Finitistic dimension conjecture

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Abstract

FDC yo!

Contents

Introduction

This is an introduction

1 finitistic dimension and conjectures

- FDC - finitistic dimesnion conjecture Finitistic dimension is always finite
- WTC - Watamatsu tilting conjecture
- GSC - Gorenstein symmetry conjecture
- NuC - Nunke condition
- SNC - strong Nakayama conjecture
- ARC - Auslander Reiten conjecture
- NC - Nakayama conjecture

1.1 Implications

$$\begin{tikzcd}
FDC \arrow[r] \arrow[d] & WTC \arrow[r] & GSC & \\
NuC \arrow[r] & SNC \arrow[r] & ARC \arrow[r] & NC
\end{tikzcd}$$

Theorem 1.1. *[Hap93, 1.2]*

i) If $findim(\Lambda) < \infty$ (FDC) then $K^b(\operatorname{inj}\Lambda)^\perp = 0$.

ii) If $K^b(\operatorname{inj}\Lambda)^\perp = 0$ then for $X \neq 0$ there exists i such that, $\operatorname{Ext}^i(D(\Lambda), X) \neq 0$ (NuC).

Proof.

i) Let $I^\bullet \in K^b(\operatorname{inj}\Lambda)^\perp$ be non-zero. Since $\mathcal{D}^b(\Lambda) \cong K^{+,b}(\operatorname{inj}\Lambda)$ we may assume $I^\bullet$ is a complex of injectives, and WLOG we may assume it concentrated in degrees $i \geq 0$, and that $d^0 : I^0 \to I^1$ is not split mono. Since if its concentrated in degrees $i \geq k$ we can just shift it, and if d^0 is split mono then replacing I^0 by 0, and I^1 be I^1/I^0 gives a homotopic complex.

$\mathrm{Hom}(D\Lambda, I^i)$ is in $\operatorname{add}\mathrm{Hom}(D\Lambda, D\Lambda) = \operatorname{add}\Lambda$ so $\mathrm{Hom}(D\Lambda, I^\bullet)$ is a complex of projectives.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & D\Lambda \arrow[r] \arrow[d, "f"] \arrow[dl, dashed] & 0 \arrow[d] \\
I^{i-1} \arrow[r, "d^{i-1}"] & I^i \arrow[r, "d^i"] & I^{i+1}
\end{tikzcd}$$

Since $I^\bullet$ is in $K^b(\operatorname{inj}\Lambda)^\perp$ and $D\Lambda$ is in $K^b(\operatorname{inj}\Lambda)$, whenever $d^i f = 0$, $f^\bullet$ is homotopic to 0. Meaning f factors through d^{i-1}. This means that $\mathrm{Hom}(D\Lambda, I^\bullet)$ is an exact complex. Further since $\mathrm{Hom}(D\Lambda, -)$ is an equivalence between $\operatorname{inj}\Lambda$ and $\operatorname{proj}\Lambda$ we have that $\mathrm{Hom}(D\Lambda, d^0)$ is not split mono.

$\operatorname{Cok}\mathrm{Hom}(D\Lambda, d^i)$ has a projective resolution of length i. This resolution is the direct sum of the minimal resolution and an acyclic bounded complex of projectives. Since bounded acyclic complexes of projectives are split and $\mathrm{Hom}(D\Lambda, d^0)$ is not, we must have that the minimal resolution has length i, and so $findim(\Lambda) = \infty$.

ii) Assume there is an $X \neq 0$ with $\mathrm{Ext}^i(D\Lambda, X) = 0$ for all $i \geq 0$. Then X considered as a stalk complex is in $K^b(\operatorname{inj}\Lambda)^\perp$. Proceed by induction: If $I[-i] \in K^b(\operatorname{inj}\Lambda)$ is a stalk complex then $\mathcal{D}^b(I[-i], X) = \mathrm{Ext}^i(I, X)$. This is 0 because $D\Lambda$ is the sum of the indecomposable injectives.

Let $I \in K^b(\operatorname{inj}\Lambda)$ be a complex of width n. WLOG assume I concentrated in degrees $0 \leq i \leq n-1$. Then

$$I^0 \to I \to I^{<0} \to I^0[1]$$

is a triangle, and $I^{<0}$ has width $n-1$. Taking the long exact sequence in $\mathcal{D}^b(-, X)$ it follows that $\mathcal{D}^b(I, X) = 0$. $\square$

1.2 Recollement

$$\begin{tikzcd}
\mathcal{D}^b(\Lambda') \arrow[rr, "i_* = i_!"] & & \mathcal{D}^b(\Lambda) \arrow[ll, bend right, "i^*"'] \arrow[ll, bend left, "i^!"] \arrow[rr, "j^! = j^*"] & & \mathcal{D}^b(\Lambda'') \arrow[ll, bend right, "j_!"'] \arrow[ll, bend left, "j_*"]
\end{tikzcd}$$

Sort of like a split exact sequence of functors. We want

1. All functors are exact/triangulated

2. $j^*i_* = 0$
3. $i^*i_* \cong i^!i_! \cong id$ (induced by unit/counit)
4. $j^!j_! \cong j^*j_* \cong id$

$$j_!j^!X \xrightarrow{\varepsilon} X \xrightarrow{\eta} i_*i^*X \longrightarrow \Sigma$$

5. $$i_!i^!X \xrightarrow{\varepsilon} X \xrightarrow{\eta} j_*j^*X \longrightarrow \Sigma$$

Are triangles in $\mathcal{D}^b(\Lambda)$

Theorem 1.2. *Given a recollement FDC holds for midlle if and only if it holds for the two others.*

Proof. write later..... Happel reduct technich [Hap93, 3.3] □

1.3 Contravariant finiteness

Definition 1.3 (Resolving)**.** A full subcategory of an abelian category is called resolving if

- It is closed under extensions
- It contains the projectives
- It is contains the kernels of its epimorphisms

Note that the subcategory of modules with finite projective dimension is resolving.

Theorem 1.4. *[AR91, 3.8] Let $\mathcal{X}$ be a contravariantly finite, resolving subcategory of* $\operatorname{mod}\Lambda$*. Let X_i be the minimal approximation of S_i. Then any $X \in \mathcal{X}$ is a direct summand of an X_i-filtered module.*

Proof. Step 1: We want to show by induction on length that any module C is in an exact sequence $0 \to Y \to X \to C \to 0$ with X X_i-filtered and $\operatorname{Ext}^1(\mathcal{X}, Y) = 0$.

Step 2: Whenever C is in $\mathcal{X}$ we get that

$\operatorname{Hom}(C, X) \longrightarrow \operatorname{Hom}(C, C) \longrightarrow \operatorname{Ext}^1(C, Y) = 0$ is exact, and thus C is a direct summand of X. □

Corollary 1.4.1. *If the subcategory of modules with finite projective dimension is contravariantly finite, then the finitistic dimension is the supremum of the projective dimension of X_i. In particular it is finite.*

2 repdimension

Definition 2.1 (dominated dimension)**.** Let $\Lambda \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$ be a minimal injective resolution of Λ. Then the dominated dimension of Λ is $\inf\{n | I_n \text{ is not projective }\}$.

Definition 2.2 (rep-dimesnion)**.** Let $A = \{\Gamma | domdim \Gamma \geq 2, \Lambda \text{ morita eqquivalent to } \operatorname{End}_\Gamma I_0(\Gamma)\}$ where $I_0(\Gamma)$ is the injective envelope of Γ. Then the repdimesnion of Λ is the minimal global dimension of $\Gamma \in A$.

Proposition 2.3. *(all modules ar right modules) Repdim is the same as minimal global dimension of* $\operatorname{End}(M)$ *for M being both a generator and cogenerator.*

Proof. Consider $\Gamma \in A$. Since $domdim \Gamma \geq 1$, $I_0(\Gamma)$ is the sum of all projective-injective modules (some probably several times).

Let $\mathcal{S}$ be the set of all Γ-modules with a copresentation

$$0 \longrightarrow X \longrightarrow I_0 \longrightarrow I_1$$

with I_i in $\operatorname{add} I_0(\Gamma)$. In particular Γ is in $\mathcal{S}$, because $domdim \Gamma \geq 2$.

The Yoneda embedding gives an equivalence

$$\operatorname{Hom}_\Gamma(-, I_0(\Gamma)) : \operatorname{add} I_0(\Gamma) \to \operatorname{proj} \operatorname{End}_\Gamma(I_0(\Gamma))^{op}$$

, and thus we get an equivalence

$$D \operatorname{Hom}_\Gamma(-, I_0(\Gamma)) : \operatorname{add} I_0(\Gamma) \to \operatorname{inj} \operatorname{End}_\Gamma(I_0(\Gamma))$$

Since $I_0(\Gamma)$ is injective $D \operatorname{Hom}(-, I_0(\Gamma))$ is exact and preserves kernels, so extends to an equivalence

$$\operatorname{Hom}_\Gamma(-, I_0(\Gamma)) : \mathcal{S} \to \operatorname{mod} \operatorname{End}_\Gamma(I_0(\Gamma))$$

Since $\operatorname{End}_\Gamma(I_0(\Gamma))$ is morita equivalent to Λ, $\mathcal{S}$ is equivalent to $\operatorname{mod} \Lambda$. $\Gamma \in \mathcal{S}$ is clearly a generator. To see that it is a cogenerator note that Γ contains all the projective-injective indecomposable objects as direct summands, so there is an injection $I_0(\Gamma) \to \Gamma^n$, and since $I_0(\Gamma)$ is a cogenerator in $\mathcal{S}$, Γ is aswell.

Thus by the equivalence $\mathcal{S} \to \operatorname{mod}\Lambda$ there is a cogenerator-generator object M such that $\operatorname{End}_\Lambda(M) = \operatorname{End}_\Gamma(\Gamma) = \Gamma$.

The last step of the proof is showing that $\operatorname{End}(M)$ is in A whenever M is a generator-cogenerator.

Let $0 \to M \to I_0(M) \to I_1(M)$ be an injective copresentation of M. Since M is a cogenerator $I_i(M)$ is in $\operatorname{add} M$, thus we get an exact sequence of projective $\operatorname{End}(M)$-modules

$$0 \to \operatorname{End}(M) \to \operatorname{Hom}(M, I_0(M)) \to \operatorname{Hom}(M, I_1(M)). \tag{1}$$

Now we have the following isomorphisms of Λ-$\operatorname{End}(M)$-bimodules

$$\begin{aligned} \operatorname{Hom}_\Lambda(M, D\Lambda) &= \\ \operatorname{Hom}_k(M \otimes \Lambda, k) &= \\ \operatorname{Hom}_k(M, k) &= \\ DM &= \\ D\operatorname{Hom}_\Lambda(\Lambda, M)& \end{aligned}$$

Since Λ is in $\operatorname{add} M$, $\operatorname{Hom}(\Lambda, M)$ is projective, and thus $D\operatorname{Hom}(\Lambda, M) = \operatorname{Hom}(M, D\Lambda)$ is injective. This means that (1) is an injective copresentation, and thus *domdim* $\operatorname{End}(M) \geq 2$.

Let $I = I_0(M)$, then $\operatorname{Hom}(I, \Lambda)$ and $I = \operatorname{Hom}(\Lambda, I)$ are bimodules. Need some kind of morita theorem here?????????????????? $\square$

I think this is specific to artin algebras

Definition 2.4. Let X be an object of $\operatorname{mod}\Lambda$ and M a contravariantaly finite subcategory.

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr, two heads] & M_2 \arrow[r] \arrow[dr, two heads] & M_1 \arrow[r] \arrow[dr, two heads] & M_0 \arrow[dr, two heads] & \\
& \Omega_M^3 X \arrow[u, hook] & \Omega_M^2 X \arrow[u, hook] & \Omega_M X \arrow[u, hook] & X
\end{tikzcd}$$

If $\twoheadrightarrow$ are minimal M-approximations (they need not be surjective), and $\hookrightarrow$ are their kernels, then this is an M-resolution of X. The M-res-dimesnion of X is the length of the sequence of (nonzero) M_i's, and the M-res-dimesnion of Λ is the supremum of the dimension on its objects.

Proposition 2.5. *Repdim-2 is the minimum of M-res-dim*$(\operatorname{mod}\Lambda)$ *for M both generator and cogenrator (assuming repdim is at least 2).*

Proof. The functor $\operatorname{Hom}(M,-)$ is an equivalence from $\operatorname{add} M$ to $\operatorname{proj}\operatorname{End}(M)$, which maps minimal M-approximations to projective covers. Let X be any module in $\operatorname{mod}\operatorname{End}(M)$ with projective dimension at least 2. Then it has a projective presentation

$$\Omega^2 X \to (M, M_1) \to (M, M_0) \to X.$$

Because of the equivalence this is induced by a map $f : M_1 \to M_0$. Since Hom is left exact we have that $\Omega^2 X \cong \operatorname{Hom}(M, \ker f)$, and so the projective dimension of X is 2 plus the M-res-dimension of $\ker f$.

Since M is a cogenerator any module Y in $\operatorname{mod}\Lambda$ has a copresentation

$$0 \longrightarrow Y \longrightarrow M_0 \xrightarrow{f} M_1.$$

Applying $\operatorname{Hom}(M,-) =: (M,-)$ we get

$$0 \longrightarrow (M,Y) \longrightarrow (M,M_0) \xrightarrow{(M,f)} (M,M_1) \longrightarrow \operatorname{Cok}(M,f) \longrightarrow 0.$$

If the projective dimension of $\operatorname{Cok}(M,f)$ is less than 2, then (M,Y) is a direct summand of (M,M_0). This means that $(M,Y) \cong (M,M')$, so the minimal M-approximation of Y is M', and $(M, \Omega_M Y) = 0$. Since M is a generator this means $\Omega_M Y = 0$ and thus the M-res-dimension of Y is 0.

So provided the projective dimension of $\operatorname{Cok}(M,f)$ is larger than or equal to 2, it equals the M-res-dimension of Y plus 2. In particular the global dimension of $\operatorname{End}(M)$ is 2 plus the M-res-dimension of $\operatorname{mod}\Lambda$, provided it is at least 2. □

2.1 The Igusa-Todorov function

Let K be the free abelian group generated by isomorphism classes of modules, modulo the relations $[A \oplus B] = [A] + [B]$ and $[P] = 0$ when P is projective. Define the linear map $L : K \to K$ by $L[A] = [\Omega A]$. For any module X, $[\operatorname{add} X]$ is a finitely generated subgroup of K. Fitting's lemma tells us that there is an integer η_X such that $L : L^m[\operatorname{add} X] \to L^{m+1}[\operatorname{add} X]$ is an isomorphism for every $m \geq \eta_X$. We define $\psi(X)$ to be $\eta_X + \sup\{\operatorname{pd} Y | Y \in \operatorname{add} \Omega^{\eta_X} X, \operatorname{pd} Y < \infty\}$.

Lemma 2.6. *[IT05, Lemma 3]*

1. $\psi(M) = \operatorname{pd} M$ *when* $\operatorname{pd} M < \infty$.
2. $\psi(M^k) = \psi(M)$
3. $\psi(M) \leq \psi(A \oplus B)$
4. *If* Z *is a direct summand of* $\Omega^n(M)$ *with finite projective dimesnion, then* $\operatorname{pd} Z + n \leq \psi(M)$.

Proof. □ prove

Theorem 2.7. *[IT05, Theorem 4]*

Theorem 2.8. *[IT05, Corollary 8,9] FDC holds for repdim* ≤ 3*,*

3 Unbounded derived category

If we go to the unbounded derived category we can get a sort of converse to theorem 1.1.

Theorem 3.1. *[Ric19, Theorem 4.3] If the localizing category of* $D\Lambda$ *is the entire unbounded derived category then* $Findim(\Lambda) < \infty$*. (Note the capital* F *meaning the finitistic dimesnion of* $\operatorname{Mod}\Lambda$*, which is bigger than or equal to that of* $\operatorname{mod}\Lambda$*).*

Proof. Assume $Findim(\Lambda) = \infty$. Then there are modules M_i with projective dimension i for every $i \geq 0$. Let P_i be the minimal projective resolution of M_i, and consider $\bigoplus P_i[-i]$ and $\prod P_i[-i]$. Both of these have homology M_i in degree i, and are concentrated in non-negative degrees.

The inclusion from the sum to the product is clearly a quasi-isomorphism. We want to show that it is not a homotopy equivalence. Assume for the sake of contradiction that it was. Then tensoring with $\Lambda/rad(\Lambda)$ would give us another homotopy equivalence. Since $\Lambda/rad(\Lambda)$ is finitely presented tensoring preserves both products and coproducts. Because all the resolutions were minimal tesnoring with $\Lambda/rad(\Lambda)$ gives us 0 differentials. In degree 0 we get

$$\bigoplus \operatorname{Tor}_i(M_i, \Lambda/rad(\Lambda)) \to \prod \operatorname{Tor}_i(M_i, \Lambda/rad(\Lambda)).$$

Since $\operatorname{Tor}_i(M_i, \Lambda/rad(\Lambda))$ is nonzero for every M_i this map is not an isomorphism, and so we don't have a homotopy equivalence.

So the cone of the inclusion $\bigoplus P_i[-i] \to \prod P_i[-i]$, C, is 0 in the derived category, but non-zero in the homotopy category. Since Λ is artinian the product of projectives is projective [Cha60, Theorem 3.3], so $\prod P_i[-i]$ consists of projectives, which means that C consists of projectives.

In other words C is an acyclic lower bounded complex of projectives that is not contractible. Tensoring with $D\Lambda$ is an equivalence from projectives to injectives, so $C \otimes D\Lambda$ is an lower bounded complex of injectives that is not contractible. Such a complex cannot be acyclic so $C \otimes D\Lambda$ has homology.

The homology of C is 0, so $K(\Lambda)(\Lambda, C[i]) = 0$. Applying the equivalence $- \otimes D\Lambda$ we get

$$\mathcal{D}(\Lambda)(D\Lambda, C \otimes D\Lambda[i]) = K(\Lambda)(D\Lambda, C \otimes D\Lambda[i]) = 0.$$

This means that $C \otimes D\Lambda$ is not in the localizing category generated by $D\Lambda$, and so that is not the entire derived category. □

Theorem 3.2. *[Ric19, Theorem 4.4] Findim*$(\Lambda) < \infty$ *if and only if* $D\Lambda^{\perp} \cap \mathcal{D}^{+}(\Lambda) = 0$.

Proof. In the theorem above we proved that when the finitistic dimension is infinite then there is a non-zero complex in $\mathcal{D}^{+}(\Lambda)$ perpendicular to $D\Lambda$.

The proof of the converse is the same as for theorem 1.1. If we have a non-zero object $X \in D\Lambda^{\perp} \cap \mathcal{D}^{+}(\Lambda) = 0$, then $\mathcal{D}(\Lambda)(D\Lambda, X)$ is a non-split complex of projectives that continue arbitrarily to the right. So the cokernels have arbitrarily big projective dimension. □

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