Finitistic dimension conjecture

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Abstract

FDC yo!

Contents

Introduction

This is an introduction

1 finitistic dimension and conjectures

- FDC - finitistic dimesnion conjecture Finitistic dimension is always finite
- WTC - Watamatsu tilting conjecture
- GSC - Gorenstein symmetry conjecture
- NuC - Nunke condition
- SNC - strong Nakayama conjecture
- ARC - Auslander Reiten conjecture
- NC - Nakayama conjecture

1.1 Implications

$$\begin{tikzcd}
FDC \arrow[r] \arrow[d] & WTC \arrow[r] & GSC & \\
NuC \arrow[r] & SNC \arrow[r] & ARC \arrow[r] & NC
\end{tikzcd}$$

Theorem 1.1. *[Hap93, 1.2]*

i) If $findim(\Lambda) < \infty$ (FDC) then $K^b(\operatorname{inj}\Lambda)^\perp = 0$.

ii) If $K^b(\operatorname{inj}\Lambda)^\perp = 0$ then for $X \neq 0$ there exists i such that, $\operatorname{Ext}^i(D(\Lambda), X) \neq 0$ (NuC).

Proof.

i) Let $I^\bullet \in K^b(\operatorname{inj}\Lambda)^\perp$ be non-zero. Since $\mathcal{D}^b(\Lambda) \cong K^{+,b}(\operatorname{inj}\Lambda)$ we may assume $I^\bullet$ is a complex of injectives, and WLOG we may assume it concentrated in degrees $i \geq 0$, and that $d^0 : I^0 \to I^1$ is not split mono. Since if its concentrated in degrees $i \geq k$ we can just shift it, and if d^0 is split mono then replacing I^0 by 0, and I^1 be I^1/I^0 gives a homotopic complex.

$\operatorname{Hom}(D\Lambda, I^i)$ is in $\operatorname{add}\operatorname{Hom}(D\Lambda, D\Lambda) = \operatorname{add}\Lambda$ so $\operatorname{Hom}(D\Lambda, I^\bullet)$ is a complex of projectives.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & D\Lambda \arrow[r] \arrow[d, "f"] \arrow[dl, dashed] & 0 \arrow[d] \\
I^{i-1} \arrow[r, "d^{i-1}"] & I^i \arrow[r, "d^i"] & I^{i+1}
\end{tikzcd}$$

Since $I^\bullet$ is in $K^b(\operatorname{inj}\Lambda)^\perp$ and $D\Lambda$ is in $K^b(\operatorname{inj}\Lambda)$, whenever $d^i f = 0$, $f^\bullet$ is homotopic to 0. Meaning f factors through d^{i-1}. This means that $\operatorname{Hom}(D\Lambda, I^\bullet)$ is an exact complex. Further since $\operatorname{Hom}(D\Lambda, -)$ is an equivalence between $\operatorname{inj}\Lambda$ and $\operatorname{proj}\Lambda$ we have that $\operatorname{Hom}(D\Lambda, d^0)$ is not split mono.

$\operatorname{Cok}\operatorname{Hom}(D\Lambda, d^i)$ has a projective resolution of length i. This resolution is the direct sum of the minimal resolution and an acyclic bounded complex of projectives. Since bounded acyclic complexes of projectives are split and $\operatorname{Hom}(D\Lambda, d^0)$ is not, we must have that the minimal resolution has length i, and so $findim(\Lambda) = \infty$.

ii) Assume there is an $X \neq 0$ with $\operatorname{Ext}^i(D\Lambda, X) = 0$ for all $i \geq 0$. Then X considered as a stalk complex is in $K^b(\operatorname{inj}\Lambda)^\perp$. Proceed by induction: If $I[-i] \in K^b(\operatorname{inj}\Lambda)$ is a stalk complex then $\mathcal{D}^b(I[-i], X) = \operatorname{Ext}^i(I, X)$. This is 0 because $D\Lambda$ is the sum of the indecomposable injectives.

Let $I \in K^b(\operatorname{inj}\Lambda)$ be a complex of width n. WLOG assume I concentrated in degrees $0 \leq i \leq n-1$. Then

$$I^0 \to I \to I^{<0} \to I^0[1]$$

is a triangle, and $I^{<0}$ has width $n-1$. Taking the long exact sequence in $\mathcal{D}^b(-, X)$ it follows that $\mathcal{D}^b(I, X) = 0$. □

2 Recollement

$$\begin{tikzcd}
\mathcal{D}^b(\Lambda') \arrow[r, "i_*=i_!"] & \mathcal{D}^b(\Lambda) \arrow[l, bend right=40, "i^*"'] \arrow[l, bend left=40, "i^!"] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"'] \arrow[r, "j^!=j^*"] & \mathcal{D}^b(\Lambda'') \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*"] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"']
\end{tikzcd}$$

Sort of like a split exact sequence of functors. We want

1. All functors are exact/triangulated

2. $j^* i_* = 0$

3. $i^* i_* \cong i^! i_! \cong id$ (induced by unit/counit)

4. $j^! j_! \cong j^* j_* \cong id$

$$j_! j^! X \xrightarrow{\varepsilon} X \xrightarrow{\eta} i_* i^* X \longrightarrow \Sigma$$

5.
$$i_! i^! X \xrightarrow{\varepsilon} X \xrightarrow{\eta} j_* j^* X \longrightarrow \Sigma$$

Are triangles in $\mathcal{D}^b(\Lambda)$

Theorem 2.1. *Given a recollement FDC holds for midlle if and only if it holds for the two others.*

Proof. ..... Happel reduct technich [Hap93, 3.3] ☐

write later

3 Contravariant finiteness

Definition 3.1 (Resolving)**.** A full subcategory of an abelian category is called resolving if

- It is closed under extensions
- It contains the projectives
- It is contains the kernels of its epimorphisms

Note that the subcategory of modules with finite projective dimension is resolving.

Theorem 3.2. *[AR91, 3.8] Let $\mathcal{X}$ be a contravariantly finite, resolving subcategory of* mod Λ*. Let X_i be the minimal approximation of S_i. Then any $X \in \mathcal{X}$ is a direct summand of an X_i-filtered module.*

Proof. Step 1: We want to show by induction on length that any module C is in an exact sequence $0 \to Y \to X \to C \to 0$ with X X_i-filtered and $\operatorname{Ext}^1(\mathcal{X}, Y) = 0$.

Step 2: Whenever C is in $\mathcal{X}$ we get that

$\mathrm{Hom}(C,X) \longrightarrow \mathrm{Hom}(C,C) \longrightarrow \mathrm{Ext}^1(C,Y)=0$ is exact, and thus C is a direct summand of X. $\square$

Corollary 3.2.1. *If the subcategory of modules with finite projective dimension is contravariantly finite, then the finitistic dimension is the supremum of the projective dimension of* X_i. *In particular it is finite.*

4 repdimension

Many results based on the survey [Opp09].

Definition 4.1 (dominated dimension)**.** Let $\Lambda \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$ be a minimal injective resolution of Λ. Then the dominated dimension of Λ is $\inf\{n|I_n$ is not projective $\}$.

Definition 4.2 (rep-dimesnion)**.** Let A be defined by

$$A = \{\Gamma|domdim\Gamma \geq 2, \Lambda \text{ morita eqquivalent to } \mathrm{End}_\Gamma(I_0(\Gamma))\}$$

where $I_0(\Gamma)$ is the injective envelope of Γ. Then the repdimesnion of Λ is the minimal global dimension of $\Gamma \in A$.

can probably reformulate this in terms of projectives and left modules... is there any significance to the distinction?

Proposition 4.3. *(all modules ar right modules) Repdim is the same as minimal global dimension of* $\mathrm{End}(M)$ *for* M *being both a generator and cogenerator.*

Proof. Consider $\Gamma \in A$. Since $domdim\Gamma \geq 1$, $I_0(\Gamma)$ is the sum of all projective-injective modules (some probably several times).

Let $\mathcal{S}$ be the set of all Γ-modules with a copresentation

$$0 \longrightarrow X \longrightarrow I_0 \longrightarrow I_1$$

with I_i in $\mathrm{add}\, I_0(\Gamma)$. In particular Γ is in $\mathcal{S}$, because $domdim\Gamma \geq 2$.

The Yoneda embedding gives an equivalence

$$\mathrm{Hom}_\Gamma(-, I_0(\Gamma)) : \mathrm{add}\, I_0(\Gamma) \to \mathrm{proj}\, \mathrm{End}_\Gamma(I_0(\Gamma))^{op}$$

, and thus we get an equivalence

$$D\,\mathrm{Hom}_\Gamma(-, I_0(\Gamma)) : \mathrm{add}\, I_0(\Gamma) \to \mathrm{inj}\, \mathrm{End}_\Gamma(I_0(\Gamma))$$

Since $I_0(\Gamma)$ is injective $D\operatorname{Hom}(-, I_0(\Gamma))$ is exact and preserves kernels, so extends to an equivalence

$$\operatorname{Hom}_\Gamma(-, I_0(\Gamma)) : \mathcal{S} \to \operatorname{mod}\operatorname{End}_\Gamma(I_0(\Gamma))$$

Since $\operatorname{End}_\Gamma(I_0(\Gamma))$ is morita equivalent to Λ, $\mathcal{S}$ is equivalent to $\operatorname{mod}\Lambda$. $\Gamma \in \mathcal{S}$ is clearly a generator. To see that it is a cogenerator note that Γ contains all the projective-injective indecomposable objects as direct summands, so there is an injection $I_0(\Gamma) \to \Gamma^n$, and since $I_0(\Gamma)$ is a cogenerator in $\mathcal{S}$, Γ is aswell.

Thus by the equivalence $\mathcal{S} \to \operatorname{mod}\Lambda$ there is a cogenerator-generator object M such that $\operatorname{End}_\Lambda(M) = \operatorname{End}_\Gamma(\Gamma) = \Gamma$.

The last step of the proof is showing that $\operatorname{End}(M)$ is in A whenever M is a generator-cogenerator.

Let $0 \to M \to I_0(M) \to I_1(M)$ be a minimal injective copresentation of M. Since M is a cogenerator $I_i(M)$ is in $\operatorname{add} M$, thus we get an exact sequence of projective $\operatorname{End}(M)$-modules

$$0 \to \operatorname{End}(M) \to \operatorname{Hom}(M, I_0(M)) \to \operatorname{Hom}(M, I_1(M)). \tag{1}$$

Now we have the following isomorphisms of Λ-$\operatorname{End}(M)$-bimodules

$$\begin{aligned} \operatorname{Hom}_\Lambda(M, D\Lambda) &= \\ \operatorname{Hom}_k(M \otimes \Lambda, k) &= \\ \operatorname{Hom}_k(M, k) &= \\ DM &= \\ D\operatorname{Hom}_\Lambda(\Lambda, M)& \end{aligned}$$

Since Λ is in $\operatorname{add} M$, $\operatorname{Hom}(\Lambda, M)$ is projective, and thus $D\operatorname{Hom}(\Lambda, M) = \operatorname{Hom}(M, D\Lambda)$ is injective. This means that (2) is an injective copresentation, and thus *domdim* $\operatorname{End}(M) \geq 2$.

Since $\operatorname{Hom}(M, I_0(M))$ is the beginning of an injective resolution of $\operatorname{End}(M)$, $I_0(\operatorname{End}(M))$, must be a direct summand. Then $\operatorname{Hom}(M, I_0(M))/I_0(\operatorname{End}(M))$ would map injectively into $\operatorname{Hom}(M, I_1(M))$, but that would mean theres a direct summand of $I_0(M)$ mapping injectively into $I_1(M)$, contradicting minimality. Thus $\operatorname{Hom}(M, I_0(M)) = I_0(\operatorname{End}(M))$.

Let $I = I_0(M)$ and $\Gamma = \operatorname{End}_\Lambda(I)$, then $D\operatorname{Hom}(-, I)$ is an exact equivalence from $\operatorname{add} I$ to $\operatorname{inj}\Gamma$. Since I is an injective cogenerator $\operatorname{add} I = \operatorname{inj}\Lambda$. Then because of exactness $D\operatorname{Hom}(-, I)$ becomes an equivalence between

$K^{+,b}(\operatorname{inj}\Lambda)$ and $K^{+,b}(\operatorname{inj}\Gamma)$. Considering only those complexes with homology in degree 0, we see that $\operatorname{mod}\Lambda$ is equivalent to $\operatorname{mod}\Gamma$. So Λ is morita equivalent to $\Gamma = \operatorname{End}(I_0(M)) = \operatorname{End}(I_0(\operatorname{End}(M)))$. □

Definition 4.4. Let X be an object of $\operatorname{mod}\Lambda$ and M a contravariantaly finite subcategory.

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr, two heads] & M_2 \arrow[r] \arrow[dr, two heads] & M_1 \arrow[r] \arrow[dr, two heads] & M_0 \arrow[dr, two heads] & \\
& \Omega^3_M X \arrow[u, hook] & \Omega^2_M X \arrow[u, hook] & \Omega_M X \arrow[u, hook] & X
\end{tikzcd}$$

If $\twoheadrightarrow$ are minimal M-approximations (they need not be surjective), and $\hookrightarrow$ are their kernels, then this is an M-resolution of X. The M-res-dimesnion of X is the length of the sequence of (nonzero) M_i's, and the M-res-dimesnion of Λ is the supremum of the dimension on its objects.

Proposition 4.5. *Repdim-2 is the minimum of M-res-dim$(\operatorname{mod}\Lambda)$ for M both generator and cogenrator (assuming repdim is at least 2).*

Proof. The functor $\operatorname{Hom}(M,-)$ is an equivalence from $\operatorname{add} M$ to $\operatorname{proj}\operatorname{End}(M)$, which maps minimal M-approximations to projective covers. Let X be any module in $\operatorname{mod}\operatorname{End}(M)$ with projective dimension at least 2. Then it has a projective presentation

$$\Omega^2 X \to (M, M_1) \to (M, M_0) \to X.$$

Because of the equivalence this is induced by a map $f : M_1 \to M_0$. Since Hom is left exact we have that $\Omega^2 X \cong \operatorname{Hom}(M, \ker f)$, and so the projective dimension of X is 2 plus the M-res-dimension of $\ker f$.

Since M is a cogenerator any module Y in $\operatorname{mod}\Lambda$ has a copresentation

$$0 \longrightarrow Y \longrightarrow M_0 \xrightarrow{f} M_1.$$

Applying $\operatorname{Hom}(M,-) =: (M,-)$ we get

$$0 \longrightarrow (M,Y) \longrightarrow (M,M_0) \xrightarrow{(M,f)} (M,M_1) \longrightarrow \operatorname{Cok}(M,f) \longrightarrow 0.$$

If the projective dimension of $\operatorname{Cok}(M,f)$ is less than 2, then (M,Y) is a direct summand of (M,M_0). This means that $(M,Y) \cong (M,M')$, so the minimal M-approximation of Y is M', and $(M, \Omega_M Y) = 0$. Since M is a generator this means $\Omega_M Y = 0$ and thus the M-res-dimension of Y is 0.

So provided the projective dimension of $\mathrm{Cok}(M, f)$ is larger than or equal to 2, it equals the M-res-dimension of Y plus 2. In particular the global dimension of $\mathrm{End}(M)$ is 2 plus the M-res-dimension of $\operatorname{mod}\Lambda$, provided it is at least 2. □

4.1 The Igusa-Todorov function

Let K be the free abelian group generated by isomorphism classes of modules, modulo the relations $[A \oplus B] = [A] + [B]$ and $[P] = 0$ when P is projective. Define the linear map $L : K \to K$ by $L[A] = [\Omega A]$. For any module X, $[\operatorname{add} X]$ is a finitely generated subgroup of K. Fitting's lemma tells us that there is an integer η_X such that $L : L^m[\operatorname{add} X] \to L^{m+1}[\operatorname{add} X]$ is an isomorphism for every $m \geq \eta_X$. We define $\psi(X)$ to be $\eta_X + \sup\{\operatorname{pd} Y | Y \in \operatorname{add} \Omega^{\eta_X} X, \operatorname{pd} Y < \infty\}$.

Lemma 4.6. *[IT05, Lemma 3]*

1. $\psi(M) = \operatorname{pd} M$ *when* $\operatorname{pd} M < \infty$.
2. $\psi(M^k) = \psi(M)$
3. $\psi(M) \leq \psi(M \oplus N)$
4. *If* Z *is a direct summand of* $\Omega^n(M)$ *where* $n \leq \eta_M$ *and* $\operatorname{pd} Z < \infty$, *then* $\operatorname{pd} Z + n \leq \psi(M)$.

Proof.

1. If $\operatorname{pd} M < \infty$ then $L^m \neq 0$ for $m < \operatorname{pd} M$, and $L^m = 0$ for $m \geq \operatorname{pd} M$.
2. $\operatorname{add} M^k = \operatorname{add} M$, and ψ is only defined in terms of additive categories.
3. $\operatorname{add} M \subseteq \operatorname{add} M \oplus N$, so if L is injective when restricted to $L^m(\operatorname{add} M \oplus N)$ then L is injective when restricted to $L^m(\operatorname{add} M)$, so $\eta_M \leq \eta_{M \oplus N}$. Further $\Omega^{\eta_{M\oplus N} - \eta_M} \operatorname{add} \Omega^{\eta_M} M \subset \operatorname{add} \Omega^{\eta_{M\oplus N}} M \oplus N$, so $\psi(M) \leq \psi(M \oplus N)$.
4. Let $p = \operatorname{pd} Z$ and $k = \eta_M - n$. Then $\Omega^k Z$ is in $\operatorname{add} \Omega^{\eta_M} M$, so $\operatorname{pd} \Omega^k Z + \eta_M \leq \psi(M)$. Thus
$$\operatorname{pd} Z + n = p + n = (p - k) + \eta_M \leq \operatorname{pd} \Omega^k Z + \eta_M \leq \psi(M).$$

□

Theorem 4.7. *[IT05, Theorem 4] Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence of modules with* $\operatorname{pd} C < \infty$*. Then* $\operatorname{pd} C \leq \psi(A \oplus B) + 1$.

Proof. Let $P_A^\bullet$ and $P_C^\bullet$ be the minimal projective resolutions of A and C. Then we get a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & P_A^0 \arrow[r] \arrow[d] & P_A^0 \oplus P_C^0 \arrow[r] \arrow[d] & P_C^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Applying the snake lemma we get $0 \to \Omega A \to \Omega B \oplus P \to \Omega C \to 0$ for some projective P. Thus for some $n \leq \operatorname{pd} C$ we have $L^n[A] = L^n[B]$, and let n be the minimal such number. Clearly $n \leq \eta_{A \oplus B}$. Let $X = \Omega^n A = \Omega^n B$, then our sequence of n-syzygies looks like

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] & X \oplus P \arrow[r] & \Omega^n C \arrow[r] & 0.
\end{tikzcd}$$

Let f be the composition $X \longrightarrow X \oplus P \xrightarrow{\pi_X} X$. Then by fittings lemma X breaks as a direct sum into two components $X = Z \oplus Y$ such that $f = f_Z \oplus f_Y$ with f_Y an isomorphism and f_Z nilpotent. In other words the sequence above can be written as

$$\begin{tikzcd}
0 \arrow[r] & Z \oplus Y \arrow[r] & Z \oplus Y \oplus P \arrow[r] & \Omega^n C \arrow[r] & 0.
\end{tikzcd}$$

with the left map being

$$\begin{bmatrix} f_Z & 0 \\ 0 & f_Y \\ * & * \end{bmatrix} \sim \begin{bmatrix} f_Z & 0 \\ 0 & f_Y \\ * & 0 \end{bmatrix}$$

So we get another short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r] & Z \oplus P \arrow[r] & \Omega^n C \arrow[r] & 0.
\end{tikzcd}$$

Let $T = \Lambda/rad(\Lambda)$and apply the long exact sequence in $\operatorname{Ext}(-, T)$. Then we get an exact sequence

$$\begin{tikzcd}
\operatorname{Ext}^k(Z, T) \arrow[r] & \operatorname{Ext}^k(Z \oplus P, T) \arrow[r] & \operatorname{Ext}^{k+1}(\Omega^n C, T)
\end{tikzcd}$$

where the left map is induced by f_Z since $\mathrm{Ext}^k(Z \oplus P, T) \cong \mathrm{Ext}^k(Z, T)$. Since f_Z is nilpotent this map is surjective if and only if $\mathrm{Ext}^k(Z, T) = 0$, and $\Omega^n C$ has finite projective dimension we have that Z has finite projective dimension. In particular $\mathrm{pd}\,\Omega^n C - 1 \leq \mathrm{pd}\,Z \leq \mathrm{pd}\,\Omega^n C$.

Since Z is a direct summand of $\Omega^n A \oplus B$ by lemma 4.6 we have that $\mathrm{pd}\,Z + n \leq \psi(A \oplus B)$, and thus $\mathrm{pd}\,\Omega^n C - 1 + n = \mathrm{pd}\,C - 1 \leq \psi(A \oplus B)$. □

Theorem 4.8. *[IT05, Corollary 8] If $\Lambda = \mathrm{End}_\Gamma(P)$ for an algebra Γ with global dimension at most 3, and P projective then $findim(\Lambda) < \infty$.*

Proof. Let X be any Λ-module with finite projective dimension. Then it has a projective presentation $(P, P_1) \to (P, P_0) \to X \to 0$ where $(P, P_i) = \mathrm{Hom}_\Gamma(P, P_i)$ with $P_i \in \mathrm{add}\,P$. Since $(P, -)$ is an equivalence from $\mathrm{add}\,P$ to $\mathrm{proj}\,\Lambda$ this corresponds to a map $P_1 \to P_0$ which we can extend to a projective resolution in Γ:

$$\begin{tikzcd} 0 \arrow[r] & P_3 \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0. \end{tikzcd}$$

Applying the exact functor $(P, -)$, we get an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & (P,P_3) \arrow[r] & (P,P_2) \arrow[r] & (P,P_1) \arrow[r] & (P,P_0) \arrow[r] & X \arrow[r] & 0. \end{tikzcd}$$

Truncating this we get a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & (P,P_3) \arrow[r] & (P,P_2) \arrow[r] & \Omega^2 X \arrow[r] & 0. \end{tikzcd}$$

Then by theorem 4.7 the projective dimension of $\Omega^2 X$ is bounded by $\psi((P, P_3) \oplus (P, P_2)) + 1$. Which means

$$\mathrm{pd}\,X \leq \psi((P, P_3) \oplus (P, P_2)) + 3 \leq \psi((P, \Gamma)) + 3$$

Since this bound doesn't depend on X, Λ has finite finitistic dimension. □

Corollary 4.8.1. *If $repdim(\Lambda) \leq 3$ then $findim(\Lambda) < \infty$.*

Proof. If Λ has rep-dimension less than or equal to 3 then by proposition 4.3 there is a generator-cogenerator M in $\mathrm{mod}\,\Lambda$ such that $\Gamma := \mathrm{End}_\Lambda(M)$ has global dimension 3 or less. Then since M is a generator Λ is in $\mathrm{add}\,M$ and so $\mathrm{Hom}_\Lambda(M, \Lambda)$ is a projective Γ-module with $\mathrm{End}_\Gamma(\mathrm{Hom}_\Lambda(M, \Lambda)) = \mathrm{End}_\Lambda(\Lambda) = \Lambda$. □

5 Unbounded derived category

If we go to the unbounded derived category we can get a sort of converse to theorem 1.1.

Theorem 5.1. *[Ric19, Theorem 4.3] If the localizing category of $D\Lambda$ is the entire unbounded derived category then $Findim(\Lambda) < \infty$. (Note the capital F meaning the finitistic dimesnion of $\operatorname{Mod}\Lambda$, which is bigger than or equal to that of $\operatorname{mod}\Lambda$).*

Proof. Assume $Findim(\Lambda) = \infty$. Then there are modules M_i with projective dimension i for every $i \geq 0$. Let P_i be the minimal projective resolution of M_i, and consider $\bigoplus P_i[-i]$ and $\prod P_i[-i]$. Both of these have homology M_i in degree i, and are concentrated in non-negative degrees.

The inclusion from the sum to the product is clearly a quasi-isomorphism. We want to show that it is not a homotopy equivalence. Assume for the sake of contradiction that it was. Then tensoring with $\Lambda/rad(\Lambda)$ would give us another homotopy equivalence. Since $\Lambda/rad(\Lambda)$ is finitely presented tensoring preserves both products and coproducts. Because all the resolutions were minimal tesnoring with $\Lambda/rad(\Lambda)$ gives us 0 differentials. In degree 0 we get

$$\bigoplus \operatorname{Tor}_i(M_i, \Lambda/rad(\Lambda)) \to \prod \operatorname{Tor}_i(M_i, \Lambda/rad(\Lambda)).$$

Since $\operatorname{Tor}_i(M_i, \Lambda/rad(\Lambda))$ is nonzero for every M_i this map is not an isomorphism, and so we don't have a homotopy equivalence.

So the cone of the inclusion $\bigoplus P_i[-i] \to \prod P_i[-i]$, C, is 0 in the derived category, but non-zero in the homotopy category. Since Λ is artinian the product of projectives is projective [Cha60, Theorem 3.3], so $\prod P_i[-i]$ is a complex of projectives, which means that C is a complex of projectives.

In other words C is an acyclic lower bounded complex of projectives that is not contractible. Tensoring with $D\Lambda$ is an equivalence from projectives to injectives, so $C \otimes D\Lambda$ is an lower bounded complex of injectives that is not contractible. Such a complex cannot be acyclic so $C \otimes D\Lambda$ has homology.

The homology of C is 0, so $K(\Lambda)(\Lambda, C[i]) = 0$. Applying the equivalence $- \otimes D\Lambda$ we get

$$\mathcal{D}(\Lambda)(D\Lambda, C \otimes D\Lambda[i]) = K(\Lambda)(D\Lambda, C \otimes D\Lambda[i]) = 0.$$

This means that $C \otimes D\Lambda$ is not in the localizing category generated by $D\Lambda$, and so that is not the entire derived category. ☐

Theorem 5.2. *[Ric19, Theorem 4.4] Findim$(\Lambda) < \infty$ if and only if $D\Lambda^{\perp} \cap \mathcal{D}^{+}(\Lambda) = 0$.*

Proof. In the theorem above we proved that when the finitistic dimension is infinite then there is a non-zero complex in $\mathcal{D}^{+}(\Lambda)$ perpendicular to $D\Lambda$.

The proof of the converse is the same as for theorem 1.1. If we have a non-zero object $X \in D\Lambda^{\perp} \cap \mathcal{D}^{+}(\Lambda) = 0$, then $\mathcal{D}(\Lambda)(D\Lambda, X)$ is a non-split complex of projectives that continue arbitrarily to the right. So the cokernels have arbitrarily big projective dimension. □

6 Personal appendix

Theorem 6.1. *The global dimension of an artin algebra is the supremum of k with $\operatorname{Ext}^{k}(T, T) \neq 0$ (T sum of simples). This is also the supremum of projective dimension and supremum of injective dimension.*

Proof. For a minimal projective resolution $\operatorname{Hom}(-, T)$ makes the differentials 0, and similarly with $\operatorname{Hom}(T, -)$ and injective resolutions. So $\operatorname{Ext}^{k}(M, T)$ is only 0 exactly when $k > \operatorname{pd} M$, similarly $\operatorname{Ext}^{k}(T, M)$ is only 0 when k is bigger than the injective dimension. Since any module is built by extensions of simples you can prove by induction, and the long exact sequence in $\operatorname{Ext}(-, T)$ you get that any module has projective dimension less than or equal to that of T. Similarly for injective dimension. □

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