Notes on the Polyakov Measure, Moduli Space and String Amplitudes

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Abstract

We give a pedagogical review of the moduli space measure in string theory and outline some aspects of string amplitudes. These notes are prepared for a talk at UBC in Summer 2015 as a part of an ongoing string theory reading group. The choice of topics is heavily inspired by D'Hoker's 1999 IAS lectures [1].

Contents

1 Motivation & Prerequisites

1.1 Motivation

The Polyakov path integral is an integral over moduli space. The first five chapters of Polchinski's first volume [3] culminates to a measure for moduli space, known as the Polyakov Measure

$$ \tag{1.1} $$

Rewriting the determinants as ghost fields, this looks like

$$ \tag{1.2} $$

The goal of the first half of these notes is to gain as much geometrical intuition as possible about where these equations come from. The second half will be concerned with computing amplitudes.

1.2 Differential Operators on Riemann Surfaces

1.2.1 Basics

Let's recall some basic notions from differential geometry on Riemann surfaces.

1.2.1.1 Riemann Surfaces

For our purposes, a Riemann surface Σ is a 1-dimensional complex manifold. We will be particularly interested in examples where Σ is compact and connected, the topology of such Σ is completely fixed by a genus g.

In string theory we deal with Riemann surfaces equipped with metrics, i.e. Hermitian manifolds (Σ, g). Being equivalent to a real Riemannian 2-manifold, (Σ, g) is conformally flat. That is, in each coordinate chart $U \subset \Sigma$, we have a diffeomophism:

$$\phi_U : U \to \mathbb{C}, \quad p \mapsto (z, \bar{z}) \tag{1.3}$$

such that

$$g = k(z, \bar{z}) dz \otimes d\bar{z} \tag{1.4}$$

1.2.1.2 The Metric Isomorphism

As usual, g gives an isomorphism $\tilde{g}$ between tangent and cotangent spaces at each point $p \in \Sigma$

$$\tilde{g} : T_p\Sigma \to T_p^*\Sigma \ , \ v \mapsto g(v, \cdot) \tag{1.5}$$

As usual, it is convenient to view the inverse g^{-1} of the g isomorphism as a rank $(2, 0)$ tensor defined in each coordinate chart by taking the matrix inverse of the components of g. In the case of Riemann surfaces, we have simply

$$g^{-1} = k^{-1}(z, \bar{z}) \partial_z \otimes \partial_{\bar{z}} \tag{1.6}$$

These in turn give isomorphisms between tensors of different ranks

$$T_{[a,b]} \xrightarrow{g} T_{[a-1,b+1]} \xrightarrow{g} \ldots \xrightarrow{g} T_{[0,a+b]} \tag{1.7}$$

$$T_{[a,b]} \xrightarrow{g^{-1}} T_{[a+1,b-1]} \xrightarrow{g^{-1}} ... \xrightarrow{g^{-1}} T_{[a+b,0]} \tag{1.8}$$

where $T_{[a,b]} \equiv (\otimes^a T\Sigma)(\otimes^b T^*\Sigma)$ is the space of rank (a,b) tensors. These sequences of isomorphisms tell us we can restrict our attention to pure tensors without losing much generality.

1.2.2 Line Bundles of Fixed-Weight Tensors

The most general rank $(0,w)$-tensor T over Σ lives in the tensor product cotangent bundle $\otimes^w T^*\Sigma$:

$$T = T\mu_1...\mu_w d\omega^{\mu_1} \otimes ... \otimes d\omega^{\mu_w} \tag{1.9}$$

with $\mu_i \in \{0,1\}$, $\omega^\mu \in \{z, \bar{z}\}$. This can always be written as

$$T = \tilde{T}(\otimes^m dz)(\otimes^n d\bar{z}) \tag{1.10}$$

For some $\tilde{T} \in \mathbb{C}$. and $m,n \in \mathbb{Z}^+$ with $m+n=w$. We say T is a (contravariant) tensor of weight (m,n), and conformal dimension w.

We call the collection of these tensors $K_0^{(mn)}$

$$K_0^{(m,n)} \equiv \{T \in \otimes^{m+n} T^*\Sigma \mid T \text{ has weight (m, n) }\} \tag{1.11}$$

From 1.10 one can see that $K^{(m,n)}$ is a 1-dimesisonal vector space, hence K defines a line bundle over Σ.

One can make a trivial remark at this point: each space $K^{(m,n)}$ with different values of (m,n) are isomorphic - namely they're all isomorphic to $\mathbb{C}$. One can then wonder how do these space differ. In our case, the only difference we will care about is the factor they gain under a pullback under a diffeomorphism. To this end, let's consider for a moment mixed tensors of the form

$$T = \tilde{T}(\otimes^m dz)(\otimes^n \partial\bar{z})(\otimes^a \partial_z)(\otimes^b \partial_{\bar{z}}) \tag{1.12}$$

Observe that under a pullback by a diffeomorphism transform exactly like the tensor

$$\tilde{T}(\otimes^{(m-a)} dz)(\otimes^{(n-b)} d\bar{z}) \tag{1.13}$$

Hence for our purposes, we will not distinguish between those two tensors. Let's give a name to this collection

Definition 1.1.

$$K^{(m,n)} \equiv \{\tilde{T}(\otimes^{m-a} dz)(\otimes^{n-b} \partial\bar{z})(\otimes^a \partial_z)(\otimes^b \partial_{\bar{z}}) | a,b \in \mathbb{Z}^+, \tilde{T} \in \mathbb{C}\}/\sim$$

where $\sim$ is the equivalence relation $A \sim B$ if they have the same scalar factor.

The quotient by the identification of course makes sure that $K^{(m,n)}$ is isomorphic to $K_0^{(m,n)}$.

Under the metric isomorphism discussed earlier, using the explicit form of the metric 1.6 (namely the fact that the diagonals vanish), we see that the the action of the metric is:

$$\tilde{T}(\otimes^m dz)(\otimes^n d\bar{z}) \mapsto \tilde{T}'(\otimes^m dz)(\otimes^{n-1} d\bar{z}) \otimes \partial_z \tag{1.14}$$

The metric g therefore gives isomorphisms

$$K^{(m,n)} \xrightarrow{g} K^{(m-1,n-1)} \xrightarrow{g} ... \xrightarrow{g} K^{(m-n,0)} \tag{1.15}$$

In particular, $K^{(m,n)} \simeq K^{(m-n,0)}$. This tells us it is often sufficient to only care about tensors that only have z-indices. We therefore introduce the abbreviated notation:

$$K^m \equiv K^{(m,0)} \tag{1.16}$$

Note that in this notation we allow m to be negative - a tensor of weight $-n$ for some $n > 0$ is defined to be one of weight $(0, n)$.

Being a complex line bundle over Σ, K^m is a complex manifold in its own right. Furthermore we will equip it with an inner product $\langle \cdot, \cdot \rangle$:

$$\langle \phi, \psi \rangle \equiv \int_\Sigma du \; g_{z\bar{z}}^{-m} \; \phi\psi \tag{1.17}$$

where du is the invariant measure on (Σ, g).

1.2.3 Differential Operators

For readability, let's slightly abuse our notation and use $K^{(m,n)}$ the space of sections over the line bundle which we previously called $K^{(m,n)}$.

Let's consider a metric connection on Σ. The condition of metric compatibility reads:

$$\nabla_X g = 0 \;\; \forall X \text{ section to } T\Sigma \tag{1.18}$$

In components, this translates to the demand that all components of $\nabla_a g_{bc}$ vanish. Using eq. 1.4, this gives us all the Christoffel symbols. In the below we only state the results needed for our purposes, for the derivations, see for example [2].

The covariant derivative can be decomposed as:

$$\nabla \equiv \nabla_{\bar{z}} + \nabla_z \tag{1.19}$$

with

$$\nabla_{\bar{z}} : K^{(m,n)} \to K^{(m,n+1)}, \quad \phi \mapsto \partial_{\bar{z}}\phi \otimes d\bar{z} \tag{1.20}$$

$$\nabla_z : K^{(m,n)} \to K^{(m+1,n)}, \quad \phi \mapsto (k^{-m}\partial_z k^m)\phi \otimes dz \tag{1.21}$$

As mentioned, we would like to restrict our attention to pure z-index tensors, so that we can look at spaces K^{a-b} as opposed to $K^{(a,b)}$. Setting $n = 0$ in the domain of the operators above.

Observe the latter of these two operators is already one which takes pure-z tensor to another pure-z one. To obtain from $\nabla_{\bar{z}}$ another operator which maps strictly to pure z-tensors, we can subsequently use the inverse metric to map $K^{(m,1)} \to K^{m-1}$. That is, we define map:

$$\nabla^z \equiv g^{-1} \circ \nabla_{\bar{z}} \tag{1.22}$$

observe this acts as:

$$\nabla^z : K^{(m,n)} \to K^{(m-1,n)}, \phi \mapsto k^{-1} \partial_{\bar{z}} \phi \otimes \partial_z \tag{1.23}$$

From now on let's fix $n = 0$ in the domain of these maps. For clarity of the below definitions we also add a superscript or subscript in brackets to denote the z-weight m of the domain of these maps, e.g. $\nabla^z_{(m)}$, $\nabla_z^{(m)}$. Note that ∇_z and ∇^z are related by adjoints with respect to the metric inner product:

$$\nabla_z^\dagger = \nabla^z \tag{1.24}$$

We can define Lapacians as compositions of these:

$$\triangle_m^+ \equiv \nabla^z_{(m+1)} \nabla_z^{(m)} \ , \ K^{m+1} \to K^{m+1} \tag{1.25}$$

$$\triangle_m^- \equiv \nabla_z^{(m-1)} \nabla_z^{(m)} \ , \ K^{m-1} \to K^{m-1} \tag{1.26}$$

All this can be summarized by a commutative diagram:

$$\begin{tikzcd}
 & K^{m-1} & \\
K^{m-1} \arrow[ur, "\triangle^-"] \arrow[r, "\nabla^z"] & K^m \arrow[u, "\nabla_z"'] \arrow[r, "\nabla^z"] & K^{m+1} \\
 & K^{m+1} \arrow[u, "\nabla_z"] \arrow[ur, "\triangle^+"'] &
\end{tikzcd} \tag{1.27}$$

2 Moduli Space

2.1 Problem Statement & Strategy

Let Σ be a Riemann surface. Let $Met(\Sigma)$ denote the space of all metrics on Σ. Define the moduli space as the quotient

$$Mod(\Sigma) \equiv Met(\Sigma)/(Diff(\Sigma) \ltimes Weyl(\Sigma)) \tag{2.1}$$

where $\ltimes$ denotes a semidirect product. Viewing this quotient space as a local product $Met(\Sigma) \xrightarrow{\pi} Mod(\Sigma)$, then locally, a metric $g \in Met(\Sigma)$ can be written as

$$g \sim (h, \sigma) \tag{2.2}$$

for $h \in Mod(\Sigma)$, and $\sigma \in Diff(\Sigma) \ltimes Weyl(\Sigma)$. In physicist terms, h lives in a gauge slice and σ live in a gauge orbit.

Observe now on $Met(\Sigma)$ we can define a natural gaussian measure $\mathcal{D}g$ with respect to any tangent space $T_gMet(\Sigma)$. Our goal is then to find a integration measure on $Mod(\Sigma)$ by "projecting" the $Met(\Sigma)$ measure $\mathcal{D}g$. More concretely, we want to decompose $\mathcal{D}g$ as

$$\mathcal{D}g = J\mathcal{D}h\mathcal{D}\sigma \tag{2.3}$$

for some Jacobian J, a measure $\mathcal{D}h$ for $Mod(\Sigma)$, and a measure $\mathcal{D}\sigma$ for $\pi^{-1}(h)$. We can subsequently perform the integration over the gauge orbits to obtain the measure on moduli space.

By this decmposition, it is clear that the main task at hand is to evaluate the Jacobian J. Evaluating J at $g \in Met(\Sigma)$ is of course the same thing as evaluating the Jacobian from a basis change at $T_gMet(\Sigma)$. Our strategy therefore is to decompose the tangent space $T_gMet(\Sigma)$.

2.2 $Met(\Sigma)$ Tangent Space Decomposition

Let $g \in Met(\Sigma)$. Physically, one can think of the tangent space $T_gMet(\Sigma)$ as variations of g

$$\delta g \equiv \varepsilon\omega \ , \ \omega \in T_gMet(\Sigma) \tag{2.4}$$

$T_gMet(\Sigma)$ is a space of rank $(0,2)$ tensors, which can be in turn decomposed by weights. Using notation introduced earlier

$$T_gMet(\Sigma) \simeq T_{[0,2]} \simeq T^{(1,1)} \oplus T^{(0,2)} \oplus T^{(2,0)} \tag{2.5}$$

We examine each space on the RHS one by one.

2.2.1 $T^{(1,1)}$

The most general element in $T^{(1,1)}$ can be written as

$$\omega = \varphi dz \otimes d\bar{z} \tag{2.6}$$

with $\varphi \in \mathbb{C}$. Now consider the metric perturbation

$$g \mapsto g + \delta g \tag{2.7}$$

In isothermal coordinates and notation from before

$$kdz \otimes d\bar{z} \mapsto kdz \otimes d\bar{z} + \delta g \tag{2.8}$$

Now consider a perturbation $\delta g = \epsilon\omega$ with $\omega \in T^{(1,1)}$, the change is

$$kdz \otimes d\bar{z} \mapsto kdz \otimes d\bar{z} + \varepsilon\varphi dz \otimes d\bar{z} \tag{2.9}$$

This is simply a scaling of the metric. In other words, this can be written as

$$g \mapsto f_\varepsilon^*(g) \tag{2.10}$$

for some map $f_\varepsilon \in Weyl(\Sigma)$. What we have just shown is the following: every $\omega \in T^{(1,1)}$ is a fundamental tensor field generated by some element in the Lie algebra of $Weyl(\Sigma)$. Conversely, every element in the Lie algebra of $Weyl(\Sigma)$ generates a vector field which evaluates to some $\omega \in T^{(1,1)}$.

To summarize this colloquially: $T^{(1,1)}$ is covered by the infinitesmall action of the Weyl group.

2.2.2 $T^{(2,0)}$ and $T^{(0,2)}$

Inspired by how we were able to identify $T^{(1,1)}$ with pullbacks of Weyl maps on the metric, let's consider the action of a diffeomorphism on the metric. Any diffeomorphism connected to the identity $f \in Diff_0(\Sigma)$ can be obtained as the flow generated by a vector field V in $T\Sigma$. The infinitesmall action of f_ϵ on the metric can be read off from the definition of the Lie derivative

$$\delta g = f_\epsilon^*(g) - g = \epsilon \mathcal{L}_V g \tag{2.11}$$

In some local coordinate chart, the Lie derivative of the metric can be nicely expressed in terms of the covariant derivative:

$$\delta g_{ab} = (\mathcal{L}_V g)_{ab} = \nabla_a V_b + \nabla_b V_a \tag{2.12}$$

where the vector indices are lowered by the metric isomorphism. By the discussion in the previous section, we know the off diagonal components correspond to Weyl scalings. Let's therefore restrict ourselves to considering "diagonal variations" $\delta_d g$ where we set the off diagonal entries of the variation to 0, the result is then

$$\frac{1}{2}\delta_d g = \nabla_z V_z dz \otimes dz + \nabla_{\bar{z}} V_{\bar{z}} d\bar{z} \otimes d\bar{z} \tag{2.13}$$

This expression tells us

$$\delta_d g \in Im(\nabla_z^{(1)}) \oplus Im(\nabla_{\bar{z}}^{(1)}) \subset T^{(2,0)} \oplus T^{(0,2)} \tag{2.14}$$

We now show that $Im(\nabla_z^{(1)}) \oplus Im(\nabla_{\bar{z}}^{(1)})$ is a strict subset of $T^{(2,0)} \oplus T^{(0,2)}$, and give explicitly the orthogonal complement of the former in the latter. To appreciate this result that we're going to prove shortly, let's take a step back and look at the geometric picture of all this.

For readability let's denote

$$D \equiv Im(\nabla_z^{(1)}) \oplus Im(\nabla_{\bar{z}}^{(1)}) \tag{2.15}$$

$$\bar{D} \equiv \text{orthogonal complement of } D \text{ in } T^{(2,0)} \oplus T^{(0,2)} \tag{2.16}$$

Now:

- We just saw D is generated by pullbacks of diffeomorphisms on Σ. In other words D is tangent to the gauge orbit.
- The orthogonal complement $\bar{D}$ is therefore tangent to the gauge slice.
- We saw in the previous section that all of $T^{(1,1)}$ is generated by Weyl maps, therefore all of $T^{(1,1)}$ is tangent to the gauge orbit.

- This leaves $\bar{D}$ as exactly the subspace of $T_g Met(\Sigma)$ that is tangent to the gauge orbit.

Our task therefore has been reduced to finding $\bar{D}$. Since $T^{(2,0)}$ and $T^{(0,2)}$ are related by complex conjugation, it is sufficient to consider $T^{(2,0)}$. The crucial result is the following

Proposition 2.1. $T^{(2,0)} \simeq Im(\nabla_z^{(1)}) \oplus Ker(\nabla_{\bar{z}}^{(2)}) \oplus CKV$

Proof. □

We can now consider a basis $(\mu_j)_{1 \leq j \leq dim(Mod(\Sigma))}$ for $Ker(\nabla_{\bar{z}}^{(2)})$. Such bases (and their complex conjugates) are called *Beltrami Differentials*.

2.3 The $Mod(\Sigma)$ Measure

3 String Amplitudes

3.1 Tree Level

3.2 Witten's Geometric Intepretation of $i\epsilon$

Here we **briefly** describe a geometric interpretation of the $i\epsilon$ prescription in String theory due to Witten [4].

3.3 Beyond Tree Level

References

[1] E. P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, and E. Witten. Quantum fields and strings. a course for mathematicians. In *Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study*. American Mathematical Society, 1999.

[2] M. Nakahara. *Geometry, Topology and Physics, Second Edition*. Graduate student series in physics. Taylor & Francis, 2003.

[3] J. Polchinski. *String Theory: Volume 1, An Introduction to the Bosonic String*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1998.

[4] E. Witten. The feynman $i\epsilon$ in string theory, 2013.