Finitistic dimension conjecture

Jacob Fjeld Grevstad

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Abstract

FDC yo!

Contents

Introduction

This is an introduction

1 The homological conjectures

- FDC - finitistic dimesnion conjecture

  Finitistic dimension is always finite

- WTC - Watamatsu tilting conjecture

  A module is called watamatsu tilting if

  - $\operatorname{Ext}^n(T,T) = 0$ for all $n > 0$.
  - There is an exact sequence

    $$\eta : 0 \to \Lambda \to T_0 \to T_1 \to \cdots$$

    where T_i is in $\operatorname{add} T$.
  - $\operatorname{Hom}(\eta, T)$ is exact. I.e. $\operatorname{Ext}^1(\operatorname{Ker} f, T) = 0$ for every f in η.

  WTC says that any watamatsu tilting module with finite projective dimension is a tilting module. I.e η can be chosen to be bounded.

- GSC - Gorenstein symmetry conjecture

  The injective dimension of ${}_\Lambda\Lambda$ is finite if and only if the projective dimension of $D(\Lambda_\Lambda)$ is finite.

- NuC - Nunke condition

  If $X \neq 0$ then there is an $n \geq 0$ such that $\operatorname{Ext}^n(D\Lambda, X) \neq 0$.

- SNC - strong Nakayama conjecture

  For every simple module S there is an $n \geq 0$ such that $\operatorname{Ext}^n(D\Lambda, S) \neq 0$.

- ARC - Auslander Reiten conjecture

  If $\operatorname{Ext}^n(M, M \oplus \Lambda) = 0$ for all $n > 0$ then M is projective.

- NC - Nakayama conjecture

  If Λ has infinite dominant dimension then Λ is self-injective.

1.1 Implications

$$\begin{tikzcd} FDC \arrow[r] \arrow[d] & WTC \arrow[r] & GSC \\ NuC \arrow[r] & SNC \arrow[r] & ARC \arrow[r] & NC \end{tikzcd}$$

Theorem 1.1. *[Hap93, 1.2]*

i) If $\operatorname{findim}(\Lambda) < \infty$ *(FDC) then* $K^b(\operatorname{inj}\Lambda)^\perp = 0$.

ii) If $K^b(\operatorname{inj}\Lambda)^\perp = 0$ *then for any* $X \neq 0$ *there exists* i *such that,* $\operatorname{Ext}^i(D(\Lambda), X) \neq 0$ *(NuC).*

Proof.

i) Let $I^\bullet \in K^b(\operatorname{inj}\Lambda)^\perp$ be non-zero. Since $\mathcal{D}^b(\Lambda) \cong K^{+,b}(\operatorname{inj}\Lambda)$ we may assume $I^\bullet$ is a complex of injectives, and WLOG we may assume it concentrated in degrees $i \geq 0$, and that $d^0 : I^0 \to I^1$ is not split mono. Since if its concentrated in degrees $i \geq k$ we can just shift it, and if d^0 is split mono then replacing I^0 by 0, and I^1 be I^1/I^0 gives a homotopic complex.

$\operatorname{Hom}(D\Lambda, I^i)$ is in $\operatorname{add}\operatorname{Hom}(D\Lambda, D\Lambda) = \operatorname{add}\Lambda$ so $\operatorname{Hom}(D\Lambda, I^\bullet)$ is a complex of projectives.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & D\Lambda \arrow[r] \arrow[d, "f"] \arrow[dl, dashed] & 0 \arrow[d] \\ I^{i-1} \arrow[r, "d^{i-1}"] & I^i \arrow[r, "d^i"] & I^{i+1} \end{tikzcd}$$

Since $I^\bullet$ is in $K^b(\operatorname{inj}\Lambda)^\perp$ and $D\Lambda$ is in $K^b(\operatorname{inj}\Lambda)$, whenever $d^i f = 0$, $f^\bullet$ is homotopic to 0. Meaning f factors through d^{i-1}. This means that $\operatorname{Hom}(D\Lambda, I^\bullet)$ is an exact complex. Further since $\operatorname{Hom}(D\Lambda, -)$ is an equivalence between $\operatorname{inj}\Lambda$ and $\operatorname{proj}\Lambda$ we have that $\operatorname{Hom}(D\Lambda, d^0)$ is not split mono.

$\operatorname{Cok}\operatorname{Hom}(D\Lambda, d^i)$ has a projective resolution of length i. This resolution is the direct sum of the minimal resolution and an acyclic bounded complex of projectives. Since bounded acyclic complexes of projectives are split and $\operatorname{Hom}(D\Lambda, d^0)$ is not, we must have that the minimal resolution has length i, and so $\operatorname{findim}(\Lambda) = \infty$.

ii) Assume there is an $X \neq 0$ with $\operatorname{Ext}^i(D\Lambda, X) = 0$ for all $i \geq 0$. Then X considered as a stalk complex is in $K^b(\operatorname{inj}\Lambda)^\perp$. Proceed by induction: If

$I[-i] \in K^b(\mathrm{inj}\,\Lambda)$ is a stalk complex then $\mathcal{D}^b(I[-i], X) = \mathrm{Ext}^i(I, X)$. This is 0 because $D\Lambda$ is the sum of the indecomposable injectives.

Let $I \in K^b(\mathrm{inj}\,\Lambda)$ be a complex of width n. WLOG assume I concentrated in degrees $0 \le i \le n-1$. Then

$$I^0 \to I \to I^{<0} \to I^0[1]$$

is a triangle, and $I^{<0}$ has width $n-1$. Taking the long exact sequence in $\mathcal{D}^b(-, X)$ it follows that $\mathcal{D}^b(I, X) = 0$. □

Proposition 1.2. *WTC $\Rightarrow$ GSC*

Proof. $D(\Lambda_\Lambda)$ is watamatsu tilting. WTC then gives us that if $D\Lambda$ has finite projective dimension then Λ has a finite injective dimension.

For the other direction assume ${}_\Lambda\Lambda$ has finite injective dimension. Then $D({}_\Lambda\Lambda)$ has finite projective dimension, so WTC gives us that Λ_Λ has finite injective dimension. Which means $D(\Lambda_\Lambda)$ has finite projective dimension. □

Proposition 1.3. *ARC is equivalent to M a generator with* $\mathrm{Ext}^n(M, M) = 0$ *for $n > 0$ implies M projective.*

Proof. Assume ARC and that M satisfies the hypothesis. Then since M is a generator Λ is in $\mathrm{add}\, M$ and thus $\mathrm{Ext}^n(M, \Lambda) = 0$. So $\mathrm{Ext}^n(M, M \oplus \Lambda) = 0$ and M is projective.

For the other direction Assume M satisfies $\mathrm{Ext}^n(M, M \oplus \Lambda) = 0$. Then $\mathrm{Ext}^n(M \oplus \Lambda, M \oplus \Lambda) = 0$, so $M \oplus \Lambda$ is projective, which means that M is projective. □

Proposition 1.4. *SNC $\Rightarrow$ ARC*

Proof. $\mathrm{Ext}^i(D\Lambda, S) = \mathrm{Ext}^i(DS, \Lambda)$, so SNC means that for every simple there is an i such that $\mathrm{Ext}^i(S, \Lambda) \neq 0$.

Assume M is a nonprojective generator such that $\mathrm{Ext}^n(M, M) = 0$ for all $n > 0$. Let Γ be $\mathrm{End}(M)^{op}$, and let

$$M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

be an injective resolution of M. Since $\mathrm{Ext}^n(M, M) = 0$ when we apply $(M, -) := \mathrm{Hom}(M, -)$ we get an exact sequence.

$$\Gamma \longrightarrow (M, I_0) \longrightarrow (M, I_1) \longrightarrow \cdots$$

By proposition 1.7 this is an injective resolution of Γ.

Since M is a non-projective generator it has every indecomposable projective as a summand and a nonprojective summand. So M has more indecomposable summands than Λ which means that Γ has more indecomposable projectives than Λ. It follows that Γ also has more injectives and thus has an injective not on the form (M, I). Let Q be such an injective and let S be its socle. Then $\mathrm{Hom}_\Gamma(S, (M, I_i)) = 0$ for all i, so $\mathrm{Ext}^i(S, \Gamma) = 0$ for all i. Thus Γ does not satisfy SNC. □

The next proposition requires part of the theory of Wedderburn projectives. The relevant theory is proven in section 1.2 below.

Proposition 1.5. *ARC $\Rightarrow$ NC*

Proof. Assume Γ has dominant dimension ∞, but is not self injective, and let

$$0 \longrightarrow \Gamma \longrightarrow I_0 \longrightarrow I_1$$

be an injective copresentation of Γ. Let P be the sum of the projective covers of all nonisomorphic simple modules in the socle of I_0. Then by proposition 1.10 we have that P is Wedderburn projective.

Let $\Lambda = \mathrm{End}(P)^{op}$ and let $M = \mathrm{Hom}(P, \Gamma)$. Then M is a nonprojective generator, we want to show that $\mathrm{Ext}^{>0}(M, M) = 0$.

We have functors $(M, -) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ and $(P, -) : \mathrm{mod}\,\Gamma \to \mathrm{mod}\,\Lambda$. By proposition 1.7 $(M, -)$ is fully faithful and $(P, -) \circ (M, -) = id_\Lambda$.

Let $0 \to M \to Q_0 \to Q_1$ be an injective copresentation of M. Applying $(M, -)$ we get an injective copresentation of Γ. We conclude that all the projective-inejctive modules are in the essential image of $(M, -)$.

In other words if $I^\bullet$ is the minimal injective resolution of Γ then $Q^\bullet := (P, I^\bullet)$ is the minimal injective resolution of M, and $(M, Q^\bullet) = I^\bullet$. This means that $(M, Q^\bullet)$ is exact away from 0, so $\mathrm{Ext}^{>0}(M, M) = 0$.

But then M is a nonprojectvie generator with $\mathrm{Ext}^{>0}(M, M) = 0$, so Λ does not satisfy ARC. □

Proposition 1.6. *[AR75] SNC $\Rightarrow$ NC*

Proof. $\mathrm{Ext}(D\Lambda, S) = \mathrm{Ext}(DS, \Lambda)$. $\mathrm{Ext}(DS, \Lambda)$ being nonzero means $I(DS)$ appears in the injective resolution of Λ. If all injectives apear in the resolution and the dominant dimension is infinity then all injectives are projective. Thus Λ is self injective. □

1.2 Wedderburn correspondence

Proposition 1.7. *Let Λ be an artin algebra and M a generator. Let $\Gamma = \mathrm{End}(M)^{op}$ and $P = (M, \Lambda)$. Then we have the following:*

- $\mathrm{End}(P)^{op} = \Lambda$ *and* $(P, \Gamma) = M$.

  Proof. By Yonedas lemma we have an equivalence $(M, -) : \operatorname{add} M \to \operatorname{add}(M, M) = \operatorname{proj} \Gamma$. Since M is a generator Λ is in $\operatorname{add} M$. So

  $$\mathrm{End}(P) = ((M, \Lambda), (M, \Lambda)) = \mathrm{End}(\Lambda) = \Lambda^{op}$$

  and

  $$(P, \Gamma) = ((M, \Lambda), (M, M)) = (\Lambda, M) = M.$$

  □

- $(P, -) \circ (M, -)$ *is the identity on* $\operatorname{mod} \Lambda$.

  Proof. Let X be a Λ-module. Since $\operatorname{add} M$ has only a finite number of indecomposables it is functorially finite. So we can take an M-resolution of X.

  $$\cdots \to M_1 \to M_0 \to X \to 0$$

  Since $\operatorname{add} M$ contains the projectives this is exact. Applying $(M, -)$ we get a projective resolution of (M, X). Since (M, X) is determined by its projective resolution and X is determined by its M-resolution we need only show that $(P, -) \circ (M, -)$ is the identity on $\operatorname{add} M$. Then again by Yonedas lemma $(P, (M, M')) = (\Lambda, M') = M'$. □

Proposition 1.8. *Let M be a module and I an injective module. If the projective cover of the socle of I is a direct summand of M, then (M, I) is an injective $\Gamma := \mathrm{End}(M)^{op}$-module.*

Proof. Let $J \leq \Gamma$ be a left ideal and let $\psi : J \to (M, I)$ be any Γ-linear map. By lemma 10.3 it is enough to show that ψ factors through Γ. Assume J is generated by f_i. If we can find $\gamma : M \to I$ such that $\gamma \circ f_i = \psi(f_i)$ then we would get our factorization by mapping $1 \in \Gamma$ to γ.

$$\begin{tikzcd}
\bigoplus M \arrow[d, "\sum f_i"'] \arrow[dr, "\sum \psi(f_i)"] & \\
M \arrow[r, dashed, "\gamma"'] & I
\end{tikzcd}$$

Next we want to show that the kernel of $\sum \psi(f_i)$ contains the kernel of $\sum f_i$. To se this let K be the kernel of $\sum f_i$ and let K' be the kernel of $\sum \psi(f_i)$. If K' does not contain K then $Q := K/K' \cap K$ is a nonzero module that is mapped injectively into I. So the socle of Q is a summand of the socle of I. Then by assumption the projective cover of the socle of Q is a direct summand of M. By the lifting property of projectives we get a map $M \to K$ such that the composition with $\sum \psi(f_i)$ is non-zero.

Let a_i be the composition $M \twoheadrightarrow K \hookrightarrow \bigoplus M \xrightarrow{\pi_i} M$. Then we get $\sum f_i \circ a_i = 0$. Applying ψ we get $\sum \psi(f_i) \circ a_i = 0$, which gives a contradiction. Thus K' contains K.

Using this we get the following commutative diagram:

$$\begin{tikzcd}
\bigoplus M \arrow[d] \arrow[dr, "\sum \psi(f)"] \arrow[dd, bend right=60, "\sum f_i"'] & \\
\bigoplus M/K \arrow[r] \arrow[d, hook] & I \\
M \arrow[ur, dashed, "\exists \gamma"'] &
\end{tikzcd}$$

Since I is injective it lifts monomorphisms so we know that γ exists. Thus (M, I) is an injective Γ-module. $\square$

Definition 1.9 (Wedderburn projective)**.** Let Γ be an artin algebra and P a finitely generated projective. Let $\Lambda = \mathrm{End}(P)^{op}$ and $M = (P, \Gamma)$. P is said to be Wedderburn projective if $\mathrm{End}(M)^{op} = \Gamma$.

Proposition 1.10. *If P contains the projective cover of all simple modules that appear in the socle of an injective copresentation of Γ, then P is Wedderburn projective.*

To prove this we first need the next proposition as a lemma.

Proposition 1.11. *Let P be a projective Γ-module, and let $\Lambda = \mathrm{End}(P)^{op}$. Then $(P, -) : \mathrm{mod}\,\Gamma \to \mathrm{mod}\,\Lambda$ is fully faithful on $\mathrm{add}\, I(P/JP)$.*

Proof. We want to show that the map $\mathrm{Hom}_\Gamma(I, I') \to \mathrm{Hom}_\Lambda((P, I), (P, I'))$ is an isomorphism. Let's first show injectivity. Let $f : I \to I'$ be a non-zero map. Then the socle of $\mathrm{Im}\, f$ is a semisimple submodule of I', so it is in $\mathrm{add}\, P/JP$. Then there exists a nonzero map from P to $\mathrm{Im}\, f$. Since P is projective this lifts to a map $\hat{f} : P \to I$. Then $f \circ \hat{f}$ is non-zero, so $\mathrm{Hom}_\Gamma(I, I') \to \mathrm{Hom}_\Lambda((P, I), (P, I'))$ is injective.

The argument for surjectivity is similar to that for proposition 1.8. Let $\psi : (P, I) \to (P, I')$ be a Λ-linear map. Let $f_i : P \to I$ generate (P, I) as a lambda module. Consider the diagram

$$\begin{tikzcd} \bigoplus P \arrow[r, "\sum f_i"] \arrow[dr, "\sum \psi(f_i)"'] & I \arrow[d, dashed, "?"] \\ & I' \end{tikzcd}$$

We wish to show that there is a map at ? completing the diagram. We wish to show that K' contains K. Assume for the sake of contradiction that it does not. Then $Q := K/K' \cap K$ is mapped injectively into I' by $\sum \psi(f_i)$. So the socle of Q is in $\mathrm{add}\, P/JP$, and we have a non-zero map $P \to Q$.

Since P is projective this extends to a map $P \to K$. Let a_i be the compositions $P \longrightarrow K \longrightarrow \bigoplus P \xrightarrow{\pi_i} P$ Then clearly $\sum f_i \circ a_i = 0$, but $\sum \psi(f_i) \circ a_i$ is non-zero. Since ψ is Λ-linear this is a contradiction, so K' contains K.

Then we get an induced diagram

$$\begin{tikzcd} \bigoplus P \arrow[d, two heads] & \\ (\bigoplus P)/K \arrow[r, hook, "\sum f_i"] \arrow[dr, "\sum \psi(f_i)"'] & I \arrow[d, dashed, "\exists"] \\ & I' \end{tikzcd}$$

Now because I' is injective we know that there is a lift, and so $\mathrm{Hom}_\Gamma(I, I') \to \mathrm{Hom}_\Lambda((P, I), (P, I'))$ is surjective, and thus an isomorphism. □

Corollary 1.11.1. *proposition 1.10*

Proof. Let $\Gamma \to I_0 \to I_1$ be a minimal injective presentation of Γ. Then by proposition 1.8we have that $(P, I_0) \to (P, I_1)$ is an injective presentation of

(P, Γ). The proposition gives us that $(P, -)$ is fully faithful on I_0 and I_1. Since the endomorphisms of Γ are exactly endomorphisms of $I_0 \to I_1$ up to homotopy this means that

$$\Gamma^{op} = \mathrm{End}_\Gamma(\Gamma) = \mathrm{End}_\Lambda((P, \Gamma))$$

So P is Wedderburn projective. $\square$

2 Recollement

Definition 2.1 (Recollement)**.** A recollement is a collection of six functors satisfying:

$$\begin{tikzcd}
\mathcal{D}^b(\Lambda') \arrow[r, "i_*=i_!"] & \mathcal{D}^b(\Lambda) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}^b(\Lambda'') \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

1. All functors are exact/triangulated
2. $j^* i_* = 0$
3. $i^* i_* \cong i^! i_! \cong id$ (induced by unit/counit)
4. $j^! j_! \cong j^* j_* \cong id$
5. For every $X \in \mathcal{D}^b(\Lambda)$ we have the following distinguished triangles:

$$j_! j^! X \xrightarrow{\varepsilon} X \xrightarrow{\eta} i_* i^* X \longrightarrow j_! j^! X[1]$$

$$i_! i^! X \xrightarrow{\varepsilon} X \xrightarrow{\eta} j_* j^* X \longrightarrow i_! i^! X[1]$$

Note that (3) and (4) are equivalent to i_*, $j_!$, and j_* being fully faithful.

Lemma 2.2. *Let* $\begin{tikzcd} \mathcal{D}^b(\Lambda') \arrow[r, bend right, "i_*"'] & \mathcal{D}^b(\Lambda) \arrow[l, bend right, "i^*"'] \end{tikzcd}$ *be exact functors with an adjoint pair* (i^*, i_*)*. Then* i^* *preserves bounded projective complexes and* i_* *preserves bounded injective complexes.*

Proof. The bounded projective complexes can be categories as the complexes P such that for any complex Y there is an integer t_Y such that $\mathrm{Hom}(P, Y[t]) = 0$ for $t \geq t_Y$.

Let P be a bounded complex of projectives in $\mathcal{D}^b(\Lambda)$. Then we want to show that i^*P is as well. Let Y be any complex in $\mathcal{D}^b(\Lambda')$. Then $\mathcal{D}^b(\Lambda')(i^*P, Y[t]) = \mathcal{D}^b(\Lambda)(P, i_*Y[t])$, so since P is a bounded complex of projectives there is t_Y such that this vanishes for $t \geq t_Y$.

The statement for injectives is exactly dual. □

Lemma 2.3. *Let* $\begin{tikzcd} \mathcal{D}^b(\Lambda') \arrow[r, "i_*"] & \mathcal{D}^b(\Lambda) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \end{tikzcd}$ *be exact functors with adjoint pairs* (i^*, i_*) *and* $(i_*, i^!)$. *Then the homology of* i_*X *is uniformly bounded for* $X \in \operatorname{mod}\Lambda'$. *I.e. there is an* r *such that* $H^j(i_*X) = 0$ *is 0 outside of* $j \in (-r, r)$.

Proof. We first prove that there is an r' such that $H^j(i_*X) = 0$ for $j \geq r'$. Let P be $i^*\Lambda \in \mathcal{D}^b(\Lambda') = K^{-,b}(\operatorname{proj}\Lambda')$. Then by lemma 2.2 P is abounded complex of projectives.

Thus there is an r' such that $P^{-j} = 0$ for $j \geq r'$. Then $\mathcal{D}^b(\Lambda')(P, X[j]) = \mathcal{D}^b(\Lambda)(\Lambda, i_*X[j]) = H^j(i_*X) = 0$ for $j \geq r'$ and any Λ'-module X.

Next we prove that there is an r'' such that $H^{-j}(i_*X) = 0$ for $j \geq r''$. The argument is completely dual. Let I be $i^!D\Lambda \in \mathcal{D}^b(\Lambda') = K^{+,b}(\operatorname{inj}\Lambda')$. Then again by lemma 2.2 I is abounded complex of injectives.

Thus there is an r'' such that $I^j = 0$ for $j \geq r''$. Then $\mathcal{D}^b(\Lambda')(X, I[j]) = \mathcal{D}^b(\Lambda)(i_*X, D\Lambda[j]) = H^{-j}(i_*X) = 0$ for $j \geq r''$ and any Λ'-module X.

Letting r be the maximum of r' and r'' we get that $H^j(X)$ is zero outside of $(-r, r)$. □

Theorem 2.4. *[Hap93, 3.3] Given a recollement FDC holds for middle if and only if it holds for the two others.*

Proof. Assume FDC holds for Λ, we begin by showing it holds for Λ'.

Let $T = \Lambda'/rad\Lambda'$. Then the projective dimension of X is the largest t for which $\mathrm{Ext}^t(X, T) \neq 0$. Let X be a module in $\operatorname{mod}\Lambda'$ with finite projective dimension. Then since X is isomorphic to its projective resolution, by lemma 2.2 i_*X is a bounded complex of projectives. Say:

$$i_*X = 0 \to P^{-s} \to \cdots \to P^{s'} \to 0$$

By lemma 2.3 we know there is an r independent of X such that $H^{-j}(X) = 0$ for $j \geq r$. Truncating i_*X at $-r$ gives a projective resolution of $\ker d^{-r}_{i_*X}$. Since Λ satisfies FDC this means that $s \leq r + \text{findim}(\Lambda)$.

Since i_*T is in $\mathcal{D}^b(\Lambda)$ it is a bounded complex, in particular there is a t_0 such that $i_*T^t = 0$ for $t \geq t_0$. Then by the bounds above $\mathcal{D}^b(\Lambda)(i_*X, i_*T[t]) = 0$ for $t \geq t_0 + r + \text{findim}(\Lambda)$. Since i_* is fully faithful this equals $\mathcal{D}^b(\Lambda')(X, T[t])$, and so $\text{findim}(\Lambda') \leq t_0 + r + \text{findim}(\Lambda)$. That is, Λ' satisfies FDC.

The proof for Λ'' is the same, just replacing i_* with $j_!$.

For the converse assume Λ' and Λ'' satisfy FDC. Let $T = \Lambda/rad\Lambda$, and X be a Λ-module with finite projective dimension. By definition 2.1 (5) we have distinguished triangles:

$$j_!j^!X \longrightarrow X \longrightarrow i_*i^*X \longrightarrow j_!j^!X[1]$$

$$i_!i^!T \longrightarrow T \longrightarrow j_*j^*T \longrightarrow i_!i^!T[1]$$

Let $(-,-)_m := \mathcal{D}^b(\Lambda)(-,-[m])$, and $X_j := j_!j^!X$, $X_i := i_*i^*X$, $T_i := i_!i^!T$, $T_j = j_*j^*T$. Then we have long exact sequences:

$$\cdots \longrightarrow (X, T_i)_m \longrightarrow (X, T)_m \longrightarrow (X, T_j)_m \longrightarrow (X, T_i)_{m+1} \longrightarrow \cdots$$

$$\cdots \longrightarrow (X_i, T_i)_m \longrightarrow (X, T_i)_m \longrightarrow (X_j, T_i)_m \longrightarrow (X_i, T_i)_{m+1} \longrightarrow \cdots$$

$$\cdots \longrightarrow (X_i, T_j)_m \longrightarrow (X, T_j)_m \longrightarrow (X_j, T_j)_m \longrightarrow (X_i, T_j)_{m+1} \longrightarrow \cdots$$

We have

$$(X_i, T_j)_m = (i_*i^*X, j_*j^*T)_m = (j^*i_*i^*X, j^*T)_m = 0$$

and

$$(X_j, T_i)_m = (j_!j^!X, i_!i^!T)_m = (j^!X, j^!i_!i^!T)_m = 0$$

which combined with long exact sequences gives us that $(X_i, T_i)_m = (X, T_i)_m$ and $(X_j, T_j)_m = (X, T_j)_m$. If we can show that $(X_i, T_i)_m$ and $(X_j, T_j)_m$ are bounded, then $(X, T_i)_m$ and $(X, T_j)_m$, and consequently $(X, T)_m$ would be bounded. Which would give a bound on the projective dimension of X.

We start by bounding $(X, T_i)_m = (X_i, T_i)_m$. First note that

$$(X_i, T_i)_m = (i_* i^* X, i_! i^! T)_m = (i^* i_* i^* X, i^! T)_m = (i^* X, i^! T)_m$$

Since X has finite projective dimension we can think of it as a bounded complex of projectives. Then by lemma 2.2 $i^* X$ is as well. By the second half of lemma 2.3 (using (i^*, i_*) instead of $(i_*, i^!)$) we have that there is an r such that $H^{-j}(i^* X) = 0$ for all $j \geq r$. This means that thinking of $i^* X$ as a complex of projectives it is 0 in degree t for all $t \leq -(r + \operatorname{pd} \ker d^{-r}_{i^* X})$, in particualr it is 0 for all $t \leq -(r + \operatorname{findim}(\Lambda'))$. Since $i^! T$ is a bounded complex, it has an upper bound, say t_0. Thus $(i^* X, i^! T)_m = 0$ for all $m \geq t_0 + r + \operatorname{findim}(\Lambda')$.

The bound on $(X, T_j)_m$ is similar, using the finitistic dimension of Λ''. Taking the maximum of these two bounds we get a bound on $(X, T)_m$, which gives a bound on the projective dimension independent of X, hence a bound on $\operatorname{findim}(\Lambda)$. □

3 Contravariant finiteness

Definition 3.1 (Resolving)**.** A full subcategory of an abelian category is called resolving if

- It is closed under extensions
- It contains the projectives
- It is contains the kernels of its epimorphisms

Note that the subcategory of modules with finite projective dimension is resolving.

Lemma 3.2. *Let $\mathcal{X}$ be coresolving. Then* $\operatorname{Ext}^1(\mathcal{X}, Y) = 0$ *implies that* $\operatorname{Ext}^i(\mathcal{X}, Y) = 0$ *for all $i \geq 1$.*

Proof. Since $\mathcal{X}$ contains the projectives ΩX is the kernel of an epimorphism in $\mathcal{X}$. Thus $\mathcal{X}$ contains all syzygies. $\operatorname{Ext}^i(X, Y) = \operatorname{Ext}^1(\Omega^{i-1} X, Y) = 0$. □

Proposition 3.3. *If $\mathcal{X}$ is resolving then $\mathcal{Y} := \ker \operatorname{Ext}^{\geq 1}(\mathcal{X}, -) = \ker \operatorname{Ext}^1(\mathcal{X}, -)$ is closed under extensions.*

Proof. Let $0 \to Y \to E \to Y' \to 0$ be an extension of objects in $\mathcal{Y}$, and let X be an object of $\mathcal{X}$. Then we get an exact sequence

$$0 = \operatorname{Ext}(X, Y) \longrightarrow \operatorname{Ext}(X, E) \longrightarrow \operatorname{Ext}(X, Y') = 0$$

Thus $\operatorname{Ext}(X, E) = 0$ and E is in $\mathcal{Y}$. $\square$

Lemma 3.4. *Let $\mathcal{X}$ be a contravariantly finite, resolving subcategory of $\operatorname{mod} \Lambda$. Then for every object $C \in \operatorname{mod} \Lambda$ there is a short exact sequence $0 \to Y \to X \to C \to 0$ with $X \to C$ minimal $\mathcal{X}$-approximation and $\operatorname{Ext}^i(\mathcal{X}, Y) = 0$ for all $i \geq 1$.*

Proof. Since $\mathcal{X}$ is contravariantly finite C has a minimal approximation $X \to C$. Since $\mathcal{X}$ contains the projective cover of C this approximation must be an epimorphism. So it is part of a short exact sequence $0 \to Y \to X \to C \to 0$. Let X' be an arbitrary object in $\mathcal{X}$. Taking the long exact sequence in $\operatorname{Ext}(X', -)$ gives us

$$\begin{tikzcd} \operatorname{Hom}(X', Y) \arrow[r] & \operatorname{Hom}(X', X) \arrow[r] & \operatorname{Hom}(X', C) \arrow[dll, out=-10, in=170] \\ \operatorname{Ext}(X', Y) \arrow[r] & \operatorname{Ext}(X', X) \arrow[r] & \operatorname{Ext}(X', C) \end{tikzcd}$$

Since $X \to C$ is an approximation we know that $\operatorname{Hom}(X', X) \to \operatorname{Hom}(X', C)$ is epi. Thus if we can prove that $\operatorname{Ext}(X', X) \to \operatorname{Ext}(X', C)$ is mono we would have that $\operatorname{Ext}(X', Y) = 0$. Assume we have an element of $\operatorname{Ext}(X', X)$ that is mapped to 0, i.e. we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C \arrow[r] & C \oplus X' \arrow[r] & X' \arrow[r] & 0 \end{tikzcd}$$

Since $\mathcal{X}$ is closed under extensions E is in $\mathcal{X}$. By composing with projection $C \oplus X' \to C$ we get a commutative triangle

$$\begin{tikzcd} X \arrow[r] \arrow[d] & E \arrow[dl] \\ C & \end{tikzcd}$$

since $X \to C$ is an approximation we get that $E \to C$ factors through X. The endomorphism $X \to E \to X$ leaves the approximation unchanged, so by minimality it must be an isomorphism. Hence $0 \to X \to E \to X' \to 0$ is split and $\mathrm{Ext}(X', X) \to \mathrm{Ext}(X', C)$ is injective. Thus $\mathrm{Ext}(X', Y) = 0$. $\square$

Theorem 3.5. *[AR91, 3.8] Let $\mathcal{X}$ be a contravariantly finite, resolving subcategory of* $\mathrm{mod}\,\Lambda$*. Let X_i be the minimal approximation of S_i. Then any $X \in \mathcal{X}$ is a direct summand of an X_i-filtered module.*

Proof. The first part of the proof is to show by induction on length that any module C is in an exact sequence $0 \to Y \to X \to C \to 0$ with X X_i-filtered and $\mathrm{Ext}^1(\mathcal{X}, Y) = 0$.

For the base case if $C = S_i$ is simple then by lemma 3.4 we have an exact sequence $0 \to Y \to X_i \to C \to 0$.

For the induction step, assume it holds for all modules of length less than n, and let C be a module of length n. Then by Jordan-Hölder C is the extension of two modules of length less than n. Say

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

Applying the induction hypothesis we get a diagram on the form

$$\begin{tikzcd} & 0 \arrow[d] & & 0 \arrow[d] & \\ & Y' \arrow[d] & & Y'' \arrow[d] & \\ & X' \arrow[d] & & X'' \arrow[d] & \\ 0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] & C'' \arrow[r] \arrow[d] & 0 \\ & 0 & & 0 & \end{tikzcd}$$

Taking the pullback of $X'' \to C''$ we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & C' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Since Y' satisfies $\mathrm{Ext}^1(\mathcal{X}, Y') = 0$ by lemma 3.2 it also satisfies $\mathrm{Ext}^2(\mathcal{X}, Y') = 0$. In particular from the long exact sequence we get that $X' \to C'$ induces an isomorphism $\mathrm{Ext}(X'', X') \to \mathrm{Ext}(X'', C)$. Thus $0 \to C' \to E \to X'' \to 0$ comes from a sequence $0 \to X' \to X \to X'' \to 0$. In other words we have a diagram

$$\begin{tikzcd}
& 0 \arrow[d] & & 0 \arrow[d] & \\
& Y' \arrow[d] & & Y'' \arrow[d] & \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] & C'' \arrow[r] \arrow[d] & 0 \\
& 0 & & 0 &
\end{tikzcd}$$

Applying the snake lemma we can fill out the diagram:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Since X is an extension of X_i-filtered modules, it is also X_i-filtered. Since Y is the extension of Y'' and Y' it follows from proposition 3.3 that $\mathrm{Ext}(\mathcal{X}, Y) =$

0.

Hence any C fits into a sequence $0 \to Y \to X \to C \to 0$ with X being X_i-filtered and $\mathrm{Ext}(\mathcal{X}, Y) = 0$.

Now suppose that C is in $\mathcal{X}$, and let $0 \to Y \to X \to C \to 0$ be as before. Then we get that

$$\mathrm{Hom}(C, X) \longrightarrow \mathrm{Hom}(C, C) \longrightarrow \mathrm{Ext}^1(C, Y) = 0$$

is exact, and thus C is a direct summand of X. So every object in $\mathcal{X}$ is a direct summand of an X_i-filtered module. □

Corollary 3.5.1. *If the subcategory of modules with finite projective dimension is contravariantly finite, then the finitistic dimension is the supremum of the projective dimension of X_i. In particular it is finite.*

4 repdimension

Many results based on the survey [Opp09].

Definition 4.1 (dominated dimension)**.** Let $\Lambda \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$ be a minimal injective resolution of Λ. Then the dominated dimension of Λ is $\inf\{n | I_n$ is not projective $\}$.

Definition 4.2 (rep-dimesnion)**.** Let A be defined by

can probably reformulate this in terms of projectives and left modules... is there any significance to the distinction?

$$A = \{\Gamma | domdim\Gamma \geq 2, \Lambda \text{ morita eqquivalent to } \mathrm{End}_\Gamma(I_0(\Gamma))\}$$

where $I_0(\Gamma)$ is the injective envelope of Γ. Then the repdimesnion of Λ is the minimal global dimension of $\Gamma \in A$.

Proposition 4.3. *(all modules ar right modules) Repdim is the same as minimal global dimension of* $\mathrm{End}(M)$ *for M being both a generator and cogenerator.*

I guess this is Auslanders original definition

Proof. Consider $\Gamma \in A$. Since $domdim\Gamma \geq 1$, $I_0(\Gamma)$ is the sum of all projective-injective modules (some probably several times).

Let $\mathcal{S}$ be the set of all Γ-modules with a copresentation

$$0 \longrightarrow X \longrightarrow I_0 \longrightarrow I_1$$

with I_i in add $I_0(\Gamma)$. In particular Γ is in $\mathcal{S}$, because *domdim*$\Gamma \geq 2$.

The Yoneda embedding gives an equivalence

$$\mathrm{Hom}_\Gamma(-, I_0(\Gamma)) : \mathrm{add}\, I_0(\Gamma) \to \mathrm{proj}\,\mathrm{End}_\Gamma(I_0(\Gamma))^{op}$$

, and thus we get an equivalence

$$D\,\mathrm{Hom}_\Gamma(-, I_0(\Gamma)) : \mathrm{add}\, I_0(\Gamma) \to \mathrm{inj}\,\mathrm{End}_\Gamma(I_0(\Gamma))$$

Since $I_0(\Gamma)$ is injective $D\,\mathrm{Hom}(-, I_0(\Gamma))$ is exact and preserves kernels, so extends to an equivalence

$$\mathrm{Hom}_\Gamma(-, I_0(\Gamma)) : \mathcal{S} \to \mathrm{mod}\,\mathrm{End}_\Gamma(I_0(\Gamma))$$

Since $\mathrm{End}_\Gamma(I_0(\Gamma))$ is morita equivalent to Λ, $\mathcal{S}$ is equivalent to $\mathrm{mod}\,\Lambda$. $\Gamma \in \mathcal{S}$ is clearly a generator. To see that it is a cogenerator note that Γ contains all the projective-injective indecomposable objects as direct summands, so there is an injection $I_0(\Gamma) \to \Gamma^n$, and since $I_0(\Gamma)$ is a cogenerator in $\mathcal{S}$, Γ is aswell.

Thus by the equivalence $\mathcal{S} \to \mathrm{mod}\,\Lambda$ there is a cogenerator-generator object M such that $\mathrm{End}_\Lambda(M) = \mathrm{End}_\Gamma(\Gamma) = \Gamma$.

The last step of the proof is showing that $\mathrm{End}(M)$ is in A whenever M is a generator-cogenerator.

Let $0 \to M \to I_0(M) \to I_1(M)$ be a minimal injective copresentation of M. Since M is a cogenerator $I_i(M)$ is in add M, thus we get an exact sequence of projective $\mathrm{End}(M)$-modules

$$0 \to \mathrm{End}(M) \to \mathrm{Hom}(M, I_0(M)) \to \mathrm{Hom}(M, I_1(M)). \tag{1}$$

Now we have the following isomorphisms of Λ-$\mathrm{End}(M)$-bimodules

$$\begin{aligned} \mathrm{Hom}_\Lambda(M, D\Lambda) &= \\ \mathrm{Hom}_k(M \otimes \Lambda, k) &= \\ \mathrm{Hom}_k(M, k) &= \\ DM &= \\ D\,\mathrm{Hom}_\Lambda(\Lambda, M) & \end{aligned}$$

Since Λ is in add M, $\mathrm{Hom}(\Lambda, M)$ is projective, and thus $D\,\mathrm{Hom}(\Lambda, M) = \mathrm{Hom}(M, D\Lambda)$ is injective. This means that (1) is an injective copresentation, and thus *domdim* $\mathrm{End}(M) \geq 2$.

Since $\mathrm{Hom}(M, I_0(M))$ is the beginning of an injective resolution of $\mathrm{End}(M)$, $I_0(\mathrm{End}(M))$, must be a direct summand. Then $\mathrm{Hom}(M, I_0(M))/I_0(\mathrm{End}(M))$

would map injectively into $\mathrm{Hom}(M, I_1(M))$, but that would mean theres a direct summand of $I_0(M)$ mapping injectively into $I_1(M)$, contradicting minimality. Thus $\mathrm{Hom}(M, I_0(M)) = I_0(\mathrm{End}(M))$.

Let $I = I_0(M)$ and $\Gamma = \mathrm{End}_\Lambda(I)$, then $D\,\mathrm{Hom}(-, I)$ is an exact equivalence from $\mathrm{add}\, I$ to $\mathrm{inj}\,\Gamma$. Since I is an injective cogenerator $\mathrm{add}\, I = \mathrm{inj}\,\Lambda$. Then because of exactness $D\,\mathrm{Hom}(-, I)$ becomes an equivalence between $K^{+,b}(\mathrm{inj}\,\Lambda)$ and $K^{+,b}(\mathrm{inj}\,\Gamma)$. Considering only those complexes with homology in degree 0, we see that $\mathrm{mod}\,\Lambda$ is equivalent to $\mathrm{mod}\,\Gamma$. So Λ is morita equivalent to $\Gamma = \mathrm{End}(I_0(M)) = \mathrm{End}(I_0(\mathrm{End}(M)))$. □

Definition 4.4. Let X be an object of $\mathrm{mod}\,\Lambda$ and M a contravariantaly finite subcategory.

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr, two heads] & M_2 \arrow[r] \arrow[dr, two heads] & M_1 \arrow[r] \arrow[dr, two heads] & M_0 \arrow[dr, two heads] & \\
& \Omega_M^3 X \arrow[u, hook] & \Omega_M^2 X \arrow[u, hook] & \Omega_M X \arrow[u, hook] & X
\end{tikzcd}$$

If $\twoheadrightarrow$ are minimal M-approximations (they need not be surjective), and $\hookrightarrow$ are their kernels, then this is an M-resolution of X. The M-res-dimesnion of X is the length of the sequence of (nonzero) M_i's, and the M-res-dimesnion of Λ is the supremum of the dimension on its objects.

Proposition 4.5. *Repdim-2 is the minimum of M-res-dim$(\mathrm{mod}\,\Lambda)$ for M both generator and cogenrator (assuming repdim is at least 2).*

Proof. The functor $\mathrm{Hom}(M, -)$ is an equivalence from $\mathrm{add}\, M$ to $\mathrm{proj}\,\mathrm{End}(M)$, which maps minimal M-approximations to projective covers. Let X be any module in $\mathrm{mod}\,\mathrm{End}(M)$ with projective dimension at least 2. Then it has a projective presentation

$$\Omega^2 X \to (M, M_1) \to (M, M_0) \to X.$$

Because of the equivalence this is induced by a map $f : M_1 \to M_0$. Since Hom is left exact we have that $\Omega^2 X \cong \mathrm{Hom}(M, \ker f)$, and so the projective dimension of X is 2 plus the M-res-dimension of $\ker f$.

Since M is a cogenerator any module Y in $\mathrm{mod}\,\Lambda$ has a copresentation

$$0 \longrightarrow Y \longrightarrow M_0 \xrightarrow{f} M_1.$$

Applying $\mathrm{Hom}(M, -) =: (M, -)$ we get

$$0 \longrightarrow (M,Y) \longrightarrow (M,M_0) \xrightarrow{(M,f)} (M,M_1) \longrightarrow \mathrm{Cok}(M,f) \longrightarrow 0.$$

If the projective dimension of $\mathrm{Cok}(M,f)$ is less than 2, then (M,Y) is a direct summand of (M,M_0). This means that $(M,Y) \cong (M,M')$, so the minimal M-approximation of Y is M', and $(M,\Omega_M Y) = 0$. Since M is a generator this means $\Omega_M Y = 0$ and thus the M-res-dimension of Y is 0.

So provided the projective dimension of $\mathrm{Cok}(M,f)$ is larger than or equal to 2, it equals the M-res-dimension of Y plus 2. In particular the global dimension of $\mathrm{End}(M)$ is 2 plus the M-res-dimension of $\mathrm{mod}\,\Lambda$, provided it is at least 2. □

Proposition 4.6. *The repdimension of an artin algebra is always finite. [Iya02]*

Theorem 4.7. *The repdimension of Λ is less than or equal to 2 if and only if Λ is representation finite.*

Proof. Assume Λ is representation finite and let M be the direct sum of all indecomposable modules (up to iso). Then M is a generator-cogenerator. Let X be an $\mathrm{End}(M)^{op}$-module with projective presentation

$$(M,M_1) \to (M,M_0) \to X \to 0.$$

Let M_2 be the kernel of $M_1 \to M_0$. Since M is the sum of all indecomposables M_2 is in $\mathrm{add}\,M$, so

$$0 \to (M,M_2) \to (M,M_1) \to (M,M_0) \to X \to 0$$

is a projective resolution of X. So Λ has repdimension at most 2.

Assume Λ has repdimension at most 2, and let M be an auslander generator. We want to show that $\mathrm{add}\,M = \mathrm{mod}\,\Lambda$. Let X be any Λ-module, and let

$$0 \to X \to I_0 \to I_1$$

be a minimal injective presentation. If $I_0 \to I_1$ is split then X is injective and thus in $\mathrm{add}\,M$. Let M_X be a minimal M-approximation of X, let $\Omega_M X$ be the kernel of the approximation, and let Y be the cokernel of $(M,I_0) \to (M,I_1)$. Then

$$(M,\Omega_M X) \to (M,M_X) \to (M,I_0) \to (M,I_1) \to Y \to 0$$

is a minimal exact sequence. Since the global dimension of $\mathrm{End}(M)^{op}$ is at most 2 this means that $(M,\Omega_M X) = 0$. Consequently we have that $\Omega_M X = 0$ and that $X = M_X$, so X is in $\mathrm{add}\,M$. Thus Λ is representation finite. □

4.1 The Igusa-Todorov function

Let K be the free abelian group generated by isomorphism classes of modules, modulo the relations $[A \oplus B] = [A] + [B]$ and $[P] = 0$ when P is projective. Define the linear map $L : K \to K$ by $L[A] = [\Omega A]$. For any module X, $[\operatorname{add} X]$ is a finitely generated subgroup of K. Fitting's lemma tells us that there is an integer η_X such that $L : L^m[\operatorname{add} X] \to L^{m+1}[\operatorname{add} X]$ is an isomorphism for every $m \geq \eta_X$. We define $\psi(X)$ to be $\eta_X + \sup\{\operatorname{pd} Y | Y \in \operatorname{add} \Omega^{\eta_X} X, \operatorname{pd} Y < \infty\}$.

Lemma 4.8. *[IT05, Lemma 3]*

1. $\psi(M) = \operatorname{pd} M$ *when* $\operatorname{pd} M < \infty$.
2. $\psi(M^k) = \psi(M)$
3. $\psi(M) \leq \psi(M \oplus N)$
4. *If* Z *is a direct summand of* $\Omega^n(M)$ *where* $n \leq \eta_M$ *and* $\operatorname{pd} Z < \infty$, *then* $\operatorname{pd} Z + n \leq \psi(M)$.

Proof.

1. If $\operatorname{pd} M < \infty$ then $L^m \neq 0$ for $m < \operatorname{pd} M$, and $L^m = 0$ for $m \geq \operatorname{pd} M$.
2. $\operatorname{add} M^k = \operatorname{add} M$, and ψ is only defined in terms of additive categories.
3. $\operatorname{add} M \subseteq \operatorname{add} M \oplus N$, so if L is injective when restricted to $L^m(\operatorname{add} M \oplus N)$ then L is injective when restricted to $L^m(\operatorname{add} M)$, so $\eta_M \leq \eta_{M \oplus N}$. Further $\Omega^{\eta_{M\oplus N} - \eta_M} \operatorname{add} \Omega^{\eta_M} M \subset \operatorname{add} \Omega^{\eta_{M\oplus N}} M \oplus N$, so $\psi(M) \leq \psi(M \oplus N)$.
4. Let $p = \operatorname{pd} Z$ and $k = \eta_M - n$. Then $\Omega^k Z$ is in $\operatorname{add} \Omega^{\eta_M} M$, so $\operatorname{pd} \Omega^k Z + \eta_M \leq \psi(M)$. Thus

$$\operatorname{pd} Z + n = p + n = (p - k) + \eta_M \leq \operatorname{pd} \Omega^k Z + \eta_M \leq \psi(M).$$

□

Theorem 4.9. *[IT05, Theorem 4] Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence of modules with* $\operatorname{pd} C < \infty$. *Then* $\operatorname{pd} C \leq \psi(A \oplus B) + 1$.

Proof. Let $P_A^\bullet$ and $P_C^\bullet$ be the minimal projective resolutions of A and C. Then we get a map of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_A^0 & \longrightarrow & P_A^0 \oplus P_C^0 & \longrightarrow & P_C^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

Applying the snake lemma we get $0 \to \Omega A \to \Omega B \oplus P \to \Omega C \to 0$ for some projective P. Thus for some $n \leq \operatorname{pd} C$ we have $L^n[A] = L^n[B]$, and let n be the minimal such number. Clearly $n \leq \eta_{A \oplus B}$. Let $X = \Omega^n A = \Omega^n B$, then our sequence of n-syzygies looks like

$$0 \longrightarrow X \longrightarrow X \oplus P \longrightarrow \Omega^n C \longrightarrow 0.$$

Let f be the composition $X \longrightarrow X \oplus P \xrightarrow{\pi_X} X$. Then by fittings lemma X breaks as a direct sum into two components $X = Z \oplus Y$ such that $f = f_Z \oplus f_Y$ with f_Y an isomorphism and f_Z nilpotent. In other words the sequence above can be written as

$$0 \longrightarrow Z \oplus Y \longrightarrow Z \oplus Y \oplus P \longrightarrow \Omega^n C \longrightarrow 0.$$

with the left map being

$$\begin{bmatrix} f_Z & 0 \\ 0 & f_Y \\ * & * \end{bmatrix} \sim \begin{bmatrix} f_Z & 0 \\ 0 & f_Y \\ * & 0 \end{bmatrix}$$

So we get another short exact sequence

$$0 \longrightarrow Z \longrightarrow Z \oplus P \longrightarrow \Omega^n C \longrightarrow 0.$$

Let $T = \Lambda/J$ and apply the long exact sequence in $\operatorname{Ext}(-, T)$. Then we get an exact sequence

$$\operatorname{Ext}^k(Z, T) \longrightarrow \operatorname{Ext}^k(Z \oplus P, T) \longrightarrow \operatorname{Ext}^{k+1}(\Omega^n C, T)$$

where the left map is induced by f_Z since $\operatorname{Ext}^k(Z \oplus P, T) \cong \operatorname{Ext}^k(Z, T)$. Since f_Z is nilpotent this map is surjective if and only if $\operatorname{Ext}^k(Z, T) = 0$, and $\Omega^n C$ has finite projective dimension we have that Z has finite projective dimension. In particular $\operatorname{pd} \Omega^n C - 1 \leq \operatorname{pd} Z \leq \operatorname{pd} \Omega^n C$.

Since Z is a direct summand of $\Omega^n A \oplus B$ by lemma 4.8 we have that $\operatorname{pd} Z + n \leq \psi(A \oplus B)$, and thus $\operatorname{pd} \Omega^n C - 1 + n = \operatorname{pd} C - 1 \leq \psi(A \oplus B)$. $\square$

Corollary 4.9.1. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of modules. If $\operatorname{pd} A < \infty$ then $\operatorname{pd} A \leq \psi(\Omega B \oplus \Omega C) + 1$, and if $\operatorname{pd} B < \infty$ then $\operatorname{pd} B \leq \psi(\Omega A \oplus \Omega^2 C) + 2$.*

Proof. Let $P_B \to B$ be a projective cover of B. Then we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & P_B & \longrightarrow & P_B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

Applying the snake lemma we get a short exact sequence $0 \to \Omega B \to \Omega C \oplus P \to A \to 0$ for some projective P. Then using the theorem we have that $\operatorname{pd} A \leq \psi(\Omega B \oplus \Omega C \oplus P) + 1 = \psi(\Omega B \oplus \Omega C) + 1$.

Applying the same reasoning to $0 \to \Omega B \to \Omega C \oplus P \to A \to 0$ gives us $\operatorname{pd} \Omega B \leq \psi(\Omega A \oplus \Omega^2 C) + 1$. Hence $\operatorname{pd} B \leq \psi(\Omega A \oplus \Omega^2 C) + 2$. □

Theorem 4.10. *[IT05, Corollary 8] If $\Lambda = \operatorname{End}_\Gamma(P)$ for an algebra Γ with global dimension at most 3, and P projective then $\operatorname{findim}(\Lambda) < \infty$.*

Proof. Let X be any Λ-module with finite projective dimension. Then it has a projective presentation $(P, P_1) \to (P, P_0) \to X \to 0$ where $(P, P_i) = \operatorname{Hom}_\Gamma(P, P_i)$ with $P_i \in \operatorname{add} P$. Since $(P, -)$ is an equivalence from $\operatorname{add} P$ to $\operatorname{proj} \Lambda$ this corresponds to a map $P_1 \to P_0$ which we can extend to a projective resolution in Γ:

$$0 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0.$$

Applying the exact functor $(P, -)$, we get an exact sequence

$$0 \longrightarrow (P, P_3) \longrightarrow (P, P_2) \longrightarrow (P, P_1) \longrightarrow (P, P_0) \longrightarrow X \longrightarrow 0.$$

Truncating this we get a short exact sequence

$$0 \longrightarrow (P, P_3) \longrightarrow (P, P_2) \longrightarrow \Omega^2 X \longrightarrow 0.$$

Then by theorem 4.9 the projective dimension of $\Omega^2 X$ is bounded by $\psi((P, P_3) \oplus (P, P_2)) + 1$. Which means

$$\operatorname{pd} X \leq \psi((P, P_3) \oplus (P, P_2)) + 3 \leq \psi((P, \Gamma)) + 3$$

Since this bound doesn't depend on X, Λ has finite finitistic dimension. $\square$

Corollary 4.10.1. *If* $repdim(\Lambda) \leq 3$ *then* $\operatorname{findim}(\Lambda) < \infty$.

Proof. If Λ has rep-dimension less than or equal to 3 then by proposition 4.3 there is a generator-cogenerator M in $\operatorname{mod} \Lambda$ such that $\Gamma := \operatorname{End}_\Lambda(M)$ has global dimension 3 or less. Then since M is a generator Λ is in $\operatorname{add} M$ and so $\operatorname{Hom}_\Lambda(M, \Lambda)$ is a projective Γ-module with $\operatorname{End}_\Gamma(\operatorname{Hom}_\Lambda(M, \Lambda)) = \operatorname{End}_\Lambda(\Lambda) = \Lambda$. $\square$

4.2 Stably hereditary algebras

In this section we will show that the class of stably hereditary algebras has repdimension at most 3, and thus that they have finite finitistic dimension.

Definition 4.11 ((co)torsionfree)**.** A module is called *torsionfree* if it is a submodule of a projective module. Dually, a module is called *cotorsionfree* if it is a factormodule of an injective.

Definition 4.12 (Stably hereditary algebra)**.** An algebra is called *stably hereditary* if any indecomposable torsionfree module is projective or simple, and any indecomposable cotorsionfree moule is injective or simple.

This generalizes the definition of hereditary algebra by also allowing simple modules to be (co)torsionfree.

Definition 4.13 (The stable category)**.** For an algebra Λ, *the stable category* $\underline{\operatorname{mod}}\Lambda$ has the same objects as $\operatorname{mod} \Lambda$, but the homsets are given by

$$\operatorname{Hom}_{\underline{\operatorname{mod}}\Lambda}(M, N) = \operatorname{Hom}_\Lambda(M, N)/\mathcal{P}(M, N)$$

where $\mathcal{P}(M, N)$ is the ideal of all morphisms factoring through a projective.

Proposition 4.14. *If for an algebra* Λ *there is a hereditary algebra* H *such that* $\underline{\operatorname{mod}}\Lambda \cong \underline{\operatorname{mod}}H$ *then* Λ *is stably hereditary.*

Proof. [AR91, Lemma 4.12] $\square$

+ a bit more...

The converse of the above proposition does not hold without more assumptions, but stably hereditary algebras generalize the idea of algebras stably equivalent to hereditary algebras.

Theorem 4.15. *[Xi02, Theorem 3.5] Stably hereditary algebras has repdimension at most 3.*

Proof. Let V be the direct sum of all the indecomposable projectives, all the indecomposable injectives, and all the simple modules. Then V is a generator-cogenerator. So by proposition 4.3 if we can show that the global dimension of $\Gamma := \mathrm{End}(V)^{op}$ is 3 or less, then we are done.

We will show that for any Λ-module M there is a short exact sequence $0 \to V_3 \to V_3 \to M \to 0$ with V_i in $\mathrm{add}\, V$, and such that $0 \to (V, V_3) \to (V, V_2) \to (V, M) \to 0$ is exact. We will use this to construct short projective resolutions for $\mathrm{mod}\,\Gamma$. To construct V_3 and V_2 let M' be the sum of the maximal injective summand of M and all simple submodules of M. Then let P be the projective cover of M/M'. Taking the pullback of $M \to M/M' \leftarrow P$ gives us the diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & M' \oplus P \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] \arrow[d] & M/M' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

I claim that $0 \to K \to M' \oplus P \to M \to 0$ is the desired sequence. Firstly $M' \oplus P$ is clearly in $\mathrm{add}\, V$ since it is the sum of an injective, a semisimple, and a projective module. Further K is a submodule of P, hence torsionfree. So since Λ is stably hereditary K is the sum of a projective and a semisimple module, so K is also in $\mathrm{add}\, V$.

Next we need to show that $0 \to (V, K) \to (V, M' \oplus P) \to (V, M) \to 0$ is exact. The only thing needed to show here is that $(V, M' \oplus P) \to (V, M)$ is surjective. We do this by showing that $(W, M' \oplus P) \to (W, M)$ is surjective for any indecomposable summand of V. If W is projective this holds by definition. If W is simple then any map from W to M factors through the socle and hence through M', so it's surjective. Lastly if W is injective then the image of W in M is a cotorsionfree module, so it is the sum of simple modules and an injective module. Hence the map from W to M factors through M'.

Now we use this to show that the global dimension of Γ is at most 3. Let N be any Γ-module. Then it has a projective presentation

$$(V, V_1) \xrightarrow{f\circ -} (V, V_0) \longrightarrow N \longrightarrow 0$$

If we let M denote the kernel of f and we choose V_3 and V_2 as above then we get a projective resolution of N by

$$0 \longrightarrow (V, V_3) \longrightarrow (V, V_2) \longrightarrow (V, V_1) \longrightarrow (V, V_0) \longrightarrow N \longrightarrow 0.$$

This shows that the projective dimension of N is at most 3, and since N was arbitrary the global dimension of Γ is at most 3. So the repdimension of Λ is at most 3. $\square$

4.3 Special biserial algebras

[EHIS04]

5 Vanishing radical powers

Throughout this section Λ is a finite dimensional algebra, and J is its radical.

Theorem 5.1. *If* $J^2 = 0$ *then* $\operatorname{findim}(\Lambda) < \infty$.

Proof. Let $d = \max\{\operatorname{pd} S_i | \operatorname{pd} S_i < \infty\}$ where S_i ranges over the simple Λ-modules. Let M be a module with $\operatorname{pd} M < \infty$. Let $P \to M$ be a projective cover. Then ΩM is contained in JP and since $J^2 P = 0$, ΩM is annihilated by J and is thus semisimple. This means $\operatorname{pd} \Omega M \leq d$, and thus $\operatorname{pd} M \leq d+1$. So $\operatorname{findim}(\Lambda) \leq d + 1 < \infty$. $\square$

Theorem 5.2. *[IT05, Corollary 6] If* $J^3 = 0$ *then* $\operatorname{findim}(\Lambda) < \infty$.

Proof. Let M be a module with $\operatorname{pd} M < \infty$, and let $P^0 \to M$ be its projective cover. Since $\Omega M \subseteq JP^0$ we have $J^2\Omega M = 0$. Let $P \to \Omega M$ be a projective cover. Since $J^2\Omega M = 0$ we can factorize this as $P \to P/J^2P \to \Omega M$, and we get a short exact sequence

$$0 \longrightarrow (\Omega^2 M + J^2P)/J^2P \longrightarrow P/J^2P \longrightarrow \Omega M \longrightarrow 0$$

Let ψ be the Igusa-Todorov function as introduced in section 4.1. Since $\Omega^2 M \subseteq JP$ we have that $(\Omega^2 M + J^2P)/J^2P$ is semisimple. Then by lemma 4.8 $\psi((\Omega^2 M + J^2P)/J^2P) = \psi(\Lambda/J)$, and $\psi(P/J^2P) = \psi(\Lambda/J^2)$.

Applying theorem 4.9 to the short exact sequence above we thus get $\operatorname{pd}\Omega M \leq \psi(\Lambda/J \oplus \Lambda/J^2) + 1$, and so $\operatorname{pd} M \leq \psi(\Lambda/J \oplus \Lambda/J^2) + 2$, and $\operatorname{findim}(\Lambda) < \infty$. □

Theorem 5.3. *[Wan94] If $J^{2l+1} = 0$ and Λ/J^l is representation finite then $\operatorname{findim}(\Lambda) < \infty$.*

Proof. Let M be a module with $\operatorname{pd} M < \infty$. We have a short exact sequence

$$0 \longrightarrow J^l\Omega M \longrightarrow \Omega M \longrightarrow \Omega M/J^l\Omega M \longrightarrow 0.$$

Since $\Omega M \subseteq JP^0_M$ we have $J^{2l}\Omega M = 0$. This means that $J^l\Omega M$ and $\Omega M/J^l\Omega M$ are Λ/J^l-modules. We will use this, the fact that Λ/J^l is representation finite, and the Igusa-Todorov function to create a bound for $\operatorname{pd} M$.

Applying corollary 4.9.1 we have that:

$$\operatorname{pd}\Omega M \leq \psi(\Omega(J^l\Omega M) \oplus \Omega^2(\Omega M/J^l\Omega M)) + 2.$$

Since Λ/J^l is representation finite there are only finitely many indecomposable Λ/J^l-modules, up to isomorphism. Let S be the sum of all of them. Then since $J^l\Omega M$ and $\Omega M/J^l\Omega M$ are in $\operatorname{add} S$, using lemma 4.8 we have that

$$\psi(\Omega(J^l\Omega M) \oplus \Omega^2(\Omega M/J^l\Omega M)) \leq \psi(\Omega S \oplus \Omega^2 S).$$

So $\operatorname{pd} M \leq \psi(\Omega S \oplus \Omega^2 S) + 3$, and thus $\operatorname{findim}(\Lambda) < \infty$. □

6 Monomial algebras

[GKK91, IZ90]

In this section we will show a particularly nice way to construct a minimal projective resolution of the right module Λ/J for a monomial algebra Λ. We will use this to compute $\mathrm{Tor}_i(\Lambda/J, M)$ and/or $\mathrm{Ext}^i(M, D\Lambda/J)$ to get a bound on the projective dimension of all modules M.

Definition 6.1 (Monomial algebra)**.** A monomial algebra is a path algebra with admissible relations that are generated by monomials. That is, we do not allow the generators for the relations to consist of nontrivial linear combinations of paths.

Definition 6.2 (m-chains)**.** [GKK91] Let $\Lambda = k\Gamma/(\rho)$ be a monomial algebra, with ρ a minimal generating set of paths. As usual we define Γ_0 to be the vertices of Γ, and Γ_1 to be the arrows. Recursively define the set of $(m-1)$-chains, Γ_m, as the paths γ with the following criteria:

- $\gamma = \beta\delta\tau$ with $\beta \in \Gamma_{m-2}$, $\beta\delta \in \Gamma_{m-1}$, and τ a non-zero path of length at least 1.
- $\delta\tau$ is 0 in Λ, i.e. it is ine the ideal of relations.
- γ is left-minimal in the sense that if $\gamma = \gamma'\sigma$ such that γ' satisfies the above conditions, then $\gamma = \gamma'$.

The Γ_m's will become the generating sets for the projectives in our projective resolution. But first we will prove some properties of them.

Lemma 6.3. *Any $\gamma \in \Gamma_m$ for $m \geq 1$ can be factored uniquely as $\gamma_1\gamma_0$ with $\gamma_1 \in \Gamma_{m-1}$, and γ_0 a non-zero path of length at least 1.*

Proof. When $m = 1$ this should be clear, since Γ_1 is the set of arrows, and Γ_0 is the set of vertices, so if $\gamma \in \Gamma_1$ is an arrow $i \to j$ then $\gamma = e_j\gamma$.

When $m > 1$ we know from the definition of Γ_m that γ can be written as $\gamma_1\gamma_0$. Assume there is another decomposition $\gamma = \gamma_1'\gamma_0'$. Then without loss of generality we may assume that γ_1' is shorter than γ_1. Then there is a σ such that $\gamma_1'\sigma = \gamma_1$. By minimality this means that $\gamma_1' = \gamma_1$, and so the decomposition is unique. $\square$

From now on we will write R for Λ/J. Let $k\Gamma_m$ be the free vectorspace generated by Γ_m. Notice that $k\Gamma_m$ has a canonical structure as a R-R-

bimodule. This means we can get projective right Λ-modules $P^m := k\Gamma_m \otimes_R \Lambda$.

Define the map $\delta_m : P^m \to P^{m-1}$ by $\delta_m(\gamma \otimes \alpha) = \gamma_1 \otimes \gamma_0\alpha$ where $\gamma_1\gamma_0$ is the unique decomposition of γ, and define $\delta_0 : k\Gamma_0 \to \Lambda/J$ by $\delta_0(e_i \otimes \alpha) = e_i\alpha + J$. Then I claim we have a minimal projective resolution of the right Λ-module Λ/J by

$$\begin{tikzcd}
\cdots \arrow[r] & P^3 \arrow[r, "\delta_3"] & P^2 \arrow[r, "\delta_2"] & P^1 \arrow[r, "\delta_1"] & P^0 \arrow[r] \arrow[d, two heads, "\delta_0"] & 0 \\
 & & & & \Lambda/J &
\end{tikzcd}$$

Proof. For all i P^i is projective and the image of δ_m is clearly contained in $P^{m-1}J$, so the only thing left to show is exactness. First we show that $\delta_m\delta_{m-1} = 0$. Let $\gamma \otimes \alpha$ be in P^m for $m \geq 2$. Then we can decompose γ uniquely as $\gamma_2\gamma_1\gamma_0$ and $\delta_m\delta_{m-1}(\gamma \otimes \alpha) = \gamma_2 \otimes \gamma_1\gamma_0\alpha$. By the way we defined Γ_m, $\gamma_1\gamma_0$ is 0 in Λ, and so $\gamma_2 \otimes \gamma_1\gamma_0\alpha = 0$.

Next we want to show that $\operatorname{Ker}\delta_{m-1} \subseteq \operatorname{Im}\delta_m$. Let $\sum \gamma^i \otimes \alpha^i$ be in $\operatorname{Ker}\delta_{m-1}$. $\square$

finish

7 Unbounded derived category

If we go to the unbounded derived category we can get a sort of converse to theorem 1.1.

Theorem 7.1. *[Ric19, Theorem 4.3] If the localizing category of $D\Lambda$ is the entire unbounded derived category then* $\operatorname{Findim}(\Lambda) < \infty$. *(Note the capital F meaning the finitistic dimesnion of* $\operatorname{Mod}\Lambda$*, which is bigger than or equal to that of* $\operatorname{mod}\Lambda$*).*

Proof. Assume $\operatorname{Findim}(\Lambda) = \infty$. Then there are modules M_i with projective dimension i for every $i \geq 0$. Let P_i be the minimal projective resolution of M_i, and consider $\bigoplus P_i[-i]$ and $\prod P_i[-i]$. Both of these have homology M_i in degree i, and are concentrated in non-negative degrees.

The inclusion from the sum to the product is clearly a quasi-isomorphism. We want to show that it is not a homotopy equivalence. Assume for the sake of contradiction that it was. Then tensoring with Λ/J would give us another homotopy equivalence. Since Λ/J is finitely presented tensoring preserves both products and coproducts. Because all the resolutions were minimal tesnoring with Λ/J gives us 0 differentials. In degree 0 we get

$$\bigoplus \mathrm{Tor}_i(M_i, \Lambda/J) \to \prod \mathrm{Tor}_i(M_i, \Lambda/J).$$

Since $\mathrm{Tor}_i(M_i, \Lambda/J)$ is nonzero for every M_i this map is not an isomorphism, and so we don't have a homotopy equivalence.

So the cone of the inclusion $\bigoplus P_i[-i] \to \prod P_i[-i]$, C, is 0 in the derived category, but non-zero in the homotopy category. Since Λ is artinian the product of projectives is projective [Cha60, Theorem 3.3], so $\prod P_i[-i]$ is a complex of projectives, which means that C is a complex of projectives.

In other words C is an acyclic lower bounded complex of projectives that is not contractible. Tensoring with $D\Lambda$ is an equivalence from projectives to injectives, so $C \otimes D\Lambda$ is an lower bounded complex of injectives that is not contractible. Such a complex cannot be acyclic so $C \otimes D\Lambda$ has homology.

The homology of C is 0, so $K(\Lambda)(\Lambda, C[i]) = 0$. Applying the equivalence $- \otimes D\Lambda$ we get

$$\mathcal{D}(\Lambda)(D\Lambda, C \otimes D\Lambda[i]) = K(\Lambda)(D\Lambda, C \otimes D\Lambda[i]) = 0.$$

This means that $C \otimes D\Lambda$ is not in the localizing category generated by $D\Lambda$, and so that is not the entire derived category. □

Theorem 7.2. *[Ric19, Theorem 4.4]* $\mathrm{Findim}(\Lambda) < \infty$ *if and only if* $D\Lambda^{\perp} \cap \mathcal{D}^{+}(\Lambda) = 0$.

Proof. In the theorem above we proved that when the finitistic dimension is infinite then there is a non-zero complex in $\mathcal{D}^{+}(\Lambda)$ perpendicular to $D\Lambda$.

The proof of the converse is the same as for theorem 1.1. If we have a non-zero object $X \in D\Lambda^{\perp} \cap \mathcal{D}^{+}(\Lambda) = 0$, then $\mathcal{D}(\Lambda)(D\Lambda, X)$ is a non-split complex of projectives that continue arbitrarily to the right. So the cokernels have arbitrarily big projective dimension. □

8 Summary

FDC holds for the following classes of algebras

- Representation finite algebras

Proof. The supremum over a finite set is finite so $\mathrm{findim}(\Lambda) < \infty$ for a representation finite algebra. □

- Monomial algebras

Proof. This was shown in section 6. □

- Gorenstein algebras

Proof. An algebra is said to be Gorenstein if all injectives have finite projective dimension and all projectives have finite injective dimension. In particular the Λ-module Λ is isomorphic to a finite injective resolution in the derived category. So Λ is in the localizing category generated by injectives. Then theorem 7.1 gives us that $\mathrm{Findim}(\Lambda) < \infty$, and therefor also $\mathrm{findim}(\Lambda) < \infty$. □

- Local algebras

Proof. Local algebras are local artinian rings. So if Λ is local then $\mathrm{findim}(\Lambda) = 0$. □

Not sure where to put this, ill put it here for now

Theorem 8.1. *Local artinian rings have finitistic dimension zero.*

Proof. Assume there is a non-projective module with finite projective dimension. Then in particular we have one with projective dimension equal to 1. Since all finitely generated projectives are free this means we have a short exact sequence

$$0 \qquad R^n \qquad R^m \qquad M \qquad 0$$

with R^n contained in JR^m. Let k be the minimal integer such that $J^k = 0$. Let a be a generator in R^n and let r be a non-zero element of J^{k-1}. Then ra is non-zero, but is mapped to something in $J^{k-1}JR^m = 0$, thus the map is not injective which gives a contradiction. □

9 Dual conjectures

Many of the cases are equivalent to their dual statements. Some are not.

- Given a recollement of the bounded derived category you get one for Λ^{op}.

  Look at examples of lates.

- Probably no link between contravariant finiteness of fin-proj-dim and fin-inj-dim.

  find counterexample.

- repdim of Λ equals the repdim of Λ^{op}.

  Proof. If M is an auslander generator for Λ then DM is an auslander generator for Λ^{op}. □

- If $J^{2l+1} = 0$ and Λ/J^l is repfinite then the same is true for Λ^{op}.
- If Λ is monomial then so is Λ^{op}.
- Injective generates implies the weaker property that projective cogenerate for the opposite algebra. This is also sufficient to prove the algebra satisfies FDC. [Ric19, Section 5]

Similarly for the weaker conjectures

- GSC says the injective dimension of Λ is finite if and only if the injective dimension of Λ^{op} is finite. This statement is symmetric with respect to Λ and Λ^{op}. So the dual is equivalent.
- NC: Certainly Λ is self injective if and only Λ^{op} is.

  Can the dominant dimension of the opposite algebra be different? Arbitrary different?

- For all the others it seems just as difficult as solving the conjecture to connect it to it's dual.

10 Personal appendix

Theorem 10.1. *The global dimension of an artin algebra is the supremum of k with $\operatorname{Ext}^k(T,T) \neq 0$ (T sum of simples). This is also the supremum of projective dimension and supremum of injective dimension.*

Proof. For a minimal projective resolution $\mathrm{Hom}(-,T)$ makes the differentials 0, and similarly with $\mathrm{Hom}(T,-)$ and injective resolutions. So $\mathrm{Ext}^k(M,T)$ is only 0 exactly when $k > \mathrm{pd}\, M$, similarly $\mathrm{Ext}^k(T,M)$ is only 0 when k is bigger than the injective dimension. Since any module is built by extensions of simples you can prove by induction, and the long exact sequence in $\mathrm{Ext}(-,T)$ you get that any module has projective dimension less than or equal to that of T. Similarly for injective dimension. $\square$

$\mathrm{findim}(\Lambda)$ need not equal $\mathrm{findim}(\Lambda^{op}) = \sup\{$injective dimension of $M | M$ has finite injective dimension$\}$.

Example 10.2. [hf] Let $\Lambda = k \left[\begin{tikzcd} 1 \arrow[loop left, "a"] \arrow[r, bend left, "b"] & 2 \arrow[l, bend left, "c"] \end{tikzcd} \right] \Big/ (a^2, ac, ba, cbc)$. Then $\mathrm{findim}(\Lambda) \geq 1$, but $\mathrm{findim}(\Lambda^{op}) = 0$.

Proof. The module $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} = P_1/P_2$ (k^2 where a acts by $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, and b and c act trivially) has projective dimension 1, so $\mathrm{findim}(\Lambda) \geq 1$. The projective/injective modules of Λ are:

$$P_1 = \begin{array}{ccc} & 1 & \\ 1 & & 2 \\ & & 1 \\ & & 2 \end{array}, \quad P_2 = \begin{array}{c} 2 \\ 1 \\ 2 \end{array}, \quad I_1 = \begin{array}{ccc} & & 1 \\ 1 & & 2 \\ & 1 & \end{array}, \quad I_2 = \begin{array}{c} 1 \\ 2 \\ 1 \\ 2 \end{array}$$

If $\mathrm{findim}(\Lambda^{op}) > 0$ there would be a module with finite non-zero injective resolution. In particular it would end with a non-split epimorphism between injectives. I claim this would mean there is a non-split epimorphism $I \to I_i$ from an injective to an indecomposable injective. Obviously we get epimorphisms by composing with the projections onto summands, so we want to show that they are not split. Assume that they are, that is the map looks like

$$\begin{tikzcd} I_i \oplus I \arrow[r, "\begin{bmatrix} 1 & 0 \\ f & g \end{bmatrix}"] \arrow[dr, "\begin{bmatrix} 1 & 0 \end{bmatrix}"'] & I_i \oplus I' \arrow[d, "\begin{bmatrix} 1 & 0 \end{bmatrix}"] \\ & I_i \end{tikzcd} .$$

We see that by changing basis in the domain we get the matrix $\begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}$. Thus I_i is mapped isomorphically to itself, which doesnt happen in a minimal resolution.

The only thing left to show is that there are no non-split epimorphisms from injective modules to I_1 and I_2. $\square$

Lemma 10.3. *[CE99, Chapter I, theorem 3.2] Let R be a noetherian ring. Then an R-module Q is injective if and only if it has the injective lifting property for inclusions of ideals into R.*

Proof. If Q is injective then Q has the lifting property for all monomorphisms, so one direction is clear. Assume we have a diagram

$$\begin{tikzcd} Q & \\ M \arrow[u, "f"] \arrow[r, hook] & N \arrow[ul, dashed] \end{tikzcd}$$

We want to show that the dashed arrow exists. Let S be the partially ordered set $\{(M', f') : M \leq M', f'|_M = f\}$. By Zorn's lemma this has a maximal element (M', f'). Assume $M' \neq N$, then there is an element $x \in N - M'$. The set of r such that $rx \in M'$ forms an ideal I. Define the map $g : I \to Q$ by $I(r) = f'(rx)$. By hypothesis g lifts to a map $\tilde{g} : R \to Q$. Let q be $\tilde{g}(1)$. Then $\tilde{f} : M' + Rx \to Q$ defined by $\tilde{f}(m + rx) = f'(m) + rq$ gives us a bigger element of S, contradicting maximality. Thus $M' = N$ and Q is injective. $\square$

Theorem 10.4. *Let R be a noetherian ring. Then an arbitrary coproduct of injectives is injective.*

Proof. By the lemma above it is enough to show the lifting property on ideals of R. Let I be an ideal and $f : I \to \bigoplus_i Q_i$ be a map to a coproduct of injectives. Since R is notherian I is finitely generated so f factors through a finite sum $I \to \bigoplus_{i=0}^n Q_i \to \bigoplus Q_i$. Since finite coproducts of injectives are injective we are done.

$$\begin{tikzcd} \bigoplus Q_i & \\ \bigoplus\limits_{i=0}^{n} Q_i \arrow[u] & \\ I \arrow[u] \arrow[r, hook] & R \arrow[ul, dashed] \end{tikzcd}$$

$\square$

Theorem 10.5. *[CE99, Chapter I, Exercise 8] Let R be a noetherian ring. Then direct limits of injectives is injective.*

Proof. By the lemma above it is enough to show the lifting property on ideals of R. Let I be an ideal and let $Q = \varinjlim Q_i$ be a direct limit of injectives.

Since R is noetherian I is finitely presented, say $R^n \to R^m \to I \to 0$. Applying $\operatorname{Hom}(-, Q)$ we get an exact sequence

$$0 \longrightarrow \operatorname{Hom}(I, Q) \longrightarrow \operatorname{Hom}(R^m, Q) \longrightarrow \operatorname{Hom}(R^n, Q)$$

Since direct limits are exact we also have an exact sequence

$$0 \longrightarrow \varinjlim \operatorname{Hom}(I, Q_i) \longrightarrow \varinjlim \operatorname{Hom}(R^m, Q_i) \longrightarrow \varinjlim \operatorname{Hom}(R^n, Q_i)$$

We also have a natural map $\varinjlim \operatorname{Hom}(-, Q_i) \to \operatorname{Hom}(-, Q)$. $\operatorname{Hom}(R^n, Q_i)$ just equals Q_i^n, so this map is an isomorphism at R^n. Then by the five lemma applied to the two sequences above we get that $\operatorname{Hom}(I, Q) \cong \varinjlim \operatorname{Hom}(I, Q_i)$ for all ideals I. So since

$$\varinjlim \operatorname{Hom}(R, Q_i) \longrightarrow \varinjlim \operatorname{Hom}(I, Q_i) \longrightarrow 0$$

is exact, we get that

$$\operatorname{Hom}(R, Q) \longrightarrow \operatorname{Hom}(I, Q) \longrightarrow 0$$

is exact. Hence Q is injective. $\square$

References

[AR75] Maurice Auslander and Idun Reiten. On a generalized version of the Nakayama conjecture. *Proc. Amer. Math. Soc.*, 52:69–74, 1975.

[AR91] Maurice Auslander and Idun Reiten. Applications of contravariantly finite subcategories. *Adv. Math.*, 86(1):111–152, 1991.

[CE99] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1999. With an appendix by David A. Buchsbaum, Reprint of the 1956 original.

[Cha60] Stephen U. Chase. Direct products of modules. *Trans. Amer. Math. Soc.*, 97:457–473, 1960.

[EHIS04] Karin Erdmann, Thorsten Holm, Osamu Iyama, and Jan Schröer. Radical embeddings and representation dimension. *Adv. Math.*, 185(1):159–177, 2004.

[GKK91] Edward L. Green, Ellen Kirkman, and James Kuzmanovich. Finitistic dimensions of finite-dimensional monomial algebras. *J. Algebra*, 136(1):37–50, 1991.

[Hap93] Dieter Happel. Reduction techniques for homological conjectures. *Tsukuba J. Math.*, 17(1):115–130, 1993.

[hf] Jacob FG (https://mathoverflow.net/users/157483/jacob fg). Is there a finite dimensional algebra with left finitistic dimension different from its right finitistic dimension? MathOverflow. URL:https://mathoverflow.net/q/376211 (version: 2020-11-11).

[IT05] Kiyoshi Igusa and Gordana Todorov. On the finitistic global dimension conjecture for Artin algebras. In *Representations of algebras and related topics*, volume 45 of *Fields Inst. Commun.*, pages 201–204. Amer. Math. Soc., Providence, RI, 2005.

[Iya02] Osamu Iyama. Finiteness of representation dimension. In *Proceedings of the 34th Symposium on Ring Theory and Representation Theory (Maebashi, 2001)*, pages 21–25. Symp. Ring Theory Represent. Theory Organ. Comm., Tsukuba, 2002.

[IZ90] Kiyoshi Igusa and Dan Zacharia. Syzygy pairs in a monomial algebra. *Proc. Amer. Math. Soc.*, 108(3):601–604, 1990.

[Opp09] Steffen Oppermann. Representation dimension of artin algebras. `https://folk.ntnu.no/opperman/repdim_survey.pdf`, 2009.

[Ric19] Jeremy Rickard. Unbounded derived categories and the finitistic dimension conjecture. *Adv. Math.*, 354:106735, 21, 2019.

[Wan94] Yong Wang. A note on the finitistic dimension conjecture. *Comm. Algebra*, 22(7):2525–2528, 1994.

[Xi02] Changchang Xi. Representation dimension and quasi-hereditary algebras. *Adv. Math.*, 168(2):193–212, 2002.