Notes on the Polyakov Measure, Moduli Space and String Amplitudes

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Abstract

We give a pedagogical review of the moduli space measure in string theory and outline some aspects of string amplitudes. These notes are prepared for a talk at UBC in Summer 2015 as a part of an ongoing string theory reading group. The exposition and choice of topics is heavily inspired by D'Hoker's 1999 IAS lectures [1]. This is a work in progress.

Contents

1 Motivation & Prerequisites

1.1 Motivation

The Polyakov path integral is an integral over moduli space. The first five chapters of Polchinski's first volume [3] culminates to a measure for moduli space, known as the Polyakov Measure

$$ \tag{1.1} $$

Rewriting the determinants as ghost fields, this looks like

$$ \tag{1.2} $$

The goal of the first half of these notes is to gain as much geometrical intuition as possible about where these equations come from. The second half will be concerned with computing amplitudes.

1.2 Differential Operators on Riemann Surfaces

1.2.1 Basics

Let's recall some basic notions from differential geometry on Riemann surfaces.

1.2.1.1 Riemann Surfaces

For our purposes, a Riemann surface Σ is a 1-dimensional complex manifold. We will be particularly interested in examples where Σ is compact and connected, the topology of such Σ is completely fixed by a genus g.

In string theory we deal with Riemann surfaces equipped with metrics, i.e. Hermitian manifolds (Σ, g). Being equivalent to a real Riemannian 2-manifold, (Σ, g) is conformally flat. That is, in each coordinate chart $U \subset \Sigma$, we have a diffeomophism:

$$\phi_U : U \to \mathbb{C}, \quad p \mapsto (z, \bar{z}) \tag{1.3}$$

such that

$$g = k(z, \bar{z}) dz \otimes d\bar{z} \tag{1.4}$$

1.2.1.2 The Metric Isomorphism

As usual, g gives an isomorphism $\tilde{g}$ between tangent and cotangent spaces at each point $p \in \Sigma$

$$\tilde{g} : T_p\Sigma \to T_p^*\Sigma \ , \ v \mapsto g(v, \cdot) \tag{1.5}$$

As usual, it is convenient to view the inverse g^{-1} of the g isomorphism as a rank $(2, 0)$ tensor defined in each coordinate chart by taking the matrix inverse of the components of g. In the case of Riemann surfaces, we have simply

$$g^{-1} = k^{-1}(z,\bar{z})\partial_z \otimes \partial_{\bar{z}} \tag{1.6}$$

These in turn give isomorphisms between tensors of different ranks

$$T_{[a,b]} \xrightarrow{g} T_{[a-1,b+1]} \xrightarrow{g} ... \xrightarrow{g} T_{[0,a+b]} \tag{1.7}$$

$$T_{[a,b]} \xrightarrow{g^{-1}} T_{[a+1,b-1]} \xrightarrow{g^{-1}} ... \xrightarrow{g^{-1}} T_{[a+b,0]} \tag{1.8}$$

where $T_{[a,b]} \equiv (\otimes^a T\Sigma)(\otimes^b T^*\Sigma)$ is the space of rank (a,b) tensors. These sequences of isomorphisms tell us we can restrict our attention to pure tensors without losing much generality.

1.2.2 Line Bundles of Fixed-Weight Tensors

The most general rank $(0,w)$-tensor T over Σ lives in the tensor product cotangent bundle $\otimes^w T^*\Sigma$:

$$T = T\mu_1...\mu_w d\omega^{\mu_1} \otimes ... \otimes d\omega^{\mu_w} \tag{1.9}$$

with $\mu_i \in \{0,1\}$, $\omega^\mu \in \{z,\bar{z}\}$. This can always be written as

$$T = \tilde{T}(\otimes^m dz)(\otimes^n d\bar{z}) \tag{1.10}$$

For some $\tilde{T} \in \mathbb{C}$. and $m,n \in \mathbb{Z}^+$ with $m+n=w$. We say T is a (contravariant) tensor of weight (m,n), and conformal dimension w.

We call the collection of these tensors $K_0^{(mn)}$

$$K_0^{(m,n)} \equiv \{T \in \otimes^{m+n} T^*\Sigma \mid T \text{ has weight (m, n) } \} \tag{1.11}$$

From 1.10 one can see that $K^{(m,n)}$ is a 1-dimesisonal vector space, hence K defines a line bundle over Σ.

One can make a trivial remark at this point: each space $K^{(m,n)}$ with different values of (m,n) are isomorphic - namely they're all isomorphic to $\mathbb{C}$. One can then wonder how do these space differ. In our case, the only difference we will care about is the factor they gain under a pullback under a diffeomorphism. To this end, let's consider for a moment mixed tensors of the form

$$T = \tilde{T}(\otimes^m dz)(\otimes^n \partial\bar{z})(\otimes^a \partial_z)(\otimes^b \partial_{\bar{z}}) \tag{1.12}$$

Observe that under a pullback by a diffeomorphism transform exactly like the tensor

$$\tilde{T}(\otimes^{(m-a)} dz)(\otimes^{(n-b)} d\bar{z}) \tag{1.13}$$

Hence for our purposes, we will not distinguish between those two tensors. Let's give a name to this collection

Definition 1.1.

$$K^{(m,n)} \equiv \{\tilde{T}(\otimes^{m-a} dz)(\otimes^{n-b}\partial\bar{z})(\otimes^a \partial_z)(\otimes^b \partial_{\bar{z}}) | a,b \in \mathbb{Z}^+, \tilde{T} \in \mathbb{C}\}/\sim$$

where $\sim$ is the equivalence relation $A \sim B$ if they have the same scalar factor.

The quotient by the identification of course makes sure that $K^{(m,n)}$ is isomorphic to $K_0^{(m,n)}$.

Under the metric isomorphism discussed earlier, using the explicit form of the metric 1.6 (namely the fact that the diagonals vanish), we see that the the action of the metric is:

$$\tilde{T}(\otimes^m dz)(\otimes^n d\bar{z}) \mapsto \tilde{T}'(\otimes^m dz)(\otimes^{n-1} d\bar{z}) \otimes \partial_z \tag{1.14}$$

The metric g therefore gives isomorphisms

$$K^{(m,n)} \xrightarrow{g} K^{(m-1,n-1)} \xrightarrow{g} ... \xrightarrow{g} K^{(m-n,0)} \tag{1.15}$$

In particular, $K^{(m,n)} \simeq K^{(m-n,0)}$. This tells us it is often sufficient to only care about tensors that only have z-indices. We therefore introduce the abbreviated notation:

$$K^m \equiv K^{(m,0)} \tag{1.16}$$

Note that in this notation we allow m to be negative - a tensor of weight $-n$ for some $n > 0$ is defined to be one of weight $(0, n)$.

Being a complex line bundle over Σ, K^m is a complex manifold in its own right. Furthermore we will equip it with an inner product $\langle \cdot, \cdot \rangle$:

$$\langle \phi, \psi \rangle \equiv \int_\Sigma du \ g_{z\bar{z}}^{-m} \ \phi\psi \tag{1.17}$$

where du is the invariant measure on (Σ, g).

1.2.3 Differential Operators

For readability, let's slightly abuse our notation and use $K^{(m,n)}$ the space of sections over the line bundle which we previously called $K^{(m,n)}$.

Let's consider a metric connection on Σ. The condition of metric compatibility reads:

$$\nabla_X g = 0 \ \ \forall X \text{ section to } T\Sigma \tag{1.18}$$

In components, this translates to the demand that all components of $\nabla_a g_{bc}$ vanish. Using eq. 1.4, this gives us all the Christoffel symbols. In the below we only state the results needed for our purposes, for the derivations, see for example [2].

The covariant derivative can be decomposed as:

$$\nabla \equiv \nabla_{\bar{z}} + \nabla_z \tag{1.19}$$

with

$$\nabla_{\bar{z}} : K^{(m,n)} \to K^{(m,n+1)}, \quad \phi \mapsto \partial_{\bar{z}}\phi \otimes d\bar{z} \tag{1.20}$$

$$\nabla_z : K^{(m,n)} \to K^{(m+1,n)}, \quad \phi \mapsto (k^{-m}\partial_z k^m)\phi \otimes dz \tag{1.21}$$

As mentioned, we would like to restrict our attention to pure z-index tensors, so that we can look at spaces K^{a-b} as opposed to $K^{(a,b)}$. Setting $n = 0$ in the domain of the operators above.

Observe the latter of these two operators is already one which takes pure-z tensor to another pure-z one. To obtain from $\nabla_{\bar{z}}$ another operator which maps strictly to pure z-tensors, we can subsequently use the inverse metric to map $K^{(m,1)} \to K^{m-1}$. That is, we define map:

$$\nabla^z \equiv g^{-1} \circ \nabla_{\bar{z}} \tag{1.22}$$

observe this acts as:

$$\nabla^z : K^{(m,n)} \to K^{(m-1,n)}, \phi \mapsto k^{-1} \partial_{\bar{z}} \phi \otimes \partial_z \tag{1.23}$$

From now on let's fix $n = 0$ in the domain of these maps. For clarity of the below definitions we also add a superscript or subscript in brackets to denote the z-weight m of the domain of these maps, e.g. $\nabla^z_{(m)}$, $\nabla^{(m)}_z$. Note that ∇_z and ∇^z are related by adjoints with respect to the metric inner product:

$$\nabla_z^\dagger = \nabla^z \tag{1.24}$$

We can define Lapacians as compositions of these:

$$\triangle^+_m \equiv \nabla^{(m)}_z \nabla^z_{(m+1)} \ , \ K^{m+1} \to K^{m+1} \tag{1.25}$$

$$\triangle^-_m \equiv \nabla^z_{(m)} \nabla^{(m-1)}_z \ , \ K^{m-1} \to K^{m-1} \tag{1.26}$$

All this can be summarized by a commutative diagram:

$$\begin{tikzcd}
 & K^{m-1} & \\
K^{m-1} \arrow[ur, "\triangle^-"] \arrow[r, "\nabla_z"] & K^m \arrow[u, "\nabla^z"'] \arrow[r, "\nabla_z"] & K^{m+1} \\
 & K^{m+1} \arrow[u, "\nabla^z"] \arrow[ur, "\triangle^+"] &
\end{tikzcd} \tag{1.27}$$

1.3 Gaussian Measures

The only result we need about measures is the fact that we can compute a measure by doing a Gaussian integral on the tangent space. Let's review how this works.

1.3.1 Jacobian on Tangent Space

Let M be a differentiable manifold, (U, x), (V, ω) be two different charts with $U \cap V \neq \emptyset$. In the overlap $U \cap V$, the measures defined by the two charts are related by a Jacobian factor

$$d^n x = J d^n \omega \tag{1.28}$$

with

$$J = det\left(\frac{\partial x^\mu}{\partial \omega^\nu}\right) \tag{1.29}$$

A crucial observation will be that J can be computed in the tangent space T_pM for some $p \in U \cap V$. This can be seen as follows. x and ω define bases $\frac{\partial}{\partial x^\mu}$ and $\frac{\partial}{\partial \omega^\mu}$. Any $V \in T_pM$ can then be written in coordinates

$$V = v^\mu \frac{\partial}{\partial x^\mu} = u^\nu \frac{\partial}{\partial \omega^\nu} \tag{1.30}$$

From this we can read off

$$v^\mu = u^\nu \frac{\partial x^\mu}{\partial \omega^\nu} \tag{1.31}$$

The Jacobian associated with a coordinate change $\frac{\partial}{\partial x^\mu} \mapsto \frac{\partial}{\partial \omega^\mu}$ on T_pM is:

$$J_T = det\left(\frac{\partial v^\mu}{\partial u^\nu}\right) = det\left(\frac{\partial x^\mu}{\partial \omega^\nu}\right) = J \tag{1.32}$$

1.3.2 Gaussian Measure

Now supposed that in the u^μ coordinates we had

$$\int d^n u \; e^{-\frac{1}{2}|u|^2} = 1 \tag{1.33}$$

We know that

$$d^n u = J d^n v \tag{1.34}$$

Subbing into previous line

$$\int d^n u \; e^{-\frac{1}{2}|u|^2} = \int J d^n v \; e^{-\frac{1}{2}|v|^2} = 1 \tag{1.35}$$

By the previous section, J is a function of (x, ω) and not of (u, v), we can pull it out. We then have

$$\int d^n v \; e^{-\frac{1}{2}|v|^2} = J^{-1} \tag{1.36}$$

This is the same J that appears in a coordinate change of the base space M

$$d^n x = J d^n \omega \tag{1.37}$$

Thus we have just computed the Jacobian for a coordinate change on M by doing a Gaussian integral in T_pM.

We can phrase all this slightly differently as follows. We start by defining the measure $\mathcal{D}v$ implicitly by the equation

$$\int \mathcal{D}v \; e^{-\frac{1}{2}|v|^2} = 1 \tag{1.38}$$

Then we say

$$\mathcal{D}v = Jd^n v \tag{1.39}$$

We then read off J by doing the gaussian integral, then use J to obtain our desired measure on the base space.

Finally, we should also remember how to do Gaussian integrals. Here's the formula

$$\int d^n x \; e^{x^T Ax+Sx} = det(A)^{-\frac{1}{2}} e^{-\frac{1}{2}SAS^{-1}} \tag{1.40}$$

2 Moduli Space

2.1 Problem Statement & Strategy

Let Σ be a Riemann surface. Let $Met(\Sigma)$ denote the space of all metrics on Σ. Define the moduli space as the quotient

$$Mod(\Sigma) \equiv Met(\Sigma)/(Diff(\Sigma) \ltimes Weyl(\Sigma)) \tag{2.1}$$

where $\ltimes$ denotes a semidirect product. Viewing this quotient space as a local product $Met(\Sigma) \xrightarrow{\pi} Mod(\Sigma)$, then locally, $Met(\Sigma)$ can be decomposed as

$$Met(\Sigma) \simeq Mod(\Sigma) \oplus (Diff(\Sigma) \ltimes Weyl(\Sigma)) \tag{2.2}$$

in particular a metric $g \in Met(\Sigma)$ can be written as

$$g \sim (m, \sigma, v) \tag{2.3}$$

for $m \in Mod(\Sigma)$, and $\sigma \in Weyl(\Sigma)$, $v \in Diff(\Sigma)$.

Observe now on $Met(\Sigma)$ we can define a natural gaussian measure $\mathcal{D}g$ with respect to any tangent space $T_g Met(\Sigma)$. Our goal is then to find a integration measure on $Mod(\Sigma)$ by "projecting" the $Met(\Sigma)$ measure $\mathcal{D}g$. More concretely, we want to decompose $\mathcal{D}g$ as

$$\mathcal{D}g = J\mathcal{D}m\mathcal{D}\sigma\mathcal{D}v \tag{2.4}$$

for some Jacobian J. We can subsequently perform the integration over the gauge orbits to obtain the measure on moduli space. By our discussion on Gaussian measures, one can compute the Jacobian J by evaluating gaussian integrals with respect to our metric inner product on the tangent space. Our next step therefore is to decompose the tangent space $T_g Met(\Sigma)$.

2.2 $Met(\Sigma)$ Tangent Space Decomposition

Let $g \in Met(\Sigma)$. Physically, one can think of the tangent space $T_g Met(\Sigma)$ as variations of g

$$\delta g \equiv \varepsilon\omega \; , \; \omega \in T_g Met(\Sigma) \tag{2.5}$$

$T_g Met(\Sigma)$ is a space of rank $(0,2)$ tensors, which can be in turn decomposed by weights. Using notation introduced earlier

$$T_g Met(\Sigma) \simeq T_{[0,2]} \simeq T^{(1,1)} \oplus T^{(0,2)} \oplus T^{(2,0)} \tag{2.6}$$

We examine each space on the RHS one by one.

2.2.1 $T^{(1,1)}$

The most general element in $T^{(1,1)}$ can be written as

$$\omega = \varphi dz \otimes d\bar{z} \tag{2.7}$$

with $\varphi \in \mathbb{C}$. Now consider the metric perturbation

$$g \mapsto g + \delta g \tag{2.8}$$

In isothermal coordinates and notation from before

$$kdz \otimes d\bar{z} \mapsto kdz \otimes d\bar{z} + \delta g \tag{2.9}$$

Now consider a perturbation $\delta g = \epsilon\omega$ with $\omega \in T^{(1,1)}$, the change is

$$kdz \otimes d\bar{z} \mapsto kdz \otimes d\bar{z} + \varepsilon\varphi dz \otimes d\bar{z} \tag{2.10}$$

This is simply a scaling of the metric. In other words, this can be written as

$$g \mapsto f_\varepsilon^*(g) \tag{2.11}$$

for some map $f_\varepsilon \in Weyl(\Sigma)$. What we have just shown is the following: every $\omega \in T^{(1,1)}$ is a fundamental tensor field generated by some element in the Lie algebra of $Weyl(\Sigma)$. Conversely, every element in the Lie algebra of $Weyl(\Sigma)$ generates a vector field which evaluates to some $\omega \in T^{(1,1)}$.

To summarize this colloquially: $T^{(1,1)}$ is covered by the infinitesmall action of the Weyl group.

2.2.2 $T^{(2,0)}$ and $T^{(0,2)}$

Inspired by how we were able to identify $T^{(1,1)}$ with pullbacks of Weyl maps on the metric, let's consider the action of a diffeomorphism on the metric. Any diffeomorphism connected to the identity $f \in Diff_0(\Sigma)$ can be obtained as the flow generated by a vector field V in $T\Sigma$. The infinitesmall action of f_ϵ on the metric can be read off from the definition of the Lie derivative

$$\delta g = f_\epsilon^*(g) - g = \epsilon \mathcal{L}_V g \tag{2.12}$$

In some local coordinate chart, the Lie derivative of the metric can be nicely expressed in terms of the covariant derivative:

$$\delta g_{ab} = (\mathcal{L}_V g)_{ab} = \nabla_a V_b + \nabla_b V_a \tag{2.13}$$

where the vector indices are lowered by the metric isomorphism. By the discussion in the previous section, we know the off diagonal components correspond to Weyl scalings. Let's therefore restrict ourselves to considering "diagonal variations" $\delta_d g$ where we set the off diagonal entries of the variation to 0, the result is then

$$\frac{1}{2}\delta_d g = \nabla_z V_z dz \otimes dz + \nabla_{\bar{z}} V_{\bar{z}} d\bar{z} \otimes d\bar{z} \tag{2.14}$$

This expression tells us

$$\delta_d g \in Im(\nabla_z^{(1)}) \oplus Im(\nabla_{\bar{z}}^{(1)}) \subset T^{(2,0)} \oplus T^{(0,2)} \tag{2.15}$$

We now show that $Im(\nabla_z^{(1)}) \oplus Im(\nabla_{\bar{z}}^{(1)})$ is a strict subset of $T^{(2,0)} \oplus T^{(0,2)}$, and give explicitly the orthogonal complement of the former in the latter. To appreciate this result that we're going to prove shortly, let's take a step back and look at the geometric picture of all this.

For readability let's denote

$$D \equiv Im(\nabla_z^{(1)}) \oplus Im(\nabla_{\bar{z}}^{(1)}) \tag{2.16}$$

$$\bar{D} \equiv \text{orthogonal complement of } D \text{ in } T^{(2,0)} \oplus T^{(0,2)} \tag{2.17}$$

Now:

- We just saw D is generated by pullbacks of diffeomorphisms on Σ. In other words D is tangent to the gauge orbit.
- The orthogonal complement $\bar{D}$ is therefore tangent to the gauge slice.
- We saw in the previous section that all of $T^{(1,1)}$ is generated by Weyl maps, therefore all of $T^{(1,1)}$ is tangent to the gauge orbit.
- This leaves $\bar{D}$ as exactly the subspace of $T_g Met(\Sigma)$ that is tangent to the gauge orbit.

Our task therefore has been reduced to finding $\bar{D}$. Since $T^{(2,0)}$ and $T^{(0,2)}$ are related by complex conjugation, it is sufficient to consider $T^{(2,0)}$. The crucial result is the following

Proposition 2.1. $T^{(2,0)} \simeq Im(\nabla_z^{(1)}) \oplus Ker(\nabla_{\bar{z}}^{(2)})$

Proof. We compute the orthogonal complement of $Im(\nabla_z^{(1)})$. Let $\nabla_z^V \in Im(\nabla_z^{(1)})$. An element ϕ in the orthogonal complement satisfies

$$\langle \nabla_z V, \phi \rangle = \langle V, \nabla_z^\dagger \phi \rangle = \langle V, \nabla_{\bar{z}}^{(2)} \phi \rangle \tag{2.18}$$

The result follows. $\square$

Let's sumarize again what we just found:

Proposition 2.2. $T_g Met(\Sigma) = [T^{(1,1)} \oplus Im(\nabla_z^{(1)})] \oplus Ker(\nabla_{\bar{z}}^{(2)}) \oplus cc.$

As argued, the subspace in the square brackets (and its complex conjugate counterparts) are tangent to the gauge orbit, while the space of holomorphic differentials $Ker(\nabla_{\bar{z}}^{(2)}) \oplus Ker(\nabla_{z}^{(2)})$ are directions along the gauge slice.

One more subtlety: in Riemann surfaces with genus < 2, there exists diffeomorphism-generating vectors $V \in Ker\nabla_z^{(1)}$. These are conformal Killing vectors. By our discussion we see that the actions of these do not contribute to the diagonal components of δg.

2.3 The $Mod(\Sigma)$ Measure

2.3.1 The Setup

Armed with the decomposition derived above, we can write the most general $\omega \in T_g Met(g)$ as

$$\omega = \varphi dz \otimes d\bar{z} + (\nabla_z V + \rho) dz \otimes dz + (\nabla_{\bar{z}} V + \bar{\rho}) d\bar{z} \otimes d\bar{z} \tag{2.19}$$

We define a measure $\mathcal{D}\omega$ by

$$\int_{T_g Met(g)} \mathcal{D}\omega e^{\frac{1}{2}|\omega|^2} = 1 \tag{2.20}$$

The norm that appears in the exponential is evaluated using our metric inner product. Using the decomposition this reads

$$|\omega|^2 = |\varphi|^2 + (|\nabla_z V|^2 + |\rho|^2 + cc.) \tag{2.21}$$

where we used the orthogonality of ρ and $\nabla_z V$.

The decomposition used here is orthogonal, this implies we can write

$$\mathcal{D}\omega = \mathcal{D}\phi \; D(\nabla_z V) \; D(\nabla_{\bar{z}} V) \; \mathcal{D}\rho \mathcal{D}\bar{\rho} \tag{2.22}$$

with each term in the product a measure defined by a Gaussian:

$$\int_{T_g Met(g)} \mathcal{D}\phi \; e^{\frac{1}{2}|\phi|^2} = 1 \tag{2.23}$$

$$\int_{T_g Met(g)} \mathcal{D}(\nabla_z V) \; e^{\frac{1}{2}|(\nabla_z V)|^2} = 1 \tag{2.24}$$

$$\int_{T_g Met(g)} \mathcal{D}\rho \; e^{\frac{1}{2}|\rho|^2} = 1 \tag{2.25}$$

where the complex conjugate counterparts are similarly defined. The Jacobian J_ω from $\mathcal{D}\omega$ is then a product of the Jacobians from these measures:

$$J_\omega = J_\phi |J_{\nabla_z V}|^2 |J_\rho|^2 \tag{2.26}$$

2.3.2 The Gauge Orbit Measures

Let's evaluate these measures one by one. First one is easy:

$$\int_{T_g Met(g)} \mathcal{D}\phi e^{\frac{1}{2}|\phi|^2} = 1 \tag{2.27}$$

$$\implies J_\phi = 1 \tag{2.28}$$

Now onto the $D(\nabla_z V)$ part. We observe

$$|\nabla_z V|^2 = \langle \nabla_z V, \nabla_z V \rangle = \langle V, \triangle^- V \rangle \tag{2.29}$$

This form makes the Gaussian integral easy to compute

$$J^{-1}_{\nabla_z V} = \int_{T_g Met(g)} \mathcal{D}(\nabla_z V)\ e^{\frac{1}{2}|(\nabla_z V)|^2} \tag{2.30}$$

$$= \int_{T_g Met(g)} \mathcal{D}(\nabla_z V)\ e^{\frac{1}{2}\langle V, \triangle^- V \rangle} \tag{2.31}$$

$$= det(\triangle^-)^{-\frac{1}{2}} \tag{2.32}$$

Multiplying by the obviously analogous result for the complex conjugate counterpart, we see

$$|J_{\nabla_z V}|^2 = det(\triangle^-) \tag{2.33}$$

2.3.3 The Gauge Slice Measure

Note that so far, where the components of ω tangent to the gauge orbit have been explicit, the component ρ tangent to the gauge orbit have been symbolic - namely ρ is an element in the complex vector space $ker \nabla^{(2)}_{\bar{z}}$. To find the ρ measure, we must choose a basis for $ker \nabla^{(2)}_{\bar{z}}$, then integrate the components of ρ in this basis. Additionally, we would like this basis to be Weyl invariant. Elements $(\mu_j)_{1 \leq j \leq dim(Mod(\Sigma))}$ in a Weyl-invariant basis for $ker \nabla^{(2)}_{\bar{z}}$ are called *Beltrami Differentials*.

In terms of the Beltrami Differentials, we can write ρ and $\bar{\rho}$ as

$$\rho = \rho^j \mu_j \tag{2.34}$$

$$\bar{\rho} = \bar{\rho}^j \bar{\mu}_j \tag{2.35}$$

$$\tag{2.36}$$

Since the Beltrami differentials are in general not an orthogonal basis, let's write the above in an orthogonal basis. Let $(\phi_j)_{1 \leq j \leq dim(Mod(\Sigma))}$ be an orthogonal basis for $ker \nabla^{(2)}_{\bar{z}}$. Freshman linear algebra tells us how to write ρ in this basis

$$\rho = \sum_j \frac{1}{|\phi_j|} \rho^k \langle \phi_j, \mu_k \rangle \phi_j \tag{2.37}$$

Taking a inner product of this with itself, we get

$$|\rho|^2 = \sum_j \rho^l \rho^k \frac{1}{|\phi_j|^2} \langle \phi_j, \mu_k \rangle \langle \phi_j, \mu_l \rangle \tag{2.38}$$

We can now compute the Gaussian

$$J_\rho^{-1} = \int e^{\frac{1}{2}|\rho|^2} \tag{2.39}$$

$$= e^{\rho^l \sum_j \frac{1}{|\phi_j|^2} \langle \phi_j, \mu_k \rangle \langle \phi_j, \mu_l \rangle \rho^k} \tag{2.40}$$

$$= det\left(\sum_j \frac{1}{|\phi_j|^2} \langle \phi_j, \mu_k \rangle \langle \phi_j, \mu_l \rangle \right)^{-\frac{1}{2}} \tag{2.41}$$

$$= \left(\frac{det(\langle \phi_j, \mu_k \rangle)^2}{det(\langle \phi_j, \phi_k \rangle)} \right)^{-\frac{1}{2}} \tag{2.42}$$

Multiplying with the complex conjugate result, we find

$$|J_\rho|^2 = \frac{|det(\langle \phi_j, \mu_k \rangle)|^2}{det(\langle \phi_j, \phi_k \rangle)} \tag{2.43}$$

2.3.4 Putting it together

Combining our Jacobians, we find

$$J_\omega = J_\phi |J_{\nabla_z V}|^2 |J_\rho|^2 = det(\triangle^-) \frac{|det(\langle \phi_j, \mu_k \rangle)|^2}{det(\langle \phi_j, \phi_k \rangle)} \tag{2.44}$$

Putting this Jacobian in the $Met(g)$ metric decomposition , we get

$$\mathcal{D}g = J \mathcal{D}m \mathcal{D}\sigma \mathcal{D}v = det(\triangle^-) \frac{|det(\langle \phi_j, \mu_k \rangle)|^2}{det(\langle \phi_j, \phi_k \rangle)} \mathcal{D}m \mathcal{D}\sigma \mathcal{D}v \tag{2.45}$$

We obtained a measure for $Met(g)$ decomposed exactly like we proposed at the beginning. In fact we can be even more explicit. Let $(m_j, \bar{m}_j)$ be a basis for $Mod(\Sigma)$, then we can write:

$$\mathcal{D}m = \prod_j dm_j d\bar{m}_j \tag{2.46}$$

We now make one final modification which will seem un-motivated at this point. The motivation will only become clear in a discussion of the Weyl transformation property of the measure in addition to an application of the Fadeev-Poppov procedure. For now we only perform the following re-writing for the sake of completeness.

Let ψ_a be an orthogonal basis for $Ker \nabla_{\bar{z}(1)}$, define:

$$Z^-_{(-1)}(g) \equiv \frac{det \triangle^-}{det\langle \phi_j, \phi_k \rangle det \langle \psi_a, \psi_b \rangle} \tag{2.47}$$

$Z^-_{(-1)}$ will turn out to be the Fadeev-Poppov determinant. We also rescale our $Diff$ orbit measure to cancel out the ψ factor we added

$$D'v = det\langle\psi_a, \psi_b\rangle Dv \tag{2.48}$$

Then our measure is rewritten as

$$\mathcal{D}g = \mathcal{D}\sigma\mathcal{D}'vZ^{-}_{(-1)}|det\langle\phi_j, \mu_k\rangle|^2 \prod_j dm_j d\bar{m}_j \tag{2.49}$$

This is what we will call the Polyakov measure.

3 String Amplitudes

3.1 Tree Level

3.2 Witten's Geometric Intepretation of $i\epsilon$

Here we **briefly** describe a geometric interpretation of the $i\epsilon$ prescription in String theory due to Witten [4].

3.3 Beyond Tree Level

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