Lecture notes for

Commutative Algebra

Jakub Paliga

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1 Preface

The following is a set of personal notes that have been taken during the lecture given by dr Joachim Jelisiejew as part of a WS2019/2020 course in commutative algebra (*Algebra przemienna*) at the faculty of Mathematics, Informatics and Mechanics, University of Warsaw.

The text of these notes was compiled by Jakub Paliga, who does not guarantee their correctness and disclaims any warranties. Indeed, he confirms any deficiency within, of which there is a nonzero number, to have been introduced by him in the process.

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1 Basics

1.1 Conventions

Rings will be understood to be commutative, associative, unitary; the ring ”0=1” is considered a ring. Ring homomorphism are assumed to preserve the unit.

k will denote a field, $\bar{k}$ its algebraic closure. Typically, this field will be $\mathbb{C}$ - the complex numbers - but not always.

Ideals will be denoted by I and J. Ideals satisfying additional properties will be written in $\mathfrak{fraktur}$ and named distinctly: prime ideals are $\mathfrak{p}$, $\mathfrak{q}$; maximal ideals, $\mathfrak{m}$, $\mathfrak{n}$.

1.2 First definitions

Definition 1.1. Let A be a ring. Then an A-*algebra* is a ring B together with a fixed homomorphism

$$A \to B.$$

This homomorphism is called the *structural homomorphism.*

Definition 1.2. Let A be a ring. A *homomorphism of* A-*algebras* is a ring homomorphism commuting with the structural maps, that is, if

$$\phi : A \to B, \quad \psi : A \to C$$

are A-algebras, then a ring homomorphism $f : B \to C$ is an algebra homomorphism if

$$\psi = f \circ \phi.$$

Example 1.3. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $z \mapsto \bar{z}$ (the complex conjugation). Then:

- f is a ring homomorphism,
- f is a real algebra homomorphism,
- f is not a complex algebra homomorphism.

Example 1.4. The polynomial ring $\mathbb{C}[x]$ becomes a $\mathbb{C}$-algebra under the inclusion map onto the zero degree component.

In this case, if I is an ideal, then the quotient map

$$\mathbb{C}[x] \to \mathbb{C}[x]/I$$

is a complex algebra homomorphism.

Definition 1.5.

1. A ring A is a domain if

$$\forall a, b \in A \ \ (ab = 0 \implies a = 0 \vee b = 0.)$$

2. An ideal I in a ring B is prime if B/I is a domain.
3. A ring A is a field if
$$\forall 0 \neq a \in A \;\; \exists b \in A \;\; ab = 1,$$
that is, every nonzero element has a multiplicative inverse.
4. An ideal I in a ring B is maximal if B/I is a field.

1.3 Motivations

Lemma 1.6. Let $f : A \to B$ be a ring homomorphism. If $\mathfrak{p} \subset B$ is prime, then $f^{-1}(\mathfrak{p}) \subset A$ is prime.

Proof. Consider the following diagram.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A/f^{-1}(\mathfrak{p}) \arrow[r] & B/\mathfrak{p} \end{tikzcd}$$

The preimage of an ideal is an ideal. Moreover, the lower horizontal map is injective; this is elementary, using but the definition of a quotient ring and the set-theoretical properties of preimages. Hence, $A/f^{-1}(\mathfrak{p}) \subseteq B/\mathfrak{p}$ is a subring. If then $\mathfrak{p}$ is prime, $B/\mathfrak{p}$ is a domain; its every subring is then a domain as well. Thus, $A/f^{-1}(\mathfrak{p})$ is a domain, and so $f^{-1}(\mathfrak{p}) \subseteq A$ is prime. $\square$

Example 1.7. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ has $f^{-1}(0) = 0$ not maximal, even though $0 \in \mathbb{Q}$ is maximal. Ergo, Lemma 1.6 does not hold with ”prime” replaced by ”maximal”.

Example 1.8. Consider

$$S^1 = \{z \in \mathbb{C} | |z| = 1\}, \quad A = C(S^1, \mathbb{R}) = \{ \text{ continuous real functions on the circle } \}.$$

Then there is a bijection between S^1 and maximal ideals in A, given by every maximal ideal being of the form
$$\mathfrak{m}_x = \{f \in A \mid f(x) = 0\}.$$

Moreover, one can recover the topology of S^1 from $A = C(S^1, \mathbb{R})$.
Indeed, for $f \in A$, consider the vanishing set

$$V(f) = \{\mathfrak{m} \text{ maximal ideal in } A \; | f \in m\}.$$

The topology is then generated by closed subsets with subbase

$$\{V(f) \mid f \in A\}.$$

1.4 The Spec() functor

We aim to repeat the previous considerations of Example 1.8 for an arbitrary ring.

Definition 1.9. Let A be a ring. The *spectrum* of A is its set of prime ideals:

$$\mathrm{Spec}(A) = \{\mathfrak{p} \text{ a prime ideal in} A\}.$$

Example 1.10. $\mathrm{Spec}(\mathbb{C}[x]) = \{0\} \cup \{(x-a) | a \in \mathbb{C}\}$.

For a ring homomorphism $f\colon A \to B$, Lemma 1.6 asserts that there is a map

$$f^*\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

given by

$$B \supseteq \mathfrak{q} \mapsto f^{-1}(\mathfrak{q}) \subseteq A.$$

We will now move towards upgrading spectra of rings (which up to this point we considered as mere sets) to topological spaces in such a way that makes the maps f^* continuous. In this, we follow the previous motivation.

Definition 1.11. For $E \subseteq A$ an arbitrary subset, we define the *vanishing locus of E* as the set

$$V(E) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid E \subseteq \mathfrak{p}\}.$$

Proposition 1.12. The sets $V(E)$, $E \subseteq A$, are the closed subsets of a topology.

Proof. One sees immediately that

$$V(1) = \emptyset, \quad V(0) = \mathrm{Spec}(A).$$

Elementary set-theoretic considerations reveal that

$$\cap_{i\in I} V(E_i) = V(\cup_{i\in I} E_i).$$

It now suffices to show that

$$V(E_1) \cup V(E_2) = V(E_1 \cdot E_2), \quad E_1 \cdot E_2 \coloneqq \{e_1 \cdot e_2 \mid e_1 \in E_1, e_2 \in E_2\}.$$

The inclusion $V(E_1) \cup V(E_2) \subseteq V(E_1 \cdot E_2)$ is easy to verify. Suppose without loss of generality that $\mathfrak{p} \in V(E_1)$, that is $\mathfrak{p} \supseteq E_1$. Then

$$E_1 \cdot E_2 \subseteq \mathfrak{p} \cdot E_2 \subseteq \mathfrak{p},$$

because $\mathfrak{p}$ is an ideal.

In order to prove the other inclusion, we need to use that $\mathfrak{p}$ is prime. Suppose that

$$\mathfrak{p} \not\supseteq E_1 \quad \text{and} \quad \mathfrak{p} \not\supseteq E_2.$$

Then there exist $e_i \in E_i \setminus \mathfrak{p}$, $i = 1, 2$. For those, $e_1 \cdot e_2 \in E_1 \cdot E_2 \setminus \mathfrak{p}$ holds because $\mathfrak{p}$ is prime. □

Definition 1.13. The topology on $\mathrm{Spec}(A)$ defined by Proposition 1.12 is called the *Zariski topology.*

Question 1.14. What is the Zariski topology for $A = \mathbb{C}[x]$?

Answer. We note that

$$\begin{aligned} V(f_1, \dots, f_r) &= \{\mathfrak{p} \text{ prime in } A \mid f_1, \dots, f_r \in \mathfrak{p}\} \\ &= \{(x-a) \text{ maximal in } A \mid f_1, \dots, f_r \in (x-a)\} \\ &= \{(x-a) \mid f_1(a) = \ldots = f_r(a) = 0\}. \end{aligned}$$

If any f_i is nonzero, then $0 \notin V(f_1, \dots, f_r)$.

Hence,

$$V(f_1, \dots, f_r) = \text{ set of common roots of } f_1, \dots, f_r$$

and one sees that $V(f_1, \dots, f_r)$ is finite.

In fact, all finite subsets of the set of maximal ideals are of the form $V(f_1, \dots, f_r)$. Note that the only closed set containing the prime ideal 0 is $\operatorname{Spec}(A)$ itself; in other words, 0 lies in every nonempty open set. This yields the cofinite topology augmented by $\{0\}$.

Note 1.15. Because the closure of $\{0\} \subseteq \operatorname{Spec}(A)$ is $\operatorname{Spec}(A)$ itself, the resulting space is not T1, let alone Hausdorff. It is, however, T0.

Proposition 1.16. Let $f \colon A \to B$ be a ring homomorphism. Then the induced map

$$f^* \colon \operatorname{Spec}(B) \to \operatorname{Spec}(A) \quad \text{with} \quad \mathfrak{q} \mapsto f^{-1}(\mathfrak{q})$$

is continuous.

Proof. It suffices to see that the preimage of every closed set is closed. Indeed, we claim that

$$(f^*)^{-1}(V(E)) = V(f(E)).$$

We unravel the definitions:

$$\begin{aligned}(f^*)^{-1}(V(E)) &= \{\mathfrak{p} \subseteq B \mid f^*(\mathfrak{p}) \in V(E)\} \\ &= \{\mathfrak{p} \subseteq B \mid f^{-1}(\mathfrak{p}) \in V(E)\} \\ &= \{\mathfrak{p} \subseteq B \mid E \subseteq f^{-1}(\mathfrak{p})\} \\ &= \{\mathfrak{p} \subseteq B \mid f(E) \subseteq \mathfrak{p}\} \\ &= V(f(E)).\end{aligned}$$

Note that the considerations here are purely set-theoretical. The condition that f be a ring homomorphism is only relevant for f^* to be well defined on spectra. □

The upshot is: when $k = \bar{k}$, $\operatorname{Spec}_{max}(k[x_1, \dots, x_n])$ is well-behaved. We will see later, in the Nullstellensatz, that all maximal ideals in $k[x_1, \dots, x_n]$ are of the form $(x_1 - a_1, \dots x_n - a_n)$.

In fact, $\operatorname{Spec}_{max}(k[x_1, \dots, x_n]) \cong k^n$ under $(x_1 - a_1, \dots, x_n - a_n) \mapsto (a_1, \dots, a_n)$.

The intuition here is: $k[x_1, \dots, x_n]$ is the ring of regular functions on k^n.

1.5 Pictures of spectra

For any I, we get

$$V(I) = \operatorname{Spec}(A/I) \hookrightarrow \operatorname{Spec}(A)$$

with a bijection:

$$\{\bar{\mathfrak{p}} \subseteq A/I\} \cong \{\mathfrak{q} \subseteq A \mid \mathfrak{q} \supseteq I\}.$$

From

$$\pi : A \to A/I$$

we get π^*, which is injective with image $V(I)$. Hence, we identify

$$\operatorname{Spec}(A/I) \quad \text{with} \quad V(I) \subseteq \operatorname{Spec}(A).$$

Example 1.17. Consider

$$\mathrm{Spec}(k[x,y]/(xy-1)) \hookrightarrow \mathrm{Spec}(k[x,y]) \supseteq \mathrm{Spec}_{\max}(k[x,y]) = k^2.$$

A point $(a,b) \in k^2$ is seen as the maximal ideal $(x-a, y-b)$; that comes from $\mathrm{Spec}(k[x,y]/(xy-1))$ if and only if $ab-1=0$.

Example 1.18. $\mathrm{Spec}(k[x,y]/xy)$ gives the "cross" $\{ab=0\}$.

Example 1.19. In the case of

$$\mathrm{Spec}(k[x,y]/(x^2+y^2+1)) \hookleftarrow \mathrm{Spec}(k[x,y]),$$

geometric interpretations transcend our $\mathbb{R}$-intuitions.

Note 1.20. In general, $\mathrm{Spec}_{\max}$ is the set of closed points in Spec, and if the ring A be of finite type (that is, finitely generated over a field or $\mathbb{Z}$), then

$$\overline{\mathrm{Spec}_{\max}(A)} = \mathrm{Spec}(A).$$

1.6 Localization

Definition 1.21. Let A be a ring. A subset $S \subseteq A$ is called *multiplicative* if the following conditions hold:

1. $1 \in S$,
2. $\forall s,t \in S \;\; st \in S$.

We wish to obtain an initial A-algebra

$$A \xrightarrow{i} S^{-1}A$$

such that the $i(s)$ are invertible; a pseudo-

$$\{\frac{a}{s} \mid a \in A, s \in S\}.$$

Indeed, in the case that A is a domain, the preceding definition is entirely satisfactory.

Construction 1.22 (localization of a ring at a multiplicative subset)**.**
Step 1. Let

$$I = \{a \in A \mid \exists s \in S \;\; sa = 0\}.$$

Note that if $a, b \in I$, then

$$\exists s \in S \;\; sa = 0, \quad \exists t \in S \;\; tb = 0.$$

Then $(st)(a \pm b) = 0$.
Upshot: I is an ideal. We let $A \xrightarrow{\pi} A/I =: A'$.
Step 2. $\pi(S) \subseteq A'$ consists of non-zerodivisors.

Indeed, suppose that $s \in S$. If $w \in A'$ is such that $w\pi(s) = 0$, then from surjectivity of π it follows that

$$\begin{aligned} \exists a \in A \quad & w = \pi(a) \\ & \pi(s)\pi(a) = 0 \\ & \implies \pi(sa) = 0 \\ & \implies sa \in I \\ & \implies \exists t \in S tsa = 0 \\ & \implies (ts)a = 0. \end{aligned}$$

But because $ts \in S$, it follows that $a \in I$ and so $\pi(a) = 0$. This means that $\pi(s)$ is not a zero divisor.

Define $S' := \pi(S)$. We want

$$(S')^{-1}A' = \{\frac{a'}{s'} \mid a' \in A', s' \in S'\}.$$

Consider pairs

$$\{(a', s') \mid a' \in A', s' \in S'\}$$

and define their relation $\sim$ thus:

$$(a'_1, s'_1) \sim (a'_2, s'_2) \iff a'_1 s'_2 = a'_2 s'_1.$$

To prove that this is an equivalence relation, we show transitivity (exercise; use that the S' are non-zerodivisors).

Step 3.

$$(S')^{-1}A' := \{(a', s') \mid a' \in A', s' \in S'\}/{\sim}$$

is a well defined set. The ring operations are given by thinking about this as

$$\{\frac{a'}{s'} \mid a' \in A', s' \in S'\}.$$

This turns the set into an associative commutative ring with unity.

Lemma 1.23. Let $a, b \in A$, $s, t \in S$. Then

$$\frac{a}{s} = \frac{b}{t} \quad \iff \quad \exists u \in S \;\; u(at - bs) = 0.$$

Proof. " $\impliedby$ ":

$$\begin{aligned} u(at - bs) = 0 &\implies uta = usb \\ &\implies \frac{uta}{1} = \frac{usb}{1} \\ &\implies \frac{a}{s} = \frac{uta}{stu} = \frac{usb}{stu} = \frac{b}{t}. \end{aligned}$$

" $\implies$ ": Observe that the map $\phi : A/I = A' \to (S')^{-1}A^{-1}$ is injective. Note that

$$\forall \bar{a}, \bar{b} \in A' \;\; \frac{\bar{a}}{1} = \frac{\bar{b}}{1} \iff \bar{a} = \bar{b} \iff 1 \cdot \bar{a} = 1 \cdot \bar{b}$$

□

Corollary 1.24. A model of $S^{-1}A$ is given as the quotient

$$\{(a,s) \mid a \in A, s \in S\}/\sim$$

under

$$(a,s) \sim (b,t) \iff \exists s \in S \;\; u(at - bs) = 0.$$

Example 1.25. For $f \in A$, let $S = \{1, f, f^2, \ldots\}$. We denote $S^{-1}A$ by A_f.

Example 1.26. $\mathfrak{p} \in \operatorname{Spec}(A)$, $S \coloneqq A \setminus \mathfrak{p}$. Then we denote $S^{-1}A$ by $A_{\mathfrak{p}}$.

Note 1.27. $S^{-1}A$ does not determine S, for instance

$$\mathbb{C}[x]_x = \mathbb{C}[x]_{x^{2019}}$$

Proposition 1.28 (universal property of localization)**.** Let $f : A \to B$ be such that $f(S)$ consists of invertible elements.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & S^{-1}A \arrow[ur, "S^{-1}f"'] & \end{tikzcd}$$

Then $\exists! S^{-1} \colon S^{-1}A \to B$ such that $f = (S^{-1}f) \circ i$.

Proof. We write $\tilde{f} \coloneqq S^{-1}f$. Suppose that $\tilde{f}$ exists. Then

$$\forall a \in A \;\; \tilde{f}(a/1) = f(a)$$

and thus

$$\forall s \in S \;\; \tilde{f}(\frac{a}{s} \cdot s) = f(\frac{a}{1}) = f(a)$$

and the formula $\tilde{f}(a/s) = f(a)/f(s)$ is recovered. One needs only to check that such a function is a homomorphism. ☐

Corollary 1.29. Consider the category of A-algebras whose structure maps invert S. Then Proposition 1.28 reads: $S^{-1}A$ is the initial object of this category.

Lemma 1.30. $A_f \cong A[x]/(fx-1)$.

Proof. Note that f is invertible in $A[x]/(fx-1)$. Hence by universal property

$$\tilde{\phi}(a/f) = \phi(a)/\phi(f) = a\bar{x}.$$

Now take $\psi \colon A[x] \to A_f$ a homomorphism of A-algebras defined by $\psi(x) = 1/f$. Then

$$\psi(fx - 1) = f \cdot 1/f - 1 = 0$$

and so

$$\exists \tilde{\psi} \colon A[x]/(xf-1) \to A_f$$

such that

$$\forall a \in A \;\; \forall n \in \mathbb{N} \;\; \tilde{\psi}(ax^n) = a/f^n.$$

The maps $\tilde{\phi}$ and $\tilde{\psi}$ are then mutual inverses. ☐

Corollary 1.31. A_f is a finitely generated A-algebra.

Lemma 1.32. For A an algebra and S a multiplicative subset:

1. $S^{-1}A = 0 \iff 0 \in S$
2. $A \xrightarrow{i} S^{-1}A$ is injective if and only if all elements of S are non-zerodivisors in A
3. $A \to S^{-1}A$ is an isomorphism if and only if all elements of S are invertible in A

Proof. 1. By Lemma 1.30:

$$\frac{1}{1} = \frac{0}{1} \iff \exists u \in S \;\; u(1 \cdot 1 - 0 \cdot 1) = 0 \iff \exists u \in Su = 0.$$

2. i is injective if and only if

$$(\{a \in A \mid \exists s \in S \;\; sa = 0\}).$$

This happens if and only if s is a non-zerodivisor.

3. If $A \to S^{-1}A$ is iso, then $\forall s \in S \;\; s/1$ is invertible. Hence, so is s.

Conversely, if all elements of S are invertible in A, then they are non-zerodivisors. Hence $A \to S^{-1}A$ is injective by the previous point. Moreover,

$$i(as^{-1}) = \frac{as^{-1}}{a} = \frac{a}{s},$$

so i is "onto". □

Intuitively, we think of A_f as the ring of functions on X_f, where $X = \operatorname{Spec}(A)$. We would expect $\operatorname{Spec}(A_f)$ to be the same as X_f. Indeed, the following holds.

Proposition 1.33. Consider the maps

$$i \colon A \to S^{-1}A$$
$$i^* \colon \operatorname{Spec}(S^{-1}A) \to \operatorname{Spec}(A).$$

Then i^* is injective and

$$\operatorname{im} i^* = \{\mathfrak{p} \mid \mathfrak{p} \cap S = \emptyset\}.$$

Before presenting the proof, for which we will require some additional facts, we note the following corollaries.

Corollary 1.34. If $S = \{1, f, f^2, \ldots\}$, then

$$\operatorname{im} i^* = \{\mathfrak{p} \mid \mathfrak{p} \not\ni f\} = (\operatorname{Spec}(A))_f.$$

Corollary 1.35. For $\mathfrak{q} \in \operatorname{Spec}(A)$, $S = A \setminus q$,

$$\operatorname{im} i^* = \{\mathfrak{p} \mid \mathfrak{p} \subseteq \mathfrak{q}\} = \operatorname{Spec}(A_p).$$

Lemma 1.36. Let $I \subseteq S^{-1}A$ be an ideal. If we let

$$J = \{a \in A \mid \frac{a}{1} \in I\},$$

then

$$I = \{\frac{j}{s} \mid j \in J, s \in S\}.$$

Proof. "$\supseteq$":

$$\forall j \in J \ \frac{j}{i} \in I \implies \frac{j}{s} = \frac{j}{1} \cdot \frac{1}{s} \in I$$

"$\subseteq$":

$$\frac{a}{s} \in I \implies \frac{a}{1} = \frac{a}{s} \cdot \frac{s}{i} \in I \implies a \in J \implies \frac{a}{s} \in \{\frac{j}{s} \mid j \in J, s \in S\}$$

□

Proposition 1.33. Let $\mathfrak{q} \in \mathrm{Spec}(S^{-1}A)$. We let

$$\mathfrak{p} = i^*(\mathfrak{q}) = i^{-1}(\mathfrak{q}) = \{a \in A \mid f \in \mathfrak{q}\}.$$

By Lemma 1.36 we can recover $\mathfrak{q}$ from $\mathfrak{p}$ alone, and so i^* is injective.

Let $\mathfrak{p} = i^*(\mathfrak{q})$. Suppose $\mathfrak{p} \cap S \neq \emptyset$. This means that

$$\frac{s}{1} \in \mathfrak{q} \text{ and } \frac{s}{t} \text{ is invertible } \implies \mathfrak{q} = (1).$$

Suppose $\mathfrak{p} \in \mathrm{Spec}(A)$, $\mathfrak{p} \cap S = \emptyset$. We wish to find $\mathfrak{q}$. We guess that

$$q \coloneqq \{\frac{p}{s} \mid p \in \mathfrak{p}, s \in S\} \subseteq S^{-1}A$$

does the job. This $\mathfrak{q}$ is an ideal, we want to see that it is prime.

Suppose that

$$\frac{a}{s} \cdot \frac{b}{t} \in \mathfrak{q}.$$

Then $ab/1 \in \mathfrak{q}$ and so □

Corollary 1.37. For $f \in A$, the following are equivalent:

1. f is nilpotent (that is, $\exists n > 0 \ f^n = 0$),
2. $f \in \bigcap\{\mathfrak{p} \mid p \in \mathrm{Spec}(A)\}$,
3. $V(f) = \mathrm{Spec}(A)$.

Proof. The equivalence between 2 and 3 is checked formally.

"$1 \implies 2$": $\forall \mathfrak{p} \ f^n = 0 \in \mathfrak{p} \implies f \in \mathfrak{p}$.

" $\neg 1 \implies \neg 2$": let $0 \notin \{1, f, f^2, \dots\}$ so $A_f \neq 0$ by Lemma 1.36. Hence, $\mathrm{Spec}(A_f) \neq \emptyset$, so by Proposition 1.33

$$\mathrm{Spec}(A_f) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid f \notin \mathfrak{p}\} \neq \emptyset.$$

□

Definition 1.38. The nilradical of a ring A is defined as

$$\mathrm{nil}(A) = \bigcap\{\mathfrak{p} \mid p \in \mathrm{Spec}(A)\} = \{f \in A \mid \exists n > 0 \ f^n = 0\}.$$

2 Modules

2.1 Modules

Definition 2.1. Let A be a ring. An A-module M is an abelian group M together with a ring homomorphism $A \to \mathrm{End}_{\mathbb{Z}}(M)$.

Equivalently, M is an abelian group together with a map

$$A \times M \to M, \quad (a, m) \mapsto am,$$

such that:

1. $\forall a \in A,\ \ m_1, m_2 \in M\ \ a(m_1 + m_2) = am_1 + am_2$
2. $(a_1 + a_2)m = a_1 m + a_2 m$
3. $1 \cdot m = m$
4. $a_1(a_2 m) = (a_1 a_2)m$

Definition 2.2. A homomorphism of A-modules $\phi\colon M \to N$ is a homomorphism of abelian groups such that

$$\forall a \in A\ \ \forall m \in M\ \ a\phi(m) = \phi(am).$$

Example 2.3. For A a ring, A is an A-module; in fact, any ideal $I \subseteq A$ is an A-module.

Example 2.4. For any A-algebra B, B is an A-module. In particular, A/I is an A-module for any ideal $I \subseteq A$.

Definition 2.5. If M is an A-module and $m_1, \ldots, m_k \in M$ its elements, then the submodule generated by $m_1, \ldots, m_k$ is

$$Am_1 + Am_2 + \ldots + Am_k = \{\sum_{i=1}^{k} a_i m_i \mid a_i \in A\}.$$

Proposition 2.6. The set

$$\mathrm{hom}_A(M, N) = \{\phi : M \to N \text{ an } A\text{-module homomorphism}\}$$

is an A-module under

$$(a.\phi)(m) \coloneqq \phi(am) = a\phi(m).$$

Definition 2.7. An A-module is finitely generated if $\exists k \in \mathbb{N},\ \ m_1, \ldots, m_k \in M$ such that

$$M = A_1 + \ldots + A_{m_k}.$$

Definition 2.8. An A-module is free if it is isomorphic to an A-module of the form

$$\bigoplus_{i \in I} A.$$

Example 2.9. All k-vector spaces are free k-modules. Not all abelian groups are free.

Lemma 2.10. An A-module M is finitely generated if and only if there exists a surjective homomorphism

$$\phi : \bigoplus_{i=1}^{k} A \to M.$$

2.2 More on modules

Note 2.11. In the following, when we write $I \cdot M$, we will mean the linear span of elements of the form $i \cdot m$.

Lemma 2.12 (adjugate matrix)**.** Let $X \in M_{n\times n}(A)$. Then

$$\exists Y \in M_{n\times n}(A) \quad \text{such that} \quad Y \cdot X = \operatorname{diag}(d), \;\; d = \det(X).$$

Concretely, Y is given by the formula

$$Y := [(-1)^{i+j} \det X_{ji}]_{ij},$$

where X_{ij} is the submatrix of X formed by excluding the i-th row and the j-th column.

We skip the proof of Lemma 2.12.

Theorem 2.13 (Cayley-Hamilton)**.** Let M be an A-module. Let $\phi : M \to M$ be a homomorphism of A-modules such that

$$\phi(m) = IM = \{\sum i_k m_k \mid i_k \in I, m_k \in M\}.$$

Then there exist elements

$$a_{n-1} \in I, a_{n-2} \in I^2, \ldots, a_0 \in I^n$$

such that

$$\phi^n + a_{n-1}\phi^{n-1} + \ldots + a_0 = 0 = \text{ the zero morphism.}$$

Proof. Let

$$\pi \colon A^{\oplus n} \to M \quad \text{via} \quad e_i \mapsto m_i.$$

Then

$$\forall i \;\; \exists u_{ij} \in I \;\; \phi(m_i) = \sum_j u_{ij} m_j$$

in face of the assumptions. Define the lift of ϕ to a self-map $\tilde{\phi}$ of the free module $A^{\oplus n}$ by

$$\tilde{\phi} \colon A^{\oplus n} \to A^{\oplus n} \quad \text{via} \quad \tilde{\phi}(e_i) = \sum_j u_{ij} e_j.$$

This can be fitted inside a commutative diagram

$$\begin{tikzcd} A^{\oplus n} \arrow[r, "\tilde{\phi}"] \arrow[d, "\pi"'] & A^{\oplus n} \arrow[d, "\pi"] \\ M \arrow[r, "\phi"] & M \end{tikzcd}$$

Let $B := A[x]$ and make $A^{\oplus n}$ into a B-module by

$$\forall f \in A^{\oplus n} \quad x.f := \tilde{\phi}(f).$$

Now,

$$X := [u_{ij}]_{ij} - \mathrm{diag}(x) \in M_{n\times n}(B).$$

View X as an endomorphism of $A^{\oplus n}$. Then

$$X.e_i = \sum_j u_{ij} e_j - x \cdot e_i = \sum_j u_{ij} e_j - \tilde{\phi}(e_j) = 0.$$

Hence X acts on $A^{\oplus n}$ as the zero endomorphism.
Now, use Lemma 2.12 to see that

$$\exists Y \;\; Y \cdot X = \mathrm{diag}(d).$$

Hence

$$\forall i \;\; 0 = Y \cdot X \cdot e_i = \mathrm{diag} d e_i = d \cdot e_i.$$

Expand the determinant by columns:

$$d = \det(X) = x^n + a_1 x^{n-1} + \ldots + a_n, \quad a_k \in I^k.$$

So as an endomorphism of $A^{\oplus n}$,

$$\tilde{\phi}^n + a_1 \tilde{\phi}^{n-1} + \ldots + a_n = 0.$$

But

$$\phi \circ \pi = \pi \circ \tilde{\phi}.$$

So for all i

$$\begin{aligned}(\phi^n + a_1\phi^{n-1} + \ldots + a_n)(m_i) &= \pi((\tilde{\phi}^n + a_1\tilde{\phi}^{n-1} + \ldots + a_n)(e_i)) \\ &= \pi(0) \\ &= 0.\end{aligned}$$

□

Lemma 2.14 (Nakayama)**.** Let M be an A-module, $I \cdot M = M$. If M is finitely generated, then

$$\exists i \in I \;\; (1 - i) \cdot M = 0.$$

Proof. Consider $\mathrm{id}_M \colon M \xrightarrow{\simeq} M$. By Section 2.2,

$$\exists a_i \;\; \mathrm{id}^n + a_1 \mathrm{id}^{n-1} + \ldots + a_n = 0$$

and then

$$\forall m \in M (1 + a_1 + \ldots + a_n) \cdot m = 0.$$

We now let

$$-i := a_1 + \ldots + a_n.$$

□

Corollary 2.15 (local Nakayama)**.** Let A be a local ring, $\mathfrak{m}$ the maximal ideal of A, M an A-module, M finitely generated, $M = \mathfrak{m} \cdot M$. Then

$$M = 0.$$

Proof. Apply Lemma 2.14:

$$\exists a \in \mathfrak{m} \ \ (1-a) \cdot M = 0, \quad (1-a) \notin \mathfrak{m}.$$

But $(1-a)$ is invertible; let its inverse be b. Then

$$M = 1 \cdot M = b(1-a)M = b \cdot 0 = 0.$$

□

Corollary 2.16. Let $N \subseteq M$, M finitely generated, A local, $\mathfrak{m}$ maximal. Let

$$M = N + \mathfrak{m} \cdot M.$$

Then $M = N$.

Proof. Write $N + M = M = N + \mathfrak{m}M$ and so $M/N = \mathfrak{m}M/N$. By Corollary 2.16, $M/N = 0$. □

Corollary 2.17. Let M be a finitely generated A-module, $(A, \mathfrak{m})$ a local ring. Suppose

$$m_1, \ldots, m_k \in M$$

be such that their images in $M/\mathfrak{m} \cdot M$ generate $M/\mathfrak{m} \cdot M$. Then $m_1, \ldots, m_k$ generate M.

Proof. Let

$$N \coloneqq \sum_{i=1}^{k} A \cdot m_i \subseteq M.$$

Then the map

$$N \to M/\mathfrak{m} \cdot M$$

is surjective by assumption, so by Corollary 2.15:

$$M = \mathfrak{m} \cdot M + N \implies M = N = \sum_{i=1}^{k} A \cdot m_i.$$

□

Example 2.18. Consider $\mathbb{Q} \in \mathbb{Z}_{\text{mod}}$. Consider $I = (2019) \subseteq \mathbb{Z}$. Certainly $\mathbb{Q} = I \cdot \mathbb{Q}$, but there does not exist an $i \in I$ such that

$$(1-i) \cdot \mathbb{Q} = 0,$$

so $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module.

Example 2.19. Another counterexample (this time local) is given by $\mathbb{Z}_{(2)}$.

Note 2.20. Localizing is a typical way of finding non-finitely generated modules.

Example 2.21. Let $X = S^1$, $A = C(S^1, \mathbb{R})$. Consider

$$\mathfrak{m}_x = \{f \mid f(x) = 0\}.$$

Then $\mathfrak{m}_x^2 = \mathfrak{m}_x$, hence $\mathfrak{m}_x$ is not a finitely generated ideal.

2.3 Tensor product

Fix A-modules M, N. We will construct their tensor product $M \otimes N$ - another A-module whose definition demands that it in some way classify “A-multiplications” with domain $M \times N$.

Definition 2.22. Let P be an A-module. A function (not a homomorphism!)

$$f \colon M \times N \to P$$

is called A-bilinear if:

1. $\forall a \in A, m \in M, n \in N \;\; af(m,n) = f(am,n) = f(m,an)$,
2. $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$,
3. $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$.

The set of such functions will be denoted by

$$\mathrm{Bilin}_{M \times N}(P).$$

Proposition 2.23. For a homomorphism of A-modules $\phi \colon P \to R$ one gets a map

$$\phi^* \colon \mathrm{Bilin}_{M \times N}(P) \to \mathrm{Bilin}_{M \times N}(R) \quad \text{via} \quad f \mapsto \phi \circ f.$$

Theorem 2.24. There exists a unique (up to isomorphism) pair

$$T \in A_{\mathrm{mod}}, \quad g \in \mathrm{Bilin}_{M \times N}(T)$$

such that

$$\forall P \in A_{\mathrm{mod}} \;\; \forall f \in \mathrm{Bilin}_{M \times N}(P) \;\; \exists! f' \colon T \to P \;\; f = f' \circ g.$$

Note 2.25. One can rephrase Theorem 2.24 as the existence of a natural isomorphism

$$\hom(T, -) \simeq \mathrm{Bilin}_{M \times N}(-).$$

Definition 2.26. The A-module T of Theorem 2.24 is denoted by

$$M \otimes_A N$$

and called the tensor product of M and N. The map g is written as

$$m \otimes n \coloneqq g(m,n).$$

Proof. (Theorem 2.24) Let

$$T_0 = A^{M \times N}.$$

Then

$$\forall P \;\; \hom(T_0, P) \simeq \mathrm{Set}(M \times N, P) \quad \text{via} \quad f \mapsto \sum a_i f(e_i)$$

as the free module functor is adjoint to the forgetful functor to Set. Denote the generator of T_0 corresponding to (m,n) by $e(m,n)$. Now define

$$T = T_0/{\sim},$$

where $\sim$ describes the relations given in Definition 2.22. We now claim that T together with the quotient of e satisfies the universal property.

Uniqueness up to isomorphism does follow from this very universal property. □

Question 2.27. Do all representable functors have left adjoints?

Answer. Yes, provided the category considered has all coproducts, as then for functors into Set the two conditions are in fact equivalent; the same holds in the case of corepresentability and having a right adjoint.

Concretely, the left adjoint to a functor represented by X is given by

$$Y \mapsto \coprod_Y X.$$

Note that this does not agree with our definition of a tensor product. Indeed, while the representable functors considered in this question are Set-valued, the one right adjoint to the tensor product is an endofunctor of the category of A-modules.

Definition 2.28. Let k be a ring, $k \to A$ a k-algebra. For $P \in A_{\mathrm{mod}}$ we define

$$\mathrm{Der}_k(A, p) = \{d\colon A \to P \mid d(a+b) = d(a) + d(b), d(ab) = ad(b) + d(a)b, d(k) = \{0\}\}.$$

For $\phi\colon P \to R$ we get pullback

$$\mathrm{Der}_k(A, P) \to \mathrm{Der}_k(A, R) \quad \text{via} \quad d \mapsto \phi \circ d$$

Theorem 2.29. There exists an A-module

$$\Omega_{A/k}, \;\; d\colon A \to \Omega_{A/k}$$

such that

$$\forall P \in A_{\mathrm{mod}} \;\; \forall f \in \mathrm{Der}_k(A, P) \;\; \exists! f'\colon \Omega_{A/k} \to P \;\; f = f' \circ d.$$

One can rephrase this as

$$\hom_{\Omega_{A/k}} \simeq \mathrm{Der}_k(A, -).$$

Proof. Define Ω as a quotient of A^A by the obvious relations, denote

$$A^A =: \{e(a) \mid a \in A\}.$$

Give $d\colon A \to \Omega$ by $d(a) = e(a)$. □

Example 2.30. Let $A = k[x_1, \ldots, x_n]$. Then $\Omega_{A/k}$ is a free module on n generators

$$d(x_1), \ldots, d(x_n).$$

Indeed, as $\Omega_{A/k}$ is generated by

$$\{\mathrm{d}f \mid f \in A\}.$$

We claim that

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i.$$

Then $\Omega_{A/k}$ is indeed generated by

$$\mathrm{d}x_1, \ldots, \mathrm{d}x_n.$$

Consider the k-linear derivation gradient

$$\nabla\colon A \to A^{\oplus n} \quad \text{via} \quad f \mapsto (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}).$$

By the universal property of localization we obtain

$$\pi\colon \Omega_{A/k} \to S^{\oplus n} \text{ such that } \forall 1 \le i \le r \;\; \pi(\mathrm{d}x_i) = e_i.$$

If we let

$$\phi\colon S^{\oplus n} \to \Omega_{A/k} \quad \text{via} \quad e_i \mapsto \mathrm{d}x_i,$$

then ϕ is onto and because $\pi \circ \phi = \mathrm{id}_{S^{\oplus n}}$, it is a bijection (and hence, an isomorphism).

2.4 Localization of modules

Definition 2.31. The rank of a free module F is

$$\mathrm{rk}(F) \coloneqq \dim(F/\mathfrak{m}F) \text{ over } A/\mathfrak{m}.$$

This does not depend on the choice of $\mathfrak{m}$.

Example 2.32. $\mathrm{rk}(A^{\oplus I}) = |I|$.

Example 2.33. $\Omega_{A/k}$ is free of rank n.

Definition 2.34. Let $S \subseteq A$ be a multiplicative subset, $M \in \;\; \mathrm{mod}\, A$. The one forms an $S^{-1}A$-module $S^{-1}M$ analogously to $S^{-1}A$ itself. That is,

$$S^{-1}M \coloneqq \{m/s \mid m \in M, s \in S\}$$

with

$$m/s = m'/s' \iff \exists t \in S \;\; t(s'm - sm') = 0.$$

Example 2.35. If

$$S = \{1, f, f^2, \dots\},$$

then we denote

$$M_f = S^{-1}M.$$

Analogously, for

$$S = A \setminus \mathfrak{p},$$

write

$$M_{\mathfrak{p}} = S^{-1}M.$$

Definition 2.36. A finitely generated k-algebra A is smooth if

$$\forall p \in \mathrm{Spec}(A) \;\; (\Omega_{A/k})_{\mathfrak{p}} \text{ is free as an } A_{\mathfrak{p}}\text{-module}.$$

Example 2.37. $k[x_1, \dots, x_n]$ is smooth.

Lemma 2.38. Let $S \subseteq A$ be a multiplicative subset, M an A-module. Then

$$S^{-1}A \otimes_A M \cong S^{-1}M$$

under the map

$$(a/s) \otimes m \mapsto am/s.$$

Proof. The given map
$$f\colon S^{-1}A \otimes M \to S^{-1}M$$
is linear by the universal property of the tensor product. We show that it is an isomorphism.

1. f is "onto":
$$f(1/s \otimes m) = m/s,$$

2. f is "into": take an element
$$n \in S^{-1}A \otimes_A M.$$
We write
$$n = \sum_{k=1}^{r} a_k/s_k \otimes m_k = \sum_{k=1}^{r} \tilde{a}_k/s \otimes \tilde{m}_k.$$
If we let
$$s = \prod_{k=1}^{r} s_k,$$
we can write
$$\sum_{k=1}^{r} 1/s \otimes \tilde{a}_k m_k = 1/s \otimes (\sum_{k=1}^{r} \tilde{a}_k m_k) =: 1/s \otimes m.$$
Since any n is of the form, we may proceed as follows. Assume $0 = f(n)$; we will show $n = 0$. Have:
$$0 = f(n) = m/s \iff \exists t \in S \;\; tm = 0.$$
Then
$$n = 1/s \otimes m = 1/(st) \otimes tm = 1/(st) \otimes 0 = 0.$$

□

Corollary 2.39.
$$S^{-1}(\oplus_{i\in I} M_i) = \oplus_{i\in I} S^{-1} M_i.$$

Proof. □

2.5 Exactness

Definition 2.40. Let M_i be A-modules, $i = 1, 2, 3$.

1. The sequence
$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$$
is exact if
$$\operatorname{im} f_1 = \ker f_2.$$

2. The sequence
$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$
is exact if and only if
$$M_1 \to M_2 \to M_3$$
is exact and f_2 is surjective.

3. The sequence
$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$
is a short exact sequence if:
   - f_1 is injective,
   - $\operatorname{im} f_1 = \ker f_2$,
   - f_2 is surjective.

Proposition 2.41 (right-exactness of the tensor product)**.** For all exact sequences
$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$
and for all $N \in \quad \operatorname{mod} A$, the induced sequence
$$M_1 \otimes_A N \xrightarrow{f_1 \otimes_A \mathrm{id}} M_2 \otimes_A N \xrightarrow{f_2 \otimes \mathrm{id}} M_3 \otimes N \to 0$$
is exact.

Proof. First, we need to check that
$$M_2 \otimes_A N \to M_3 \otimes_A N$$
is "onto". This is checked on elements: $M_3 \otimes_A N$ is generated by elements of the form
$$m_3 \otimes n \quad \text{such that} \quad m_3 \in M_3, n \in N.$$
The claim then follows by surjectivity of f_2 itself.

Exactness in the middle means that
$$M_2 \otimes N/(f_1 \otimes \mathrm{id})(M_1 \otimes N) \to M_3 \otimes N$$
is an isomorphism. We already checked that it is "onto"; now we need to know that there exists a section, that is, a one-sided inverse.

Have
$$M_2/f_1(M_1) \cong M_3 \times N \to M_2 \otimes_A N/(f_1 \otimes \mathrm{id})(M_1 \otimes_A N) \text{ via } (m_3, n) \mapsto \overline{m_2 \otimes n}.$$
This is bilinear if it is well-defined. Hence, it yields a map
$$s\colon M_3 \otimes_A N \to M_2 \otimes N$$
and indeed, the equality
$$s \circ \pi = \mathrm{id}_{M_3 \otimes N}$$
holds, and so π is injective and further, an isomorphism. □

Note 2.42*.* We have shown that
$$M_3 \cong M_2/f_1(M_1).$$
Hence, we get
$$M_2/(f_1(M_1) \otimes N \cong M_2 \otimes_A N/(f_1(M_1) \otimes_A N).$$

Example 2.43. $A/I \otimes_A N \cong N/IN$.

And in fact, quotients and localizations of modules can be computed by means of the tensor product.

Note 2.44*.* The notions of right-exactness as defined above in Proposition 2.41 coincide with the preserving of colimits. The definition of Proposition 2.41 is valid in the case of preadditive categories.

2.6 Why do we care?

First of all, as noted before, we have the folowing isomorphisms:

$$M \otimes_A A/I \cong M/IM,$$

$$M \otimes_A S^{-1}A \cong S^{-1}M.$$

Moreover:

Proposition 2.45. Tensoring with an A-algebra B is left adjoint to the forgetful functor

$$B_{\text{mod}} \to A_{\text{mod}}.$$

Proposition 2.46. Suppose A-algebras B, C are given. Then

$$B \otimes_A C$$

is an A-algebra with multiplication

$$(b_1 \otimes c_1) \cdot (b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2).$$

We claim that

$$B \otimes_A C$$

is in fact the coproduct of B and C.

Note 2.47. In the same setting, one has the diagram

$$\begin{tikzcd}
& \mathrm{Spec}(B) \arrow[dl] & \\
\mathrm{Spec}(A) & & \mathrm{Spec}(B \otimes_k C) \arrow[ul] \arrow[dl] \\
& \mathrm{Spec}(C) \arrow[ul] &
\end{tikzcd}$$

Note that then

$$\mathrm{Spec}(B \otimes_k C) \neq \mathrm{Spec}(B) \times \mathrm{Spec}(C);$$

we only get the arrows demanded that make the diagram commute.

We note the following special cases of Proposition 2.46.

Corollary 2.48. Let $A = k$ a field,

$$B = k[x_1, \ldots, x_s], \quad C = k[y_1, \ldots, y_k].$$

Then

$$B \otimes_k C \cong k[x_1, \ldots, x_s, y_1, \ldots, y_t].$$

Proof. Note that B and C are free k-modules, so $B \otimes_k C$ is a free k-module with basis

$$x_1^{a_1} \otimes \ldots \otimes x_s^{a_s} \otimes y_1^{b_1} \otimes \ldots \otimes y_t^{b_t}.$$

One checks that the obvious map defined by the universal property is an isomorphism. $\square$

Corollary 2.49. Suppose

$$B = k[x_1, \dots, x_s]/I,$$
$$C = k[y_1, \dots, y_t]/J.$$

Then

$$B \otimes_k C \cong k[x_1, \dots, x_s, y_1, \dots, y_t]/(I + J).$$

In differential geometry, the fiber of a map $f\colon X \to Y$ over a point is not typically a manifold. In contrast, in the Spec-world, we have the following.

Definition 2.50. Let $f\colon A \to B$ be a homomorphism, $\mathfrak{m} \subseteq A$ a maximal ideal. Then the *fiber* of f over $\mathfrak{m}$ is

$$B/\mathfrak{m}B.$$

Proposition 2.51. In the above setting,

$$\operatorname{Spec}(B/\mathfrak{m}B) \cong \{\mathfrak{p} \subseteq B \mid f^*(\mathfrak{p}) = \{\mathfrak{m}\}\}.$$

Note that

$$\operatorname{Spec}(A_\mathfrak{p}) = \{\mathfrak{q} \subseteq \mathfrak{p}\}, \quad \operatorname{Spec}(A/\mathfrak{p}) = \{\mathfrak{q} \supseteq \mathfrak{p}\}.$$

Definition 2.52. More generally, if $f\colon A \to B$ is a homomorphism, $\mathfrak{p} \subseteq B$ a prime ideal, then the *fiber* of f over $\mathfrak{p}$ is given by

$$B \otimes_A \kappa(\mathfrak{p}),$$

where

$$\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$$

is the *residue field*.

Proposition 2.53. For any prime ideal $\mathfrak{p} \subseteq B$, $\kappa(\mathfrak{p})$ is a field.

Note 2.54. If $\mathfrak{m}$ is maximal, then

$$\kappa(\mathfrak{m}) \cong A/\mathfrak{m}.$$

Proposition 2.55. There is a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(B) \arrow[d, "f^*"] & \operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p})) \arrow[l] \arrow[d] \\
\operatorname{Spec}(A) & \operatorname{Spec}(\kappa(\mathfrak{p}) \arrow[l]
\end{tikzcd}$$

3 Properties of rings and modules

3.1 Noetherian modules

Proposition 3.1. Let $M \in \mod A$. Then the following conditions are equivalent:

1. every submodule of M is finitely generated,
2. every sequence of submodules
$$M_1 \subseteq M_2 \subseteq \ldots \subseteq M$$
stabilises, that is,
$$\exists n_0 \in \mathbb{N} \ \forall n \geq n_0 M_n - M_{n_0},$$
3. every family of submodules of M has a maximal element with respect to inclusion.

Proof. (Proposition 3.1): The implication "2 $\implies$ 3" follows from Kuratowski-Zorn.

For "3 $\implies$ 2", take $\{M_i\}_{i\in\mathbb{N}}$ as the family in statement of 3; then the maximal element is also necessarily the one on which the sequence stabilises.

"2 $\implies$ 1": choose $N \subseteq M$ a submodule. Further, take

$$n_1 \in N \setminus \{0\}, \ \ n_2 \in N \setminus A_{n_1}, \ \ n_3 \in N \setminus A_{n_2} \oplus A_{n_3}, \ \ \ldots.$$

Either one can do this for all $k \in \mathbb{N}$ or not. In the latter case, N is necessarily finitely generated. In the former, we get the sequence

$$A_{n_1} \subseteq A_{n_1} + A_{n_2} \subseteq A_{n_1} + A_{n_2} + A_{n_3} \subseteq \ldots$$

that does not stabilize.

"1 $\implies$ 2": take a sequence
$$M_1 \subseteq M_2 \subseteq \ldots;$$
we want to stabilise it. To that end, consider

$$N = \bigcup_{n\in\mathbb{N}} M_n.$$

Since the sequence is increasing, this is a submodule of M.

By 1, this is finitely generated, and by a finite set of elements from N. Then this very set is already contained in one of the M_k, and then for $n > k$, $M_n = M_k$. □

Definition 3.2. An A-module M is called Noetherian if it satisfies the equivalent conditions of Proposition 3.1.

Definition 3.3. A ring A is Noetherian if and only if the A-module A is Noetherian.

Proposition 3.4. Let A be a ring. The following conditions are equivalent:

1. A is Noetherian,

2. every ideal of A is finitely generated,
3. every prime ideal of A is finitely generated.

■

The implication "3 $\implies$ 1" of Proposition 3.4 is due to Cohen, 1950.

Proposition 3.5. Let $N \subseteq M$ be A-modules. Then the following conditions are equivalent:

1. M is Noetherian,
2. N and M/N are Noetherian.

Proof. We start with "1 $\implies$ 2". First, we show that N is Noetherian; indeed, if

$$N_1 \subseteq N_2 \subseteq \ldots \subseteq N \subseteq M$$

793124 is a sequence of submodules of N, it is also a sequence of submodules of M and the claim follows.

To show that M/N is Noetherian, consider a sequence of submodules

$$P_1 \subseteq P_2 \subseteq M/N.$$

If then $\pi\colon M \to M/N$ denotes the quotient map, then the sequence

$$\pi^{-1}(P_1) \subseteq \pi^{-1}(P_2) \subseteq \ldots \subseteq M$$

stabilizes, and since π is surjective, we have

$$\pi(\pi^{-1}(P_n)) = P_n,$$

ending the proof of "1 $\implies$ 2".

For "2 $\implies$ 1", pick a sequence of submodules

$$M_1 \subseteq M_2 \subseteq \ldots \subseteq M$$

and define

$$N_k = M_k \cap N, \quad P_k = \pi(M_k).$$

The sequences defined by the N_k and P_k stabilize, say at common n_0.

We will show that

$$\forall n > n_0 \;\; M_n = M_{n_0}.$$

First, show $M_n \subseteq M_{n_0}$. Take $m \in M_n$, then

$$\pi(m) \in P_n = P_{n_0} = M_{n_0}/\ker\pi \cap M_{n_0}$$

Pick $\tilde{m} \in M_{n_0}$ such that $\pi(m) = \pi(\tilde{m})$. Then we have

$$m - \tilde{m} \in \ker(\pi) \cap M_n = N_n = N_{n_0} \subseteq M_{n_0}.$$

Since $m - \tilde{m} \in M_{n_0}$, we have that in fact

$$m = \tilde{m} + m - \tilde{m} \in M_{n_0}$$

and the claim follows since m was arbitrary in M_n. □

Note that this translates to the claim that in the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

the middle term is Noetherian if and only if the other ones are.

Corollary 3.6. If A is a Noetherian ring, then all finitely generated A-modules are Noetherian.

Proof. First, we show that all finitely generated free modules are Noetherian. This follows easily by induction since we get short exact sequences

$$0 \to A \to A^{\oplus n} \to A^{\oplus(n-1)} \to 0.$$

If then K is a finitely generated A-module, say generated by k elements, we get a short exact sequence

$$0 \to \ker \alpha \to A^{\oplus k} \xrightarrow{\alpha} K \to 0,$$

and the claim follows from Proposition 3.5 again. □

Example 3.7. Some Noetherian rings include:

1. fields,
2. principal ideal domains (e.g. $\mathbb{Z}$, $k[x]$, $\mathbb{Z}[x]$).

Theorem 3.8 (Hilbert basis theorem)**.** If A is Noetherian, then

$$A[x_1, \ldots, x_n]$$

is also a Noetherian ring.

Note the difference: when considered as a $\mathbb{C}$-module, $\mathbb{C}[x]$ is not Noetherian, as it is not finitely generated. However, Theorem 3.8 states that as a module over itself, it is Noetherian.

Corollary 3.9. If A is Noetherian and B is a finitely generated A-algebra, then B is also Noetherian.

Proof. Immediate, since

$$B = A[x_1, \ldots, x_n]/I.$$

□

Question 3.10. Are tensor products of Noetherian modules also Noetherian?

Answer. No, since

$$\bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$$

is not Noetherian. In fact,

$$\operatorname{Spec}(\bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}) \simeq \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

is a natural isomorphism of sets (and not of groups).

Lemma 3.11. If the ring A is Noetherian, then so is A/I for all ideals $I \subseteq A$.

Lemma 3.12. If A is Noetherian, then for any multiplicative subset $S \subseteq A$, the localization $S^{-1}A$ is also Noetherian.

Proof. Consider $I \subseteq S^{-1}A$, $J = i^{-1}(I) \subseteq A$. Write

$$I = \{\frac{j}{s} \mid j \in J, s \in S\}.$$

The ideal $J \subseteq A$ is finitely generated, say

$$J = A(j_1, \dots, j_r).$$

Then

$$\forall j \in J \;\; \exists a_1, \dots, a_r \;\; j = \sum_{k=1}^{r} a_k j_k.$$

We then have

$$\frac{j}{s} = \sum_{k=1}^{r} \frac{a_k}{s} \cdot \frac{j_k}{1},$$

and so I is generated by

$$\frac{j_1}{1}, \dots, \frac{j_r}{1}.$$

□

Note that we have used the previous characterization of Proposition 3.1; indeed, as the submodules of a ring are exactly the ideals.

Proof. (Hilbert basis theorem) Let $I \subseteq A[x]$. Write

$$J = \{a \in A \mid \exists f \in I \;\; f = ax^n + a_1x^{n-1} + \dots\},$$

that is, J is the set of polynomials with leading term ax^n.

One claims that J is an ideal: indeed, let $j_1, j_2 \in J$. Put

$$r = j_1 - j_2.$$

If $r = 0$, we are done; in the other case, proceed. Take $f_1, f_2 \in I$ with leading term $f_i = j_i x^{n_i}$. Then the leading term of

$$x^{n_2} f_1 - x^{n_1} f_2$$

is

$$(j_1 - j_2)x^{n_1+n_2}.$$

Hence, J is an abelian subgroup.

If $j \in J, a \in A$, and so J is an ideal.

Since $J \subseteq A$ is an ideal of a Noetherian ring, it is finitely generated, say by elements

$$j_1, \dots, j_r.$$

One then has

$$\exists f_1, \dots, f_r \in I$$

with leading terms of f_i equal to $j_i x^{n_i}$.

If we let $n = \max(n_i)$, one can modify the f_i to get leading term $f_i = j_i x^n$.

Let

$$SP = I \cap A[x]_{<n}.$$

This is not an ideal, but an A-module; it is isomorphic to $A^{\oplus n}$, and so finitely generated, we write

$$SP = Ag_1 + \ldots + Ag_t.$$

We claim that

$$I = (f_1, \ldots, f_r, g_1, \ldots, g_t)$$

as an ideal in $A[x]$.

Clearly,

$$I \supseteq (f_1, \ldots, f_r, g_1, \ldots, g_t).$$

For the other inclusion, we pick $h \in I$ and proceed by induction on degree.

Write the leading term of h as

$$bx^{\deg(h)}.$$

Now, if $\deg(h) < n$, h must necessarily be an element of SP.

In the other case,

$$\deg(h) \geq n$$

and we can divide with remainder. Write

$$b = \sum_{s=1}^{r} a_s j_s$$

and consider

$$h' = h - \sum a_s f_s x^{\deg(h)-n}.$$

Then

$$\deg(h') < \deg(h) \implies h' \in (f_1, \ldots, f_r, g_1, \ldots, g_t)$$

and so also

$$h \in (f_1, \ldots, f_r, g_1, \ldots, g_t).$$

This ends the proof. □

3.2 Finite and integral ring extensions

Consider B an A-algebra.

Definition 3.13. An element $b \in B$ is *integral* over A if

$$\exists a_{n-1}, \ldots, a_0 \in A \;\; b^n + a_{n-1}b^{n-1} + \ldots + a_0 = 0.$$

B is integral over A if all its elements are integral over A.

Definition 3.14. An A-algebra B is *finite* over A if it is finitely generated as an A-module, that is,

$$\exists b_1 \ldots, b_k \;\; B = Ab_1 + \ldots + Ab_k.$$

Example 3.15. The following hold:

1. $\mathbb{Z} \to \mathbb{Z}[i]$ is finite,
2. $\mathbb{Z} \to \mathbb{Z}/n$ is finite,
3. more generally, $A \to A/I$ is finite,

4. $\mathbb{Z} \to \mathbb{Z}[x]$ is not finite.

Lemma 3.16. If $A \to B$ is finite, then it is integral.

Proof. Let $b \in B$ and consider the multiplication map

$$\phi\colon B \to B, \quad \phi(a) = a \cdot b.$$

This is an A-module homomorphism. By Cayley-Hamilton with $I = (1)$, one has

$$\exists a_{n-1}, \dots, a_0 \in A \;\; b^n + a_{n-1}b^{n-1} + \ldots + a_0 = 0.$$

□

Lemma 3.17. If $A \to B$ is integral and B is a finitely generated A-algebra, then $A \to B$ is finite.

Proof. Let

$$B = A[x_1, \dots, x_n]/I.$$

Then every $x_i \in B$ is integral over A and so

$$\exists n \;\; \forall i \;\; \exists a_0^{(i)} \;\; x_i^n + a_{n-1}^{(i)} + \ldots + a_0^{(i)} = 0.$$

We check that B is generated as an A-module by the finitely many monomials

$$\{x_1^{c_1}, \dots, x_n^{c_n}, 0 \leq c_i \leq n\}.$$

□

Example 3.18. The extension

$$\mathbb{Q} \to \bar{\mathbb{Q}}$$

is integral, but not finite.

Definition 3.19. For B and A-algebra, $b \in B$ an element, $A[b]$ will denote the smallest A-algebra contained in B and containing $b \in B$.

Proposition 3.20. Let $A \to B$ be an A-algebra, $b \in B$. The following conditions are equivalent:

1. v is integral over A,
2. $A[b]$ is a finitely generated A-module,
3. $A[b]$ is contained in a finitely generated A-module

Proof. "1 $\implies$ 2 $\implies$ 3" is formal: the first one uses the same trick as in Lemma 3.17.
For "3 $\implies$ 1, fix a finitely generated A-module $A[b] \subseteq C$. Then the multiplication map

$$\phi_b\colon C \to C$$

is an $A-$module homomorphism, so Cayley-Hamilton implies

$$\exists n \exists a_{n-1}, \dots, a_0 \in A \;\; b^n + a_{n-1}b^{n-1} + \ldots + a_0 = 0.$$

□

Definition 3.21. Let B be an A-algebra. Then the *integral closure* of A in B is

$$\bar{A} = \{b \in B \mid b \text{ is integral in } A\}.$$

The *normalization* of a domain A is its integral closure in the field of fractions $\mathrm{Frac}(A)$.

Corollary 3.22. $\bar{A}$ is an A-algebra.

Proof. Consider $x, y \in \bar{A}$. By Proposition 3.20, $A[x]$ and $A[y]$ are finitely generated A-modules. In fact, by the proof of Lemma 3.17, $A[x, y]$ is a finitely generated A-module.

One has

$$A[x-y] \subseteq A[x,y], \quad A[xy] \subseteq A[x,y],$$

so by Proposition 3.20, point 3,

$$x - y, xy \in \bar{A}.$$

Hence, $\bar{A}$ is a ring. One has also the map

$$A \to \bar{A}$$

which makes $\bar{A}$ an A-algebra; indeed, an A-subalgebra of B. □

Example 3.23. Let $\mathbb{Z} \to \mathbb{Q} \to K$ with $\mathbb{Q} \to K$ finite. Then one has also

$$O_K = \bar{Z} \to K$$

- the ring of algebraic integers in K. This is Noetherian.

Theorem 3.24 (Nagata)**.** If A is finitely generated over k, then the normalization of A is as well. Moreover,

$$A \to \bar{A}$$

is finite.

3.3 Further properties of integral extensions

Recall that we have talked about the fiber and noted an isomorphism of A-algebras $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ and $\mathrm{Frac}(A/\mathfrak{p})$; we denoted this result by $\kappa(\mathfrak{p})$.

As motivation for the following, consider the diagram

$$\begin{tikzcd} A \arrow[r, "\text{integral}"] \arrow[d] & B \arrow[d] \\ A/\mathfrak{p} \arrow[r, "\text{integral}"] & B/\mathfrak{q} \end{tikzcd}$$

with $\mathfrak{p}$, $\mathfrak{q}$ prime.

Lemma 3.25. Let $f\colon A \hookrightarrow B$ be an integral extension and let f be injective. Suppose that A and B are domains. Then A is a field if and only if B is a field.

Proof. Assume that A is a field. Take $0 \neq b \in B$ and minimum n such that

$$b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0.$$

Then a_0 cannnot be zero since n is minimal. Thus, it is invertible (as A is a field). One can then write

$$b(b^{n-1} + \cdots + a_2 b + a_1) = -a_0,$$

so that b is invertible. Hence, A is a field.

Conversely, supose B is a field. Let $a \in A \setminus \{0\}$, so that for some $b \in B$, $ab = 1$ holds. Let

$$b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0$$

and multiply both sides by a^{n-1}. We then have

$$b(ab)^{n-1} + a_{n-1}(ab)^{n-1} + \cdots + a_1(ab)a^{n-2} + a_0 a^{n-1} = 0.$$

Thus, for some $a' \in A$, one has $b + a = 0$. Thus $b \in A$ and so a is invertible in A. □

Proposition 3.26. Suppose $f \colon A \hookrightarrow B$ is integral with f injective. Then

$$f^* \colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$$

is surjective.

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(A)$, $S = A \setminus \mathfrak{p}$. An exercise shows that

$$S^{-1}f \colon S^{-1}A \hookrightarrow S^{-1}B$$

is integral and injective. In particular, $S^{-1}B$ is nonzero. Thus, there exists a maximal ideal $\mathfrak{m} \subseteq S^{-1}B$. If we now let $\mathfrak{p}'$ be the preimage of $\mathfrak{p}$, then by Lemma 3.25, $S^{-1}A/\mathfrak{p}'$ is a field.

In particular, $A_\mathfrak{p} = S^{-1}A$ has only one maximal ideal $\mathfrak{p}A_\mathfrak{p}$ and we get $\mathfrak{p}' = mathfrakpA_\mathfrak{p}$.

If we now tahe $\mathfrak{m} \subseteq B$ to be the preimage of $\mathfrak{m} \subseteq S^{-1}B$, then $f^*(\mathfrak{m}) = \mathfrak{p}$. □

Note 3.27. For $f \colon A \to A/\mathfrak{p}$ integral, $\operatorname{Spec}(A) \leftarrow \operatorname{Spec}(A/\mathfrak{p}$ is not usually surjective.

Proposition 3.28 (incomparability)**.** Suppose $f \colon A \to B$ is integral, $\mathfrak{p} \in \operatorname{Spec}(A)$. Let $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$, $\mathfrak{q}_i \in \operatorname{Spec}(B)$ such that $f^*(\mathfrak{q}_1) = f^*(\mathfrak{q}_2) = \mathfrak{p}$. Then $\mathfrak{q}_1 = \mathfrak{q}_2$.

Proof. Take $D \coloneqq \kappa(\mathfrak{p}) \otimes_A B$. Consider the integral extensions

$$\kappa(\mathfrak{p}) \to D \to D/\mathfrak{q}_i D.$$

By Lemma 3.25, from $\kappa(\mathfrak{p}$ being a field it follows that $D/\mathfrak{q}_i D$ is also a field, and hence $\mathfrak{q}_i D \subseteq D$ is maximal. But $\mathfrak{q}_1 D \subseteq \mathfrak{q}_2 D$ and so $\mathfrak{q}_1 D = \mathfrak{q}_2 D$.

Then the claim follows, as $\mathfrak{q}_i = g^{-1}(\mathfrak{q}_i D)$, where $g \colon B \to D$ is the natural map. □

3.4 Krull dimension

Definition 3.29. Let A be a ring. A *chain of prime ideals* in A is a sequence of proper inclusions

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_s$$

of prime ideals.

The *length* of such a chain is the number of those inclusions, that is, a number smaller by one than the number of ideals in the chain.

Definition 3.30. The *Krull dimension* of A is

$$\dim A = \sup\{\text{ length of a chain of prime ideals in } A\}.$$

Example 3.31.

1. If k is a field, then $\dim k = 0$.
2. If A is a principal ideal domain, then $\dim A = 1$, since nonzero prime ideals are maximal.

The following result is more difficult to prove.

Theorem 3.32. If k is a field, then

$$\dim(k[x_1, \dots, x_n]) = n.$$

A chain of length n is given as

$$(0) \subseteq (x_1) \subseteq (x_1, x_2) \subseteq \dots \subseteq (x_1, \dots, x_n).$$

However, $k[x_1, \dots, x_n]$ has many more prime ideals than those.

Example 3.33. Let $f = x_n^3 + x_n^2 x_1$. One can take $x_i' = x_i$ for $i = 1, 2, \dots, n-1$. Then

$$x_n^3 + x_1' x_n^2 - f = 0$$

and the extension $k[x_1', \dots, x_{n-1}', x_n \cdot f] \subseteq k[x_1, \dots, x_n]$ is finite.

This approach can be extended in a way allowing us to prove Theorem 3.32.

Lemma 3.34 (Nagata's trick)**.** Let $f \in S = k[x_1, \dots, x_n]$, $f \notin k$. Then there exist $x_1', x_2', \dots, x_{n-1}' \in S$ such that

$$k[x_1', x_2', \dots, x_{n-1}' \cdot f] \subseteq k[x_1, \dots, x_n]$$

is a finite extension.

Proof. Fix $e > \deg(f)$ and assume without loss of generality that x_n appears in f. Consider a k-algebra homomorphism

$$\phi\colon S \to S \quad \text{via} \quad x_i \mapsto x_i - x_n^{e^{n+1-i}}, \;\; x_n \mapsto x_n.$$

We claim that ϕ is an isomorphism with inverse

$$\phi^{-1}\colon S \to S \quad \text{via} \quad x_i \mapsto x_i + x_n^{e^{n+1-i}} \quad x_n \to x_n.$$

Moreover, $\phi(f) = \pm x_n^D +$ monomials of a smaller total degree.

Indeed, one sees that distinct monomials are mapped by ϕ to polynomials with different top degree of x_n. More specifically, if $m = x_1^{a_1} \cdot \dots \cdot x_n^{a_n}$ is such that $a_n > 0$ and m is lexicographically the greatest possible of all monomials in f, then

$$D = \sum_{i=0}^{n} a_i e^{n-i}.$$

The process is akin to coding the variables in base e, and our claim is that decoding remains possible.

One takes $x_i' = \phi^{-1}(x_i)$. Then all x_i are integral over $k[x_1', \dots, x_{n-1}', \phi(f)$. One may look at the diagram

$$\begin{tikzcd}
k[x_1, \dots, x_{n-1}, \phi(f)] \arrow[r, "\text{integral}"] \arrow[d, "\phi^{-1}"] & k[x_1, \dots, x_n] \arrow[d, "\phi^{-1}"] \\
k[x_1', \dots, x_{n-1}', f] \arrow[r] & k[x_1, \dots, x_n]
\end{tikzcd}$$

The horizontal inclusions are integral, and because B is finitely generated over A, the extension must be finite by Lemma 3.17. $\square$

Proposition 3.35. If $A \hookrightarrow B$ is integral, then $\dim A = \dim B$.

Proof. First, we show $\dim A \leq \dim B$. Take a chain of prime ideals in A

$$p_0 \subsetneq p_1 \subsetneq \ldots \subsetneq \mathfrak{p}_s.$$

We will lift it to a chain in B. We shall proceed by induction:

- The base case follows from surjectivity of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$.
- For the induction step, consider a partial lift

  $$\mathfrak{q}_0 \subsetneq \ldots \subsetneq \mathfrak{q}_k.$$

  We wish to find $\mathfrak{q}_{k+1}$ such that $\mathfrak{q}_{k+1} \cap A = \mathfrak{p}_{k+1}$. Consider $A/\mathfrak{p}_k \to B/\mathfrak{q}_k$; this is injective, and so it is surjective by . Then one picks a preimage of p_{k+1} in $B/\mathfrak{q}_k$.

For $\dim A \geq \dim B$, take

$$\mathfrak{q}_0 \subsetneq \ldots \subsetneq q_s$$

a chain in B; take $p_i = \mathfrak{q}_i \cap A$. Then

$$\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_s$$

is a chain in A, as if $\mathfrak{p}_i = \mathfrak{p}_{i+1}$, $\mathfrak{q}_i \cap A = \mathfrak{q}_{i+1} \cap A$ and then by incomparability $\mathfrak{q}_i = \mathfrak{q}_{i+1}$. □

Proof. (Theorem 3.32) Proceed by induction on n. Base $n = 0$ is fine.

Now assume that we have shown the claim for $n-1$ want to prove for n. Suppose that this is false, and so we get a chain of length $n+1$. For $i \geq 1$, pick $f \in \mathfrak{p}_i$. Use Nagata's trick to get a finite extension

$$A = k[x'_1, \ldots, x'_{n-1} \cdot f] \subseteq k[x_1, \ldots, x_n].$$

Take a preimage of a chain on the right side in the left side. Consider a quotient map

$$k[y_1, \ldots y_{n-1}] \to k[x'_1, \ldots, x'_n \cdot f]/(f).$$

Put $\bar{\mathfrak{q}_i} = \mathfrak{q}_i/f$. If we let $\mathfrak{r}_i$ to be the preimage of $\bar{\mathfrak{q}_i}$ in $k[y_1, \ldots, y_{n-1}]$, then

$$\mathfrak{r}_1 \subsetneq \mathfrak{r}_2 \subsetneq \ldots \subsetneq \mathfrak{r}_{n+1}$$

is a chain of length n, in contradiction of the inductive assumption. □