TOROIDAL COMPACTIFICATION AND T-DUALITY OF CLOSED STRINGS

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Toroidal Compactification consists of periodically identifying one or more dimensions of spacetime. More elegantly, we think of it in terms of fiber bundles in geometry. The name comes from the fact that these compact dimensions will form a k-torus, $\mathbb{T}^k = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ with k being the number of compact dimensions. These are not the most realistic compactifications that occur in string phenomenology. In fact, Calabi-Yau compactifications are studied because they produce string spectra whose force charges and masses are more phenomenologically realistic.

We will find that compactifications to a large radius yield identical physics to those with a small radius. We call this T-duality and we will study it in the context of closed string theory. Finally, we will mention a family of elegant dualities known as mirror symmetries and S-dualities.

1. Kaluza-Klein Unification in Field Theory

Consider an arbitrary d dimensional (pseudo-) Riemannian manifold $\mathbb{M}$. For our purposes, let $\mathbb{M}$ be non-compact with coordinates x^μ, index μ ranging from 0 to $d-1$. We endow $\mathbb{M}$ with the added structure of a circle bundle to construct a new manifold $\mathbb{M}\times\mathbb{S}^1$. Recall that this means that we've attached a copy of $\mathbb{S}^1$ above each point on $\mathbb{M}$. The simple product structure of $\mathbb{M}\times\mathbb{S}^1$ implies that it has been constructed in such a way that there is no "twisting" as we go from one circle to a nearby circle.

Let the circular fiber have radius R and let the coordinate along the circular dimension be x^d. We express the periodicity condition as,

$$x^d = x^d + 2\pi R. \tag{1}$$

This is how compactification goes in string theory, in general: start with a base space $\mathbb{M}$ and endow it with a bundle structure with fiber $\mathcal{C}$, some compact space. This generates the product space $\mathbb{M}\times\mathcal{C}$. But what if there is "twisting" between the fibers? In other words, what if the copies of $\mathcal{C}$

are oriented differently above different points of the base space? Such a structure would produce different physics at different points in the base space, so we rule it out on general principles.

Such general constructions as described above are one the most beautiful parts of geometry and physics. They require an grasp of complex geometry and algebraic geometry. It appears as though a good source would be the last two chapters of the second volume of Green, Schwarz, and Witten.

Returning to the "cylinder world" $\mathbb{M} \times \mathbb{S}^1$,the invariant interval of this spacetime is given as,

$$ds^2 = G^D_{MN} dx^M dx^N = G_{\mu\nu} dx^\mu dx^\nu + G_{dd}(dx^d + A_\mu dx^\mu)^2. \tag{2}$$

Capital Roman indices M, N range over the entire spacetime $0, \dots d$ while the Greek indices range over only the non-compact dimensions $0, \dots d-1$ within $\mathbb{M}$. It's important to keep in mind that we raise and lower indices on the full space using G_{MN} while we raise and lower indices on the non-compact base space using $G_{\mu\nu}$.

If we're probing length scales much larger than R, then motion through $\mathbb{S}^1$ can be neglected. We think of this as the long-distance/low-energy limit, since this is equivalent to probing energies much smaller than $1/R$. Therefore, we'll be left with a low-energy effective theory by making all fields independent of x^d. This will occupy us in the next section, after which we will account for dependence on the periodic coordinate.

1.1. **Low-Energy Effective Action.** We ask the question, "what physics will appear to a being living in $\mathbb{M}$?" Despite the fact that the quantum fields this being studies are independent of x^d, the compact dimension will be made manifest to him.

Consider a change of variables on the compactified coordinate,

$$x^d \to x^d + \lambda(x^\mu). \tag{3}$$

We would like ds^2 to be invariant under this transformation since none of our fields depend on x^d. Clearly, the term $G_{\mu\nu} dx^\mu dx^\nu$ is already invariant since it doesn't depend on x^d. Given the above transformation of x^d, the differential transforms as,

$$dx^d \to dx^d + (\partial_\mu \lambda) dx^\mu. \tag{4}$$

Therefore, the second term transforms as,

$$G_{dd}(dx^d + A_\mu dx^\mu)^2 \longrightarrow G_{dd}\big(dx^d + (A_\mu + \partial_\mu \lambda) dx^\mu\big)^2. \tag{5}$$

Clearly, if we want this to be invariant, we need an induced transformation on the gauge field,

(6) $$A_\mu \to A_\mu - \partial_\mu \lambda$$

So the familiar gauge transformations from electromagnetism arise here in $\mathbb{M}$ thanks to higher dimensional coordinate transformations. This shouldn't come as much of a surprise because our bundle structure can easily be interpreted as a $U(1)$ gauge theory on the base space $\mathbb{M}$. The transformations of the compactified coordinate given in (3) are local in the sense the they depend on where we are in the base space.

The Ricci scalar on the full spacetime is given as,

(7) $$\mathcal{R}^{(D)} = \mathcal{R} - 2G_{dd}^{-1/2}\nabla^2 G_{dd}^{1/2} - \frac{1}{4}G_{dd}F_{\mu\nu}F^{\mu\nu},$$

where $\mathcal{R}$ refers to the Ricci scalar on the non-compact subspace $\mathbb{M}$. We consider Einstein gravity on the full spacetime and show that it dimensionally reduces, in the following sense,

(8) $$S_G = \frac{1}{2\kappa_0^2}\int d^D x\sqrt{-G^{(D)}}\mathcal{R}^{(D)} = \frac{\pi R}{\kappa_0^2}\int d^d x\sqrt{-G}\sqrt{G_{dd}}\Big(\mathcal{R} - \frac{1}{4}G_{dd}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\sigma\partial^\mu\sigma\Big),$$

where we define σ by $G_{dd} = e^{2\sigma}$.

So we see that Einstein gravity in the full D dimensions, reduces to Einstein gravity in d dimensions coupled to a $U(1)$ gauge theory and a massless scalar field σ. In other words, the graviton field in D dimensions appears to a lower dimensional observer as a graviton field in d dimensions, a vector field A_μ and a scalar field σ.

In addition to the graviton, the full theory in D dimensions also contains a scalar dilaton Φ and an anti-symmetric tensor fields B_{MN}. The scalar dilaton in D dimensions is still a scalar field in d dimensions. The anti-symmetric tensor field B_{MN} decomposes into an anti-symmetric tensor field $B_{\mu\nu}$ on the subspace and another vector field $A'_\mu = B_{d\mu}$.

When the smoke clears, the low-energy quantum fields on the space $\mathbb{M}$ are a metric $G_{\mu\nu}$, an anti-symmetric tensor field $B_{\mu\nu}$, two $U(1)$ gauge fields A_μ and A'_μ and two scalar fields Φ and $G_{dd} = e^{2\sigma}$.

$$G^{(D)}_{MN} \longrightarrow \left(\begin{array}{c|c} G_{\mu\nu}+G_{dd}A_\mu A_\nu & \\ \hline G_{dd}A_\mu & G_{dd}\end{array}\right); \quad B^{(D)}_{MN} \longrightarrow \left(\begin{array}{c|c} B_{\mu\nu} & \\ \hline A'_\mu \equiv B_{d\mu} & 0\end{array}\right); \quad \Phi \to \Phi$$

FIGURE 1. The Dimensional Reduction of the theory on the full spacetime to a low-energy effective theory on the base space $\mathbb{M}$. Notice that we acquire two gauge vector fields.

1.2. **High-Energy Action.** We now consider physics at energies on the order of $1/R$ or equivalently, lengths on the order of R. No longer can we neglect motion in the compactified dimension; we must let our fields depend on the coordinate x^d. However, this dependence has to be very simple because any function on the circle can be decomposed into its Fourier modes. Let us focus on a scalar field,

$$\Phi(x^\mu, x^d) = \sum_{n=-\infty}^{\infty} \Phi_n(x^\mu) e^{inx^d/R}. \tag{9}$$

Notice that the momentum in the d direction is quantized: $p^d = n/R$ with $n \in \mathbb{Z}$. The kinetic action for such a scalar field is,

$$\int d^D x \partial_M \Phi \partial^M \Phi = 2\pi R \int d^d x \sum_{n=-\infty}^{\infty} \left(\partial_\mu \Phi_n \partial^\mu \Phi_{-n} + \frac{n^2}{R^2} |\Phi_n|^2 \right), \tag{10}$$

where we require that $\Phi_n^* = \Phi_{-n}$. The above formula is quite profound. We observe that a single scalar field on $\mathbb{M} \times \mathbb{S}^1$ gives rise to an infinite tower of scalar fields on $\mathbb{M}$, indexed by n, with mass,

$$M_n^2 = \frac{n^2}{R^2}. \tag{11}$$

If R is on the order of the Planck length, then all of these massive Kaluza-Klein modes will be astronomically heavy, except when $n = 0$. Therefore, if we're only interested in low-energy/long-distance physics, we can ignore the Kaluza-Klein modes.

2. Winding States of a String

Notice that what we've done so far, has not used string theory one bit. Rather, we've merely been studying quantum field theory on a compactified spacetime. We now want to study closed strings propagating on such a spacetime. There is an important new phenomena which arises that wasn't present when we were studying strings propagating on non-compact backgrounds. Namely, closed strings can "wind around" the new compact dimension.

Our string coordinate X^d is now allowed the more general boundary condition,

$$X^d(\sigma + 2\pi) = X^d(\sigma) + 2\pi R m. \tag{12}$$

It's very easy to get confused by a formula like this. The confusion arises from possibly failing to distinguish between the string coordinates X^μ and the spacetime coordinates x^μ. It is certainly true that $x^d = x^d + 2\pi R m$, however the string embedding coordinates don't have to obey the same formula!

The momentum in the compact dimension is quantized, as we would expect,

$$p^d = \frac{n}{R}, \qquad n \in \mathbb{Z}. \tag{13}$$

It's important to realize that this quantization of momenta would have been present in the field theory; the winding states are the truly new, stringy phenomena. Consider now just the mode expansion of the periodic string coordinate $X^d(\sigma, \tau)$. We will need to define the left and right moving momenta, respectively as,

$$p_L = \frac{n}{R} + \frac{mR}{\alpha'} \quad , \quad p_R = \frac{n}{R} - \frac{mR}{\alpha'}, \tag{14}$$

which then gives our mode expansion to be,

$$\begin{aligned} X^d_L(\sigma^+) &= \frac{1}{2}x^d + \frac{1}{2}\alpha' p_L \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^d_n e^{-in\sigma^+} \\ X^d_R(\sigma^-) &= \frac{1}{2}x^d + \frac{1}{2}\alpha' p_R \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha^d_n e^{-in\sigma^-}. \end{aligned} \tag{15}$$

Unlike the non-compact case, our level-matching formula now reads $N - \tilde{N} = nm$. Finally, the mass of the string state is given by,

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2). \tag{16}$$

The final term in our expression for M^2 is familiar from before. The first term tells us that a string with momentum $p^d = n/R$ in the compact dimension has a contribution of n^2/R^2 to its mass. Finally, the middle term tells us that a non-zero winding number also contributes to the mass. To motivate this, recall that the energy of a string is roughly the string tension T times the length. For a string with winding number m, the length is simply $2\pi Rm$ and the tension is related to α' through the formula, $T = 1/2\pi\alpha'$. Therefore, the energy of such a string is mR/α', which explains the second term in the formula for the mass.

2.1. **Aside: Brandenberger-Vafa Theory.** We allow for $m \in \mathbb{Z}$ such that, for example, $m = 2$ and $m = -2$ both describe a string wound twice around the cylinder, but with orientations in opposite directions. It is straightforward to see that scattering of two such strings can produce an unwound string with $m = 0$. This has striking consequences to cosmological phenomenology.

At this point, we should be asking the question, "why do we observe exactly three extended spatial dimensions?" One prevailing theory, due to Vafa and Brandenberger, is that right after the birth of the universe, spacetime consisted entirely of compactified dimensions. As time went on, eventually precisely three dimensions "unraveled" or un-compactified. Their simple and elegant argument went roughly as follows:

When the universe was a purely compact manifold, it was held as such thanks to a sea of strings with tension wrapped around it. As we saw above, these strings can unwind if strings with opposite winding numbers collide. Imagine two objects whose trajectories through spacetime have dimensions d and d', respectively. For example, a point-particle has $d = 1$ and a string has $d = 2$. If two such objects are moving in an ambient spacetime of dimension D, they will be statistically likely to collide if and only if $d + d' \geq D$. Since for two strings, $d = d' = 2$, $d + d' = 4$ and we see that strings with opposite winding numbers are likely to collide within some subspace that is four dimensional; three space and one time. Thus, according to Brandenberger and Vafa, it was essentially statistically inevitable for three spatial dimensions to unravel as we observe in our universe.

3. Massless States and Enhanced Gauge Symmetry

In looking at equation (16) we see that we can achieve massless states by taking $n = m = 0$ and $N = \tilde{N} = 1$. We see that there are four possible types of states,

$$\begin{aligned}
&\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}|0,p\rangle \\
&\left(\alpha^{d}_{-1}\alpha^{\mu}_{-1} + \alpha^{\mu}_{-1}\alpha^{d}_{-1}\right)|0,p\rangle \\
&\left(\alpha^{d}_{-1}\alpha^{\mu}_{-1} - \alpha^{\mu}_{-1}\alpha^{d}_{-1}\right)|0,p\rangle \\
&\alpha^{d}_{-1}\alpha^{d}_{-1}|0,p\rangle
\end{aligned} \tag{17}$$

There is a clear interpretation to each of these states. The first one decomposes into a symmetric tensor $G_{\mu\nu}$, an anti-symmetric tensor $B_{\mu\nu}$ and a scalar dilaton field Φ. The second term can be associated with the gauge vector field A_μ and the third can be identified with the gauge vector field $A'_\mu = B_{d\mu}$ coming from the anti-symmetric tensor. Finally, the last term can be associated to the scalar field σ defined by $G_{dd} = e^{2\sigma}$. So we see that we've accounted for all of the massless states that we depict in Figure 1.

We've found two massless vector particles and they're both charged under A_μ and A'_μ. Therefore, these two particles fill out a representation of U(1)×U(1).

At the so-called "self-dual radius" $R = \sqrt{\alpha'}$ (named as such for reasons we'll see shortly) we have even more massless states. In addition to all the states that are already present, we also have:

- There are scalar particles with $N = \tilde{N} = 0, m = 0, n = \pm 2$ and $N = \tilde{N} = 0, m = \pm 2, n = 0$. In both cases, these states are charged under the U(1) × U(1) gauge symmetry.

  Of more interest to us, are four new vector particles which arise only at this special radius:

- With $N = 1, \tilde{N} = 0$ we can have $n = m = \pm 1$. Likewise, with $N = 0, \tilde{N} = 1$ we can have $n = -m = \pm 1$. Accounting for the $\pm$ this gives us four new vector particles.

So at the self-dual radius, we have a total of 6 massless, charged vector particles. Recall that su(2)×su(2)=so(4). Furthermore, recall that so(4) is a six-dimensional Lie Algebra. Our six massless vectors fill out the adjoint representation of su(2)×su(2). We therefore, have an "enhanced gauge symmetry" at this special radius,

$$U(1) \times U(1) \longrightarrow SU(2) \times SU(2) \tag{18}$$

This is, of course, a non-Abelian gauge theory which is to be expected because we cannot describe massless, *charged* vector particles with an abelian gauge theory.

4. Toroidal Compactification

We now generalize to the case of k periodic dimensions, given by,

$$x^m = x^m + 2\pi R \quad , \quad D - k \leq m \leq D - 1, \tag{19}$$

where $d \equiv D - k$ is the number of non-compact dimensions. Given an arbitrary non-compact (pseudo-) Riemann manifold $\mathbb{M}$ of dimension d, the full spacetime is simply the trivial product bundle $\mathbb{M} \times \mathbb{T}^k$.

Recall that for $k = 1$, we had a topologically distinct string state for each element of $\mathbb{Z}$. It is not an accident that the fundamental group of the circle is $\pi_1(\mathbb{S}^1) = \mathbb{Z}$. We can think of each topologically distinct string as forming a representation of the fundamental group.

For $k = 2$, since $\mathbb{T}^2$ now consists of two copies of the circle, there are two linearly independent ways to wind once around the torus; one for each of the two circles. In other words, a string on the torus is determined by two integers (m, n) such that we wind m times around one of the circles and n times around the other. Thus, topologically, our closed strings form a representation of the fundamental group of the torus $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$.

5. T-Duality of Closed Strings

There is a startling consequence of the mass spectrum (16). Notice the spectrum is invariant under the simultaneous interchanges,

$$R \leftrightarrow \frac{\alpha'}{R} \quad \text{and} \quad n \leftrightarrow m. \tag{20}$$

This is known as T-duality. We have two completely distinct spacetimes, one with a small radius and another with a large radius. By interchanging winding states with ordinary momentum states, string theory has hidden from us which spacetime we're actually living in. Tong describes this phenomena by saying that the strings are "confused" about what spacetime they're propagating in.

If we were to incorporate supersymmetry, we would conclude that the a Type IIA string theory on a circle of radius R is identical to a Type IIB string theory on a circle of radius α'/R. Likewise, an $\mathbb{E}_8 \times \mathbb{E}_8$ string theory on a circle of radius R is identical to a SO(32) string theory on a circle of radius α'/R.

Note that we haven't proven T-duality here, although it is not too hard to do so. A duality, by definition, is a mapping between two theories acting on the operators within that theory. This mapping must leave all expectation values invariant. If such a mapping exists, then there is no experiment one can carry out to determine which of the two theories is true. Thus, a duality gives two distinct mathematical formulations of the same physical theory.

Recall that α' acts as sort of a "pixel" of string theory. That is to say, $\sqrt{\alpha'}$ is the smallest conceivable distance in the theory. Here we have further evidence of this: for what value of R will we have $R = \alpha'/R$? Clearly, $R = \sqrt{\alpha'}$, this is the self-dual radius introduced above. Therefore, thanks to T-duality, as a string begins to shrink below the impossible threshold of $\sqrt{\alpha'}$, we can think of it as actually beginning to *grow* such that winding states and momentum states are interchanged.

This gives one possible interpretation of T-duality. We give another below.

6. Dualities and the Finiteness of String Theory

Quantum Field Theory is a bit of a tease of a theory; it models spacetime as a smooth manifold, but when you ask the natural question of what's happening within tiny neighborhoods on the manifold, it blows up in your face and yields an infinite result. String Theory, we're told, is finite. As we noted above, we shouldn't be asking the question of what spacetime looks like below length scales of $\sqrt{\alpha'}$. The entire concept is fuzzy when we probe this small a length scale. Therefore, when we postulate a string theory on a cylinder with radius smaller than $\sqrt{\alpha'}$ we might expect the theory to explode. Rather, string theory politely places us at a larger radius thanks to T-duality! In this sense, we can think of T-duality as imposing the finiteness of string theory and forbidding us from asking the questions which we ought not.

This same principle shows up elsewhere in string theory with respect to other dualities. We really need to have SUSY under our belts to discuss S-duality and mirror symmetry proper. But perhaps we'll mention them briefly just to make a point.

Before Witten's talk at Strings 1995 at the University of Southern California, physicists thought that there were five distinct string theories. It was presumed that one of them was the correct one, and the others were merely mathematical curiosities. Then Witten ignited the so-called the "second string revolution" or the "duality revolution." Each of the theories has its own coupling constant. Of course, at high coupling, the theories would yield infinite answers. Luckily, thanks to S-dualities, each of the five string theories at strong coupling is dual to another string theory at low coupling. We refer to these informally as "strong-weak dualities" with $g \leftrightarrow 1/g$. Therefore, just as we saw above, S-duality comes to the rescue and enforces the finiteness of string theory! When embarking on what would be a divergent calculation at strong coupling, we can use the dualities to compute the same observable at weak coupling and arrive at a sensible, finite answer. These dualities represent the birth of M-theory.

So in a sense, we can think of dualities as enforcing the finiteness of string theory. Earlier, we described the finiteness of string theory as owing to the fact that strings as one-dimensional objects "spread out" the divergences that plague point-particle theories. What we see here is closely related: thanks to their one-dimensional nature strings get "confused" as to what manifold they're moving on. In T-duality, they can't tell whether they're on small circles or large circles. In superstring theory, they can't tell whether they're on a given Calabi-Yau three-fold, or its mirror pair. This is the essence of mirror symmetry.

It should be noted, T-duality is not a conjecture but rather a theorem. In other words, we can exhibit an explicit isomorphism taking a physical theory at radius R to a distinct physical theory at radius α'/R leaving observables invariant. On the other hand, the above S-dualities are merely a conjecture that nearly every string theorist believes are true.

We mention in passing that it appears a proof of the S-duality might require number theory. After watching a talk on a number theoretic derivation of the Polyakov measure of string theory, Witten bought all the books he could find on number theory. It also appears that Michael Atiyah warned Witten in the 1970's that number theory would soon become important in physics. Sure enough, today many brilliant minds in both math and physics are interested in the Geometric Langlands program which appears to be related to S-dualities. Physics looks like it may be tending in the direction of number theory, specifically p-adic and adelic analysis, class field theory, and Galois representation theory.