Relations

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Reference

- https://math.libretexts.org/Bookshelves/Combinatorics_and_Discrete_Mathematics

Motivation

- Connection between relations, graphs, matrix ...
- Preliminary to studying knowledge graphs
- Where does the term "transitive closures" taught in algorithms come from?

Relation from A to B

Definition. Let A and B be sets. A relation from A to B is any subset of $A \times B$.

Example. $A = \{1, 2, 3\}$, $B = \{8, 9\}$. Then, each of the followings is a relation from A to B

- $A \times B = \{(1, 8), (1, 9), (2, 8), (2, 9), (3, 8), (3, 9)\}$
- $\{(1, 8), (1, 9)\}$
- $\{(1, 8), (2, 9), (3, 8)\}$ as known as a function from A to B

Surprisingly, a function is also a relation with some conditions: A function f from A to B is a relation from A to B such that for each a in A, there exists only one b in B.

So a function is nothing but a set.

Relation on a set A

Definition. A relation from a set A into itself is called a relation on A.

Example. $A = \{1, 2, 3\}$. Each of the followings is a relation on A:

- $A \times A = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$
- $\{(1,2),(1,3),(2,1),(2,2)\}$
- $\{(1,2),(2,1),(3,3)\}$ as known as a function from A to A

Notation

If (a, b) is an element of a relation R, a standard notation is by $a \sim b$. We often denote the relation itself by ~.

Some textbooks use another notation: aRb

Properties of relation

Def. A relation ~ on A is reflexive if $a \sim a$ for all $a \in A$.

Def. A relation ~ on A is symmetric if $a \sim b$ implies that $b \sim a$ for all $a, b \in A$ and $a \neq b$.

Def. A relation ~ on A is transitive if $a \sim b$ and $b \sim c$ implies that $a \sim c$, for all $a, b, c \in A$.

Def. A relation ~ on A is antisymmetric if $a \sim b$ implies that $b \sim a$ is false for all $a, b \in A$ and $a \neq b$.

Examples in real life

- Consider $a \sim b$ if and only if a and b are roommates (symmetric)
- Consider $a \sim b$ if and only if $a \leq b$ (transitive, but not symmetric)

Equivalence relation

Def. An equivalence relation on A is a relation on A that is reflexive, symmetric and transitive.

Real life example. $x \sim y$ if and only if x and y have the same birthday.

- $x \sim x$ is true
- $x \sim y$ is true, then $y \sim x$ is true
- $x \sim y$ and $y \sim z$ are true, then $x \sim z$

Def. The equivalence class of a determined by ~ is the subset of A, denoted by $[a]$, consisting of all the elements of A that are equivalent to a. That is,

$$[a] = \{x \in A, x \sim a\}$$

We say that $[a]$ is "the equivalence class of a" or as "bracket a"

Congruent modulo n

Notation. $a \equiv b \pmod 3$ means that (a mod 3 = b mod 3). e.g.: $5 \equiv 2 \pmod 3$

Let $a, b \in A$ where A is a set of integers. Consider $a \sim b$ if and only if $a \equiv b \pmod 3$.
Then, ~ is an equivalence relation, because

- $a \sim a$ is true for all a in A
- $a \sim b$ is true, then $b \sim a$ is true for all a, b in A and $a \neq b$
- $a \sim b$ and $b \sim c$ are true, then $a \sim c$ for all $a, b, c \in A$

We can partition a set of integers into three equivalence classes of [0], [1], [2].

In this case, [a] is called the congruence class of a modulo n

- i.e., [0] is the congruence class of 0 modulo 3.

Theorem. If R is an equivalence relation on any non-empty set A, then the distinct set of equivalence classes of R forms a partition of A.

Appendix: Asymptotic notation in Algorithms

https://en.wikipedia.org/wiki/Big_O_notation

Consider a relation: $f \sim g$ if and only if $f(n) = O(g(n))$.
The big O notation gives an upper bound of f, kind of $f \leq g$.

- $f \sim f$ is true
- If $f \sim g$ is true, it does not imply that $g \sim f$. (e.g., 3≤4 doesn't imply 4≤3)

Def. We say that $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$

Consider a relation of two functions: $f \sim g$ if and only if $f(n) = \Theta(g(n))$.

Exercise. Show that this relation gives an *equivalence relation*.

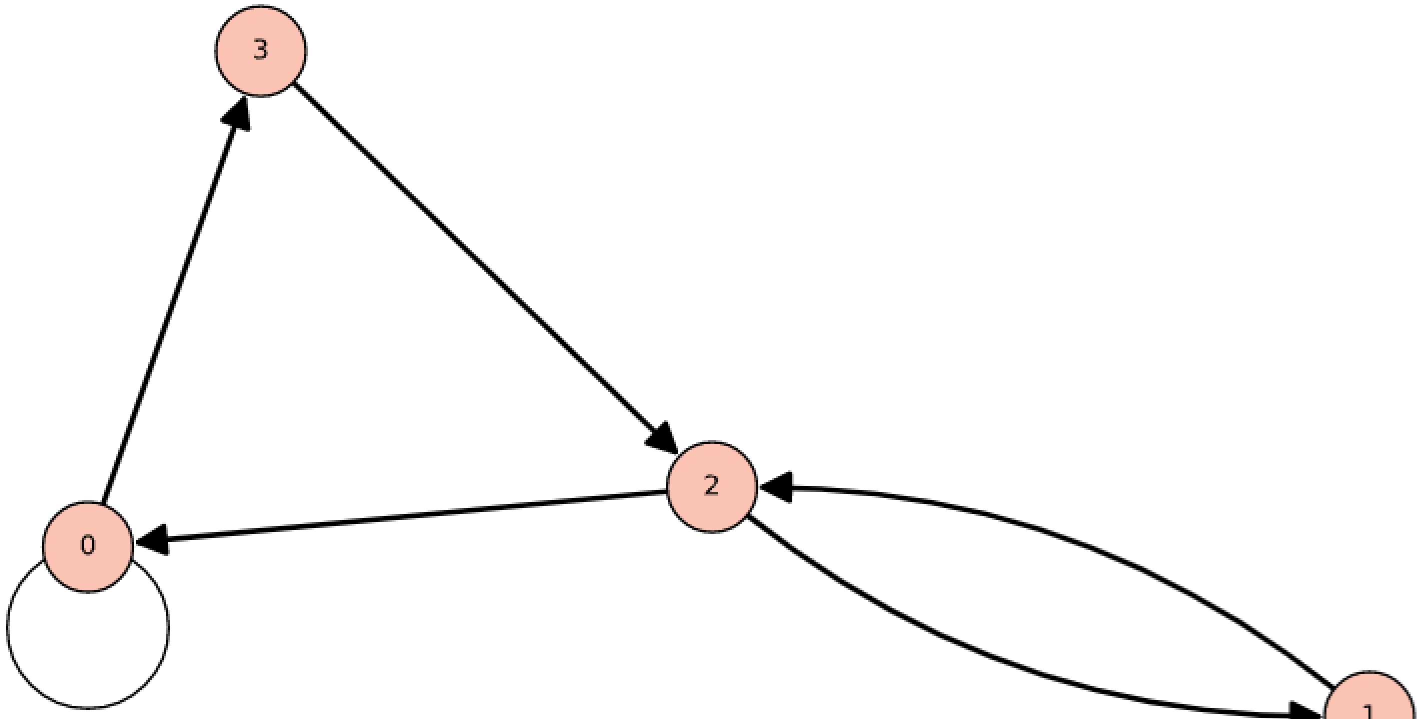
Relations and graphs

Graph representation of relations on a set A

Let $A = \{0, 1, 2, 3\}$ and a relation R on A:

$R = \{(0,0), (0,3), (1,2), (2,1), (3,2), (2,0)\}$

We can represent a relation by a graph:



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Matrix representation of relations

A relation ~ on A can be represented by the n x n matrix defined by:

$$R_{ij} = \begin{cases} 1 & \text{if} \quad a_i \sim a_j \\ 0 & \text{otherwise} \end{cases}$$

where $A = \{a_1, \ldots, a_n\}$.

Example. Let $A = \{2, 5, 6\}$ and a relation $R = \{(2,2), (2,5), (5,6), (6,6)\}$. Its matrix representation is

$$R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Composition of two relations can be regarded as the multiplication of their corresponding matrices.

Matrix representation of relations

- If all of a matrix A's diagonal elements are non-zeros, the corresponding relation is reflexive
- If a matrix A is symmetric, the corresponding relation is also symmetric
- If a matrix A satisfies $A^2 \le A$, the corresponding relation is transitive
  Here, ≤ is defined as an element-wise comparison (i.e., $A^2_{i,j} \le A_{i,j}$ for all i,j)

If there is a path of length 2 (i -> k -> j) in the relation, then there must also be a direct path (i -> j) for the relation to be transitive

The matrix multiplication A^2 captures all the paths of length 2, and the inequality $A^2 \le A$ ensures that any such path of length 2 also has a corresponding direct path

Closure operations

Def. (Transitive closure). Let A be a set and R be a relation on A. The transitive closure of R is the **smallest** transitive relation that contains R as a subset.

> In general, "smallest" helps ensure that the definition is well-defined and provides a unique solution. It also allows us to focus on the essential parts of the concept without introducing unnecessary complexity or ambiguity. e.g., convex hull, ...

> A set S is said to be closed under the operation * if whenever we apply the operation * to an arbitrary element of S, the result is also an element of S. The set of integers are closed under + and x, but not closed under /. The closure of S under the operation * is the smallest superset of S that is closed under the operation *.

Example. Let $A = \{1, 2, 3, 4\}$, and let $R = \{(1, 2), (2, 3), (3, 4)\}$ be a relation on A.

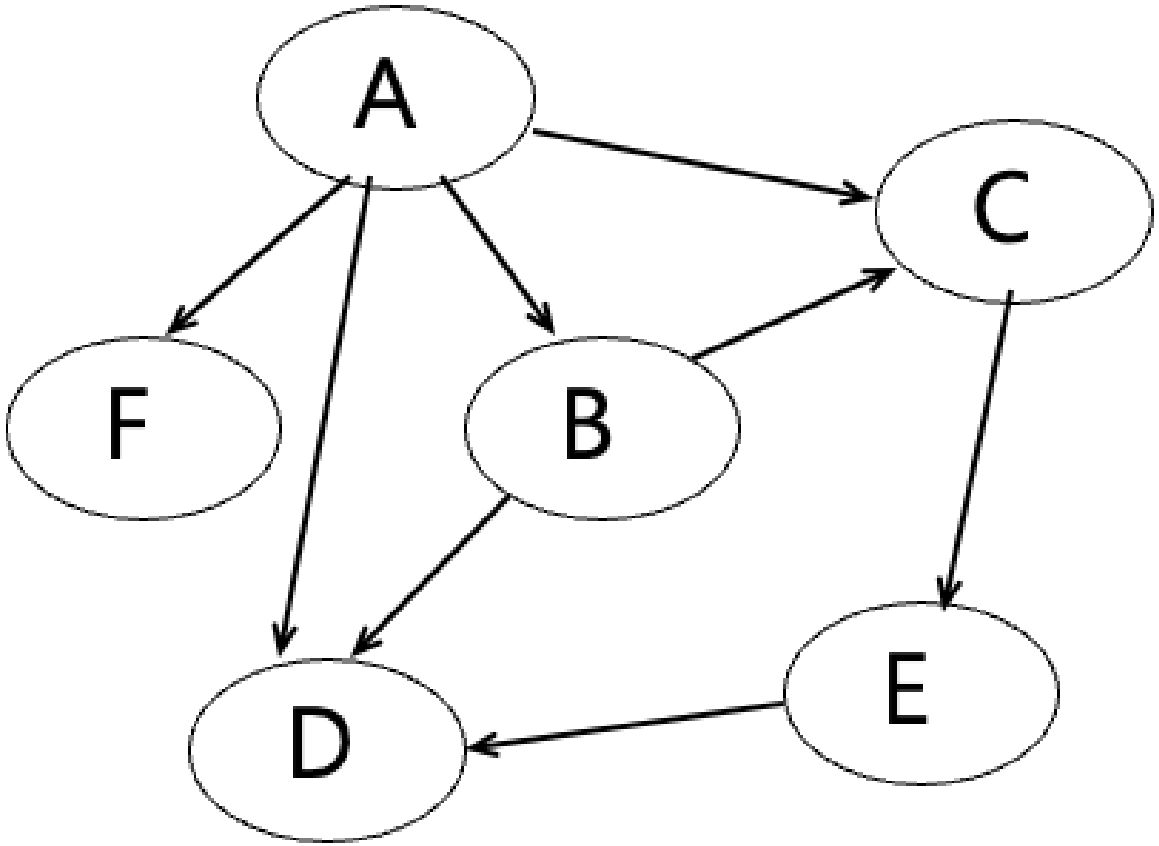
The transitive closure of R is $\{(1, 2), (2, 3), (3, 4), (1, 3), (2, 4), (1, 4)\}$

The composition

- $R^2 = \{(1, 3), (2, 4)\}$
- $R^3 = \{(1, 4)\}$

Example

```
>>> import numpy as np
>>> A = np.array([[0,1,1,1,0,1],[0,0,1,1,0,0],
[0,0,0,0,1,0],[0,0,0,0,0,0],
[0,0,0,1,0,0],[0,0,0,0,0,0,]])
>>> A
array([[0, 1, 1, 1, 0, 1],
       [0, 0, 1, 1, 0, 0],
       [0, 0, 0, 0, 1, 0],
       [0, 0, 0, 0, 0, 0],
       [0, 0, 0, 1, 0, 0],
       [0, 0, 0, 0, 0, 0]])
```



In A^2, (i,j) element represents a path whose length is 2.

```
>>> A.dot(A) # length 2
array([[0, 0, 1, 1, 1, 0],
       [0, 0, 0, 0, 1, 0],
       [0, 0, 0, 1, 0, 0],
       [0, 0, 0, 0, 0, 0],
       [0, 0, 0, 0, 0, 0],
       [0, 0, 0, 0, 0, 0]])
>>> A.dot(A).dot(A) # length 3
array([[0, 0, 0, 1, 1, 0],
       [0, 0, 0, 1, 0, 0],
       [0, 0, 0, 0, 0, 0],
       [0, 0, 0, 0, 0, 0],
       [0, 0, 0, 0, 0, 0],
       [0, 0, 0, 0, 0, 0]])
>>> A+A.dot(A) # at most length 2
array([[0, 1, 2, 2, 1, 1],
       [0, 0, 1, 1, 1, 0],
       [0, 0, 0, 1, 1, 0],
       [0, 0, 0, 0, 0, 0],
       [0, 0, 0, 1, 0, 0],
       [0, 0, 0, 0, 0, 0]])
```