Finitistic dimension conjecture

Jacob Fjeld Grevstad

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Abstract

FDC yo!

Contents

Introduction

This is an introduction

1 finitistic dimension and conjectures

- FDC - finitistic dimesnion conjecture Finitistic dimension is always finite
- WTC - Watamatsu tilting conjecture
- GSC - Gorenstein symmetry conjecture
- NuC - Nunke condition
- SNC - strong Nakayama conjecture
- ARC - Auslander Reiten conjecture
- NC - Nakayama conjecture

1.1 Implications

$$\begin{tikzcd} FDC \arrow[r] \arrow[d] & WTC \arrow[r] & GSC & \\ NuC \arrow[r] & SNC \arrow[r] & ARC \arrow[r] & NC \end{tikzcd}$$

Theorem 1.1. *[Hap93, 1.2]*

i) If $findim(\Lambda) < \infty$ *(FDC) then* $K^b(\mathrm{inj}\,\Lambda)^\perp = 0$.

ii) If $K^b(\mathrm{inj}\,\Lambda)^\perp = 0$ *then for* $X \neq 0$ *there exists* i *such that,* $\mathrm{Ext}^i(D(\Lambda), X) \neq 0$ *(NuC).*

Proof.

i) Let $I^\bullet \in K^b(\mathrm{inj}\,\Lambda)^\perp$ be non-zero. Since $D^b(\Lambda) \cong K^{+,b}(\mathrm{inj}\,\Lambda)$ we may assume $I^\bullet$ is a complex of injectives, and WLOG we may assume it concentrated in degrees $i \geq 0$.

$\mathrm{Hom}(D\Lambda, I^i)$ is in $add\,\mathrm{Hom}(D\Lambda, D\Lambda) = add\Lambda$ so $\mathrm{Hom}(D\Lambda, I^\bullet)$ is a complex of projectives.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & D\Lambda \arrow[r] \arrow[d, "f"] \arrow[dl, dashed] & 0 \arrow[d] \\
I^{i-1} \arrow[r, "d^{i-1}"] & I^i \arrow[r, "d^i"] & I^{i+1}
\end{tikzcd}$$

Since $I^\bullet$ is in $K^b(\operatorname{inj}\Lambda)^\perp$ and $D\Lambda$ is in $K^b(\operatorname{inj}\Lambda)$, whenever $d^i f = 0$, $f^\bullet$ is homotopic to 0. Meaning f factors through d^{i-1}. This means that $\operatorname{Hom}(D\Lambda, I^\bullet)$ is an exact complex.

$\operatorname{Cok}\operatorname{Hom}(D\Lambda, d^i)$ has a projective resolution of length i. This resolution is minimal??????? Hence $findim(\Lambda)$ is infinite.

Probably want to use I indecomposable or something like that

ii) Assume there is an $X \neq 0$ with $\operatorname{Ext}^i(D\Lambda, X) = 0$ for all $i \geq 0$. Then X considered as a stalk complex is in $K^b(\operatorname{inj}\Lambda)^\perp$. Proceed by induction: If $I[-i] \in K^b(\operatorname{inj}\Lambda)$ is a stalk complex then $D^b(I[-i], X) = \operatorname{Ext}^i(I, X)$. This is 0 because $D\Lambda$ is the sum of the indecomposable injectives.

Let $I \in K^b(\operatorname{inj}\Lambda)$ be a complex of width n. WLOG assume I concentrated in degrees $0 \leq i \leq n-1$. Then

$$I^0 \to I \to I^{<0} \to I^0[1]$$

is a triangle, and $I^{<0}$ has width $n-1$. Taking the long exact sequence in $D^b(-, X)$ it follows that $D^b(I, X) = 0$. □

1.2 Recollement

$$\begin{tikzcd}[column sep=huge]
D^b(\Lambda') \arrow[r, "i_*=i_!"] & D^b(\Lambda) \arrow[l, bend right=40, "i^*"'] \arrow[l, bend left=40, "i^!"] \arrow[r, "j^!=j^*"] & D^b(\Lambda'') \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*"]
\end{tikzcd}$$

Sort of like a split exact sequence of functors. We want

1. All functors are exact/triangulated
2. $j^* i_* = 0$
3. $i^* i_* \cong i^! i_! \cong id$ (induced by unit/counit)
4. $j^! j_! \cong j^* j_* \cong id$

$$j_!j^!X \xrightarrow{\varepsilon} X \xrightarrow{\eta} i_*i^*X \longrightarrow \Sigma$$

5.
$$i_!i^!X \xrightarrow{\varepsilon} X \xrightarrow{\eta} j_*j^*X \longrightarrow \Sigma$$

Are triangles in $D^b(\Lambda)$

Theorem 1.2. *Given a recollement FDC holds for midlle if and only if it holds for the two others.*

Proof. write later..... Happel reduct technich [Hap93, 3.3] □

1.3 Contravariant finiteness

Definition 1.3 (Resolving)**.** A full subcategory of an abelian category is called resolving if

- It is closed under extensions
- It contains the projectives
- It is contains the kernels of its epimorphisms

Note that the subcategory of modules with finite projective dimension is resolving.

Theorem 1.4. *[AR91, 3.8] Let $\mathcal{X}$ be a contravariantly finite, resolving subcategory of* $\operatorname{mod}\Lambda$*. Let X_i be the minimal approximation of S_i. Then any $X \in \mathcal{X}$ is a direct summand of an X_i-filtered module.*

Proof. Step 1: We want to show by induction on length that any module C is in an exact sequence $0 \to Y \to X \to C \to 0$ with X X_i-filtered and $\operatorname{Ext}^1(\mathcal{X}, Y) = 0$.

Step 2: Whenever C is in $\mathcal{X}$ we get that

$\operatorname{Hom}(C, X) \longrightarrow \operatorname{Hom}(C, C) \longrightarrow \operatorname{Ext}^1(C, Y) = 0$ is exact, and thus C is a direct summand of X. □

Corollary 1.4.1. *If the subcategory of modules with finite projective dimension is contravariantly finite, then the finitistic dimension is the supremum of the projective dimension of X_i. In particular it is finite.*

2 repdimension

Definition 2.1 (dominated dimension)**.** Let $\Lambda \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$ be a minimal injective resolution of Λ. Then the dominated dimension of Λ is $\inf\{n|I_n$ is not projective $\}$.

Definition 2.2 (rep-dimesnion)**.** Let $A = \{\Gamma|domdim\Gamma \geq 2, \Lambda$ morita eqquivalent to $\operatorname{End}_\Gamma I_0(\Gamma)\}$ where $I_0(\Gamma)$ is the injective envelope of Γ. Then the repdimesnion of Λ is the minimal global dimension of $\Gamma \in A$.

Proposition 2.3. *(all modules ar right modules) Repdim is the same as minimal global dimension of* $\operatorname{End}(M)$ *for* M *being both a generator and cogenerator.*

Proof. Consider $\Gamma \in A$. Since $domdim\Gamma \geq 1$, $I_0(\Gamma)$ is the sum of all projective-injective modules (some probably several times).

Let $\mathcal{S}$ be the set of all Γ-modules with a copresentation

$$0 \longrightarrow X \longrightarrow I_0 \longrightarrow I_1$$

with I_i in $\operatorname{add} I_0(\Gamma)$. In particular Γ is in $\mathcal{S}$, because $domdim\Gamma \geq 2$.

The Yoneda embedding gives an equivalence

$$\operatorname{Hom}_\Gamma(-, I_0(\Gamma)) : \operatorname{add} I_0(\Gamma) \to \operatorname{proj} \operatorname{End}_\Gamma(I_0(\Gamma))^{op}$$

, and thus we get an equivalence

$$D \operatorname{Hom}_\Gamma(-, I_0(\Gamma)) : \operatorname{add} I_0(\Gamma) \to \operatorname{inj} \operatorname{End}_\Gamma(I_0(\Gamma))$$

Since $I_0(\Gamma)$ is injective $D \operatorname{Hom}(-, I_0(\Gamma))$ is exact and preserves kernels, so extends to an equivalence

$$\operatorname{Hom}_\Gamma(-, I_0(\Gamma)) : \mathcal{S} \to \operatorname{mod} \operatorname{End}_\Gamma(I_0(\Gamma))$$

Since $\operatorname{End}_\Gamma(I_0(\Gamma))$ is morita equivalent to Λ, $\mathcal{S}$ is equivalent to $\operatorname{mod} \Lambda$. $\Gamma \in \mathcal{S}$ is clearly a generator. To see that it is a cogenerator note that Γ contains all the projective-injective indecomposable objects as direct summands, so there is an injection $I_0(\Gamma) \to \Gamma^n$, and since $I_0(\Gamma)$ is a cogenerator in $\mathcal{S}$, Γ is aswell.

Thus by the equivalence $\mathcal{S} \to \operatorname{mod} \Lambda$ there is a cogenerator-generator object M such that $\operatorname{End}_\Lambda(M) = \operatorname{End}_\Gamma(\Gamma) = \Gamma$.

The last step of the proof is showing that $\mathrm{End}(M)$ is in A whenever M is a generator-cogenerator.

Let $0 \to M \to I_0(M) \to I_1(M)$ be an injective copresentation of M. Since M is a cogenerator $I_i(M)$ is in $\operatorname{add} M$, thus we get an exact sequence of projective $\mathrm{End}(M)$-modules

$$0 \to \mathrm{End}(M) \to \mathrm{Hom}(M, I_0(M)) \to \mathrm{Hom}(M, I_1(M)). \tag{1}$$

Now we have the following isomorphisms of Λ-$\mathrm{End}(M)$-bimodules

$$\begin{aligned}\mathrm{Hom}_\Lambda(M, D\Lambda) &= \\ \mathrm{Hom}_k(M \otimes \Lambda, k) &= \\ \mathrm{Hom}_k(M, k) &= \\ DM &= \\ D\,\mathrm{Hom}_\Lambda(\Lambda, M)\end{aligned}$$

Since Λ is in $\operatorname{add} M$, $\mathrm{Hom}(\Lambda, M)$ is projective, and thus $D\,\mathrm{Hom}(\Lambda, M) = \mathrm{Hom}(M, D\Lambda)$ is injective. This means that (1) is an injective copresentation, and thus $\mathit{domdim}\,\mathrm{End}(M) \geq 2$.

Let $I = I_0(M)$, then $\mathrm{Hom}(I, \Lambda)$ and $I = \mathrm{Hom}(\Lambda, I)$ are bimodules. Need some kind of morita theorem here?????????????????? □

I think this is specific to artin algebras

Definition 2.4. Let X be an object of $\operatorname{mod}\Lambda$ and M a contravariantaly finite subcategory.

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr, two heads] & M_2 \arrow[r] \arrow[dr, two heads] & M_1 \arrow[r] \arrow[dr, two heads] & M_0 \arrow[dr, two heads] & \\
& \Omega_M^3 X \arrow[u, hook] & \Omega_M^2 X \arrow[u, hook] & \Omega_M X \arrow[u, hook] & X
\end{tikzcd}$$

If $\twoheadrightarrow$ are minimal M-approximations (they need not be surjective), and $\hookrightarrow$ are their kernels, then this is an M-resolution of X. The M-res-dimesnion of X is the length of the sequence of (nonzero) M_is, and the M-res-dimesnion of Λ is the supremum of the dimension on its objects.

Proposition 2.5. *repdim* $-\,2$ *is the minimum of M-res-dim*$(\operatorname{mod}\Lambda)$ *for M both generator and cogenrator.*

Theorem 2.6. *FDC holds for repdim ¡ 3, [IT05, cor 8,9]*

References

[AR91] Maurice Auslander and Idun Reiten. Applications of contravariantly finite subcategories. *Adv. Math.*, 86(1):111–152, 1991.

[Hap93] Dieter Happel. Reduction techniques for homological conjectures. *Tsukuba J. Math.*, 17(1):115–130, 1993.

[IT05] Kiyoshi Igusa and Gordana Todorov. On the finitistic global dimension conjecture for Artin algebras. In *Representations of algebras and related topics*, volume 45 of *Fields Inst. Commun.*, pages 201–204. Amer. Math. Soc., Providence, RI, 2005.