Finitistic dimension conjecture

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2020

Abstract

FDC yo!

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Introduction

This is an introduction

1 finitistic dimension and conjectures

- FDC finitistic dimesnion conjecture Finitistic dimension is always finite
- WTC Watamatsu tilting conjecture
- GSC Gorenstein symmetry conjecture
- NuC Nunke condition
- SNC strong Nakayama conjecture
- ARC Auslander Reiten conjecture
- NC Nakayama conjecture

1.1 Implications

$$\begin{array}{c} FDC \longrightarrow WTC \longrightarrow GSC \\ \downarrow \\ NuC \longrightarrow SNC \longrightarrow ARC \longrightarrow NC \end{array}$$

Theorem 1.1. /Hap93, 1.2/

- i) If $findim(\Lambda) < \infty$ (FDC) then $K^b(inj \Lambda)^{\perp} = 0$.
- ii) If $K^b(\operatorname{inj} \Lambda)^{\perp} = 0$ then for $X \neq 0$ there exists i such that, $\operatorname{Ext}^i(D(\Lambda), X) \neq 0$ (NuC).

Proof.

i) Let $I^{\bullet} \in K^b(\operatorname{inj} \Lambda)^{\perp}$ be non-zero. Since $D^b(\Lambda) \cong K^{+,b}(\operatorname{inj} \Lambda)$ we may assume I^{\bullet} is a complex of injectives, and WLOG we may assume it concentrated in degrees $i \geq 0$.

 $\operatorname{Hom}(D\Lambda, I^i)$ is in $add \operatorname{Hom}(D\Lambda, D\Lambda) = add\Lambda$ so $\operatorname{Hom}(D\Lambda, I^{\bullet})$ is a complex of projectives.

$$\begin{array}{cccc}
0 & \longrightarrow & D\Lambda & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
I^{i-1} & \stackrel{d^{i-1}}{\xrightarrow{d^{i-1}}} & I^{i} & \stackrel{d^{i}}{\xrightarrow{d^{i}}} & I^{i+1}
\end{array}$$

Since I^{\bullet} is in $K^b(\operatorname{inj} \Lambda)^{\perp}$ and $D\Lambda$ is in $K^b(\operatorname{inj} \Lambda)$, whenever $d^i f = 0$, f^{\bullet} is homotopic to 0. Meaning f factors through d^{i-1} . This means that $\operatorname{Hom}(D\Lambda, I^{\bullet})$ is an exact complex.

Cok Hom $(D\Lambda, d^i)$ has a projective resolution of length i. This resolution is minimal??????? Hence $findim(\Lambda)$ is infinite.

ii) Assume there is an $X \neq 0$ with $\operatorname{Ext}^i(D\Lambda, X) = 0$ for all $i \geq 0$. Then X considered as a stalk complex is in $K^b(\operatorname{inj}\Lambda)^{\perp}$. Proceed by induction: If $I[-i] \in K^b(\operatorname{inj}\Lambda)$ is a stalk complex then $D^b(I[-i], X) = \operatorname{Ext}^i(I, X)$. This is 0 because $D\Lambda$ is the sum of the indecomposable injectives.

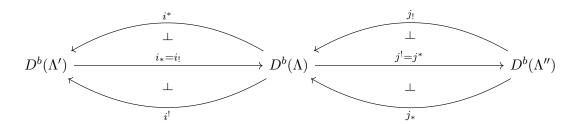
Probably want to use I indecomposable or something like that

Let $I \in K^b(\text{inj }\Lambda)$ be a complex of width n. WLOG assume I concentrated in degrees $0 \le i \le n-1$. Then

$$I^0 \to I \to I^{<0} \to I^0[1]$$

is a triangle, and $I^{<0}$ has width n-1. Taking the long exact sequence in $D^b(-,X)$ it follows that $D^b(I,X)=0$.

1.2 Recollement



Sort of like a split exact sequence of functors. We want

- 1. All functors are exact/triangulated
- 2. $j^*i_* = 0$
- 3. $i^*i_* \cong i^!i_! \cong id$ (induced by unit/counit)
- 4. $j!j! \cong j^*j_* \cong id$

$$j_! j^! X \xrightarrow{\varepsilon} X \xrightarrow{\eta} i_* i^* X \longrightarrow \Sigma$$

 $i_! i^! X \xrightarrow{\varepsilon} X \xrightarrow{\eta} j_* j^* X \longrightarrow \Sigma$

Are triangles in $D^b(\Lambda)$

Theorem 1.2. Given a recollement FDC holds for midlle if and only if it holds for the two others.

Proof. write later.... Happel reduct technich [Hap93, 3.3]

1.3 Contravariant finiteness

Definition 1.3 (Resolving). A full subcategory of an abelian category is called resolving if

- It is closed under extensions
- It contains the projectives
- It is contains the kernels of its epimorphisms

Note that the subcategory of modules with finite projective dimension is resolving.

Theorem 1.4. [AR91, 3.8] Let \mathcal{X} be a contravariantly finite, resolving subcategory of mod Λ . Let X_i be the minimal approximation of S_i . Then any $X \in \mathcal{X}$ is a direct summand of an X_i -filtered module.

Proof. Step 1: We want to show by induction on length that any module C is in an exact sequence $0 \to Y \to X \to C \to 0$ with X X_i -filtered and $\operatorname{Ext}^1(\mathcal{X},Y) = 0$.

Step 2: Whenever C is in \mathcal{X} we get that

$$\operatorname{Hom}(C,X) \longrightarrow \operatorname{Hom}(C,C) \longrightarrow \operatorname{Ext}^1(C,Y) = 0$$
 is exact, and thus C is a direct summand of X .

Corollary 1.4.1. If the subcategory of modules with finite projective dimension is contravariantly finite, then the finitistic dimension is the supremum of the projective dimension of X_i . In particular it is finite.

2 repdimension

Definition 2.1 (dominated dimension). Let $\Lambda \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$ be a minimal injective resolution of Λ . Then the dominated dimension of Λ is $\inf\{n|I_n \text{ is not projective }\}.$

Definition 2.2 (rep-dimesnion). Let $A = \{\Gamma | domdim\Gamma \geq 2, \Lambda \text{ morita eqquivalent to } \operatorname{End}_{\Gamma} I_0(\Gamma)\}$ where $I_0(\Gamma)$ is the injective envelope of Γ . Then the repdimesnion of Λ is the minimal global dimension of $\Gamma \in A$.

Proposition 2.3. (all modules ar right modules) Repdim is the same as minimal global dimension of $\operatorname{End}(M)$ for M being both a generator and cogenerator.

Proof. Consider $\Gamma \in A$. Since $domdim\Gamma \geq 1$, $I_0(\Gamma)$ is the sum of all projective-injective modules (some probably several times).

Let S be the set of all Γ -modules with a copresentation

$$0 \longrightarrow X \longrightarrow I_0 \longrightarrow I_1$$

with I_i in add $I_0(\Gamma)$. In particular Γ is in \mathcal{S} , because $domdim\Gamma \geq 2$.

The Yoneda embedding gives an equivalence

$$\operatorname{Hom}_{\Gamma}(-, I_0(\Gamma)) : \operatorname{add} I_0(\Gamma) \to \operatorname{proj} \operatorname{End}_{\Gamma}(I_0(\Gamma))^{op}$$

, and thus we get an equivalence

$$D \operatorname{Hom}_{\Gamma}(-, I_0(\Gamma)) : \operatorname{add} I_0(\Gamma) \to \operatorname{inj} \operatorname{End}_{\Gamma}(I_0(\Gamma))$$

Since $I_0(\Gamma)$ is injective $D \operatorname{Hom}(-, I_0(\Gamma))$ is exact and preserves kernels, so extends to an equivalence

$$\operatorname{Hom}_{\Gamma}(-, I_0(\Gamma)) : \mathcal{S} \to \operatorname{mod} \operatorname{End}_{\Gamma}(I_0(\Gamma))$$

Since $\operatorname{End}_{\Gamma}(I_0(\Gamma))$ is morita equivalent to Λ , \mathcal{S} is equivalent to $\operatorname{mod} \Lambda$. $\Gamma \in \mathcal{S}$ is clearly a generator. To see that it is a cogenerator note that Γ contains all the projective-injective indecomposable objects as direct summands, so there is an injection $I_0(\Gamma) \to \Gamma^n$, and since $I_0(\Gamma)$ is a cogenerator in \mathcal{S} , Γ is aswell.

Thus by the equivalence $S \to \operatorname{mod} \Lambda$ there is a cogenerator-generator object M such that $\operatorname{End}_{\Lambda}(M) = \operatorname{End}_{\Gamma}(\Gamma) = \Gamma$.

The last step of the proof is showing that End(M) is in A whenever M is a generator-cogenerator.

Let $0 \to M \to I_0(M) \to I_1(M)$ be an injective copresentation of M. Since M is a cogenerator $I_i(M)$ is in add M, thus we get an exact sequence of projective $\operatorname{End}(M)$ -modules

$$0 \to \operatorname{End}(M) \to \operatorname{Hom}(M, I_0(M)) \to \operatorname{Hom}(M, I_1(M)). \tag{1}$$

Now we have the following isomorphisms of Λ -End(M)-bimodules

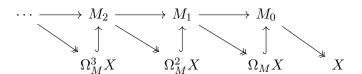
$$\operatorname{Hom}_{\Lambda}(M, D\Lambda) =$$
 $\operatorname{Hom}_{k}(M \otimes \Lambda, k) =$
 $\operatorname{Hom}_{k}(M, k) =$
 $DM =$
 $D \operatorname{Hom}_{\Lambda}(\Lambda, M)$

Since Λ is in add M, $\operatorname{Hom}(\Lambda, M)$ is projective, and thus $D \operatorname{Hom}(\Lambda, M) = \operatorname{Hom}(M, D\Lambda)$ is injective. This means that (1) is an injective copresentation, and thus $\operatorname{domdim} \operatorname{End}(M) \geq 2$.

Let $I = I_0(M)$, then $\operatorname{Hom}(I, \Lambda)$ and $I = \operatorname{Hom}(\Lambda, I)$ are bimodules. Need some kind of morita theorem here????????????

Definition 2.4. Let X be an object of mod Λ and M a contravariantaly finite subcategory.

I think this is specific to artin algebras



If \to are minimal M-approximations (they need not be surjective), and \hookrightarrow are their kernels, then this is an M-resolution of X. The M-resolution of X is the length of the sequence of (nonzero) M_i s, and the M-resolution of Λ is the supremum of the dimension on its objects.

Proposition 2.5. repdim - 2 is the minimum of M-res-dim(mod Λ) for M both generator and cogenrator.

Theorem 2.6. FDC holds for repdim; 3, [IT05, cor 8,9]

References

- [AR91] Maurice Auslander and Idun Reiten. Applications of contravariantly finite subcategories. *Adv. Math.*, 86(1):111–152, 1991.
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