

Finitistic dimension conjecture

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Abstract

FDC yo!

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Introduction

This is an introduction

1 finitistic dimension and conjectures

- FDC - finitistic dimesnion conjecture Finitistic dimension is always finite
- WTC - Watamatsu tilting conjecture
- GSC - Gorenstein symmetry conjecture
- NuC - Nunke condition
- SNC - strong Nakayama conjecture
- ARC - Auslander Reiten conjecture
- NC - Nakayama conjecture

1.1 Implications

$$\begin{array}{ccccccc}
 FDC & \longrightarrow & WTC & \longrightarrow & GSC & & \\
 \downarrow & & & & & & \\
 NuC & \longrightarrow & SNC & \longrightarrow & ARC & \longrightarrow & NC
 \end{array}$$

Theorem 1.1. [Hap93, 1.2]

- i) If $\text{findim}(\Lambda) < \infty$ (FDC) then $K^b(\text{inj } \Lambda)^\perp = 0$.
- ii) If $K^b(\text{inj } \Lambda)^\perp = 0$ then for $X \neq 0$ there exists i such that, $\text{Ext}^i(D(\Lambda), X) \neq 0$ (NuC).

Proof.

- i) Let $I^\bullet \in K^b(\text{inj } \Lambda)^\perp$ be non-zero. Since $\mathcal{D}^b(\Lambda) \cong K^{+,b}(\text{inj } \Lambda)$ we may assume I^\bullet is a complex of injectives, and WLOG we may assume it concentrated in degrees $i \geq 0$, and that $d^0 : I^0 \rightarrow I^1$ is not split mono. Since if its concentrated in degrees $i \geq k$ we can just shift it, and if d^0 is split mono then replacing I^0 by 0, and I^1 be I^1/I^0 gives a homotopic complex.

$\text{Hom}(D\Lambda, I^i)$ is in $\text{add Hom}(D\Lambda, D\Lambda) = \text{add } \Lambda$ so $\text{Hom}(D\Lambda, I^\bullet)$ is a complex of projectives.

$$\begin{array}{ccccc} 0 & \longrightarrow & D\Lambda & \longrightarrow & 0 \\ \downarrow & \swarrow \text{dashed} & \downarrow f & & \downarrow \\ I^{i-1} & \xrightarrow{d^{i-1}} & I^i & \xrightarrow{d^i} & I^{i+1} \end{array}$$

Since I^\bullet is in $K^b(\text{inj } \Lambda)^\perp$ and $D\Lambda$ is in $K^b(\text{inj } \Lambda)$, whenever $d^i f = 0$, f^\bullet is homotopic to 0. Meaning f factors through d^{i-1} . This means that $\text{Hom}(D\Lambda, I^\bullet)$ is an exact complex. Further since $\text{Hom}(D\Lambda, -)$ is an equivalence between $\text{inj } \Lambda$ and $\text{proj } \Lambda$ we have that $\text{Hom}(D\Lambda, d^0)$ is not split mono.

$\text{Cok Hom}(D\Lambda, d^i)$ has a projective resolution of length i . This resolution is the direct sum of the minimal resolution and an acyclic bounded complex of projectives. Since bounded acyclic complexes of projectives are split and $\text{Hom}(D\Lambda, d^0)$ is not, we must have that the minimal resolution has length i , and so $\text{findim}(\Lambda) = \infty$.

ii) Assume there is an $X \neq 0$ with $\text{Ext}^i(D\Lambda, X) = 0$ for all $i \geq 0$. Then X considered as a stalk complex is in $K^b(\text{inj } \Lambda)^\perp$. Proceed by induction: If $I[-i] \in K^b(\text{inj } \Lambda)$ is a stalk complex then $\mathcal{D}^b(I[-i], X) = \text{Ext}^i(I, X)$. This is 0 because $D\Lambda$ is the sum of the indecomposable injectives.

Let $I \in K^b(\text{inj } \Lambda)$ be a complex of width n . WLOG assume I concentrated in degrees $0 \leq i \leq n-1$. Then

$$I^0 \rightarrow I \rightarrow I^{<0} \rightarrow I^0[1]$$

is a triangle, and $I^{<0}$ has width $n-1$. Taking the long exact sequence in $\mathcal{D}^b(-, X)$ it follows that $\mathcal{D}^b(I, X) = 0$. \square

2 Recollement

$$\begin{array}{ccccc} & & i^* & & j^! \\ & \swarrow & \perp & \searrow & \\ \mathcal{D}^b(\Lambda') & \xrightarrow{i_* = i_!} & \mathcal{D}^b(\Lambda) & \xrightarrow{j^! = j^*} & \mathcal{D}^b(\Lambda'') \\ & \nwarrow & \perp & \swarrow & \\ & & i^! & & j_* \end{array}$$

Sort of like a split exact sequence of functors. We want

1. All functors are exact/triangulated

2. $j^*i_* = 0$
3. $i^*i_* \cong i^!i_! \cong id$ (induced by unit/counit)
4. $j^!j_! \cong j^*j_* \cong id$

$$j_!j^!X \xrightarrow{\varepsilon} X \xrightarrow{\eta} i_*i^*X \longrightarrow \Sigma$$

5. $i_!i^!X \xrightarrow{\varepsilon} X \xrightarrow{\eta} j_*j^*X \longrightarrow \Sigma$

Are triangles in $\mathcal{D}^b(\Lambda)$

Theorem 2.1. *Given a recollement FDC holds for middle if and only if it holds for the two others.*

Proof. Happel reduct technich [Hap93, 3.3]

□

write
later

3 Contravariant finiteness

Definition 3.1 (Resolving). A full subcategory of an abelian category is called resolving if

- It is closed under extensions
- It contains the projectives
- It contains the kernels of its epimorphisms

Note that the subcategory of modules with finite projective dimension is resolving.

Theorem 3.2. [AR91, 3.8] *Let \mathcal{X} be a contravariantly finite, resolving subcategory of $\text{mod } \Lambda$. Let X_i be the minimal approximation of S_i . Then any $X \in \mathcal{X}$ is a direct summand of an X_i -filtered module.*

Proof. Step 1: We want to show by induction on length that any module C is in an exact sequence $0 \rightarrow Y \rightarrow X \rightarrow C \rightarrow 0$ with X X_i -filtered and $\text{Ext}^1(\mathcal{X}, Y) = 0$.

Step 2: Whenever C is in \mathcal{X} we get that

$\text{Hom}(C, X) \longrightarrow \text{Hom}(C, C) \longrightarrow \text{Ext}^1(C, Y) = 0$ is exact, and thus C is a direct summand of X . \square

Corollary 3.2.1. *If the subcategory of modules with finite projective dimension is contravariantly finite, then the finitistic dimension is the supremum of the projective dimension of X_i . In particular it is finite.*

4 repdimension

Definition 4.1 (dominated dimension). Let $\Lambda \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \dots$ be a minimal injective resolution of Λ . Then the dominated dimension of Λ is $\inf\{n \mid I_n \text{ is not projective}\}$.

Definition 4.2 (rep-dimesnion). Let A be defined by

$$A = \{\Gamma \mid \text{domdim} \Gamma \geq 2, \Lambda \text{ morita equivalent to } \text{End}_\Gamma(I_0(\Gamma))\}$$

where $I_0(\Gamma)$ is the injective envelope of Γ . Then the repdimesnion of Λ is the minimal global dimension of $\Gamma \in A$.

Proposition 4.3. *(all modules are right modules) Repdim is the same as minimal global dimension of $\text{End}(M)$ for M being both a generator and cogenerator.*

Proof. Consider $\Gamma \in A$. Since $\text{domdim} \Gamma \geq 1$, $I_0(\Gamma)$ is the sum of all projective-injective modules (some probably several times).

Let \mathcal{S} be the set of all Γ -modules with a copresentation

$$0 \longrightarrow X \longrightarrow I_0 \longrightarrow I_1$$

with I_i in $\text{add } I_0(\Gamma)$. In particular Γ is in \mathcal{S} , because $\text{domdim} \Gamma \geq 2$.

The Yoneda embedding gives an equivalence

$$\text{Hom}_\Gamma(-, I_0(\Gamma)) : \text{add } I_0(\Gamma) \rightarrow \text{proj } \text{End}_\Gamma(I_0(\Gamma))^{\text{op}}$$

, and thus we get an equivalence

$$D \text{Hom}_\Gamma(-, I_0(\Gamma)) : \text{add } I_0(\Gamma) \rightarrow \text{inj } \text{End}_\Gamma(I_0(\Gamma))$$

Since $I_0(\Gamma)$ is injective $D \text{Hom}(-, I_0(\Gamma))$ is exact and preserves kernels, so extends to an equivalence

$$\text{Hom}_\Gamma(-, I_0(\Gamma)) : \mathcal{S} \rightarrow \text{mod } \text{End}_\Gamma(I_0(\Gamma))$$

can probably reformulate this in terms of projectives and left modules... is there any significance to the distinction?

Since $\text{End}_\Gamma(I_0(\Gamma))$ is morita equivalent to Λ , \mathcal{S} is equivalent to $\text{mod } \Lambda$. $\Gamma \in \mathcal{S}$ is clearly a generator. To see that it is a cogenerator note that Γ contains all the projective-injective indecomposable objects as direct summands, so there is an injection $I_0(\Gamma) \rightarrow \Gamma^n$, and since $I_0(\Gamma)$ is a cogenerator in \mathcal{S} , Γ is aswell.

Thus by the equivalence $\mathcal{S} \rightarrow \text{mod } \Lambda$ there is a cogenerator-generator object M such that $\text{End}_\Lambda(M) = \text{End}_\Gamma(\Gamma) = \Gamma$.

The last step of the proof is showing that $\text{End}(M)$ is in A whenever M is a generator-cogenerator.

Let $0 \rightarrow M \rightarrow I_0(M) \rightarrow I_1(M)$ be an injective copresentation of M . Since M is a cogenerator $I_i(M)$ is in $\text{add } M$, thus we get an exact sequence of projective $\text{End}(M)$ -modules

$$0 \rightarrow \text{End}(M) \rightarrow \text{Hom}(M, I_0(M)) \rightarrow \text{Hom}(M, I_1(M)). \quad (1)$$

Now we have the following isomorphisms of Λ - $\text{End}(M)$ -bimodules

$$\begin{aligned} \text{Hom}_\Lambda(M, D\Lambda) &= \\ \text{Hom}_k(M \otimes \Lambda, k) &= \\ \text{Hom}_k(M, k) &= \\ DM &= \\ D \text{Hom}_\Lambda(\Lambda, M) \end{aligned}$$

Since Λ is in $\text{add } M$, $\text{Hom}(\Lambda, M)$ is projective, and thus $D \text{Hom}(\Lambda, M) = \text{Hom}(M, D\Lambda)$ is injective. This means that (1) is an injective copresentation, and thus $\text{domdim } \text{End}(M) \geq 2$.

Let $I = I_0(M)$, then $\text{Hom}(I, \Lambda)$ and $I = \text{Hom}(\Lambda, I)$ are bimodules. Need some kind of morita theorem here????????????????? □

Definition 4.4. Let X be an object of $\text{mod } \Lambda$ and M a contravariantly finite subcategory.

I think this is specific to artin algebras

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M_2 & \longrightarrow & M_1 & \longrightarrow & M_0 \\ & \searrow & \uparrow & \searrow & \uparrow & \searrow & \uparrow \\ & & \Omega_M^3 X & & \Omega_M^2 X & & \Omega_M X & \longrightarrow & X \end{array}$$

If \rightarrow are minimal M -approximations (they need not be surjective), and \hookrightarrow are their kernels, then this is an M -resolution of X . The M -res-dimension of X is the length of the sequence of (nonzero) M_i 's, and the M -res-dimension of Λ is the supremum of the dimension on its objects.

Proposition 4.5. *Repdim-2 is the minimum of M -res-dim(mod Λ) for M both generator and cogenerator (assuming repdim is at least 2).*

Proof. The functor $\text{Hom}(M, -)$ is an equivalence from $\text{add } M$ to $\text{proj End}(M)$, which maps minimal M -approximations to projective covers. Let X be any module in $\text{mod End}(M)$ with projective dimension at least 2. Then it has a projective presentation

$$\Omega^2 X \rightarrow (M, M_1) \rightarrow (M, M_0) \rightarrow X.$$

Because of the equivalence this is induced by a map $f : M_1 \rightarrow M_0$. Since Hom is left exact we have that $\Omega^2 X \cong \text{Hom}(M, \ker f)$, and so the projective dimension of X is 2 plus the M -res-dimension of $\ker f$.

Since M is a cogenerator any module Y in $\text{mod } \Lambda$ has a copresentation

$$0 \longrightarrow Y \longrightarrow M_0 \xrightarrow{f} M_1.$$

Applying $\text{Hom}(M, -) =: (M, -)$ we get

$$0 \longrightarrow (M, Y) \longrightarrow (M, M_0) \xrightarrow{(M, f)} (M, M_1) \longrightarrow \text{Cok}(M, f) \longrightarrow 0.$$

If the projective dimension of $\text{Cok}(M, f)$ is less than 2, then (M, Y) is a direct summand of (M, M_0) . This means that $(M, Y) \cong (M, M')$, so the minimal M -approximation of Y is M' , and $(M, \Omega_M Y) = 0$. Since M is a generator this means $\Omega_M Y = 0$ and thus the M -res-dimension of Y is 0.

So provided the projective dimension of $\text{Cok}(M, f)$ is larger than or equal to 2, it equals the M -res-dimension of Y plus 2. In particular the global dimension of $\text{End}(M)$ is 2 plus the M -res-dimension of $\text{mod } \Lambda$, provided it is at least 2. \square

4.1 The Igusa-Todorov function

Let K be the free abelian group generated by isomorphism classes of modules, modulo the relations $[A \oplus B] = [A] + [B]$ and $[P] = 0$ when P is projective. Define the linear map $L : K \rightarrow K$ by $L[A] = [\Omega A]$. For any module X , $[\text{add } X]$ is a finitely generated subgroup of K . Fitting's lemma tells us that there is an integer η_X such that $L : L^m[\text{add } X] \rightarrow L^{m+1}[\text{add } X]$ is an isomorphism for every $m \geq \eta_X$. We define $\psi(X)$ to be $\eta_X + \sup\{\text{pd } Y \mid Y \in \text{add } \Omega^{\eta_X} X, \text{pd } Y < \infty\}$.

Lemma 4.6. *[IT05, Lemma 3]*

1. $\psi(M) = \text{pd } M$ when $\text{pd } M < \infty$.
2. $\psi(M^k) = \psi(M)$
3. $\psi(M) \leq \psi(M \oplus N)$
4. If Z is a direct summand of $\Omega^n(M)$ where $n \leq \eta_M$ and $\text{pd } Z < \infty$, then $\text{pd } Z + n \leq \psi(M)$.

Proof.

1. If $\text{pd } M < \infty$ then $L^m \neq 0$ for $m < \text{pd } M$, and $L^m = 0$ for $m \geq \text{pd } M$.
2. $\text{add } M^k = \text{add } M$, and ψ is only defined in terms of additive categories.
3. $\text{add } M \subseteq \text{add } M \oplus N$, so if L is injective when restricted to $L^m(\text{add } M \oplus N)$ then L is injective when restricted to $L^m(\text{add } M)$, so $\eta_M \leq \eta_{M \oplus N}$. Further $\Omega^{\eta_{M \oplus N} - \eta_M} \text{add } \Omega^{\eta_M} M \subset \text{add } \Omega^{\eta_{M \oplus N}} M \oplus N$, so $\psi(M) \leq \psi(M \oplus N)$.
4. Let $p = \text{pd } Z$ and $k = \eta_M - n$. Then $\Omega^k Z$ is in $\text{add } \Omega^{\eta_M} M$, so $\text{pd } \Omega^k Z + \eta_M \leq \psi(M)$. Thus

$$\text{pd } Z + n = p + n = (p - k) + \eta_M \leq \text{pd } \Omega^k Z + \eta_M \leq \psi(M).$$

□

Theorem 4.7. *[IT05, Theorem 4] Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of modules with $\text{pd } C < \infty$. Then $\text{pd } C \leq \psi(A \oplus B) + 1$.*

Proof. Let P_A^\bullet and P_C^\bullet be the minimal projective resolutions of A and C . Then we get a map of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_A^0 & \longrightarrow & P_A^0 \oplus P_C^0 & \longrightarrow & P_C^0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \end{array}$$

Applying the snake lemma we get $0 \rightarrow \Omega A \rightarrow \Omega B \oplus P \rightarrow \Omega C \rightarrow 0$ for some projective P . Thus for some $n \leq \text{pd } C$ we have $L^n[A] = L^n[B]$, and let n be the minimal such number. Clearly $n \leq \eta_{A \oplus B}$. Let $X = \Omega^n A = \Omega^n B$, then our sequence of n -syzygies looks like

$$0 \longrightarrow X \longrightarrow X \oplus P \longrightarrow \Omega^n C \longrightarrow 0.$$

Let f be the composition $X \longrightarrow X \oplus P \xrightarrow{\pi_X} X$. Then by fittings lemma X breaks as a direct sum into two components $X = Z \oplus Y$ such that $f = f_Z \oplus f_Y$ with f_Y an isomorphism and f_Z nilpotent. In other words the sequence above can be written as

$$0 \longrightarrow Z \oplus Y \longrightarrow Z \oplus Y \oplus P \longrightarrow \Omega^n C \longrightarrow 0.$$

with the left map being

$$\begin{bmatrix} f_Z & 0 \\ 0 & f_Y \\ * & * \end{bmatrix} \sim \begin{bmatrix} f_Z & 0 \\ 0 & f_Y \\ * & 0 \end{bmatrix}$$

So we get another short exact sequence

$$0 \longrightarrow Z \longrightarrow Z \oplus P \longrightarrow \Omega^n C \longrightarrow 0.$$

Let $T = \Lambda/\text{rad}(\Lambda)$ and apply the long exact sequence in $\text{Ext}(-, T)$. Then we get an exact sequence

$$\text{Ext}^k(Z, T) \longrightarrow \text{Ext}^k(Z \oplus P, T) \longrightarrow \text{Ext}^{k+1}(\Omega^n C, T)$$

where the left map is induced by f_Z since $\text{Ext}^k(Z \oplus P, T) \cong \text{Ext}^k(Z, T)$. Since f_Z is nilpotent this map is surjective if and only if $\text{Ext}^k(Z, T) = 0$, and $\Omega^n C$ has finite projective dimension we have that Z has finite projective dimension. In particular $\text{pd } \Omega^n C - 1 \leq \text{pd } Z \leq \text{pd } \Omega^n C$.

Since Z is a direct summand of $\Omega^n A \oplus B$ by lemma 4.6 we have that $\text{pd } Z + n \leq \psi(A \oplus B)$, and thus $\text{pd } \Omega^n C - 1 + n = \text{pd } C - 1 \leq \psi(A \oplus B)$. \square

Theorem 4.8. [IT05, Corollary 8] *If $\Lambda = \text{End}_\Gamma(P)$ for an algebra Γ with global dimension at most 3, and P projective then $\text{findim}(\Lambda) < \infty$.*

Proof. Let X be any Λ -module with finite projective dimension. Then it has a projective presentation $(P, P_1) \rightarrow (P, P_0) \rightarrow X \rightarrow 0$ where $(P, P_i) = \text{Hom}_\Gamma(P, P_i)$ with $P_i \in \text{add } P$. Since $(P, -)$ is an equivalence from $\text{add } P$ to $\text{proj } \Lambda$ this corresponds to a map $P_1 \rightarrow P_0$ which we can extend to a projective resolution in Γ :

$$0 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0.$$

Applying the exact functor $(P, -)$, we get an exact sequence

$$0 \longrightarrow (P, P_3) \longrightarrow (P, P_2) \longrightarrow (P, P_1) \longrightarrow (P, P_0) \longrightarrow X \longrightarrow 0.$$

Truncating this we get a short exact sequence

$$0 \longrightarrow (P, P_3) \longrightarrow (P, P_2) \longrightarrow \Omega^2 X \longrightarrow 0.$$

Then by theorem 4.7 the projective dimension of $\Omega^2 X$ is bounded by $\psi((P, P_3) \oplus (P, P_2)) + 1$. Which means

$$\text{pd } X \leq \psi((P, P_3) \oplus (P, P_2)) + 3 \leq \psi((P, \Gamma)) + 3$$

Since this bound doesn't depend on X , Λ has finite finitistic dimension. \square

Corollary 4.8.1. *If $\text{repdim}(\Lambda) \leq 3$ then $\text{findim}(\Lambda) < \infty$.*

Proof. If Λ has rep-dimension less than or equal to 3 then by proposition 4.3 there is a generator-cogenerator M in $\text{mod } \Lambda$ such that $\Gamma := \text{End}_\Lambda(M)$ has global dimension 3 or less. Then since M is a generator Λ is in $\text{add } M$ and so $\text{Hom}_\Lambda(M, \Lambda)$ is a projective Γ -module with $\text{End}_\Gamma(\text{Hom}_\Lambda(M, \Lambda)) = \text{End}_\Lambda(\Lambda) = \Lambda$. \square

5 Unbounded derived category

If we go to the unbounded derived category we can get a sort of converse to theorem 1.1.

Theorem 5.1. *[Ric19, Theorem 4.3] If the localizing category of $D\Lambda$ is the entire unbounded derived category then $\text{Findim}(\Lambda) < \infty$. (Note the capital F meaning the finitistic dimesnion of $\text{Mod } \Lambda$, which is bigger than or equal to that of $\text{mod } \Lambda$).*

Proof. Assume $\text{Findim}(\Lambda) = \infty$. Then there are modules M_i with projective dimension i for every $i \geq 0$. Let P_i be the minimal projective resolution of M_i , and consider $\bigoplus P_i[-i]$ and $\prod P_i[-i]$. Both of these have homology M_i in degree i , and are concentrated in non-negative degrees.

The inclusion from the sum to the product is clearly a quasi-isomorphism. We want to show that it is not a homotopy equivalence. Assume for the sake of contradiction that it was. Then tensoring with $\Lambda/\text{rad}(\Lambda)$ would give us another homotopy equivalence. Since $\Lambda/\text{rad}(\Lambda)$ is finitely presented tensoring preserves both products and coproducts. Because all the resolutions were minimal tensoring with $\Lambda/\text{rad}(\Lambda)$ gives us 0 differentials. In degree 0 we get

$$\bigoplus \text{Tor}_i(M_i, \Lambda/\text{rad}(\Lambda)) \rightarrow \prod \text{Tor}_i(M_i, \Lambda/\text{rad}(\Lambda)).$$

Since $\text{Tor}_i(M_i, \Lambda/\text{rad}(\Lambda))$ is nonzero for every M_i this map is not an isomorphism, and so we don't have a homotopy equivalence.

So the cone of the inclusion $\bigoplus P_i[-i] \rightarrow \prod P_i[-i]$, C , is 0 in the derived category, but non-zero in the homotopy category. Since Λ is artinian the product of projectives is projective [Cha60, Theorem 3.3], so $\prod P_i[-i]$ is a complex of projectives, which means that C is a complex of projectives.

In other words C is an acyclic lower bounded complex of projectives that is not contractible. Tensoring with $D\Lambda$ is an equivalence from projectives to injectives, so $C \otimes D\Lambda$ is a lower bounded complex of injectives that is not contractible. Such a complex cannot be acyclic so $C \otimes D\Lambda$ has homology.

The homology of C is 0, so $K(\Lambda)(\Lambda, C[i]) = 0$. Applying the equivalence $- \otimes D\Lambda$ we get

$$\mathcal{D}(\Lambda)(D\Lambda, C \otimes D\Lambda[i]) = K(\Lambda)(D\Lambda, C \otimes D\Lambda[i]) = 0.$$

This means that $C \otimes D\Lambda$ is not in the localizing category generated by $D\Lambda$, and so that is not the entire derived category. \square

Theorem 5.2. [Ric19, Theorem 4.4] *Finitistic dimension of Λ is finite if and only if $D\Lambda^\perp \cap \mathcal{D}^+(\Lambda) = 0$.*

Proof. In the theorem above we proved that when the finitistic dimension is infinite then there is a non-zero complex in $\mathcal{D}^+(\Lambda)$ perpendicular to $D\Lambda$.

The proof of the converse is the same as for theorem 1.1. If we have a non-zero object $X \in D\Lambda^\perp \cap \mathcal{D}^+(\Lambda) \neq 0$, then $\mathcal{D}(\Lambda)(D\Lambda, X)$ is a non-split complex of projectives that continue arbitrarily to the right. So the cokernels have arbitrarily big projective dimension. \square

6 Personal appendix

Theorem 6.1. *The global dimension of an artin algebra is the supremum of k with $\text{Ext}^k(T, T) \neq 0$ (T sum of simples). This is also the supremum of projective dimension and supremum of injective dimension.*

Proof. For a minimal projective resolution $\text{Hom}(-, T)$ makes the differentials 0, and similarly with $\text{Hom}(T, -)$ and injective resolutions. So $\text{Ext}^k(M, T)$ is only 0 exactly when $k > \text{pd } M$, similarly $\text{Ext}^k(T, M)$ is only 0 when k is bigger than the injective dimension. Since any module is built by extensions of simples you can prove by induction, and the long exact sequence in $\text{Ext}(-, T)$ you get that any module has projective dimension less than or equal to that of T . Similarly for injective dimension. \square

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