy VC. $\square$

Before we can prove the equivalence between the generalized Nakayama conjecture and the Auslander–Reiten Conjecture we will need the following proposition.

Proposition 1.10. *Let M be a module and I an injective module. If the projective cover of the socle of I is in* $\operatorname{add} M$, *then (M, I) is an injective $\Gamma := \operatorname{End}(M)^{\mathrm{op}}$-module. In particular if M is a generator then $(M, -)$ preserves injectives.*

Proof. Let $J \leq \Gamma$ be a left ideal and let $\psi\colon J \to (M, I)$ be any Γ-linear map. By Lemma A.1 it is enough to show that ψ factors through Γ to conclude that (M, I) is injective. Assume J is generated by $\{f_i\}$. If we can find $\gamma\colon M \to I$ such that $\gamma \circ f_i = \psi(f_i)$ then we would get our factorization of ψ by $J \hookrightarrow \Gamma \xrightarrow{\gamma\circ -} (M, I)$. To construct such a γ we consider the following diagram.

[...] In the appendix

$$\begin{tikzcd} \bigoplus M \arrow[d, "\sum f_i"'] \arrow[dr, "\sum \psi(f_i)"] & \\ M \arrow[r, dashed, "\gamma"'] & I \end{tikzcd}$$

We want to show that the kernel of $\sum \psi(f_i)$ contains the kernel of $\sum f_i$, so that we can use the injective property of I. To see this let K be the kernel of $\sum f_i$ and let K' be the kernel of $\sum \psi(f_i)$. If K' does not contain K, then $Q := K/K' \cap K$ is a nonzero module that is mapped injectively into I. So the socle of Q is a summand of the socle of I. Then by assumption the projective cover of the socle of Q is in $\operatorname{add} M$, so there is a non-zero projective map $M \to Q$. By the lifting property of projectives we get a map $M \to K$ such that the composition with $\sum \psi(f_i)$ is non-zero.

Let a_i be the composition $M \longrightarrow K \hookrightarrow \bigoplus M \xrightarrow{\pi_i} M$. Then we get $\sum f_i \circ a_i = 0$. Applying ψ we get $\sum \psi(f_i) \circ a_i = 0$, which gives a contradiction since a_i was explicitly constructed such that $\sum \psi(f_i) \circ a_i$ is non-zero. Thus K' contains K.

Using this we get the following commutative diagram:

$$
\begin{tikzcd}
\bigoplus M \arrow[d] \arrow[dr, "\sum \psi(f_i)"] \arrow[dd, bend right=60, "\sum f_i"'] & \\
\left(\bigoplus M\right)/K \arrow[r] \arrow[d, hook] & I \\
M \arrow[ur, dashed, "\exists\gamma"'] &
\end{tikzcd}
$$

Since I is injective it lifts monomorphisms so we know that γ exists. Thus (M, I) is an injective Γ-module. $\square$

Theorem 1.11. *The generalized Nakayama conjecture implies the Auslander–Reiten conjecture.*

Proof. The proof goes by contraposition. Assume Λ does not satisfy ARC. Then we have a nonprojective generator M such that $\mathrm{Ext}^n(M, M) = 0$ for all $n > 0$. We wish to show that $\Gamma := \mathrm{End}(M)^{\mathrm{op}}$ does not satisfy GNC. Let

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

be an injective resolution of M. Since $\mathrm{Ext}^n(M, M) = 0$, when we apply $(M, -) := \mathrm{Hom}(M, -)$ we get an exact sequence.

$$0 \longrightarrow \Gamma \longrightarrow (M, I_0) \longrightarrow (M, I_1) \longrightarrow \cdots$$

By Proposition 1.10 this is an injective resolution of Γ.

Since M is a non-projective generator it has every indecomposable projective as a summand and a nonprojective summand. So M has more indecomposable summands than Λ which means that Γ has more indecomposable projectives than Λ. It follows that Γ also has more injectives and thus has an injective not on the form (M, I). Since all modules that appear in the injective resolution of Γ are on the form (M, I), not all indecomposable injectives appear in the resolution. Therefore by Proposition 1.4 we have that Γ does not satisfy GNC. $\square$

Theorem 1.12. *[Yam96, Theorem 3.4.3] The Auslander–Reiten conjecture implies the generalized Nakayama conjecture.*

Proof. Assume that ARC holds, and let Γ be a finite dimensional algebra. We wish to show that Γ satisfies GNC. Let the minimal injective resolution of Γ be given by

$$0 \longrightarrow \Gamma \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

Let I be the minimal injective module such that each I_i is in $\operatorname{add} I$. If we can show that I is a cogenerator, then it will follow that Γ satisfies GNC. Let $P = DI$ be the projective right Γ-module dual to I, and let $\Lambda = \operatorname{End}_\Gamma(P)$ be its endomorphism ring.

Using the Hom-Tensor adjunction we see that

$$\begin{aligned} D(P \otimes_\Gamma X) &= \operatorname{Hom}_k(P \otimes_\Gamma X, k) \\ &= \operatorname{Hom}_\Gamma(P, \operatorname{Hom}_k(X, k)) \\ &= \operatorname{Hom}_\Gamma(P, DX) \end{aligned}$$

In particular we have that $D(P \otimes_\Gamma I) = \operatorname{End}_\Gamma(P) = \Lambda$ as right Λ-modules, and so $P \otimes_\Gamma I = D\Lambda$.

Now let $\mathcal{S}$ be the subcategory of $\operatorname{mod}\Gamma$ of modules that have a copresentation of modules in $\operatorname{add} I$. Then we claim there is an equivalence of categories

$$\begin{tikzcd} \mathcal{S} \arrow[r, bend left, "P\otimes_\Gamma -"] & \operatorname{mod}\Lambda \arrow[l, bend left, "\operatorname{Hom}_\Lambda(P,-)"] \end{tikzcd}$$

To see this we first note the following identities

$$\begin{aligned} \operatorname{Hom}_\Lambda(P, P \otimes_\Gamma I) &= \operatorname{Hom}_\Lambda(P, D\Lambda) \\ &= \operatorname{Hom}_k(\Lambda \otimes_\Lambda P, k) \\ &= DP = I \end{aligned}$$

$$\begin{aligned} P \otimes_\Gamma \operatorname{Hom}_\Lambda(P, D\Lambda) &= P \otimes_\Gamma DP \\ &= D\Lambda \end{aligned}$$

Since P_Γ is projective $P \otimes_\Gamma -$ is exact, so both functors are left exact. This means they induce equivalences between the subcategories with copresentations in $\operatorname{add} I$ and $\operatorname{add} D\Lambda$ respectively. Thus we get our wanted equivalence.

Now if we apply $P \otimes_\Gamma -$ to the injective resolution $I_\bullet$, we get an injective resolution of $P \otimes_\Gamma \Gamma = P$ as a Λ-module. Applying $\operatorname{Hom}_\Lambda(P, -)$ gives us back the complex $I_\bullet$ and thus we have that $\operatorname{Ext}^n_\Lambda(P, P) = 0$ for all $n > 0$.

Since $\operatorname{Hom}_\Lambda(P, -)$ is non-vanishing, P is a generator in $\operatorname{mod}\Lambda$. Since by assumption ARC holds, we get that P is projective as a Λ-module. Thus $\operatorname{Hom}_\Lambda(P, -)$ is right exact. Since $\operatorname{Hom}_\Lambda(P, P) = \operatorname{Hom}_\Lambda(P, P \otimes \Gamma) = \Gamma$ we get that $\mathrm{Hom}_\Lambda(P, -)$ induces an equivalence between modules with a presentation in $\mathrm{add}\, P$ and modules with a presentation in $\mathrm{add}\,\Gamma$. We conclude that $\mathcal{S} = \mathrm{mod}\,\Gamma$, and thus that I is a cogenerator.

Since I is a cogenerator all indecomposable injective modules appear in the resolution of Γ, an thus Γ satisfies GNC. □

Proposition 1.13. *[AR75] The generalized Nakayama conjecture implies the Nakayama conjecture*

Proof. Assume Λ satisfies GNC and that the dominant dimension of Λ is ∞. As shown in Proposition 1.4 if $\mathrm{Ext}^\bullet(S, \Lambda)$ is nonzero that means the injective envelope $I(S)$ appears in the minimal injective resolution of Λ. If all injectives appear in the resolution and the dominant dimension is infinity then all injectives are projective. Thus Λ is self injective, and hence Λ satisfies NC. □

The proofs above do not necessarily work on the level of individual algebras. For example for the proof that WTC implies GSC we need to assume that WTC holds for both Λ and Λ^{op} to prove that Λ satisfies GSC. We list the relationships between the conjectures for individual algebras.

Proposition 1.14. *The implications between the conjectures on the level of individual algebras can be described as follows:*

i) If Λ satisfies FDC, then Λ also satisfies WTC.

ii) If both Λ and Λ^{op} satisfy WTC, then both Λ and Λ^{op} also satisfy GSC.

iii) The implications FDC $\Rightarrow$ VC $\Rightarrow$ NuC, both hold on the level of individual algebras.

iv) An algebra Λ satisfies Nuc if and only if Λ^{op} satisfies SNC.

v) The implications SNC $\Rightarrow$ GNC $\Rightarrow$ NC, both gold on the level of individual algebras.

vi) If Γ satisfies GNC whenever $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$ for a generator M in $\mathrm{mod}\,\Lambda$, then Λ satisfies ARC.x

vii) If $\mathrm{End}(I)^{\mathrm{op}}$ satisfies ARC, where $\mathrm{add}\, I$ contains every injective in the minimal resolution of Λ, then Λ satisfies GNC.

viii) If $\mathrm{End}(P)^{\mathrm{op}}$ satisfies ARC, where P is the projective cover of $\mathrm{soc}\, I$ where I is the sum of all indecomposable projective-injective Λ-modules, then Λ satisfies NC.

2 Recollement

In this section we will discuss a reduction technique known as recollement. The idea of reduction techniques is to reduce the work of proving an algebra has finite finitistic dimension to proving the same for "simpler" algebras. In Section 2.1 we will consider a reduction technique of triangular matrix algebras. This turns out to be a special case of a recollement, but we give a different proof due to Fossum–Griffith–Reiten, that predate the introduction of recollements. This also gives a stricter bound on the finitistic dimension that is specific to triangular matrix algebras.

Definition 2.1 (Recollement)**.** A *recollement* between triangulated categories $\mathcal{T}'$, $\mathcal{T}$ and $\mathcal{T}''$ is a collection of six functors satisfying:

$$\begin{tikzcd}[column sep=huge]
\mathcal{T}' \arrow[r, "i_*=i_!" description] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \mathcal{T} \arrow[l, bend right=40, "i^*"'] \arrow[l, bend left=40, "i^!"] \arrow[r, "j^!=j^*" description] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \mathcal{T}'' \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*"]
\end{tikzcd}$$

(i) All functors are exact/triangulated.

(ii) The composition $j^*i_* = 0$ vanishes.

(iii) We have natural isomorphisms $i^*i_* \cong i^!i_! \cong \mathrm{id}_{\mathcal{T}'}$ induced by the units and counits of the adjunctions.

(iv) We have similar isomorphisms $j^!j_! \cong j^*j_* \cong \mathrm{id}_{\mathcal{T}''}$, also induced by the units and counits.

(v) For every $X \in \mathcal{T}$ we have the following distinguished triangles:

$$j_!j^!X \xrightarrow{\varepsilon} X \xrightarrow{\eta} i_*i^*X \longrightarrow j_!j^!X[1]$$

$$i_!i^!X \xrightarrow{\varepsilon} X \xrightarrow{\eta} j_*j^*X \longrightarrow i_!i^!X[1].$$

Note that (iii) and (iv) are equivalent to i_*, $j_!$, and j_* being fully faithful.

We are specifically interested in recollements when the triangulated categories in question are (bounded) derived categories of finite dimensional algebras.

Lemma 2.2. *Let* $\mathscr{D}^b(\Lambda') \underset{i_*}{\overset{i^*}{\leftrightarrows}} \mathscr{D}^b(\Lambda)$ *be exact functors with an adjoint pair* (i^*, i_*)*. Then* i^* *preserves bounded projective complexes and* i_* *preserves bounded injective complexes.*

Proof. The bounded projective complexes can be characterized up to isomorphism as the complexes P such that for any complex Y there is an integer t_Y with $\mathscr{D}^b(\Lambda)(P, Y[t]) = 0$ for $t \geq t_Y$. One can see this by using the equivalence $\mathscr{D}^b(\Lambda) \cong K^{-,b}(\operatorname{proj} \Lambda)$.

Let P be a bounded complex of projectives in $\mathscr{D}^b(\Lambda)$. Then we want to show that i^*P is as well. Let Y be any complex in $\mathscr{D}^b(\Lambda')$. Then $\mathscr{D}^b(\Lambda')(i^*P, Y[t]) = \mathscr{D}^b(\Lambda)(P, i_*Y[t])$, so since P is a bounded complex of projectives there is t_Y such that this vanishes for $t \geq t_Y$.

The statement for injectives is exactly dual, and so we do not write it out here, but leave it to the reader. $\square$

Lemma 2.3. *Let*
$$\begin{tikzcd} \mathscr{D}^b(\Lambda') \arrow[r, "i_*"] & \mathscr{D}^b(\Lambda) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \end{tikzcd}$$
be exact functors with adjoint pairs (i^*, i_*) *and* $(i_*, i^!)$*. Then the homology of* i_*X *is uniformly bounded for* $X \in \operatorname{mod} \Lambda'$ *considered as a complex concentrated in degree 0. I.e. there is an* r*, independent of* X*, such that* $H^j(i_*X) = 0$ *for* $j \notin (-r, r)$*.*

Proof. We first prove that there is an r', independent of X, such that $H^j(i_*X) = 0$ for $j \geq r'$. Let P be $i^*\Lambda \in \mathscr{D}^b(\Lambda')$. Then by Lemma 2.2 P is a bounded complex of projectives.

Thus there is an r' such that $P^{-j} = 0$ for $j \geq r'$. Then

$$\mathscr{D}^b(\Lambda')(P, X[j]) = \mathscr{D}^b(\Lambda)(\Lambda, i_*X[j]) = H^j(i_*X) = 0$$

for $j \geq r'$ and any Λ'-module X, when considered as a complex concentrated in degree 0.

Next we prove that there is an r'' such that $H^{-j}(i_*X) = 0$ for $j \geq r''$. The argument is completely dual. Let I be $i^!D\Lambda \in \mathscr{D}^b(\Lambda') \cong K^{+,b}(\operatorname{inj} \Lambda')$. Then again by Lemma 2.2 I is a bounded complex of injectives.

Thus there is an r'' such that $I^j = 0$ for $j \geq r''$. Then

$$\mathscr{D}^b(\Lambda')(X, I[j]) = \mathscr{D}^b(\Lambda)(i_*X, D\Lambda[j]) = H^{-j}(i_*X) = 0$$

for $j \geq r''$ and any Λ'-module X, when considered as a complex concentrated in degree 0.

Letting r be the maximum of r' and r'' we get that $H^j(X)$ is zero outside of $(-r, r)$. $\square$

Now that we have a good understanding of how the functors in a recollement interact with homology we can use this to say something about the projective dimension of modules, and thus about the finitistic dimension.

Theorem 2.4. *[Hap93, 3.3] Given a recollement between bounded derived categories*

$$\begin{tikzcd}
\mathscr{D}^b(\Lambda') \arrow[rr, "i_*=i_!"] \arrow[rr, phantom, shift left=4, "\perp"] \arrow[rr, phantom, shift right=4, "\perp"] && \mathscr{D}^b(\Lambda) \arrow[ll, bend right, "i^*"'] \arrow[ll, bend left, "i^!"] \arrow[rr, "j^!=j^*"] \arrow[rr, phantom, shift left=4, "\perp"] \arrow[rr, phantom, shift right=4, "\perp"] && \mathscr{D}^b(\Lambda''), \arrow[ll, bend right, "j_!"'] \arrow[ll, bend left, "j_*"]
\end{tikzcd}$$

then we have that $\operatorname{findim}(\Lambda) < \infty$ *if and only if we have that* $\operatorname{findim}(\Lambda') < \infty$ *and* $\operatorname{findim}(\Lambda'') < \infty$*.*

Proof. Assume $\operatorname{findim}(\Lambda) < \infty$. We begin by showing that $\operatorname{findim}(\Lambda') < \infty$.

Let $T = \Lambda'/\operatorname{rad}\Lambda'$ be the sum of all simple Λ'-modules. Then the projective dimension of X is the largest t for which $\operatorname{Ext}^t(X, T) \neq 0$. Let X be a module in $\operatorname{mod}\Lambda'$ with finite projective dimension. We consider X as a complex concentrated in degree 0. Then since X is isomorphic to its projective resolution, by Lemma 2.2 i_*X is a bounded complex of projectives. Say:

$$i_*X = 0 \to P^{-s} \to \cdots \to P^{s'} \to 0$$

By Lemma 2.3 we know there is an r independent of X such that $H^{-j}(X) = 0$ for $j \geq r$. Truncating i_*X at $-r$ gives a projective resolution of $\ker d_{i_*X}^{-r}$. So $\ker d_{i_*X}^{-r}$ has projective dimension $-r-(-s) = s-r$. Since $\operatorname{findim}(\Lambda) < \infty$ this means that $s \leq r + \operatorname{findim}(\Lambda)$.

Since i_*T is in $\mathscr{D}^b(\Lambda)$ it is a bounded complex, in particular there is a t_0 such that $i_*T^t = 0$ for $t \geq t_0$. Then by the bounds above $\mathscr{D}^b(\Lambda)(i_*X, i_*T[t]) = 0$ for $t \geq t_0 + s \geq t_0 + r + \operatorname{findim}(\Lambda)$. Since i_* is fully faithful this equals $\mathscr{D}^b(\Lambda')(X, T[t])$, and so $\operatorname{findim}(\Lambda') \leq t_0 + r + \operatorname{findim}(\Lambda)$. In particular it is finite.

The proof for findim(Λ'') is the same, just replacing i_* with $j_!$. We leave writing out the details of this to the reader.

For the converse assume Λ' and Λ'' both have finite finitistic dimension. Let $T = \Lambda/\operatorname{rad}\Lambda$, and X be a Λ-module with finite projective dimension, and consider both modules as a complex concentrated in degree 0. By Definition 2.1(v) we have distinguished triangles:

$$j_!j^!X \longrightarrow X \longrightarrow i_*i^*X \longrightarrow j_!j^!X[1]$$

$$i_!i^!T \longrightarrow T \longrightarrow j_*j^*T \longrightarrow i_!i^!T[1].$$

We write $(-,-)_m$ instead of $\mathscr{D}^b(\Lambda)(-,-[m])$, and make the following abbreviation:

$$\begin{aligned} X_j &:= j_!j^!X & \qquad X_i &:= i_*i^*X \\ T_i &:= i_!i^!T & \qquad T_j &:= j_*j^*T. \end{aligned}$$

Taking the long exact sequence in homfuntors we get the long exact sequences:

$$\cdots \longrightarrow (X,T_i)_m \longrightarrow (X,T)_m \longrightarrow (X,T_j)_m \longrightarrow (X,T_i)_{m+1} \longrightarrow \cdots$$

$$\cdots \longrightarrow (X_i,T_i)_m \longrightarrow (X,T_i)_m \longrightarrow (X_j,T_i)_m \longrightarrow (X_i,T_i)_{m+1} \longrightarrow \cdots$$

$$\cdots \longrightarrow (X_i,T_j)_m \longrightarrow (X,T_j)_m \longrightarrow (X_j,T_j)_m \longrightarrow (X_i,T_j)_{m+1} \longrightarrow \cdots$$

From this we conclude that

$$(X_i,T_j)_m = (i_*i^*X, j_*j^*T)_m = (j^*i_*i^*X, j^*T)_m = 0$$

and

$$(X_j,T_i)_m = (j_!j^!X, i_!i^!T)_m = (j^!X, j^!i_!i^!T)_m = 0.$$

Combining this with the long exact sequences gives us that

$$(X_i,T_i)_m = (X,T_i)_m \text{ and } (X_j,T_j)_m = (X,T_j)_m.$$

If we can show that $(X_i,T_i)_m$ and $(X_j,T_j)_m$ are bounded, then $(X,T_i)_m$ and $(X,T_j)_m$ would be bounded as well. Consequently we would have that

$(X,T)_m$ is bounded. This would give us a bound on the projective dimension of X.

We start by bounding $(X,T_i)_m = (X_i,T_i)_m$. First note that

$$(X_i,T_i)_m = (i_*i^*X, i_!i^!T)_m = (i^*i_*i^*X, i^!T)_m = (i^*X, i^!T)_m$$

Since X has finite projective dimension we can think of it as a bounded complex of projectives. Then by Lemma 2.2 i^*X is as well. By the second half of Lemma 2.3 (using (i^*, i_*) instead of $(i_*, i^!)$) we have that there is an r such that $H^{-j}(i^*X) = 0$ for all $j \geq r$. This means that thinking of i^*X as a complex of projectives, it is 0 in degree $-t$ for all $t \geq r + \operatorname{pd}\ker d^{-r}_{i^*X}$, in particular it is 0 for all $t \geq r + \operatorname{findim}(\Lambda')$. Since $i^!T$ is a bounded complex, it has an upper bound, say t_0. Thus $(i^*X, i^!T)_m = 0$ for all $m \geq t_0 + r + \operatorname{findim}(\Lambda')$.

The bound on $(X,T_j)_m$ is similar, using the finitistic dimension of Λ''. Taking the maximum of these two bounds we get a bound on $(X,T)_m$, which gives a bound on the projective dimension independent of X, hence a bound on $\operatorname{findim}(\Lambda)$. □

2.1 Triangular matrix rings and vertex removal

In this section we will relate the finitistic dimension of the triangular matrix ring $\begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$ to the finitistic dimension of R and S. The finitistic dimension of $\begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$ can be shown to be finite if and only if the finitistic dimensions of both R and S are finite, by constructing a recollement. In this section we use a different argument which gives an explicit bound on the finitistic dimension of the matrix ring.

Definition 2.5 (Comma category)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor. Then the *comma category* $(F, \mathcal{B})$ has as objects triplets (A, B, f) with $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $f\colon FA \to B$ a morphism in $\mathcal{B}$. The morphisms are pairs $(\alpha, \beta)\colon (A,B,f) \to (A',B',f')$ with $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ such that the following diagram commutes:

$$\begin{tikzcd} FA \arrow[r, "f"] \arrow[d, "F\alpha"'] & B \arrow[d, "\beta"] \\ FA' \arrow[r, "f'"] & B'. \end{tikzcd}$$

Proposition 2.6. *If $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and F is right exact, then the comma category $(F, \mathcal{B})$ is abelian.*

Proof. We need to show that $(F, \mathcal{B})$ has kernels, has cokernels, and that for any map the image equals the coimage. First we show that it contains kernels. Let $(\alpha, \beta)\colon (A, B, f) \to (A', B', f')$ be a morphism in the comma category. Then we have a diagram:

$$\begin{tikzcd} & F\ker\alpha \arrow[r, "F\iota_\alpha"] \arrow[d, dashed, "\theta"] & FA \arrow[r, "F\alpha"] \arrow[d, "f"] & FA' \arrow[d, "f'"] \\ 0 \arrow[r] & \ker\beta \arrow[r, "\iota_\beta"] & B \arrow[r, "\beta"] & B' \end{tikzcd}$$

Since $\beta f F\iota_\alpha = f' F\alpha F\iota_\alpha = 0$ there is a unique θ making the diagram commute. I claim the kernel of (α, β) is $(\ker\alpha, \ker\beta, \theta)$. Indeed if (α', β') is any map such that $(\alpha, \beta) \circ (\alpha', \beta') = 0$ then $\alpha\alpha' = 0$ and $\beta\beta' = 0$ so both α' and β' factor uniquely through ι_α and ι_β.

$$\begin{tikzcd} FA'' \arrow[r, "\alpha''"] \arrow[d, "f''"] & F\ker\alpha \arrow[r, "F\iota_\alpha"] \arrow[d, dashed, "\theta"] & FA \arrow[d, "f"] \\ B'' \arrow[r, "\beta''"] & \ker\beta \arrow[r, "\iota_\beta"] & B \end{tikzcd}$$

The only thing left to verify is that the left square commutes. This follows from the outer rectangle commuting, and that ι_β is a monomorphism.

Showing that cokernels exists is similar, but relies on F being right exact. The construction is completely dual, but to verify commutativity at the end instead of using that ι_β is mono we must use that $F\pi_\alpha\colon FA' \to F\operatorname{Cok}\alpha$ is an epimorphism. This follows from F being right exact. We leave the details to the reader.

Now the image equaling the coimage follows from $\mathcal{A}$ and $\mathcal{B}$ being abelian, and the way we constructed the kernels and cokernels. $\square$

For the rest of this section we assume F is a right exact functor between abelian catgeories so that the comma category is abelian. We also assume $\mathcal{A}$ and $\mathcal{B}$ has enough projectives. In particular we are interested in the case when $\mathcal{A}$ and $\mathcal{B}$ are module categories over finite dimensional algebras.

Definition 2.7. For $\mathcal{A}$ and $\mathcal{B}$ abelian categories and F right exact we define the following functors:

$$\begin{aligned} T\colon \mathcal{A}\times\mathcal{B} &\longrightarrow (F,\mathcal{B})\\ (A,B) &\longmapsto (A, B\oplus FA, FA\hookrightarrow FA\oplus B)\\ (\alpha,\beta) &\longmapsto (\alpha, F\alpha\oplus\beta) \end{aligned}$$

$$\begin{aligned} U\colon (F,\mathcal{B}) &\longrightarrow \mathcal{A}\times\mathcal{B} & \qquad C\colon (F,\mathcal{B}) &\longrightarrow \mathcal{A}\times\mathcal{B}\\ (A,B,f) &\longmapsto (A,B) & (A,B,f) &\longmapsto (A,\operatorname{Cok} f)\\ (\alpha,\beta) &\longmapsto (\alpha,\beta) & (\alpha,\beta) &\longmapsto (\alpha,\hat{\beta}) \end{aligned}$$

$$\begin{aligned} Z\colon \mathcal{A}\times\mathcal{B} &\longrightarrow (F,B)\\ (A,B) &\longmapsto (A,B,0)\\ (\alpha,\beta) &\longmapsto (\alpha,\beta) \end{aligned}$$

Proposition 2.8. *With the definitions above U and Z become exact functors.*

Proof. Using the characterization of exact sequences shown in Proposition 2.6 a short exact sequence in $(F,\mathcal{B})$ is a commutative diagram

$$\begin{tikzcd} & FA'' \arrow[r, "F\alpha'"] \arrow[d, "f''"] & FA \arrow[r, "F\alpha"] \arrow[d, "f"] & FA' \arrow[r] \arrow[d, "f'"] & 0\\ 0 \arrow[r] & B'' \arrow[r, "\beta'"] & B \arrow[r, "\beta"] & B' \arrow[r] & 0 \end{tikzcd}$$

such that the sequences

$$0 \longrightarrow A'' \xrightarrow{\alpha'} A \xrightarrow{\alpha} A' \longrightarrow 0$$

$$0 \longrightarrow B'' \xrightarrow{\beta'} B \xrightarrow{\beta} B' \longrightarrow 0$$

are short exact. Since when we apply U we simply get the product of these two sequences, U is exact.

Similarly for Z since the two sequences we start with are assumed to be exact the resulting sequence will be exact by the characterization in Proposition 2.6. □

Proposition 2.9. *[FGR75, Proposition 1.3] The pairs of functors (T,U) and (C,Z) form adjoint pairs.*

Proof. We want to establish an isomorphism

$$\mathrm{Hom}(T(A,B),(A',B',FA'\to B'))\cong \mathrm{Hom}((A,B),(A',B')).$$

A morphism in $\mathrm{Hom}(T(A,B),(A',B',FA'\to B'))$ is given by a commutative diagram

$$\begin{tikzcd} FA \arrow[r, "\begin{bmatrix}0\\1\end{bmatrix}"] \arrow[d, "F\alpha"'] & B\oplus FA \arrow[d, "\begin{bmatrix}\beta & \gamma\end{bmatrix}"] \\ FA' \arrow[r, "f"] & B'. \end{tikzcd}$$

The isomorphism is then given by sending this to (α,β). This is clearly surjective.

For injectivity assume $(\alpha,\beta)=0$, then $\gamma=\begin{bmatrix}\beta & \gamma\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=fF\alpha=0$. So the map is injective, and (T,U) is an adjoint pair.

Next we consider (C,Z). We want an isomorphism

$$\begin{aligned}\mathrm{Hom}(C(A,B,f),(A',B')) &= \mathrm{Hom}((A,\operatorname{Cok} f),(A',B'))\\ &\cong \mathrm{Hom}((A,B,f),(A',B',0)).\end{aligned}$$

A morphism in $\mathrm{Hom}((A,B,f),(A',B',0))$ is a commutative diagram

$$\begin{tikzcd} FA \arrow[r, "f"] \arrow[d, "F\alpha"'] & B \arrow[d, "\beta"] \\ FA' \arrow[r, "0"] & B' \end{tikzcd}$$

Since $\beta f=0$, we have that β factors through the cokernel of f uniquely. Let the factorization be given by the map $\beta'\colon \operatorname{Cok} f\to B'$. Then we send this diagram to (α,β'). Since the choice of β' was unique this is an isomorphism, so (C,Z) is an adjoint pair. □

Corollary 2.9.1. *The functors T and C preserve projective objects.*

Proof. What we need to check is that for projective objects P and Q in $(\mathcal{A}\times\mathcal{B})$ and $(F,\mathcal{B})$ respectively we have that $\mathrm{Hom}(TP,-)$ and $\mathrm{Hom}(CQ,-)$ are exact. By adjointness these are equal to $\mathrm{Hom}(P,U-)$ and $\mathrm{Hom}(Q,Z-)$ respectively. Since U and Z are exact this holds, and so T and C preserve projective objects. □

Proposition 2.10. *[FGR75, Corollary 1.6c] For a projective object P in $(F, \mathcal{B})$ we have that $T(C(P)) \cong P$, in particular all projectives are of the form $T(P')$ for a projective $P' \in \mathcal{A} \times \mathcal{B}$.*

Proof. Let P be given by $f\colon FA \to B$. Applying C we get $(A, \operatorname{Cok} f)$. We have morphisms $P \to ZC(P)$ and $TC(P) \to ZC(P)$ given by the following diagram

$$\begin{tikzcd}
FA \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, two heads] \\
FA \arrow[r, "0"] \arrow[d, equal] & \operatorname{Cok} f \\
FA \arrow[r, hook] & \operatorname{Cok} f \oplus FA. \arrow[u, two heads]
\end{tikzcd}$$

By the projective property of P there is some morphism β factorizing the map $P \to ZC(P)$, which gives us the diagram:

$$\begin{tikzcd}
FA \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, "\beta"] \\
FA \arrow[r, hook] \arrow[d, equal] & \operatorname{Cok} f \oplus FA \arrow[d, two heads] \\
FA \arrow[r, "0"] & \operatorname{Cok} f.
\end{tikzcd}$$

Since $FA \hookrightarrow \operatorname{Cok} f \oplus FA$ is split mono, f is split mono, and consequently β is an isomorphism. So we have $P \cong TC(P)$. $\square$

Proposition 2.11. *[FGR75, Lemma 4.16] Let $X = (A, B, f)$ be an object in the comma category. Then $\operatorname{pd} X \geq \operatorname{pd} A$, and if $A = 0$ then $\operatorname{pd} X = \operatorname{pd} B$.*

Proof. We first show that $\operatorname{pd} X \geq \operatorname{pd} A$. Note that $\operatorname{pd} C(X) = \max\{\operatorname{pd} A, \operatorname{pd} \operatorname{Cok} f\}$ so we always have $\operatorname{pd} C(X) \geq \operatorname{pd} A$. If $\operatorname{pd} X = \infty$ then the statement holds so let us assume $\operatorname{pd} X = n < \infty$. We proceed by induction on n. If $n = 0$ then $C(X)$ is projective so $\operatorname{pd} X = \operatorname{pd} C(X) = \operatorname{pd} A = 0$. Next assume the statement holds whenever the projective dimension is less than n. Let $P \to A$ and $P' \to \operatorname{Cok} f$ be epimorphisms from projectives. Then we have an epimorphism $T(P, P') \to X$. If we let ΩA be the kernel of $P \to A$ and $X' = (\Omega A, K, \theta)$ be the kernel of $T(P, P') \to X$ as shown in the following diagram,

$$\begin{tikzcd}
 & F\Omega A \arrow[r] \arrow[d, "\theta"'] & FP \arrow[r] \arrow[d, hook] & FA \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K \arrow[r] & P' \oplus FP \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

then we have $\operatorname{pd} A \leq \operatorname{pd} \Omega A + 1$ and $\operatorname{pd} X = \operatorname{pd} X' + 1$. By induction we have that $\operatorname{pd} X' \geq \operatorname{pd} \Omega A$ and so $\operatorname{pd} X \geq \operatorname{pd} \Omega A + 1 \geq \operatorname{pd} A$.

If $A = 0$ then we can associate $C(X) = (0, B)$ with B. Any projective resolution $P_B^\bullet$ of B gives a resolution of X by $T(0, P_B^\bullet)$, and any resolution $P_X^\bullet$ of X gives a resolution of $(0, B)$ by $C(P_X^\bullet)$. Thus $\operatorname{pd} X = \operatorname{pd} B$. □

Theorem 2.12. *[FGR75, Theorem 4.20] The finitistic dimension of the comma category $(F, \mathcal{B})$ is bounded above by* $\operatorname{findim}(\mathcal{A}) + \operatorname{findim}(\mathcal{B}) + 1$.

Proof. Let $X = (A, B, f)$ be an element of the comma category with finite projective dimension. Let $P_A^\bullet$ be a projective resolution of A shorter than $\operatorname{findim}(\mathcal{A})$. Similar to what we did in Proposition 2.11 define P_X^0 to be $T(P_A^0, P(\operatorname{Cok} f))$ where $P(\operatorname{Cok} f)$ is a projective module with an epimorphism onto $\operatorname{Cok} f$. Then we have that the kernel of $P_X^0 \to X$ is $F\Omega A \xrightarrow{\theta^0} K^0$. We continue inductively defining P_X^n to be $T(P_A^n, \operatorname{Cok} \theta^{n-1})$. Then $\Omega^{\operatorname{findim}(\mathcal{A})+1} X = (0, K^{\operatorname{findim}(\mathcal{A})}, 0)$. Then by Proposition 2.11 we know that $\operatorname{pd} \Omega^{\operatorname{findim}(\mathcal{A})+1} X = \operatorname{pd} K^{\operatorname{findim}(\mathcal{A})} \leq \operatorname{findim}(\mathcal{B})$. So

$$\operatorname{pd} X \leq \operatorname{findim}(\mathcal{A}) + \operatorname{findim}(\mathcal{B}) + 1.$$

□

Example 2.13. If k is a field, $\mathcal{A} = \mathcal{B} = \operatorname{mod} k$ and F is the identity, then the comma category $(F, \mathcal{B})$ is equivalent to the category of finite dimensional representations of A_2 over k. Then $\mathcal{A}$ and $\mathcal{B}$ both have finitistic dimension 0 while $(F, \mathcal{B})$ has finitistic dimension 1. So the bound shown above is tight.

Definition 2.14 (Triangular matrix ring)**.** Let R and S be rings, and let M be an S-R-bimodule. Then the *triangular matrix ring* $\begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$ is the ring of all matricies $\begin{bmatrix} r & 0 \\ m & s \end{bmatrix}$ with $r \in R$, $s \in S$, and $m \in M$. The multplication is given by

$$\begin{bmatrix} r & 0 \\ m & s \end{bmatrix} \begin{bmatrix} r' & 0 \\ m' & s' \end{bmatrix} = \begin{bmatrix} rr' & 0 \\ mr' + sm' & ss' \end{bmatrix}.$$

Notice, if N is a module over the matrix ring $\begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$, then as an abelian group N splits as a direct sum into

$$N = N_R \oplus N_S := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} N \oplus \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} N.$$

By restriction of scalars we can think of N_R as an R-module and N_S as an S-module. Further multiplication by $\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$ is 0 on N_S and maps N_R into N_S. So N consists of an R-module N_R, an S-module N_S and a S-R-linear map $M \to \operatorname{Hom}_{\mathbb{Z}}(N_R, N_S)$, or equivalently a S-linear map $M \otimes_R N_R \to N_S$. This means that mod $\begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$ is equivalent to the comma category $(\operatorname{mod} R, \operatorname{mod} S, M \otimes_R -)$. So we have that

$$\operatorname{findim} \begin{pmatrix} R & 0 \\ M & S \end{pmatrix} \leq \operatorname{findim}(R) + \operatorname{findim}(S) + 1.$$

Example

3 Contravariant finiteness

Results are generalized in [Trl01]

Definition 3.1 (Resolving)**.** A full subcategory of an abelian category is called *resolving* if

i) It is closed under extensions.

ii) It contains the projectives.

iii) It is contains the kernels of its epimorphisms.

Note that the subcategory of modules with finite projective dimension is resolving.

Lemma 3.2. *Let $\mathcal{X}$ be resolving. Then $\operatorname{Ext}^1(\mathcal{X}, Y) = 0$ implies that $\operatorname{Ext}^i(\mathcal{X}, Y) = 0$ for all $i \geq 1$.*

Proof. Since $\mathcal{X}$ contains the projectives, ΩX is the kernel of an epimorphism in $\mathcal{X}$. Thus $\mathcal{X}$ contains all syzygies. $\operatorname{Ext}^i(X, Y) = \operatorname{Ext}^1(\Omega^{i-1} X, Y) = 0$. □

Proposition 3.3. *If $\mathcal{X}$ is resolving, then $\mathcal{Y} := \ker \operatorname{Ext}^{\geq 1}(\mathcal{X}, -) = \ker \operatorname{Ext}^1(\mathcal{X}, -)$ is closed under extensions.*

Proof. Let $0 \to Y \to E \to Y' \to 0$ be an extension of objects in $\mathcal{Y}$, and let X be an object of $\mathcal{X}$. Then we get an exact sequence

$$0 = \mathrm{Ext}^i(X, Y) \longrightarrow \mathrm{Ext}^i(X, E) \longrightarrow \mathrm{Ext}^i(X, Y') = 0$$

Thus $\mathrm{Ext}^i(X, E) = 0$ for all $i \geq 1$ and E is in $\mathcal{Y}$. $\square$

Lemma 3.4. *Let $\mathcal{X}$ be a contravariantly finite, resolving subcategory of $\operatorname{mod}\Lambda$. Then for every object $C \in \operatorname{mod}\Lambda$ there is a short exact sequence*

$$0 \to Y \to X \to C \to 0$$

with $X \to C$ minimal $\mathcal{X}$-approximation and $\mathrm{Ext}^i(\mathcal{X}, Y) = 0$ for all $i \geq 1$.

Proof. Since $\mathcal{X}$ is contravariantly finite, C has a minimal $\mathcal{X}$-approximation $X \to C$. Since $\mathcal{X}$ contains the projective cover of C this approximation must be an epimorphism. So it is part of a short exact sequence

$$0 \to Y \to X \to C \to 0.$$

Let X' be an arbitrary object in $\mathcal{X}$. Taking the long exact sequence in $\mathrm{Ext}(X', -)$ gives us

$$\begin{tikzcd}
\mathrm{Hom}(X', Y) \arrow[r] & \mathrm{Hom}(X', X) \arrow[r] & \mathrm{Hom}(X', C) \arrow[dll] \\
\mathrm{Ext}^1(X', Y) \arrow[r] & \mathrm{Ext}(X', X)^1 \arrow[r] & \mathrm{Ext}^1(X', C)
\end{tikzcd}$$

Since $X \to C$ is an approximation, we know that $\mathrm{Hom}(X', X) \to \mathrm{Hom}(X', C)$ is epi. Thus if we can prove that $\mathrm{Ext}^1(X', X) \to \mathrm{Ext}^1(X', C)$ is mono we would have that $\mathrm{Ext}^1(X', Y) = 0$. Assume we have an element of $\mathrm{Ext}^1(X', X)$ that is mapped to 0, i.e. we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] & C \oplus X' \arrow[r] & X' \arrow[r] & 0
\end{tikzcd}$$

Since $\mathcal{X}$ is closed under extensions E is in $\mathcal{X}$. By composing with projection $C \oplus X' \to C$ we get a commutative triangle

$$\begin{array}{ccc} X & \longrightarrow & E \\ \downarrow & \swarrow & \\ C & & \end{array}$$

since $X \to C$ is an approximation we get that $E \to C$ factors through X. The endomorphism $X \to E \to X$ leaves the approximation unchanged, so by minimality it must be an isomorphism. Hence

$$0 \to X \to E \to X' \to 0$$

is split and $\mathrm{Ext}(X', X) \to \mathrm{Ext}(X', C)$ is injective. Thus $\mathrm{Ext}(X', Y) = 0$. $\square$

Theorem 3.5. *[AR91, 3.8] Let $\mathcal{X}$ be a contravariantly finite, resolving subcategory of* $\mathrm{mod}\,\Lambda$*. Let X_i be the minimal approximation of S_i. Then any $X \in \mathcal{X}$ is a direct summand of an X_i-filtered module.*

Proof. The first part of the proof is to show by induction on length that any module C is in an exact sequence $0 \to Y \to X \to C \to 0$ with X X_i-filtered and $\mathrm{Ext}^1(\mathcal{X}, Y) = 0$.

For the base case if $C = S_i$ is simple then by Lemma 3.4 we have an exact sequence $0 \to Y \to X_i \to C \to 0$ with the desired properties stated above.

For the induction step, assume it holds for all modules of length less than n, and let C be a module of length n. Then by Jordan-Hölder C is the extension of two modules of length less than n. Say

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

Applying the induction hypothesis we get a diagram on the form

$$\begin{array}{ccccccccc} & & 0 & & & & 0 & & \\ & & \downarrow & & & & \downarrow & & \\ & & Y' & & & & Y'' & & \\ & & \downarrow & & & & \downarrow & & \\ & & X' & & & & X'' & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & 0 & & & & 0 & & \end{array}$$

Taking the pullback of $X'' \to C''$ we get a diagram

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Since Y' satisfies $\mathrm{Ext}^1(\mathcal{X}, Y') = 0$ by Lemma 3.2 it also satisfies $\mathrm{Ext}^2(\mathcal{X}, Y') = 0$. In particular from the long exact sequence

$$0 = \mathrm{Ext}^1(X'', Y) \to \mathrm{Ext}^1(X'', X') \to \mathrm{Ext}^1(X'', C) \to \mathrm{Ext}^2(X'', Y) = 0$$

we get that $X' \to C'$ induces an isomorphism $\mathrm{Ext}^1(X'', X') \to \mathrm{Ext}^1(X'', C)$. Thus the short exact sequence $0 \to C' \to E \to X'' \to 0$ must come from a sequence $0 \to X' \to X \to X'' \to 0$. This gives us a diagram

$$\begin{tikzcd} & 0 \arrow[d] & & 0 \arrow[d] & \\ & Y' \arrow[d] & & Y'' \arrow[d] & \\ 0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] & C'' \arrow[r] \arrow[d] & 0 \\ & 0 & & 0 & \end{tikzcd}$$

Applying the Snake Lemma we can fill out the diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since X is an extension of X_i-filtered modules, it is also X_i-filtered. Since Y is the extension of Y'' and Y' it follows from Proposition 3.3 that $\mathrm{Ext}(\mathcal{X}, Y) = 0$.

Hence any C fits into a sequence $0 \to Y \to X \to C \to 0$ with X being X_i-filtered and $\mathrm{Ext}^{\geq 1}(\mathcal{X}, Y) = 0$.

Now suppose that C is in $\mathcal{X}$, and let $0 \to Y \to X \to C \to 0$ be as before. Then we get that

$$\mathrm{Hom}(C, X) \longrightarrow \mathrm{Hom}(C, C) \longrightarrow \mathrm{Ext}^1(C, Y) = 0$$

is exact, and thus C is a direct summand of X. So every object in $\mathcal{X}$ is a direct summand of an X_i-filtered module. □

Corollary 3.5.1. *If the subcategory of modules with finite projective dimension is contravariantly finite, then the finitistic dimension is the supremum of the projective dimension of X_i. In particular it is finite.*

Example 3.6. [IST90, Proposition 2.3] Let Λ be the path algebra of

$$\begin{tikzcd}
1 \arrow[r, bend left, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[l, bend left, "\gamma"]
\end{tikzcd}$$

with relations $\alpha\gamma$, $\beta\gamma$, and $\gamma\alpha$ over an algebraically closed field k. Then $\mathrm{findim}(\Lambda) = 1$, but the subcategory of modules with finite projective dimension is not contravariantly finite.

Proof. The indecomposable projective Λ-modules are given by the following quivers

$$
\begin{tikzcd}
& 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
2 & & 2 \arrow[d, "\gamma"] \\
& & 1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
2 \arrow[d, "\gamma"] \\
1
\end{tikzcd}
$$

Note that both the indecomposable projectives have even dimension, so any projective module has even dimension. Then if X is a module with finite projective dimension, since $\dim X = \sum(-1)^i \dim P_X^i$ the dimension of X is also even. In particular the two simple modules have infinite projective dimension.

The radical of P_1 is P_2 and the radical of P_2 is S_1, so the radical of an arbitrary projective looks like $P_2^n \oplus S_1^m$. Let $P \to X$ be the projective cover of a module with finite projective dimension. Then ΩX is a submodule of $JP = P_2^n \oplus S_1^m$. Let M be an indecomposable summand of ΩX, and consider the composition $M \to JP \to P_2$ for any possible projection to P_2. If this is epi then we must have $M = P_2$. If none of these are epi then M is contained in $JP_2^n \oplus S_1^m = S_1^{m+n}$. This would mean $M = S_1$, but S_1 has infinite projective dimension. Thus we must have ΩX projective, and so $\operatorname{pd} X \leq 1$.

Nextly we want to show that S_1 has no minimal approximation by modules with finite projective dimension. Assume for the sake of contradiction that $X \to S_1$ is such a minimal approximation. Then we claim that P_2 is not a submodule of X. Since $\operatorname{Hom}(P_2, S_1) = 0$ if this were the case then $X' = X/P_2$ would give an approximation of shorter length, because X' would also have finite projective dimension. Which can be seen in the diagram below.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & P_X^1 \arrow[r] \arrow[d, equal] & P_X^1 \oplus P_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_X^1 \arrow[r] & P_X^0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & & X' \arrow[r, equal] \arrow[d] & X' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

make a notation chapter setting notation for projective cover / resolution

This means that $\gamma X = 0$, because if there was an element $x \in X$ with $\gamma x \neq 0$, then $(e_2 x)$ would be a submodule of X isomorphic to P_2. So X is a $\Lambda/(\gamma)$ module.

The algebra $\Lambda/(\gamma)$ is the path algebra of the 2-Kronecker quiver, whose representation theory is well understood. Specifically $\Lambda/(\gamma)$ can be associated with the subquiver highlighted below.

$$\begin{tikzcd}
1 \arrow[r, bend left, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[l, bend left, "\gamma"]
\end{tikzcd}$$

The indecomposable modules are as given in the table below.

$$\begin{tikzcd}
k^n \arrow[r, bend left, "\begin{bmatrix} I_n \\ 0 \end{bmatrix}"] \arrow[r, "\begin{bmatrix} 0 \\ I_n \end{bmatrix}"'] & k^{n+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
k^n \arrow[r, bend left, "J(n{,}\lambda)"] \arrow[r, "I_n"'] & k^n \\
k^n \arrow[r, bend left, "I_n"] \arrow[r, "J(n{,}0)"'] & k^n
\end{tikzcd}
\qquad
\begin{tikzcd}
k^{n+1} \arrow[r, bend left, "\begin{bmatrix} I_n & 0 \end{bmatrix}"] \arrow[r, "\begin{bmatrix} 0 & I_n \end{bmatrix}"'] & k^n
\end{tikzcd}$$

preprojective regular preinjective

We see that the preprojective and preinjective modules both have odd dimension, so they will have infinite projective dimension as Λ-modules. We can easily verify that the $\Lambda/(\gamma)$-modules $k \underset{1}{\overset{\lambda}{\rightrightarrows}} k$ all have finite projective dimension as Λ-modules and that they have a nonzero map onto S_1. So each of these modules would need to have a nonzero map to X. But it is easy to verify that there is a nonzero homomorphism between the regular modules only if they have the same value of λ. So for it to be possible for X to factorize all these maps we would need X to have an infinite amount of direct summands. Since we are working with finitely generated modules this is impossible, hence S_1 has no approximation, and the subcategory is not contravariantly finite. $\square$

In the next example we look at the opposite algebra of Λ to show that there is not necessarily any link between the contravariant finiteness for Λ and for Λ^{op}.

Example 3.7. Let Γ be the opposite algebra of the one in Example 3.6. That is, Γ is the path algebra of

$$\begin{tikzcd} 2 \arrow[r, bend left=50, "\hat{\alpha}"] \arrow[r, "\hat{\beta}"] & 1 \arrow[l, bend left=50, "\hat{\gamma}"] \end{tikzcd}$$

with relations $\hat{\gamma}\hat{\alpha}$, $\hat{\gamma}\hat{\beta}$, and $\hat{\alpha}\hat{\gamma}$. Then the subcategory of modules with finite projective dimension is contravariantly finite. In other words the subcategory of Λ-modules with finite injective dimension is covariantly finite.

Proof. The indecomposable projective Γ-modules are given by the following quivers

$$\begin{tikzcd} 1 \arrow[d, "\hat{\gamma}"] \\ 2 \arrow[d, "\hat{\beta}"] \\ 1 \end{tikzcd} \qquad \begin{tikzcd} & 2 \arrow[dl, "\hat{\alpha}"'] \arrow[dr, "\hat{\beta}"] & \\ 1 & & 1 \end{tikzcd}$$

Similar to before before, notice that the indecomposable projective modules are 3-dimensional and thus every module with finite projective dimension will have a k-dimension that is a multiple of 3. So in particular the simple modules have infinite projective dimension.

Let X be a module with finite projective dimension, and let P be its projective cover. We have that ΩX is a submodule of JP. Notice that $\hat{\alpha}J = \hat{\gamma}J = 0$, so ΩX is a $\Gamma/(\hat{\alpha}, \hat{\gamma})$-module. But $\Gamma/(\hat{\alpha}, \hat{\gamma})$ is simply isomorphic to the path algebra of $2 \longrightarrow 1$, over which there are just 3 indecomposable modules. We already know that the simple modules cannot be summands of ΩX, because they have infinite projective dimension. The non-simple module $k \xrightarrow{1} k$ is 2-dimensional and thus also has infinite projective dimension over Γ. So we conclude that $\Omega X = 0$, so X is projective.

So the only modules with finite projective dimension are the projectives themselves. In particular there are only a finite number of indecomposable modules with finite projective dimension. So the subcategory is contravariantly finite. $\square$

4 The Igusa-Todorov functions

In this section we let K_0 be the abelian group generated by isomorphism classes of modules in $\operatorname{mod}\Lambda$, with the relations that $[A \oplus B] - [A] - [B] = 0$ for any modules A and B, and $[P] = 0$ when P is projective. We define the linear map $L\colon K \to K$ by $L[A] = [\Omega A]$. For any module X, we let $[\operatorname{add} X]$ be the finitely generated subgroup of K_0 generated by modules in $\operatorname{add} X$. Fitting's lemma tells us that there is an integer η_X such that $L\colon L^m[\operatorname{add} X] \to L^{m+1}[\operatorname{add} X]$ is an isomorphism for every $m \geq \eta_X$. We use this to define two important functions from $\operatorname{mod}\Lambda$ to $\mathbb{N}$.

maybe make appendix

Definition 4.1 (The Igusa–Todorov functions)**.** We define two functions ϕ and ψ from $\operatorname{mod}\Lambda$ to $\mathbb{N}$. For a module $M \in \operatorname{mod}\Lambda$ we define $\phi(M)$ to be the integer η_M coming from Fitting's lemma, as explained above. In other words, $\phi(M)$ is the smallest integer such that

$$L\colon L^m[\operatorname{add} M] \to L^{m+1}[\operatorname{add} M]$$

is an isomorphism for every $m \geq \phi(M)$. We define $\psi(M)$ in a similar way, but adding on an extra term to account for the structure of $\Omega^{\phi(M)}M$.

$$\psi(M) = \phi(M) + \sup\left\{\operatorname{pd} Z \;\middle|\; \operatorname{pd} Z < \infty, Z \in \operatorname{add}\Omega^{\phi(M)}M\right\}$$

Lemma 4.2. *[IT05, Lemma 3]*

i) $\psi(M) = \operatorname{pd} M$*, when* $\operatorname{pd} M < \infty$.

ii) $\psi(M^k) = \psi(M)$.

iii) $\psi(M) \leq \psi(M \oplus N)$.

iv) *If* Z *is a direct summand of* $\Omega^n(M)$ *where* $n \leq \phi(M)$ *and* $\operatorname{pd} Z < \infty$, *then* $\operatorname{pd} Z + n \leq \psi(M)$.

Proof.

i) If $\operatorname{pd} M < \infty$, then $L^m \neq 0$ for $m < \operatorname{pd} M$, and $L^m = 0$ for $m \geq \operatorname{pd} M$. So $\psi(M) = \phi(M) = \operatorname{pd} M$.

ii) The subcategory $\operatorname{add} M^k = \operatorname{add} M$, and ψ is defined only in terms of the additive subcategory $\operatorname{add} M$.

iii) The subcategory $\operatorname{add} M$ is contained in $\operatorname{add} M \oplus N$, so if L is injective when restricted to $L^m(\operatorname{add} M \oplus N)$ then L is injective when restricted to $L^m(\operatorname{add} M)$. Thus we have $\phi(M) \leq \phi(M \oplus N)$. Further

$$\Omega^{\phi(M\oplus N)-\phi(M)} \left(\operatorname{add} \Omega^{\phi(M)} M\right) \subseteq \operatorname{add} \Omega^{\phi(M\oplus N)} M \oplus N,$$

so $\psi(M) \leq \psi(M \oplus N)$.

iv) Let $p = \operatorname{pd} Z$ and $k = \phi(M) - n$. Then $\Omega^k Z$ is in $\operatorname{add} \Omega^{\phi(M)} M$, so $\operatorname{pd} \Omega^k Z + \phi(M) \leq \psi(M)$. Thus

$$\operatorname{pd} Z + n = p + n = (p - k) + \phi(M) \leq \operatorname{pd} \Omega^k Z + \phi(M) \leq \psi(M).$$

□

Theorem 4.3. *[IT05, Theorem 4] Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence of modules with* $\operatorname{pd} C < \infty$. *Then* $\operatorname{pd} C \leq \psi(A \oplus B) + 1$.

Proof. Let $P_A^\bullet$ and $P_C^\bullet$ be the minimal projective resolutions of A and C. Then we get a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_A^0 & \longrightarrow & P_A^0 \oplus P_C^0 & \longrightarrow & P_C^0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Applying the Snake Lemma we get $0 \to \Omega A \to \Omega B \oplus P \to \Omega C \to 0$ for some projective module P. Thus for some $n \leq \operatorname{pd} C$ we have $L^n[A] = L^n[B]$, and let n be the minimal such number. Clearly $n \leq \phi(A \oplus B)$. Let $X = \Omega^n A = \Omega^n B$, then our sequence of n-syzygies looks like

$$0 \longrightarrow X \longrightarrow X \oplus P \longrightarrow \Omega^n C \longrightarrow 0.$$

Let f be the composition $X \longrightarrow X \oplus P \xrightarrow{\pi_X} X$. Then by Fitting's lemma X breaks as a direct sum into two components $X = Z \oplus Y$ such that $f = f_Z \oplus f_Y$ with f_Y an isomorphism and f_Z nilpotent. In other words the sequence above can be written as

$$0 \longrightarrow Z \oplus Y \longrightarrow Z \oplus Y \oplus P \longrightarrow \Omega^n C \longrightarrow 0.$$

with the left map being

$$\begin{bmatrix} f_Z & 0 \\ 0 & f_Y \\ * & * \end{bmatrix} \sim \begin{bmatrix} f_Z & 0 \\ 0 & 1_Y \\ * & 0 \end{bmatrix}$$

So by changing basis this restricts to another short exact sequence

$$0 \longrightarrow Z \longrightarrow Z \oplus P \longrightarrow \Omega^n C \longrightarrow 0.$$

Let $T = \Lambda/J$ and apply the long exact sequence in $\mathrm{Ext}(-,T)$. Then we get an exact sequence

$$\mathrm{Ext}^k(Z,T) \longrightarrow \mathrm{Ext}^k(Z \oplus P,T) \longrightarrow \mathrm{Ext}^{k+1}(\Omega^n C,T)$$

where the left map is induced by f_Z since $\mathrm{Ext}^k(Z\oplus P,T) \cong \mathrm{Ext}^k(Z,T)$. Since f_Z is nilpotent this map is surjective if and only if $\mathrm{Ext}^k(Z,T) = 0$. We know that, since $\Omega^n C$ has finite projective dimension, $\mathrm{Ext}^{k+1}(\Omega^n C,T)$ is 0 for k large enough. Then we must have that $\mathrm{Ext}^k(Z,T) = 0$, and thus Z has finite projective dimension. Specifically we have $\operatorname{pd}\Omega^n C - 1 \leq \operatorname{pd} Z \leq \operatorname{pd}\Omega^n C$.

Since Z is a direct summand of $\Omega^n(A \oplus B)$, by Lemma 4.2 we have that $\operatorname{pd} Z + n \leq \psi(A \oplus B)$, and thus $\operatorname{pd}\Omega^n C - 1 + n = \operatorname{pd} C - 1 \leq \psi(A \oplus B)$. □

Corollary 4.3.1. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of modules.*

i) If $\operatorname{pd} A < \infty$, then $\operatorname{pd} A \leq \psi(\Omega B \oplus \Omega C) + 1$.

ii) If $\operatorname{pd} B < \infty$ then $\operatorname{pd} B \leq \psi(\Omega A \oplus \Omega^2 C) + 2$.

Proof. Let $P_B \to B$ be a projective cover of B. Then we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & P_B \arrow[r] \arrow[d] & P_B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Applying the Snake Lemma we get a short exact sequence

$$0 \to \Omega B \to \Omega C \oplus P \to A \to 0$$

for some projective module P. Then using the theorem we have that if $\operatorname{pd} A \leq \infty$, then $\operatorname{pd} A \leq \psi(\Omega B \oplus \Omega C \oplus P) + 1 = \psi(\Omega B \oplus \Omega C) + 1$.

Applying the same reasoning to $0 \to \Omega B \to \Omega C \oplus P \to A \to 0$ gives us that if $\operatorname{pd} B \leq \infty$, then $\operatorname{pd} \Omega B \leq \psi(\Omega A \oplus \Omega^2 C) + 1$. Hence $\operatorname{pd} B \leq \psi(\Omega A \oplus \Omega^2 C) + 2$. $\square$

In the following sections we will apply this theory to show that $\operatorname{findim}(\Lambda) < \infty$ for some families of algebras.

4.1 Representation dimension

In this section we look at the representation dimension of an algebra. This is another useful homological invariant of the representation theory for a finite dimensional algebra. The representation dimension is less than or equal to 2 if and only if Λ is representation finite, so it is natural to think that the representation dimension in some sense measures the *complexity* of $\operatorname{mod} \Lambda$. In Corollary 4.9.1 we show that $\operatorname{findim}(\Lambda) < \infty$ when $\operatorname{repdim}(\Lambda) \leq 3$, and in section* we give examples of classes of algebras that satisfy this.

Definition 4.4 (Representation dimension)**.** Let Λ be a finite dimensional algebra. The *representation dimension* of Λ, denoted $\operatorname{repdim}(\Lambda)$, is the minimal global dimension of $\operatorname{End}(M)^{\mathrm{op}}$ for M a generator-cogenerator in $\operatorname{mod} \Lambda$. We call a generator-cogenerator that achieves this minimum an *Auslander-generator.*

Definition 4.5 ($\mathcal{M}$-resolutions)**.** Let X be an object in $\operatorname{mod} \Lambda$ and $\mathcal{M}$ a contravariantly finite subcategory. We consider a diagram as the one below.

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr, two heads] & M_2 \arrow[r] \arrow[dr, two heads] & M_1 \arrow[r] \arrow[dr, two heads] & M_0 \arrow[dr, two heads] & \\
& \Omega^3_M X \arrow[u, hook] & \Omega^2_M X \arrow[u, hook] & \Omega_M X \arrow[u, hook] & X
\end{tikzcd}$$

If the maps $M_n \twoheadrightarrow \Omega^n_M X$ are minimal right $\mathcal{M}$-approximations for $n \geq 0$ (they need not be surjective), and $\Omega^{n+1}_M \hookrightarrow M_n$ are their kernels, then this is a minimal *$\mathcal{M}$-resolution* of X. The *$\mathcal{M}$-res-dimension* of X is the length of this sequence of (nonzero) M_i's, and the $\mathcal{M}$-res-dimension of Λ is the supremum of the dimension on its objects.

Proposition 4.6. *If the representation dimension of Λ is at least 2, then $\operatorname{repdim}(\Lambda)-2$ equals the minimum of M-res-dim$(\operatorname{mod}\Lambda)$ for M both generator and cogenerator. In fact, for any generator-cogenerator, M-res-dim$(\operatorname{mod}\Lambda)$ is two less than the global dimension of $\operatorname{End}(M)^{\mathrm{op}}$.*

Proof. Let M be a generator-cogenerator. We first show that the global dimension of $\operatorname{End}(M)^{\mathrm{op}}$ is less than or equal to M-res-dim$(\operatorname{mod}\Lambda) + 2$.

The functor $\operatorname{Hom}(M,-)$ is an equivalence from $\operatorname{add} M$ to $\operatorname{proj}\operatorname{End}(M)^{\mathrm{op}}$, which maps minimal M-approximations to projective covers. Let X be any module in $\operatorname{mod}\operatorname{End}(M)^{\mathrm{op}}$ with projective dimension at least 2. Then it has a projective presentation

$$\Omega^2 X \to (M, M_1) \to (M, M_0) \to X.$$

Because of the equivalence this is induced by a map $f\colon M_1 \to M_0$. Since $\operatorname{Hom}(M,-)$ is left exact we have that $\Omega^2 X \cong \operatorname{Hom}(M, \ker f)$, and so the projective dimension of X is 2 plus the M-res-dimension of $\ker f$. Hence the global dimension $\operatorname{End}(M)^{\mathrm{op}}$ is less than or equal to M-res-dim$(\operatorname{mod}\Lambda) + 2$.

Next we prove the other inequality.

Since M is a cogenerator any module Y in $\operatorname{mod}\Lambda$ has a copresentation

$$0 \longrightarrow Y \longrightarrow M_0 \xrightarrow{f} M_1.$$

Applying $(M,-) := \operatorname{Hom}(M,-)$ we get

$$0 \longrightarrow (M,Y) \longrightarrow (M,M_0) \xrightarrow{(M,f)} (M,M_1) \longrightarrow \operatorname{Cok}(M,f) \longrightarrow 0.$$

If the projective dimension of $\operatorname{Cok}(M,f)$ is less than 2, then (M,Y) is a direct summand of (M,M_0). This means that $(M,Y) \cong (M,M')$, so the minimal M-approximation of Y is M', and $(M,\Omega_M Y) = 0$. Since M is a generator this means $\Omega_M Y = 0$ and thus the M-res-dimension of Y is 0.

So provided the projective dimension of $\operatorname{Cok}(M,f)$ is larger than or equal to 2, it equals the M-res-dimension of Y plus 2. In particular the global dimension of $\mathrm{End}(M)^{\mathrm{op}}$ is larger than or equal to M-res-dim$(\mathrm{mod}\,\Lambda) + 2$. Hence they are equal. $\square$

Proposition 4.7. *The representation dimension of an artin algebra is always finite. [Iya02]*

Theorem 4.8. *The representation dimension of Λ is less than or equal to 2 if and only if Λ is representation finite.*

Proof. Assume Λ is representation finite and let M be the direct sum of all indecomposable modules (up to iso). Then M is a generator-cogenerator. Let X be an $\mathrm{End}(M)^{\mathrm{op}}$-module with projective presentation

$$(M, M_1) \to (M, M_0) \to X \to 0.$$

Let M_2 be the kernel of $M_1 \to M_0$. Since M is the sum of all indecomposables M_2 is in $\mathrm{add}\, M$, so

$$0 \to (M, M_2) \to (M, M_1) \to (M, M_0) \to X \to 0$$

is a projective resolution of X. So Λ has representation dimension at most 2.

Assume Λ has representation dimension at most 2, and let M be an Auslander-generator. We want to show that $\mathrm{add}\, M = \mathrm{mod}\,\Lambda$. Let X be any Λ-module, and let

$$0 \to X \to I_0 \to I_1$$

be a minimal injective presentation. If $I_0 \to I_1$ is split then X is injective and thus in $\mathrm{add}\, M$. Let M_X be a minimal M-approximation of X, let $\Omega_M X$ be the kernel of the approximation, and let Y be the cokernel of $(M, I_0) \to (M, I_1)$. Then

$$(M, \Omega_M X) \to (M, M_X) \to (M, I_0) \to (M, I_1) \to Y \to 0$$

is a minimal exact sequence. Since the global dimension of $\mathrm{End}(M)^{\mathrm{op}}$ is at most 2 this means that $(M, \Omega_M X) = 0$. Consequently we have that $\Omega_M X = 0$ and that $X = M_X$, so X is in $\mathrm{add}\, M$. Thus Λ is representation finite. $\square$

Theorem 4.9. *[IT05, Corollary 8] If $\Lambda = \mathrm{End}_\Gamma(P)^{\mathrm{op}}$ for an algebra Γ with global dimension at most 3, and P projective, then* $\mathrm{findim}(\Lambda) < \infty$.

Proof. Let X be any Λ-module with finite projective dimension. Then it has a projective presentation $(P, P_1) \to (P, P_0) \to X \to 0$ where $(P, P_i) = \mathrm{Hom}_\Gamma(P, P_i)$ with $P_i \in \mathrm{add}\, P$. Since $(P, -)$ is an equivalence from $\mathrm{add}\, P$ to $\mathrm{proj}\,\Lambda$ this corresponds to a map $P_1 \to P_0$ which we can extend to a projective resolution in Γ:

$$0 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0.$$

Applying the exact functor $(P, -)$, we get an exact sequence

$$0 \longrightarrow (P, P_3) \longrightarrow (P, P_2) \longrightarrow (P, P_1) \longrightarrow (P, P_0) \longrightarrow X \longrightarrow 0.$$

Truncating this we get a short exact sequence

$$0 \longrightarrow (P, P_3) \longrightarrow (P, P_2) \longrightarrow \Omega^2 X \longrightarrow 0.$$

Then by Theorem 4.3 the projective dimension of $\Omega^2 X$ is bounded by $\psi((P, P_3) \oplus (P, P_2)) + 1$. Which means

$$\operatorname{pd} X \leq \psi((P, P_3) \oplus (P, P_2)) + 3 \leq \psi((P, \Gamma)) + 3$$

Since this bound doesn't depend on X, Λ has finite finitistic dimension. $\square$

Corollary 4.9.1. *If* $\operatorname{repdim}(\Lambda) \leq 3$ *then* $\operatorname{findim}(\Lambda) < \infty$.

Proof. If Λ has rep-dimension less than or equal to 3 then by **??** there is a generator-cogenerator M in $\operatorname{mod} \Lambda$ such that $\Gamma := \operatorname{End}_\Lambda(M)$ has global dimension 3 or less. Then since M is a generator Λ is in $\operatorname{add} M$ and so $\operatorname{Hom}_\Lambda(M, \Lambda)$ is a projective Γ-module with $\operatorname{End}_\Gamma(\operatorname{Hom}_\Lambda(M, \Lambda)) = \operatorname{End}_\Lambda(\Lambda) = \Lambda$. $\square$

4.2 Stably hereditary algebras

In this section we will show that the class of stably hereditary algebras has repdimension at most 3, and thus that they have finite finitistic dimension.

Definition 4.10 ((co)torsionfree)**.** A module is called *torsionfree* if it is a submodule of a projective module. Dually, a module is called *cotorsionfree* if it is a factormodule of an injective.

Definition 4.11 (Stably hereditary algebra)**.** An algebra is called *stably hereditary* if any indecomposable torsionfree module is projective or simple, and any indecomposable cotorsionfree moule is injective or simple.

This generalizes the definition of hereditary algebra by also allowing simple modules to be (co)torsionfree.

Definition 4.12 (The stable category)**.** For an algebra Λ, *the stable category* $\underline{\text{mod}}\Lambda$ has the same objects as $\text{mod}\,\Lambda$, but the sets of homomorphisms are given by

$$\text{Hom}_{\underline{\text{mod}}\Lambda}(M, N) = \text{Hom}_\Lambda(M, N)/\mathcal{P}(M, N)$$

where $\mathcal{P}(M, N)$ is the ideal of all morphisms factoring through a projective.

Proposition 4.13. *If for an algebra Λ there is a hereditary algebra H such that $\underline{\text{mod}}\Lambda \cong \underline{\text{mod}}H$ then Λ is stably hereditary.*

Proof. [AR73, Chapter IV, Theorem 1.5] □ write

The converse of the above proposition does not hold without more assumptions, but stably hereditary algebras generalize the idea of algebras stably equivalent to hereditary algebras.

Theorem 4.14. *[Xi02, Theorem 3.5] Stably hereditary algebras has repdimension at most 3.*

Proof. Let V be the direct sum of all the indecomposable projectives, all the indecomposable injectives, and all the simple modules. Then V is a generator-cogenerator. So if we show that the global dimension of $\Gamma := \text{End}(V)^{op}$ is 3 or less, then we are done.

We will show that for any Λ-module M there is a short exact sequence $0 \to V_3 \to V_3 \to M \to 0$ with V_i in $\text{add}\, V$, and such that $0 \to (V, V_3) \to (V, V_2) \to (V, M) \to 0$ is exact. We will use this to construct short projective resolutions for $\text{mod}\,\Gamma$. To construct V_3 and V_2 let M' be the sum of the maximal injective summand of M and all simple submodules of M. Then let P be the projective cover of M/M'. Taking the pullback of $M \to M/M' \leftarrow P$ gives us the diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & M' \oplus P \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] \arrow[d] & M/M' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

I claim that $0 \to K \to M' \oplus P \to M \to 0$ is the desired sequence. Firstly $M' \oplus P$ is clearly in $\operatorname{add} V$ since it is the sum of an injective, a semisimple, and a projective module. Further K is a submodule of P, hence torsionfree. So since Λ is stably hereditary K is the sum of a projective and a semisimple module, so K is also in $\operatorname{add} V$.

Next we need to show that $0 \to (V, K) \to (V, M' \oplus P) \to (V, M) \to 0$ is exact. The only thing needed to show here is that $(V, M' \oplus P) \to (V, M)$ is surjective. We do this by showing that $(W, M' \oplus P) \to (W, M)$ is surjective for any indecomposable summand of V. If W is projective this holds by definition. If W is simple then any map from W to M factors through the socle and hence through M', so it's surjective. Lastly if W is injective then the image of W in M is a cotorsionfree module, so it is the sum of simple modules and an injective module. Hence the map from W to M factors through M'.

Now we use this to show that the global dimension of Γ is at most 3. Let N be any Γ-module. Then it has a projective presentation

$$(V, V_1) \xrightarrow{f \circ -} (V, V_0) \longrightarrow N \longrightarrow 0$$

If we let M denote the kernel of f and we choose V_3 and V_2 as above then we get a projective resolution of N by

$$0 \longrightarrow (V, V_3) \longrightarrow (V, V_2) \longrightarrow (V, V_1) \longrightarrow (V, V_0) \longrightarrow N \longrightarrow 0.$$

This shows that the projective dimension of N is at most 3, and since N was arbitrary the global dimension of Γ is at most 3. So the repdimension of Λ is at most 3. $\square$

4.3 Special biserial algebras

[EHIS04]

5 Vanishing radical powers

Throughout this section Λ is a finite dimensional algebra, and J is its radical.

Theorem 5.1. *If $J^2 = 0$ then* $\operatorname{findim}(\Lambda) < \infty$.

Proof. Let $d = \max\{\operatorname{pd} S_i \mid \operatorname{pd} S_i < \infty\}$ where S_i ranges over the simple Λ-modules. Let M be a module with $\operatorname{pd} M < \infty$. Let $P \to M$ be a projective cover. Then ΩM is contained in JP and since $J^2P = 0$, ΩM is annihilated by J and is thus semisimple. This means $\operatorname{pd} \Omega M \leq d$, and thus $\operatorname{pd} M \leq d+1$. So $\operatorname{findim}(\Lambda) \leq d + 1 < \infty$. $\square$

Theorem 5.2. *[IT05, Corollary 6] If $J^3 = 0$ then* $\operatorname{findim}(\Lambda) < \infty$.

Proof. Let M be a module with $\operatorname{pd} M < \infty$, and let $P^0 \to M$ be its projective cover. Since $\Omega M \subseteq JP^0$ we have $J^2\Omega M = 0$. Let $P \to \Omega M$ be a projective cover. Since $J^2\Omega M = 0$ we can factorize this as $P \to P/J^2P \to \Omega M$, and we get a short exact sequence

$$0 \longrightarrow (\Omega^2 M + J^2P)/J^2P \longrightarrow P/J^2P \longrightarrow \Omega M \longrightarrow 0$$

Let ψ be the Igusa-Todorov function as introduced in Section 4. Since $\Omega^2 M \subseteq JP$ we have that $(\Omega^2 M + J^2P)/J^2P$ is semisimple. Then by Lemma 4.2 $\psi((\Omega^2 M + J^2P)/J^2P) \leq \psi(\Lambda/J)$, and $\psi(P/J^2P) \leq \psi(\Lambda/J^2)$.

Applying Theorem 4.3 to the short exact sequence above we thus get $\operatorname{pd} \Omega M \leq \psi(\Lambda/J \oplus \Lambda/J^2) + 1$, and so $\operatorname{pd} M \leq \psi(\Lambda/J \oplus \Lambda/J^2) + 2$, and $\operatorname{findim}(\Lambda) < \infty$. $\square$

Theorem 5.3. *[Wan94] If $J^{2l+1} = 0$ and Λ/J^l is representation finite, then* $\operatorname{findim}(\Lambda) < \infty$.

Proof. Let M be a module with $\operatorname{pd} M < \infty$. We have a short exact sequence

$$0 \longrightarrow J^l\Omega M \longrightarrow \Omega M \longrightarrow \Omega M/J^l\Omega M \longrightarrow 0.$$

Since $\Omega M \subseteq JP_M^0$ we have $J^{2l}\Omega M = 0$. This means that $J^l\Omega M$ and $\Omega M/J^l\Omega M$ are Λ/J^l-modules. We use this, the fact that Λ/J^l is representation finite, and the Igusa-Todorov function to create a bound for $\operatorname{pd} M$.

Applying Corollary 4.3.1 (ii) we have that:

$$\operatorname{pd}\Omega M \leq \psi(\Omega(J^l\Omega M) \oplus \Omega^2(\Omega M/J^l\Omega M)) + 2.$$

Since Λ/J^l is representation finite, there are only finitely many indecomposable Λ/J^l-modules, up to isomorphism. Let S be the sum of all of them. Then since $J^l\Omega M$ and $\Omega M/J^l\Omega M$ are in $\operatorname{add} S$, using Lemma 4.2 we have that

$$\psi(\Omega(J^l\Omega M) \oplus \Omega^2(\Omega M/J^l\Omega M)) \leq \psi(\Omega S \oplus \Omega^2 S).$$

So $\operatorname{pd} M \leq \psi(\Omega S \oplus \Omega^2 S) + 3$, and thus $\operatorname{findim}(\Lambda) < \infty$. □

6 Monomial algebras

[GKK91, IZ90]

In this section we show a particularly nice way to construct a minimal projective resolution of the right module Λ/J for a monomial algebra Λ. We use this to compute $\operatorname{Tor}_i(\Lambda/J, M)$ and/or $\operatorname{Ext}^i(M, D\Lambda/J)$ to get a bound on the projective dimension of all modules M.

Definition 6.1 (Monomial algebra)**.** A *monomial algebra* is a path algebra with admissible relations that are generated by monomials. That is, we do not allow the generators for the relations to consist of nontrivial linear combinations of paths.

We may assume rho contains J2

Definition 6.2 (m-chains)**.** [GKK91] Let $\Lambda = k\Gamma/(\rho)$ be a monomial algebra, with ρ a minimal generating set of paths. As usual we define Γ_0 to be the vertices of Γ, and Γ_1 to be the arrows. Recursively define the set of $(m-1)$-chains, Γ_m, as the paths γ with the following criteria:

i) $\gamma = \beta\delta\tau$ with $\beta \in \Gamma_{m-2}$, $\beta\delta \in \Gamma_{m-1}$, and τ a non-zero path of length at least 1.

ii) $\delta\tau$ is 0 in Λ, i.e. it is in the ideal of relations.

iii) γ is left-minimal in the sense that if $\gamma = \gamma'\sigma$ such that γ' satisfies the above conditions, then $\gamma = \gamma'$.

The sets of m-chains will become the generating sets for the projectives in our projective resolution. But first we prove some properties of them.

Lemma 6.3. *Any $\gamma \in \Gamma_m$ for $m \geq 1$ can be factored uniquely as $\gamma_1\gamma_0$ with $\gamma_1 \in \Gamma_{m-1}$, and γ_0 a non-zero path of length at least 1.*

Proof. When $m = 1$ this should be clear, since Γ_1 is the set of arrows, and Γ_0 is the set of vertices, so if $\gamma \in \Gamma_1$ is an arrow $i \to j$ then $\gamma = e_j\gamma$.

When $m > 1$ we know from the definition of Γ_m that γ can be written as $\gamma_1\gamma_0$. Assume there is another decomposition $\gamma = \gamma_1'\gamma_0'$. Then without loss of generality we may assume that γ_1' is shorter than γ_1. Then there is a σ such that $\gamma_1'\sigma = \gamma_1$. By minimality this means that $\gamma_1' = \gamma_1$, and so the decomposition is unique. □

From now on we write R for the ring Λ/J, which we identify with the subring of Λ generated by the paths of length 0. Let $k\Gamma_m$ be the free vector space generated by Γ_m. Notice that $k\Gamma_m$ has a canonical structure as a $R-R$-bimodule. This means we can get projective right Λ-modules $P^m := k\Gamma_m \otimes_R \Lambda$.

Proposition 6.4. *Define the map $\delta_m \colon P^m \to P^{m-1}$ by $\delta_m(\gamma \otimes \alpha) = \gamma_1 \otimes \gamma_0\alpha$ where $\gamma_1\gamma_0$ is the unique decomposition of γ, and define $\delta_0 \colon P^0 \to \Lambda/J$ by $\delta_0(e_i \otimes \alpha) = e_i\alpha + J$. Then we have a minimal projective resolution of the right Λ-module Λ/J by*

$$\begin{tikzcd} \cdots \arrow[r] & P^3 \arrow[r, "\delta_3"] & P^2 \arrow[r, "\delta_2"] & P^1 \arrow[r, "\delta_1"] & P^0 \arrow[r] \arrow[d, two heads, "\delta_0"] & 0 \\ & & & & \Lambda/J & \end{tikzcd}$$

Before proving this proposition we require a lemma.

Lemma 6.5. *[GKK91, Lemma 2.1] Let M be a Λ-module, and x an element in the kernel of of $\delta_m \otimes M \colon k\Gamma_m \otimes_R M \to k\Gamma_{m-1} \otimes_R M$. Write x on the form*

$$x = \sum_j \sum_{k=0}^{n_j} \gamma_j \gamma_j^k \otimes m_j^k$$

with $\gamma_i \in \Gamma_{m-1}$ and $\gamma_i \neq \gamma_j$ and $\gamma_j^k \neq \gamma_j^l$ when $i \neq j$ and $k \neq l$. Then

$$\sum_{k=0}^{n_j} \gamma_j \gamma_j^k \otimes m_j^k$$

is also in the kernel for each j.

Proof. Let x be as given above. Applying $\delta_m \otimes M$ we get that

$$\sum_j \gamma_j \otimes \sum_{k=0}^{n_j} \gamma_j^k m_j^k = 0.$$

Since the γ_j's are distinct we can deduce that

$$\sum_{k=0}^{n_j} \gamma_j^k m_j^k = 0.$$

From this it follows that

$$\sum_{k=0}^{n_j} \gamma_j \gamma_j^k \otimes m_j^k$$

is also in the kernel of $\delta_m \otimes M$. □

Proof of Proposition 6.4. For all i the module P^i is projective as a right Λ-module and the image of δ_m is clearly contained in $P^{m-1}J$, so the only thing left to show is exactness. First we show that $\delta_m \delta_{m-1} = 0$. Let $\gamma \otimes \alpha$ be in P^m for $m \geq 2$. Then we can decompose γ uniquely as $\gamma_2\gamma_1\gamma_0$ and $\delta_m\delta_{m-1}(\gamma \otimes \alpha) = \gamma_2 \otimes \gamma_1\gamma_0\alpha$. By the way we defined Γ_m, $\gamma_1\gamma_0$ is 0 in Λ, and so $\gamma_2 \otimes \gamma_1\gamma_0\alpha = 0$.

Next we want to show that $\operatorname{Ker}\delta_{m-1} \subseteq \operatorname{Im}\delta_m$. Let x be in the kernel of δ_{m-1}. By Lemma 6.5 it is sufficient to assume x is of the form

$$\sum_k \gamma\gamma_k \otimes \alpha_k$$

with $\gamma \in \Gamma_{m-2}$ and the γ_k's all distinct. Then $\sum_k \gamma_k\alpha_k = 0$. Since Λ only has monomial relations, and by the minimality of the γ_k's none of them divide each other, we have that $\gamma_k\alpha_k = 0$.

Because of this we have that $\gamma\gamma_k\alpha_k = \zeta_k\sigma_k$ for some m-chain ζ_k and some path σ_k (possibly of length 0). This gives us that x is the image of

$$\sum_k \zeta_k \otimes \sigma_k$$

by δ_m. Hence $\operatorname{Ker}\delta_{m-1} \subseteq \operatorname{Im}\delta_m$, and the sequence is exact. So this gives a minimal projective resolution of Λ/J as a right Λ-module. □

Definition 6.6. We call a path τ in Γ a *special segment* for $\Lambda = k\Gamma/(\rho)$ if there is a path γ such that $\gamma\tau$ is a minimal relation.

Note that when we decompose an m-chain γ in Lemma 6.3 into $\gamma_1\gamma_0$ then γ_0 is a special segment, and that the set of special segments is finite.

Lemma 6.7. *[GKK91, Theorem 2.2] Let d be the number of special segments for Λ. If $s \geq d+3$ and γ is in Γ_s, then for any integer N there is an $n \geq N$ and a $\hat{\gamma} \in \Gamma_n$ such that for any path τ and any integer $r \geq 1$ we have $\gamma\tau \in \Gamma_{s+r}$ if and only if $\hat{\gamma}\tau \in \Gamma_{n+r}$.*

Proof. Applying Lemma 6.3 recursively we get that γ can be written as $\gamma = \tau_0\tau_1 \cdots \tau_{s-1}$ where $\tau_0\tau_1 \cdots \tau_{i-1} \in \Gamma_i$. In particular each τ_i is a special segment.

Since $s \geq d+3$ we must have that there exists i and j, $1 \leq i < j \leq s-1$ such that $\tau_i = \tau_j$. Let $\beta = \tau_{i+1}\tau_{i+2} \cdots \tau_j$. Then

$$\gamma_k := \tau_0\tau_1 \cdots \tau_{j-1}\tau_j\beta^k\tau_{j+1} \cdots \tau_{s-1} \in \Gamma_{s+k(j-i)}$$

where β^k means β repeated k times. If we now choose k large enough such that $s + k(j-i) \geq N$ we can choose $n = s + k(j-i)$ and $\hat{\gamma} = \gamma_k$. Then we see that for any path τ, the composition $\gamma\tau$ is in Γ_{s+r} if and only if $\hat{\gamma}\tau$ is in Γ_{n+r}. $\square$

Theorem 6.8. *[GKK91, Corollary 2.4] Let $\Lambda = k\Gamma/(\rho)$ be a monomial relation algebra. Then* $\operatorname{findim}(\Lambda) \leq d+3$ *where d is the number of special segments for Λ.*

Proof. Let M be a module of finite projective dimension and let N be $\operatorname{pd} M$. The projective dimension of M can be characterized as the largest integer c such that $\operatorname{Tor}_c(\Lambda/J, M) \neq 0$. We show that this is at most $d+3$. Let $s \geq d+3$ be an integer. Then we want to show that $\operatorname{Tor}_{s+1}(\Lambda/J, M) = 0$. We compute this by taking the projective resolution of Λ/J found in Proposition 6.4 and tensoring with M.

$$\cdots \longrightarrow k\Gamma_{s+2} \otimes M \xrightarrow{\delta_{s+2}\otimes M} k\Gamma_{s+1} \otimes M \xrightarrow{\delta_{s+1}\otimes M} k\Gamma_s \otimes M \longrightarrow \cdots$$

Let x be in the kernel of $\delta_{s+1} \otimes M$. Then by Lemma 6.5 we may assume x is on the form

$$x = \sum_j \gamma\gamma_j \otimes m_j$$

with γ in Γ_s and all the γ_j's distinct. Then Lemma 6.7 gives us that there is an $n \geq N$ and a $\hat{\gamma} \in \Gamma_n$ such that $\gamma\tau$ is in Γ_{s+r} if and only if $\hat{\gamma}\tau$ is in Γ_{n+r}.

Then $\hat{x} = \sum \hat{\gamma}\gamma_j \otimes m_j$ is in the kernel of $\delta_{n+1} \otimes M$. Since $n+1 > N = \operatorname{pd} M$ the complex is exact at $n+1$. This means that there are elements γ_j^k and

m_j^k such that

$$\hat{x} = \delta_{n+2}\left(\sum_j \sum_{k=0}^{n_j} \hat{\gamma}\gamma_j\gamma_j^k \otimes m_j^k\right) = \sum_j \sum_{k=0}^{n_j} \hat{\gamma}\gamma_j \otimes \gamma_j^k m_j^k$$

Since $\hat{\gamma}\gamma_j\gamma_j^k$ is in Γ_{n+2} if and only if $\gamma\gamma_j\gamma_j^k$ is in Γ_{s+2} we have that

$$x = \delta_{s+2}\left(\sum_j \sum_{k=0}^{n_j} \gamma\gamma_j\gamma_j^k \otimes m_j^k\right)$$

and thus $\operatorname{Tor}_{s+1}(\Lambda/J, M) = 0$ so $\operatorname{pd} M \leq d+3$. □

7 Unbounded derived category

So far we have been focused on the finite dimensional version of the finitistic dimension, known as the little finitistic dimension. Namely

$$\operatorname{findim}(\Lambda) = \sup\{\operatorname{pd} M \mid M \in \operatorname{mod}\Lambda, \operatorname{pd} M < \infty\}.$$

In this section we will consider infinite dimensional modules, and thus it is natural for us to look at the infinite dimensional version of the finitistic dimension, known as the big finitistic dimension. It is defined, as you would expect, by considering not just finite dimensional modules, but all Λ-modules:

$$\operatorname{Findim}(\Lambda) = \sup\{\operatorname{pd} M \mid M \in \operatorname{Mod}\Lambda, \operatorname{pd} M < \infty\}.$$

Note that $\operatorname{findim}(\Lambda) \leq \operatorname{Findim}(\Lambda)$ and so if we can show that $\operatorname{Findim}(\Lambda) < \infty$ we have also shown that $\operatorname{findim}(\Lambda) < \infty$.

In Theorem 1.8 we showed that if $\operatorname{findim}(\Lambda) < \infty$, then $D\Lambda$ becomes a generator in $\mathscr{D}^b(\Lambda)$. In this section we show that if we instead consider the unbounded derived category of all Λ-modules, then we get an analogous converse result.

Theorem 7.1. *[Ric19, Theorem 4.3] If the localizing subcategory generated by $D\Lambda$ is the entire unbounded derived category, then* $\operatorname{Findim}(\Lambda) < \infty$.

Proof. Assume $\operatorname{Findim}(\Lambda) = \infty$. Then there are modules M_i with projective dimension i for every $i \geq 0$. Let P_i be the minimal projective resolution of

M_i, and consider $\bigoplus P_i[-i]$ and $\prod P_i[-i]$. Both of these have homology M_i in degree i, and are concentrated in non-negative degrees.

The inclusion from the sum to the product is clearly a quasi-isomorphism. We want to show that it is not a homotopy equivalence. Assume for the sake of contradiction that it was. Then tensoring with Λ/J would give us another homotopy equivalence. Since Λ/J is finitely presented tensoring preserves both products and coproducts. Because all the resolutions were minimal tensoring with Λ/J gives us 0 differentials. In degree 0 we get

$$\bigoplus \mathrm{Tor}_i(\Lambda/J, M_i) \to \prod \mathrm{Tor}_i(\Lambda/J, M_i).$$

Since $\mathrm{Tor}_i(\Lambda/J, M_i)$ is nonzero for every M_i this map is not an isomorphism, and so we don't have a homotopy equivalence.

So the cone of the inclusion $\bigoplus P_i[-i] \to \prod P_i[-i]$, C, is 0 in the derived category, but non-zero in the homotopy category. Since Λ is artinian the product of projectives is projective [Cha60, Theorem 3.3], so $\prod P_i[-i]$ is a complex of projectives, which means that C is a complex of projectives.

In other words C is an acyclic lower bounded complex of projectives that is not contractible. Tensoring with $D\Lambda$ is an equivalence from projectives to injectives with inverse $\mathrm{Hom}(D\Lambda, -)$, so $D\Lambda \otimes C$ is a lower bounded complex of injectives that is not contractible. Such a complex cannot be acyclic so $D\Lambda \otimes C$ has homology, and is thus non-zero in $\mathscr{D}(\Lambda)$.

Theorem A.4 in appendix

The homology of C is 0, so $K(\Lambda)(\Lambda, C[i]) = 0$. Applying the equivalence $D\Lambda \otimes -$ we get

$$0 = K(\Lambda)(D\Lambda, D\Lambda \otimes C[i]) = \mathscr{D}(\Lambda)(D\Lambda, D\Lambda \otimes C[i]).$$

This means that $D\Lambda \otimes C$ is not in the localizing category generated by $D\Lambda$, and so that can not be the entire derived category. □

Theorem 7.2. *[Ric19, Theorem 4.4]* $\mathrm{Findim}(\Lambda) < \infty$ *if and only if* $D\Lambda^{\perp} \cap \mathscr{D}^{+}(\Lambda) = 0$.

Proof. In the theorem above we proved that when the finitistic dimension is infinite then there is a non-zero complex in $\mathscr{D}^{+}(\Lambda)$ perpendicular to $D\Lambda$.

The proof of the converse is the same as for Theorem 1.8. If we have a non-zero object $X \in D\Lambda^{\perp} \cap \mathscr{D}^{+}(\Lambda)$, then $\mathscr{D}(\Lambda)(D\Lambda, X)$ is an acyclic minimal complex of projectives that continue arbitrarily to the right. So the cokernels have arbitrarily big projective dimension. □

We see this by taking injective resolution of X

8 Summary

FDC holds for the following classes of algebras

- **Big FDC:**
- Representation finite algebras

  Proof. The supremum over a finite set is finite so $\operatorname{findim}(\Lambda) < \infty$ for a representation finite algebra. □

- Monomial algebras

  Proof. This was shown in Section 6. □

- Gorenstein algebras

  Proof. An algebra is said to be Gorenstein if all injectives have finite projective dimension and all projectives have finite injective dimension. In particular the Λ-module Λ is isomorphic to a finite injective resolution in the derived category. So Λ is in the localizing category generated by injectives. Then Theorem 7.1 gives us that $\operatorname{Findim}(\Lambda) < \infty$, and therefor also $\operatorname{findim}(\Lambda) < \infty$. □

- Finite global dimension
- Self injective
- $J^2 = 0$
- Derived equivalent to the above
- Local algebras

  Proof. Local algebras are local artinian rings. So if Λ is local then $\operatorname{findim}(\Lambda) = 0$. □

- **only small FDC is known?:**
- Stably hereditary algebras
- Special biserial algebras
- "half rep-finite" algebras, i.e. Λ/J^l rep-finite $J^{2l+1} = 0$.

Not sure where to put this, ill put it here for now

Theorem 8.1. *Local artinian rings have finitistic dimension zero.*

Proof. Assume there is a non-projective module with finite projective dimension. Then in particular we have one with projective dimension equal to 1. Since all finitely generated projectives are free this means we have a short exact sequence

$$0 \qquad R^n \qquad R^m \qquad M \qquad 0$$

with R^n contained in JR^m. Let k be the minimal integer such that $J^k = 0$. Let a be a generator in R^n and let r be a non-zero element of J^{k-1}. Then ra is non-zero, but is mapped to something in $J^{k-1}JR^m = 0$, thus the map is not injective which gives a contradiction. □

9 Dual conjectures

Many of the cases are equivalent to their dual statements. Some are not.

- Given a recollement of the bounded derived category you get one for Λ^{op}.

Look at examples of reccolement to see how it translates.

- Just because the subcategory of modules with finite projective dimension is contravariantly finite does not mean the subcategory of modules with finite injective dimension has to be covariantly finite. See Example 3.6.
- repdim of Λ equals the repdim of Λ^{op}.

  Proof. If M is an auslander generator for Λ then DM is an auslander generator for Λ^{op}. □
- If $J^{2l+1} = 0$ and Λ/J^l is repfinite then the same is true for Λ^{op}.
- If Λ is monomial then so is Λ^{op}.
- Injective generates implies the weaker property that projective cogenerate for the opposite algebra. This is also sufficient to prove the algebra satisfies FDC. [Ric19, Section 5]

Similarly for the weaker conjectures

- GSC says the injective dimension of Λ is finite if and only if the injective dimension of Λ^{op} is finite. This statement is symmetric with respect to Λ and Λ^{op}. So the dual is equivalent.
- NC: Certainly Λ is self injective if and only Λ^{op} is.
- For all the others it seems just as difficult as solving the conjecture to connect it to it's dual.

Can the dominant dimension of the opposite algebra be different? Arbitrary different?

Appendices

A Appendix: Homological algebra

In this section we collect relevant theorems from homological algebra that would be distracting within the text itself.

Lemma A.1. *[CE99, Chapter I, theorem 3.2] Let R be a noetherian ring. Then an R-module Q is injective if and only if it has the injective lifting property for inclusions of ideals into R.*

Proof. If Q is injective then Q has the lifting property for all monomorphisms, so one direction is clear. Assume we have a diagram

$$\begin{tikzcd} Q & \\ M \arrow[u, "f"] \arrow[r, hook] & N \arrow[ul, dashed] \end{tikzcd}$$

We want to show that the dashed arrow exists. Let S be the partially ordered set $\{(M', f') : M \leq M', f'|_M = f\}$. By Zorn's lemma this has a maximal element (M', f'). Assume $M' \neq N$, then there is an element $x \in N - M'$. The set of r such that $rx \in M'$ forms an ideal I. Define the map $g : I \to Q$ by $I(r) = f'(rx)$. By hypothesis g lifts to a map $\tilde{g} : R \to Q$. Let q be $\tilde{g}(1)$. Then $\tilde{f} : M' + Rx \to Q$ defined by $\tilde{f}(m + rx) = f'(m) + rq$ gives us a bigger element of S, contradicting maximality. Thus $M' = N$ and Q is injective. $\square$

Theorem A.2. *Let R be a noetherian ring. Then an arbitrary coproduct of injectives is injective.*

Proof. By the lemma above it is enough to show the lifting property on ideals of R. Let I be an ideal and $f : I \to \bigoplus_i Q_i$ be a map to a coproduct of injectives. Since R is notherian I is finitely generated so f factors through a finite sum $I \to \bigoplus_{i=0}^n Q_i \to \bigoplus Q_i$. Since finite coproducts of injectives are injective we are done.

$$\begin{tikzcd}
\bigoplus Q_i & \\
\bigoplus\limits_{i=0}^{n} Q_i \arrow[u] & \\
I \arrow[u] \arrow[r, hook] & R \arrow[ul, dashed]
\end{tikzcd}$$

□

Theorem A.3. *[CE99, Chapter I, Exercise 8] Let R be a noetherian ring. Then direct limits of injectives is injective.*

Proof. By the lemma above it is enough to show the lifting property on ideals of R. Let I be an ideal and let $Q = \varinjlim Q_i$ be a direct limit of injectives.

Since R is noetherian I is finitely presented, say $R^n \to R^m \to I \to 0$. Applying $\mathrm{Hom}(-, Q)$ we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(I, Q) \longrightarrow \mathrm{Hom}(R^m, Q) \longrightarrow \mathrm{Hom}(R^n, Q)$$

Since direct limits are exact we also have an exact sequence

$$0 \longrightarrow \varinjlim \mathrm{Hom}(I, Q_i) \longrightarrow \varinjlim \mathrm{Hom}(R^m, Q_i) \longrightarrow \varinjlim \mathrm{Hom}(R^n, Q_i)$$

We also have a natural map $\varinjlim \mathrm{Hom}(-, Q_i) \to \mathrm{Hom}(-, Q)$. $\mathrm{Hom}(R^n, Q_i)$ just equals Q_i^n, so this map is an isomorphism at R^n. Then by the five lemma applied to the two sequences above we get that $\mathrm{Hom}(I, Q) \cong \varinjlim \mathrm{Hom}(I, Q_i)$ for all ideals I. So since

$$\varinjlim \mathrm{Hom}(R, Q_i) \longrightarrow \varinjlim \mathrm{Hom}(I, Q_i) \longrightarrow 0$$

is exact, we get that

$$\operatorname{Hom}(R,Q) \longrightarrow \operatorname{Hom}(I,Q) \longrightarrow 0$$

is exact. Hence Q is injective. $\square$

Theorem A.4. *Let Λ be an artin algebra. Then we have an equivalence of categories*

$$\begin{tikzcd} \operatorname{Proj}\Lambda \arrow[r, bend left, "D\Lambda\otimes -"] & \operatorname{Inj}\Lambda \arrow[l, bend left, "\operatorname{Hom}(D\Lambda,-)"] \end{tikzcd}$$

where the tensor product is over Λ, and $\operatorname{Hom}(D\Lambda, X)$ is considered as a Λ-module by $\Lambda \cong \operatorname{End}(D\Lambda)^{\mathrm{op}}$.

Proof. First we note the following isomorphisms when evaluating the functors at Λ and $D\Lambda$

$$\begin{aligned} \operatorname{Hom}(D\Lambda, D\Lambda \otimes \Lambda) &\cong \operatorname{End}(D\Lambda) \\ &\cong \operatorname{End}(\Lambda_\Lambda) \\ &\cong \Lambda \end{aligned}$$

and

$$\begin{aligned} D\Lambda \otimes \operatorname{Hom}(D\Lambda, D\Lambda) &\cong D\Lambda \otimes \Lambda \\ &\cong D\Lambda. \end{aligned}$$

Since $D\Lambda$ is finitely presented $D\Lambda \otimes -$ and $\operatorname{Hom}(D\Lambda, -)$ preserve both products and coproducts. Then since $\operatorname{Proj}\Lambda = \operatorname{Add}\Lambda$ and $\operatorname{Inj}\Lambda = \operatorname{Prod} D\Lambda$ it follows from the equations above that $\operatorname{Hom}(D\Lambda, D\Lambda \otimes -)$ and $D\Lambda \otimes \operatorname{Hom}(D\Lambda, -)$ are isomorphic to the identity on $\operatorname{Proj}\Lambda$ and $\operatorname{Inj}\Lambda$ respectively.

Lastly we verify that the maps are well defined. Since Λ is an artin algebra each injective module is the injective envelope of its socle. Since the socle is semisimple it is the direct sum of simple modules. Thus each injective is the sum of indecomposable injective modules, and hence we have that $\operatorname{Add} D\Lambda = \operatorname{Inj}\Lambda$. It is true for any ring that $\operatorname{Add}\Lambda = \operatorname{Proj}\Lambda$, and so we have the following:

$$D\Lambda \otimes (\operatorname{Proj}\Lambda) = D\Lambda \otimes (\operatorname{Add}\Lambda) = \operatorname{Add} D\Lambda = \operatorname{Inj}\Lambda,$$

and

$$\operatorname{Hom}(D\Lambda, \operatorname{Inj}\Lambda) = \operatorname{Hom}(D\Lambda, \operatorname{Add} D\Lambda) = \operatorname{Add}\Lambda = \operatorname{Proj}\Lambda.$$

So the maps induce an equivalence of categories. $\square$

Theorem A.5 (Fitting's Lemma)**.** *Let R be a ring, M an R-module, and $L\colon M \to M$ an endomorphism. If X is a noetherian submodule of M, then there exists a positive integer η_X such that $L|_{L^n(X)}\colon L^n(X) \to M$ is injective for all $n \geq \eta_X$.*

Proof. We have an increasing sequence of submodules of X given by:

$$\ker L \cap X \subseteq \ker L^2 \cap X \subseteq \ker L^3 \cap X \subseteq \cdots$$

Since X is noetherian this sequence stabilizes, i.e. there is an integer η_X such that $\ker L^n \cap X = \ker L^{n+1} \cap X$ for all $n \geq \eta_X$. We know that $L^n(X) \cong X/\ker L^n \cap X$, and that through this isomorphism the map $L\colon L^n(X) \to M$ is induced by $L^{n+1}\colon X/\ker L^n \cap X \to L^{n+1}(X) \subseteq M$. Since for $n \geq \eta_X$ we have that $\ker L^n \cap X = \ker L^{n+1} \cap X$ this map is injective, and so the theorem holds. □

Interesting examples of Fitting's Lemma comes from R being a noetherian ring and X being a finitely generated modules. In particular the case when $R = \mathbb{Z}$ appears in Section 4. An important special case of Fitting's Lemma that comes up when working with artinian rings is when $X = M$ and X has finite length. Remember that over an artin ring all finitely generated modules have finite length.

Corollary A.5.1. *Let X be a module of finite length, and let $L\colon X \to X$ be an endomorphism. Then L splits as a direct sum $L_1 \oplus L_2\colon X_1 \oplus X_2 \to X_1 \oplus X_2$ such that L_1 is nilpotent and L_2 is an isomorphism.*

Proof. Since X has finite length it is noetherian, thus we can apply Fitting's Lemma. Let n be the positive integer we get from Fitting's Lemma, and let K be $\ker L^n$. We wish to show that X is the direct sum of K and $L^n(X)$. Note that since L is inejctive when restricted to $L^n(X)$ we have that $K \cap L^n(X) = 0$, so all we have to show is that $X = K + L^n(X)$.

We have a short exact sequence

$$0 \longrightarrow K \longrightarrow X \longrightarrow L^n(X) \longrightarrow 0.$$

From this we conclude that the length of $L^n(X)$ is equal to the length of X minus the length of K. Since $\ker L^n = \ker L^{2n}$ we also have that the length of $L^n(X)$ and $L^{2n}(X)$ are equal. Since $L^{2n}(X)$ is a submodule of $L^n(X)$ this means that $L^n(X) = L^{2n}(X)$. Thus L restricts to an automorphism on $L^n(X)$. Let ψ be its inverse. Then for any $x \in X$ we have $x = \psi L^n(x) +$

$x - \psi L^n(x)$. Clearly $\psi L^n(x)$ is in $L^n(X)$. Applying L^n to $x - \psi L^n(x)$ we get

$$\begin{aligned} L^n(x - \psi L^n(x)) &= L^n(x) - L^n \psi L^n(x) \\ &= L^n(x) - L^n(x) \\ &= 0 \end{aligned}$$

Thus $x - \psi L^n(x)$ is in the kernel and so $X = K \oplus L^n(X)$. Then we see that L breaks down as a direct sum $L = L_1 \oplus L_2$ with $L_1 \colon K \to K$ nilpotent and $L_2 \colon L^n(X) \to L^n(X)$ an isomorphism. $\square$

10 Personal appendix

Theorem 10.1. *The global dimension of an artin algebra is the supremum of k with $\operatorname{Ext}^k(T,T) \neq 0$ (T sum of simples). This is also the supremum of projective dimension and supremum of injective dimension.*

Proof. For a minimal projective resolution $\operatorname{Hom}(-,T)$ makes the differentials 0, and similarly with $\operatorname{Hom}(T,-)$ and injective resolutions. So $\operatorname{Ext}^k(M,T)$ is only 0 exactly when $k > \operatorname{pd} M$, similarly $\operatorname{Ext}^k(T,M)$ is only 0 when k is bigger than the injective dimension. Since any module is built by extensions of simples you can prove by induction, and the long exact sequence in $\operatorname{Ext}(-,T)$ you get that any module has projective dimension less than or equal to that of T. Similarly for injective dimension. $\square$

$\operatorname{findim}(\Lambda)$ need not equal $\operatorname{findim}(\Lambda^{\mathrm{op}}) = \sup\{$injective dimension of $M | M$ has finite injective dimension$\}$.

Example 10.2. [hf] Let $\Lambda = k \left[a \circlearrowright 1 \underset{c}{\overset{b}{\rightleftarrows}} 2 \right] \Big/ (a^2, ac, ba, cbc)$. Then $\operatorname{findim}(\Lambda) \geq 1$, but $\operatorname{findim}(\Lambda^{\mathrm{op}}) = 0$.

Proof. The module $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} = P_1/P_2$ (k^2 where a acts by $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, and b and c act trivially) has projective dimension 1, so $\operatorname{findim}(\Lambda) \geq 1$. The projective/injective modules of Λ are:

$$P_1 = \begin{array}{ccc} & 1 & \\ 1 & & 2 \\ & & 1 \\ & & 2 \end{array}, \quad P_2 = \begin{array}{c} 2 \\ 1 \\ 2 \end{array}, \quad I_1 = \begin{array}{ccc} & & 1 \\ 1 & & 2 \\ & 1 & \end{array}, \quad I_2 = \begin{array}{c} 1 \\ 2 \\ 1 \\ 2 \end{array}$$

If $\operatorname{findim}(\Lambda^{\mathrm{op}}) > 0$ there would be a module with finite non-zero injective resolution. In particular it would end with a non-split epimorphism between injectives. I claim this would mean there is a non-split epimorphism $I \to I_i$ from an injective to an indecomposable injective. Obviously we get epimorphisms by composing with the projections onto summands, so we want to show that they are not split. Assume that they are, that is the map looks like

$$\begin{tikzcd}
I_i \oplus I \arrow[r, "\begin{bmatrix} 1 & 0 \\ f & g \end{bmatrix}"] \arrow[dr, "\begin{bmatrix} 1 & 0 \end{bmatrix}"'] & I_i \oplus I' \arrow[d, "\begin{bmatrix} 1 & 0 \end{bmatrix}"] \\
 & I_i
\end{tikzcd} .$$

We see that by changing basis in the domain we get the matrix $\begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}$. Thus I_i is mapped isomorphically to itself, which doesn't happen in a minimal resolution.

The only thing left to show is that there are no non-split epimorphisms from injective modules to I_1 and I_2. □

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