

Lecture notes for

Commutative Algebra

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October 31, 2019

1 Preface

The following is a set of personal notes that have been taken during the lecture given by dr Joachim Jelisiejew as part of a WS2019/2020 course in commutative algebra (*Algebra przemienna*) at the faculty of Mathematics, Informatics and Mechanics, University of Warsaw.

The text of these notes was compiled by Jakub Paliga, who does not guarantee their correctness and disclaims any warranties when applicable. Indeed, he asserts any deficiency within, of which there is a nonzero number, to have been introduced by him in the process.

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1 Basics

1.1 Conventions

Rings will be understood to be commutative, associative, unitary; the ring "0=1" is considered a ring. Ring homomorphisms are assumed to preserve the unit.

k will denote a field, \bar{k} its algebraic closure. Typically, this field will be \mathbb{C} - the complex numbers - but not always.

Ideals will be denoted by I and J . Ideals satisfying additional properties will be written in **fraktur** and named distinctly: prime ideals are \mathfrak{p} , \mathfrak{q} ; maximal ideals, \mathfrak{m} , \mathfrak{n} .

1.2 First definitions

Definition 1.1. Let A be a ring. Then an A -algebra is a ring B together with a fixed homomorphism

$$A \rightarrow B.$$

This homomorphism is called the *structural homomorphism*.

Definition 1.2. Let A be a ring. A *homomorphism of A -algebras* is a ring homomorphism commuting with the structural maps, that is, if

$$\phi : A \rightarrow B, \quad \psi : A \rightarrow C$$

are A -algebras, then a ring homomorphism $f : B \rightarrow C$ is an algebra homomorphism if

$$\psi = f \circ \phi.$$

Example 1.3. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be given by $z \mapsto \bar{z}$ (the complex conjugation). Then:

- f is a ring homomorphism,
- f is a real algebra homomorphism,
- f is not a complex algebra homomorphism.

Example 1.4. The polynomial ring $\mathbb{C}[x]$ becomes a \mathbb{C} -algebra under the inclusion map onto the zero degree component.

In this case, if I is an ideal, then the quotient map

$$\mathbb{C}[x] \rightarrow \mathbb{C}[x]/I$$

is a complex algebra homomorphism.

Definition 1.5.

1. A ring A is a domain if

$$\forall a, b \in A \quad (ab = 0 \implies a = 0 \vee b = 0.)$$

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2. An ideal I in a ring B is prime if B/I is a domain.

3. A ring A is a field if

$$\forall 0 \neq a \in A \quad \exists b \in A \quad ab = 1,$$

that is, every nonzero element has a multiplicative inverse.

4. An ideal I in a ring B is maximal if B/I is a field.

1.3 Motivations

Lemma 1.6. Let $f : A \rightarrow B$ be a ring homomorphism. If $\mathfrak{p} \subset B$ is prime, then $f^{-1}(\mathfrak{p}) \subset A$ is prime.

Proof. Consider the following diagram.

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A/f^{-1}(\mathfrak{p}) & \longrightarrow & B/\mathfrak{p} \end{array}$$

The preimage of an ideal is an ideal; moreover, the lower vertical map is injective. Hence, $A/f^{-1}(\mathfrak{p}) \subseteq B/\mathfrak{p}$ is a subring. If then \mathfrak{p} is prime, B/\mathfrak{p} is a domain; its every subring is then a domain as well. Thus, $A/f^{-1}(\mathfrak{p})$ is a domain, and so $f^{-1}(\mathfrak{p}) \subset A$ is prime. \square

Example 1.7. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ has $f^{-1}(0) = 0$ not maximal, even though $0 \in \mathbb{Q}$ is maximal. Ergo, Lemma 1.6 does not hold with "prime" replaced by "maximal".

Example 1.8. Consider

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}, \quad A = C(S^1, \mathbb{R}) = \{ \text{continuous real functions on the circle} \}.$$

Then there is a bijection between S^1 and maximal ideals in A , given by every maximal ideal being of the form

$$\mathfrak{m}_x = \{f \in A \mid f(x) = 0\}.$$

Moreover, one can recover the topology of S^1 from $A = C(S^1, \mathbb{R})$.

Indeed, for $f \in A$, consider the vanishing set

$$V(f) = \{\mathfrak{m} \text{ maximal ideal in } A \mid f \in \mathfrak{m}\}.$$

The topology is then generated by closed subsets with subbase

$$\{V(f) \mid f \in A\}.$$

1.4 The Spec() functor

We aim to repeat the previous considerations of Example 1.8 for an arbitrary ring.

Definition 1.9. Let A be a ring. The *spectrum* of A is its set of prime ideals:

$$\text{Spec}(A) = \{\mathfrak{p} \text{ a prime ideal in } A\}.$$

Example 1.10. $\text{Spec}(\mathbb{C}[x]) = \{0\} \cup \{(x - a) \mid a \in \mathbb{C}\}$.

For a ring homomorphism $f: A \rightarrow B$, Lemma 1.6 asserts that there is a map

$$f^*: \text{Spec}(B) \rightarrow \text{Spec}(A)$$

given by

$$B \supseteq \mathfrak{q} \mapsto f^{-1}(\mathfrak{q}) \subseteq A.$$

We will now move towards upgrading spectra of rings (which up to this point we considered as mere sets) to topological spaces in such a way that makes the maps f^* continuous. In this, we follow the previous motivation.

Definition 1.11. For $E \subseteq A$ an arbitrary subset, we define the *vanishing locus* of E as the set

$$V(E) = \{\mathfrak{p} \in \text{Spec}(A) \mid E \subseteq \mathfrak{p}\}.$$

Proposition 1.12. The sets $V(E)$, $E \subseteq A$, are the closed subsets of a topology.

Proof. One sees immediately that

$$V(1) = \emptyset, \quad V(0) = \text{Spec}(A).$$

Elementary set-theoretic considerations reveal that

$$\bigcap_{i \in I} V(E_i) = V(\bigcup_{i \in I} E_i).$$

It now suffices to show that

$$V(E_1) \cup V(E_2) = V(E_1 \cdot E_2), \quad E_1 \cdot E_2 := \{e_1 \cdot e_2 \mid e_1 \in E_1, e_2 \in E_2\}.$$

The inclusion $V(E_1) \cup V(E_2) \subseteq V(E_1 \cdot E_2)$ is easy to verify. Suppose without loss of generality that $\mathfrak{p} \in V(E_1)$, that is $\mathfrak{p} \supseteq E_1$. Then

$$E_1 \cdot E_2 \subseteq \mathfrak{p} \cdot E_2 \subseteq \mathfrak{p},$$

because \mathfrak{p} is an ideal.

In order to prove the other inclusion, we need to use that \mathfrak{p} is prime. Suppose that

$$\mathfrak{p} \not\supseteq E_1 \quad \text{and} \quad \mathfrak{p} \not\supseteq E_2.$$

Then there exist $e_i \in E_i \setminus \mathfrak{p}$, $i = 1, 2$. For those

$$e_1 \cdot e_2 \in E_1 \cdot E_2 \setminus \mathfrak{p}$$

holds because \mathfrak{p} is prime. □

Definition 1.13. The topology on $\text{Spec}(A)$ defined by Proposition 1.12 is called the *Zariski topology*.

Question 1.14. What is the Zariski topology for $A = \mathbb{C}[x]$?

Answer. We note that

$$\begin{aligned} V(f_1, \dots, f_r) &= \{\mathfrak{p} \text{ prime in } A \mid f_1, \dots, f_r \in \mathfrak{p}\} \\ &= \{(x - a) \text{ maximal in } A \mid f_1, \dots, f_r \in (x - a)\} \\ &= \{(x - a) \mid f_1(a) = \dots = f_r(a) = 0\}. \end{aligned}$$

If any f_i is nonzero, then $0 \notin V(f_1, \dots, f_r)$.

Hence,

$$V(f_1, \dots, f_r) = \text{set of common roots of } f_1, \dots, f_r$$

and one sees that $V(f_1, \dots, f_r)$ is finite.

In fact, all finite subsets of the set of maximal ideals are of the form $V(f_1, \dots, f_r)$. Note that the only closed set containing the prime ideal 0 is $\text{Spec}(A)$ itself; in other words, 0 lies in every nonempty open set. This yields the cofinite topology augmented by $\{0\}$.

Note 1.15. Because the closure of $\{0\} \subseteq \text{Spec}(A)$ is $\text{Spec}(A)$ itself, the resulting space is not T1, let alone Hausdorff. It is, however, T0.

Proposition 1.16. Let $f: A \rightarrow B$ be a ring homomorphism. Then the induced map

$$f^*: \text{Spec}(B) \rightarrow \text{Spec}(A) \quad \text{with} \quad \mathfrak{q} \mapsto f^{-1}(\mathfrak{q})$$

is continuous.

Proof. It suffices to see that the preimage of every closed set is closed. Indeed, we claim that

$$(f^*)^{-1}(V(E)) = V(f(E)).$$

We unravel the definitions:

$$\begin{aligned} (f^*)^{-1}(V(E)) &= \{\mathfrak{p} \subseteq B \mid f^*(\mathfrak{p}) \in V(E)\} \\ &= \{\mathfrak{p} \subseteq B \mid f^{-1}(\mathfrak{p}) \in V(E)\} \\ &= \{\mathfrak{p} \subseteq B \mid E \subseteq f^{-1}(\mathfrak{p})\} \\ &= \{\mathfrak{p} \subseteq B \mid f(E) \subseteq \mathfrak{p}\} \\ &= V(f(E)). \end{aligned}$$

Note that the considerations here are purely set-theoretical. The condition that f be a ring homomorphism is only relevant for f^* to be well defined on spectra. \square

The upshot is: when $k = \bar{k}$, $\text{Spec}_{\max}(k[x_1, \dots, x_n])$ is well-behaved. We will see later, in the Nullstellensatz, that all maximal ideals in $k[x_1, \dots, x_n]$ are of the form $(x_1 - a_1, \dots, x_n - a_n)$.

In fact, $\text{Spec}_{\max}(k[x_1, \dots, x_n]) \cong k^n$ under $(x_1 - a_1, \dots, x_n - a_n) \mapsto (a_1, \dots, a_n)$.

The intuition here is: $k[x_1, \dots, x_n]$ is the ring of regular functions on k^n .

1.5 Pictures of spectra

For any I , we get

$$V(I) = \text{Spec}(A/I) \hookrightarrow \text{Spec}(A)$$

with a bijection:

$$\{\bar{\mathfrak{p}} \subseteq A/I\} \simeq \{\mathfrak{q} \subseteq A \mid \mathfrak{q} \supseteq I\}.$$

From

$$\pi : A \rightarrow A/I$$

we get π^* , which is injective with image $V(I)$. Hence, we identify

$$\operatorname{Spec}(A/I) \quad \text{with} \quad V(I) \subseteq \operatorname{Spec}(A).$$

Example 1.17. Consider

$$\operatorname{Spec}(k[x, y]/(xy - 1)) \hookrightarrow \operatorname{Spec}(k[x, y]) \supseteq \operatorname{Spec}_{\max}(k[x, y]) = k^2.$$

A point $(a, b) \in k^2$ is seen as the maximal ideal $(x - a, y - b)$; that comes from $\operatorname{Spec}(k[x, y]/(xy - 1))$ if and only if $ab - 1 = 0$.

Example 1.18. $\operatorname{Spec}(k[x, y]/xy)$ gives the "cross" $\{ab = 0\}$.

Example 1.19. In the case $\operatorname{Spec}(k[x, y]/(x^2 + y^2 + 1)) \hookrightarrow \operatorname{Spec}(k[x, y])$ we do not know how to draw.

Note 1.20. In this case we do not lose anything by thinking about $\operatorname{Spec}_{\max}$ instead of Spec .

1.6 Localization

Definition 1.21. Let A be a ring. $S \subseteq A$ is called a multiplicative subset if the both conditions hold:

1. $1 \in S$,
2. $\forall s, t \in S \quad st \in S$.

We wish to obtain a ring $S^{-1}A \xleftarrow{i} A$ such that the $i(s)$ are invertible; a pseudo-

$$\left\{ \frac{a}{s} \mid a \in A, s \in S \right\}.$$

Construction 1.22 (localization of a ring at a multiplicative subset).

Step 1. Let

$$I = \{a \in A \mid \exists s \in S \quad sa = 0\}.$$

Note that if $a, b \in I$, then

$$\exists s \in S \quad sa = 0, \quad \exists t \in S \quad tb = 0.$$

Then $(st)(a \pm b) = 0$.

Upshot: I is an ideal. We let $A \xrightarrow{\pi} A/I =: A'$.

Step 2. $\pi(S) \subseteq A'$ consists of non-zero-divisors.

Indeed, suppose that $s \in S$, $\pi(s)$ is a zero divisor, that is

$$\begin{aligned} \exists 0 \neq a \in A \quad \pi(s)\pi(a) &= 0 \\ \implies \pi(sa) &= 0 \\ \implies sa &\in I \\ \implies \exists t \in S \quad tsa &= 0 \\ \implies (ts)a &= 0. \end{aligned}$$

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But

$$ts \in S \implies a \in I \implies \pi(a) = 0.$$

Which means that $\pi(s)$ was not really a zero divisor.

Define $S' := \pi(S)$. We want

$$(S')^{-1}A' = \left\{ \frac{a'}{s'} \mid a' \in A', s' \in S' \right\}.$$

Consider pairs

$$\{(a', s') \mid a' \in A', s' \in S'\}.$$

Define a relation

$$(a'_1, s'_1) \sim (a'_2, s'_2) \iff a'_1 s'_2 = a'_2 s'_1.$$

To prove that this is an equivalence relation, we show transitivity (exercise; use that the S' are non-zero-divisors).

Step 3.

$$(S')^{-1}A' := \{(a', s') \mid a' \in A', s' \in S'\} / \sim$$

is a well-defined set. The ring operations are given by thinking about this as

$$\left\{ \frac{a'}{s'} \mid a' \in A', s' \in S' \right\}.$$

This turns the set into an associative commutative ring with unity.

Lemma 1.23. Let $a, b \in A, s, t \in S$. Then

$$a/s = b/t \iff \exists u \in S \ u(at - bs) = 0.$$

Proof. “ (\Leftarrow) ”:

$$\begin{aligned} u(at - bs) &= 0 \\ \implies uta &= usb \\ \implies uta/1 &= usb/1 \\ \implies a/s &= uta/stu = usb/stu = b/t \end{aligned}$$

“ (\Rightarrow) ”: Observe: the map $\phi : A/I = A' \rightarrow S'^{-1}A^{-1}$ is injective. Note that

$$\forall \bar{a}, \bar{b} \in A' \quad \bar{a}/1 = \bar{b}/1 \iff \bar{a} = \bar{b} \iff 1 \cdot \bar{a} = 1 \cdot \bar{b}$$

□

Corollary 1.24. We could define $S^{-1}A$ as the set of points

$$\{(a, s) \mid a \in A, s \in S\} / \sim$$

under

$$(a, s) \sim (b, t) \iff \exists s \in S \ u(at - bs) = 0.$$

Example 1.25. For $f \in A$, let $S = \{1, f, f^2, \dots\}$. We denote $S^{-1}A$ by A_f .

Example 1.26. $\mathfrak{p} \in \text{Spec}(A)$, $S := A \setminus \mathfrak{p}$. Then we denote $S^{-1}A$ by $A_{\mathfrak{p}}$.

Note 1.27. $S^{-1}A$ does not determine S , for instance

$$\mathbb{C}[x]_x = \mathbb{C}[x]_{x^{2019}}$$

Proposition 1.28 (universal property of localization). Let $f : A \rightarrow B$ be such that $f(S)$ consists of invertible elements.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow i & \nearrow S^{-1}f \\ & S^{-1}A & \end{array}$$

Then $\exists! S^{-1} : S^{-1}A \rightarrow B$ such that $f = (S^{-1}f) \circ i$.

Proof. We write $\tilde{f} := S^{-1}f$. Suppose that \tilde{f} exists. Then

$$\forall a \in A \quad \tilde{f}(a/1) = f(a)$$

and thus

$$\forall s \in S \quad \tilde{f}\left(\frac{a}{s} \cdot s\right) = f\left(\frac{a}{1}\right) = f(a)$$

and the formula $\tilde{f}(a/s) = f(a)/f(s)$ is recovered. One needs only to check that such a function is a homomorphism. \square

Lemma 1.29. $A_f \cong A[x]/(fx - 1)$.

Proof. Note that f is invertible in $A[x]/(fx - 1)$. Hence by universal property

$$\tilde{\phi}(a/f) = \phi(a)/\phi(f) = a\bar{x}.$$

Now take $\psi : A[x] \rightarrow A_f$ a homomorphism of A -algebras defined by $\psi(x) = 1/f$. Then

$$\psi(fx - 1) = f \cdot 1/f - 1 = 0$$

and so

$$\exists \tilde{\psi} : A[x]/(fx - 1) \rightarrow A_f$$

such that

$$\forall a \in A, n \in \mathbb{N} \quad \tilde{\psi}(ax^n) = a/f^n.$$

The maps $\tilde{\phi}$ and $\tilde{\psi}$ are then mutual inverses. \square

Corollary 1.30. If A is a finitely generated algebra, then A_f is also a finitely generated algebra.

Lemma 1.31. For A an algebra and S a multiplicative subset:

1. $S^{-1}A = 0 \iff 0 \in S$
2. $A \xrightarrow{i} S^{-1}A$ is injective if and only if all elements of S are non-zero-divisors in A
3. $A \rightarrow S^{-1}A$ is an isomorphism if and only if all elements of S are invertible in A

Proof. 1. By Lemma 1.29:

$$\frac{1}{1} = \frac{0}{1} \iff \exists u \in S \quad u(1 \cdot 1 - 0 \cdot 1) = 0 \iff \exists u \in S \quad u = 0.$$

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2. i is injective if and only if

$$(\{a \in A \mid \exists s \in S \ sa = 0\}).$$

This happens if and only if s is a non-zero-divisor.

3. If $A \rightarrow S^{-1}A$ is iso, then $\forall s \in S \ s/1$ is invertible. Hence, so is s .

Conversely, if all elements of S are invertible in A , then they are non-zero-divisors. Hence $A \rightarrow S^{-1}A$ is injective by the previous point. Moreover,

$$i(as^{-1}) = \frac{as^{-1}}{a} = \frac{a}{s},$$

so i is “onto”.

□

Intuitively, we think of A_f as the ring of functions on X_f , where $X = \text{Spec}(A)$.

We would expect $\text{Spec}(A_f)$ to be the same as X_f . Indeed, the following holds.

Proposition 1.32. Consider the maps

$$\begin{aligned} i: A &\rightarrow S^{-1}A \\ i^*: \text{Spec}(S^{-1}A) &\rightarrow \text{Spec}(A). \end{aligned}$$

Then i^* is injective and

$$\text{im } i^* = \{\mathfrak{p} \mid \mathfrak{p} \cap S = \emptyset\}.$$

Before presenting the proof, for which we will require some additional facts, we note the following corollaries.

Corollary 1.33. If $S = \{1, f, f^2, \dots\}$, then

$$\text{im } i^* = \{\mathfrak{p} \mid \mathfrak{p} \not\ni f\} = (\text{Spec}(A))_f.$$

Corollary 1.34. For $\mathfrak{q} \in \text{Spec}(A)$, $S = A \setminus \mathfrak{q}$,

$$\text{im } i^* = \{\mathfrak{p} \mid \mathfrak{p} \subseteq \mathfrak{q}\} = \text{Spec}(A_{\mathfrak{p}}).$$

Lemma 1.35. Let $I \subseteq S^{-1}A$ be an ideal. If we let

$$J = \{a \in A \mid \frac{a}{1} \in I\},$$

then

$$I = \{\frac{j}{s} \mid j \in J, s \in S\}.$$

Proof. “ \supseteq ”:

$$\forall j \in J \frac{j}{i} \in I \implies \frac{j}{s} = \frac{j}{1} \cdot \frac{1}{s} \in I$$

“ \subseteq ”:

$$\frac{a}{s} \in I \implies \frac{a}{1} = \frac{a}{s} \cdot \frac{s}{i} \in I \implies a \in J \implies \frac{a}{s} \in \{\frac{j}{s} \mid j \in J, s \in S\}$$

□

Proposition 1.32. Let $\mathfrak{q} \in \text{Spec}(S^{-1}A)$. We let

$$\mathfrak{p} = i^*(\mathfrak{q}) = i^{-1}(\mathfrak{q}) = \{a \in A \mid f \in \mathfrak{q}\}.$$

By Lemma 1.35 we can recover \mathfrak{q} from \mathfrak{p} alone, and so i^* is injective.

Let $\mathfrak{p} = i^*(\mathfrak{q})$. Suppose $\mathfrak{p} \cap S \neq \emptyset$. This means that

$$\frac{s}{1} \in \mathfrak{q} \text{ and } \frac{s}{t} \text{ is invertible} \implies \mathfrak{q} = (1).$$

Suppose $\mathfrak{p} \in \text{Spec}(A)$, $\mathfrak{p} \cap S = \emptyset$. We wish to find \mathfrak{q} . We guess that

$$\mathfrak{q} := \left\{ \frac{p}{s} \mid p \in \mathfrak{p}, s \in S \right\} \subseteq S^{-1}A$$

does the job. This \mathfrak{q} is an ideal, we want to see that it is prime.

Suppose that

$$\frac{a}{s} \cdot \frac{b}{t} \in \mathfrak{q}.$$

Then $ab/1 \in \mathfrak{q}$ and so

□

Corollary 1.36. For $f \in A$, the following are equivalent:

1. f is nilpotent (that is, $\exists n > 0 \ f^n = 0$),
2. $f \in \cap \{\mathfrak{p} \mid p \in \text{Spec}(A)\}$,
3. $V(f) = \text{Spec}(A)$.

Proof. The equivalence between 2 and 3 is checked formally.

1 \implies 2: $\forall \mathfrak{p} \ f^n = 0 \in \mathfrak{p} \implies f \in \mathfrak{p}$.

Negation of 1 implies negation of 2: let $0 \notin \{1, f, f^2, \dots\}$ so $A_f \neq 0$ by Lemma 1.35. Hence, $\text{Spec}(A_f) \neq \emptyset$, so by Proposition 1.32

$$\text{Spec}(A_f) = \{\mathfrak{p} \in \text{Spec}(A) \mid f \notin \mathfrak{p}\} \neq \emptyset$$

□

Definition 1.37. The nilradical of a ring A is defined as

$$\text{nil}(A) = \cap \{\mathfrak{p} \mid p \in \text{Spec}(A)\} = \{f \in A \mid \exists n > 0 \ f^n = 0\}.$$

2 Modules

2.1 Modules

Definition 2.1. Let A be a ring. An A -module M is an abelian group M together with a ring homomorphism $A \rightarrow \text{End}_{\mathbb{Z}}(M)$.

Equivalently, M is an abelian group together with a map

$$A \times M \rightarrow M, \quad (a, m) \mapsto am,$$

such that:

1. $\forall a \in A, m_1, m_2 \in M \quad a(m_1 + m_2) = am_1 + am_2$
2. $(a_1 + a_2)m = a_1m + a_2m$
3. $1 \cdot m = m$
4. $a_1(a_2m) = (a_1a_2)m$

Definition 2.2. A homomorphism of A -modules $\phi: M \rightarrow N$ is a homomorphism of abelian groups such that

$$\forall a \in A, m \in M \quad a\phi(m) = \phi(am).$$

Example 2.3. For A a ring, A is an A -module; in fact, any ideal $I \subseteq A$ is an A -module.

Example 2.4. For any A -algebra B , B is an A -module. In particular, A/I is an A -module for any ideal $I \subseteq A$.

Definition 2.5. If M is an A -module and $m_1, \dots, m_k \in M$ its elements, then the submodule generated by m_1, \dots, m_k is

$$Am_1 + Am_2 + \dots + Am_k = \left\{ \sum_{i=1}^k a_i m_i \mid a_i \in A \right\}.$$

Note 2.6. The set

$$\text{hom}_A(M, N) = \{ \phi : M \rightarrow N \text{ an } A\text{-module homomorphism} \}$$

is an A -module under

$$(a \cdot \phi)(m) := \phi(am) = a\phi(m).$$

Definition 2.7. An A -module is finitely generated if $\exists k \in \mathbb{N}, m_1, \dots, m_k \in M$ such that

$$M = Am_1 + \dots + Am_k.$$

Definition 2.8. An A -module is free if it is isomorphic to an A -module of the form

$$\bigoplus_{i \in I} A.$$

Example 2.9. All k -vector spaces are free k -modules. Not all abelian groups are free.

Lemma 2.10. An A -module M is finitely generated if and only if

$$\exists \phi : \bigoplus_{i=1}^k A \rightarrow M \text{ surjective.}$$

Proof. □

Theorem 2.11 (Cayley-Hamilton). Let M be an A -module. Let $\phi : M \rightarrow M$ be a homomorphism of A -modules such that

$$\phi(m) = IM = \left\{ \sum i_k m_k \mid i_k \in I, m_k \in M \right\}.$$

Then there exist elements

$$a_{n-1} \in I, a_{n-2} \in I^2, \dots, a_0 \in I^n$$

such that

$$\phi^n + a_{n-1}\phi^{n-1} + \dots + a_0 = 0 \text{ (the zero morphism).}$$

2.2 More on modules

Note 2.12. When we write $I \cdot M$, we mean the linear span of elements of the form $i \cdot m$.

Lemma 2.13 (adjugate matrix). Let $X \in M_{n \times n}(A)$. Then

$$\exists Y \in M_{n \times n}(A) \text{ such that } Y \cdot X = \text{diag}(d), \quad d = \det(X).$$

Concretely, Y is given by the formula

$$Y := [(-1)^{i+j} \det X_{ji}]_{ij},$$

where X_{ij} is the submatrix of X formed by excluding the i -th row and the j -th column.

Proof. Let

$$\pi : A^{\oplus n} \rightarrow M \quad \text{via} \quad e_i \mapsto m_i.$$

Then

$$\forall i \quad \exists u_{ij} \in I \quad \phi(m_i) = \sum_j u_{ij} m_j$$

in face of the assumptions. Define the lift of ϕ to a self-map $\tilde{\phi}$ of the free module $A^{\oplus n}$ by

$$\tilde{\phi} : A^{\oplus n} \rightarrow A^{\oplus n} \quad \text{via} \quad \tilde{\phi}(e_i) = \sum_j u_{ij} e_j.$$

This can be fitted inside a commutative diagram

$$\begin{array}{ccc} A^{\oplus n} & \xrightarrow{\tilde{\phi}} & A^{\oplus n} \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\phi} & M \end{array}$$

2 Modules

Let $B := A[x]$ and make $A^{\oplus n}$ into a B -module by

$$\forall f \in A^{\oplus n} \quad x.f := \tilde{\phi}(f).$$

Now,

$$X := [u_{ij}]_{ij} - \text{diag}(x) \in M_{n \times n}(B).$$

View X as an endomorphism of $A^{\oplus n}$. Then

$$X.e_i = \sum_j u_{ij}e_j - x \cdot e_i = \sum_j u_{ij}e_j - \tilde{\phi}(e_j) = 0.$$

Hence X acts on $A^{\oplus n}$ as the zero endomorphism.

Now, use Lemma 2.13 to see that

$$\exists Y \quad Y \cdot X = \text{diag}(d).$$

Hence

$$\forall i \quad 0 = Y \cdot X \cdot e_i = \text{diag} d e_i = d \cdot e_i.$$

Expand the determinant by columns:

$$d = \det(X) = x^n + a_1 x^{n-1} + \dots + a_n, \quad a_k \in I^k.$$

So as an endomorphism of $A^{\oplus n}$,

$$\tilde{\phi}^n + a_1 \tilde{\phi}^{n-1} + \dots + a_n = 0.$$

But

$$\phi \circ \pi = \pi \circ \tilde{\phi}.$$

So for all i

$$\begin{aligned} (\phi^n + a_1 \phi^{n-1} + \dots + a_n)(m_i) &= \pi((\tilde{\phi}^n + a_1 \tilde{\phi}^{n-1} + \dots + a_n)(e_i)) \\ &= \pi(0) \\ &= 0. \end{aligned}$$

□

Lemma 2.14 (Nakayama). Let M be an A -module, $I \cdot M = M$. If M is finitely generated, then

$$\exists i \in I \quad (1 - i) \cdot M = 0.$$

Proof. Consider $\text{id}_M: M \xrightarrow{\sim} M$. By Section 2.2,

$$\exists a_i \quad \text{id}^n + a_1 \text{id}^{n-1} + \dots + a_n = 0$$

and then

$$\forall m \in M \quad (1 + a_1 + \dots + a_n) \cdot m = 0.$$

We now let

$$-i := a_1 + \dots + a_n.$$

□

Corollary 2.15 (local Nakayama). Let A be a local ring, \mathfrak{m} the maximal ideal of A , M an A -module, M finitely generated, $M = \mathfrak{m} \cdot M$. Then

$$M = 0.$$

Proof. Apply Lemma 2.14:

$$\exists a \in \mathfrak{m} \quad (1 - a) \cdot M = 0, \quad (1 - a) \notin \mathfrak{m}.$$

But $(1 - a)$ is invertible; let its inverse be b . Then

$$M = 1 \cdot M = b(1 - a)M = b \cdot 0 = 0.$$

□

Corollary 2.16. Let $N \subseteq M$, M finitely generated, A local, \mathfrak{m} maximal. Let

$$M = N + \mathfrak{m} \cdot M.$$

Then $M = N$.

Proof. $(M/N) = (\mathfrak{m} \cdot M)/N$. By Corollary 2.15 $M/N = 0$. □

Corollary 2.17. Let M be a finitely generated A -module, (A, \mathfrak{m}) a local ring. Suppose

$$m_1, \dots, m_k \in M$$

be such that their images in $M/\mathfrak{m} \cdot M$ generate $M/\mathfrak{m} \cdot M$. Then m_1, \dots, m_k generate M .

Proof. Let

$$N := \sum_{i=1}^k A \cdot m_i \subseteq M.$$

Then the map

$$N \rightarrow M/\mathfrak{m} \cdot M$$

is surjective by assumption, so by Corollary 2.15:

$$M = \mathfrak{m} \cdot M + N \implies M = N = \sum_{i=1}^k A \cdot m_i.$$

Note that

$$A/\mathfrak{m}$$

is a field, so

$$M/\mathfrak{m} \cdot M$$

is a vector space. This finishes the proof. □

Example 2.18. Consider $\mathbb{Q} \in \mathbb{Z}_{\text{mod}}$. Consider $I = (2019) \subseteq \mathbb{Z}$. Certainly $\mathbb{Q} = I \cdot \mathbb{Q}$, but there does not exist an $i \in I$ such that

$$(1 - i) \cdot \mathbb{Q} = 0,$$

so \mathbb{Q} is not a finitely generated \mathbb{Z} -module.

Example 2.19. Another counterexample (this time local) is given by $\mathbb{Z}_{(2)}$.

Note 2.20. Localizing is a typical way of finding non-finitely generated modules.

Example 2.21. Let $X = S^1$, $A = C(S^1, \mathbb{R})$. Consider

$$\mathfrak{m}_x = \{f \mid f(x) = 0\}.$$

Then $\mathfrak{m}_x^2 = \mathfrak{m}_x$, hence \mathfrak{m}_x is not a finitely generated ideal.

2.3 Tensor product

Fix A -modules M, N . We will construct another A -module, denoted by

$$M \otimes_A N,$$

and study its properties.

Definition 2.22. Let P be an A -module. A function (not a homomorphism!)

$$f: M \times N \rightarrow P$$

is called bilinear if:

1. $\forall a \in A, m \in M, n \in N \quad af(m, n) = f(am, n) = f(m, an),$
2. $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n),$
3. $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2).$

The set of such functions will be denoted by

$$\text{Bilin}_{M \times N}(P).$$

Intuitively, we will think about f as a “multiplication”.

Example 2.23. For a homomorphism

$$\phi: P \rightarrow R,$$

where R is an A -module, we get

$$\phi^*: \text{Bilin}_{M \times N}(P) \rightarrow \text{Bilin}_{M \times N}(R) \quad \text{via} \quad f \mapsto \phi \circ f.$$

Theorem 2.24. In Example 2.23, there exists

$$T \in A_{\text{mod}}, g \in \text{Bilin}_{M \times N}(T) \quad \text{such that} \quad \forall P \in A_{\text{mod}} \quad \forall f \in \text{Bilin}_{M \times N}(P) \quad \exists! f': T \rightarrow P$$

such that

$$f = f' \circ g.$$

Moreover, this T is unique.

Note 2.25. One can rephrase the theorem as the existence of a natural isomorphism

$$\text{hom}(T, -) \simeq \text{Bilin}_{M \times N}(-).$$

Definition 2.26. The A -module T of Theorem 2.24 is denoted by

$$M \otimes_A N$$

and called a tensor product of M and N . The map g is written as

$$m \otimes n := g(m, n).$$

Theorem 2.24. Let

$$T_0 = A^{M \times N}.$$

Then

$$\forall p \quad \text{hom}(T_0, P) \simeq \text{Set}(M \times N, P) \quad \text{via} \quad f \mapsto \sum a_i f(e_i),$$

that is, the free module functor is adjoint to the forgetful functor to Set . Denote the generator of T_0 corresponding to (m, n) by $e(m, n)$. Now define

$$T = T_0 / \sim,$$

where \sim describes the relations given in Definition 2.22. We now claim that T together with the quotient of e satisfies the universal property.

Now, for uniqueness: adjoints are unique by the faithfulness of the Yoneda embedding.

Alternatively, apply standard abstract nonsense for universal properties. \square

Question 2.27. Do all representable functors have left adjoints?

Answer. Yes, provided the category considered has all coproducts, as then for functors into Set the two conditions are in fact equivalent; the same holds in the case of corepresentability and having a right adjoint.

Concretely, the left adjoint to a functor represented by X is given by

$$Y \mapsto \bigsqcup_Y X.$$

Definition 2.28. Let k be a ring, $k \rightarrow A$ a k -algebra. For $P \in A_{\text{mod}}$ we define

$$\text{Der}_k(A, P) = \{d: A \rightarrow P \mid d(a+b) = d(a) + d(b), d(ab) = ad(b) + d(a)b, d(k) = \{0\}\}.$$

For $\phi: P \rightarrow R$ we get pullback

$$\text{Der}_k(A, P) \rightarrow \text{Der}_k(A, R) \quad \text{via} \quad d \mapsto \phi \circ d$$

Theorem 2.29. There exists an A -module

$$\Omega_{A/k}, \quad d: A \rightarrow \Omega_{A/k}$$

such that

$$\forall P \in A_{\text{mod}} \forall f \in \text{Der}_k(A, P) \exists ! f': \Omega_{A/k} \rightarrow P \quad \text{such that} \quad f = f' \circ d.$$

One can rephrase this as

$$\text{hom}_{\Omega_{A/k}} \simeq \text{Der}_k(A, -).$$

Proof. Define Ω as a quotient of A^A by the obvious relations, denote

$$A^A =: \{e(a) \mid a \in A\}.$$

Give $d: A \rightarrow \Omega$ by $d(a) = e(a)$. \square

2 Modules

Example 2.30. Let $A = k[x_1, \dots, x_n]$. Then $\Omega_{A/k}$ is a free module on n generators

$$d(x_1), \dots, d(x_n).$$

Indeed, as $\Omega_{A/k}$ is generated by

$$\{df \mid f \in A\}.$$

We claim that

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i.$$

Then $\Omega_{A/k}$ is indeed generated by

$$dx_1, \dots, dx_n.$$

Consider the k -linear derivation gradient

$$\nabla: A \rightarrow A^{\oplus n} \quad \text{via} \quad f \mapsto \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

By the universal property of localization we obtain

$$\pi: \Omega_{A/k} \text{ such that } \forall 1 \leq i \leq n \quad \pi(dx_i) = e_i.$$

If we let

$$\phi: S^{\oplus n} \rightarrow \Omega_{A/k} \quad \text{via} \quad e_i \mapsto dx_i.$$

Then ϕ is onto and since

$$\pi \circ \phi = \text{id}_{S^{\oplus n}},$$

ϕ is a bijection (and hence, an isomorphism).

2.4 Localization of modules

Definition 2.31. The rank of a free module F is

$$\text{rk}(F) := \dim(F/\mathfrak{m}F) \text{ over } A/\mathfrak{m}.$$

This does not depend on the choice of \mathfrak{m} .

Example 2.32. $\text{rk}(A^{\oplus I}) = |I|$.

Example 2.33. $\Omega_{A/k}$ is free of rank n .

Definition 2.34. Let $S \subseteq A$ be a multiplicative subset, $M \in \text{Mod } A$. The one forms an $S^{-1}A$ -module $S^{-1}M$ analogously to $S^{-1}A$ itself. That is,

$$S^{-1}M := \{m/s \mid m \in M, s \in S\}$$

with

$$m/s = m'/s' \iff \exists t \in S \quad t(s'm - sm') = 0.$$

Example 2.35. If

$$S = \{1, f, f^2, \dots\},$$

then we denote

$$M_f = S^{-1}M.$$

Analogously, for

$$S = A \setminus \mathfrak{p},$$

write

$$M_{\mathfrak{p}} = S^{-1}M.$$

Definition 2.36. A finitely generated k -algebra A is smooth if

$$\forall p \in \text{Spec}(A) \ (\Omega_{A/k})_{\mathfrak{p}} \text{ is free as an } A_{\mathfrak{p}}\text{-module.}$$

Example 2.37. $k[x_1, \dots, x_n]$ is smooth.

Lemma 2.38. Let $S \subseteq A$ be a multiplicative subset, M an A -module. Then

$$S^{-1}A \otimes_A M \cong S^{-1}M$$

under the map

$$(a/s) \otimes m \mapsto am/s.$$

Proof. The given map

$$f: S^{-1}A \otimes M \rightarrow S^{-1}M$$

is linear by the universal property of the tensor product. We show that it is an isomorphism.

1. f is “onto”:

$$f(1/s \otimes m) = m/s,$$

2. f is “into”: take an element

$$n \in S^{-1}A \otimes_A M.$$

We write

$$n = \sum_{k=1}^r a_k/s_k \otimes m_k = \sum_{k=1}^r \tilde{a}_k/s \otimes \tilde{m}_k.$$

If we let

$$s = \prod_{k=1}^r s_k,$$

we can write

$$\sum_{k=1}^r 1/s \otimes \tilde{a}_k m_k = 1/s \otimes \left(\sum_{k=1}^r \tilde{a}_k m_k \right) =: 1/s \otimes m.$$

Since any n is of the form, we may proceed as follows. Assume $0 = f(n)$; we will show $n = 0$. Have:

$$0 = f(n) = m/s \iff \exists t \in S \ tm = 0.$$

Then

$$n = 1/s \otimes m = 1/(st) \otimes tm = 1/(st) \otimes 0 = 0.$$

□

Corollary 2.39.

$$S^{-1}(\oplus_{i \in I} M_i) = \oplus_{i \in I} S^{-1}M_i.$$

Proof.

□

2.5 Exactness

Definition 2.40. Let M_i be A -modules, $i = 1, 2, 3$.

1. The sequence

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$$

is exact if

$$\text{im } f_1 = \ker f_2.$$

2. The sequence

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \rightarrow 0$$

is exact if and only if

$$M_1 \rightarrow M_2 \rightarrow M_3$$

is exact and f_2 is surjective.

3. The sequence

$$0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \rightarrow 0$$

is a short exact sequence if:

- f_1 is injective,
- $\text{im } f_1 = \ker f_2$,
- f_2 is surjective.

Proposition 2.41 (right-exactness of the tensor product). For all exact sequences

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \rightarrow 0$$

and for all $N \in \text{mod } A$, the induced sequence

$$M_1 \otimes_A N \xrightarrow{f_1 \otimes_A \text{id}} M_2 \otimes_A N \xrightarrow{f_2 \otimes \text{id}} M_3 \otimes N \rightarrow 0$$

is exact.

Proof. First, we need to check that

$$M_2 \otimes_A N \rightarrow M_3 \otimes_A N$$

is “onto”. This is checked on elements: $M_3 \otimes_A N$ is generated by elements of the form

$$m_3 \otimes n \quad \text{such that} \quad m_3 \in M_3, n \in N.$$

The claim then follows by surjectivity of f_2 itself.

Exactness in the middle means that

$$M_2 \otimes N / (f_1 \otimes \text{id})(M_1 \otimes N) \rightarrow M_3 \otimes N$$

is an isomorphism. We already checked that it is “onto”; now we need to know that there exists a section, that is, a one-sided inverse.

Have

$$M_2 / f_1(M_1) \cong M_3 \times N \rightarrow M_2 \otimes_A N / (f_1 \otimes \text{id})(M_1 \otimes_A N) \quad \text{via} \quad (m_3, n) \mapsto \overline{m_2 \otimes n}.$$

This is bilinear if it is well-defined. Hence, it yields a map

$$s: M_3 \otimes_A N \rightarrow M_2 \otimes N$$

and indeed, the equality

$$s \circ \pi = \text{id}_{M_3 \otimes N}$$

holds, and so π is injective and further, an isomorphism. \square

Note 2.42. We have shown that

$$M_3 \cong M_2/f_1(M_1).$$

Hence, we get

$$M_2/(f_1(M_1) \otimes N) \cong M_2 \otimes_A N/(f_1(M_1) \otimes_A N).$$

Example 2.43. $A/I \otimes_A N \cong N/IN$.

And in fact, quotients and localizations of modules can be computed by means of the tensor product.

Note 2.44. The notions of right-exactness as defined above in Proposition 2.41 coincide with the preserving of colimits. The definition of Proposition 2.41 is valid in the case of preadditive categories.

2.6 Why do we care?

First of all, as noted before, we have the following isomorphisms:

$$M \otimes_A A/I \cong M/IM,$$

$$M \otimes_A S^{-1}A \cong S^{-1}M.$$

Moreover:

Proposition 2.45. Tensoring with an A -algebra B is left adjoint to the forgetful functor

$$B_{\text{mod}} \rightarrow A_{\text{mod}}.$$

Proposition 2.46. Suppose A -algebras B, C are given. Then

$$B \otimes_A C$$

is an A -algebra with multiplication

$$(b_1 \otimes c_1) \cdot (b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2).$$

We claim that

$$B \otimes_A C$$

is in fact the coproduct of B and C .

2 Modules

Note 2.47. In the same setting, one has the diagram

$$\begin{array}{ccc}
 & \text{Spec}(B) & \\
 \swarrow & & \searrow \\
 \text{Spec}(A) & & \text{Spec}(B \otimes_k C) \\
 \swarrow & & \searrow \\
 & \text{Spec}(C) &
 \end{array}$$

Note that then

$$\text{Spec}(B \otimes_k C) \neq \text{Spec}(B) \times \text{Spec}(C);$$

we only get the arrows demanded that make the diagram commute.

We note the following special cases of Proposition 2.46.

Corollary 2.48. Let $A = k$ a field,

$$B = k[x_1, \dots, x_s], \quad C = k[y_1, \dots, y_t].$$

Then

$$B \otimes_k C \cong k[x_1, \dots, x_s, y_1, \dots, y_t].$$

Proof. Note that B and C are free k -modules, so $B \otimes_k C$ is a free k -module with basis

$$x_1^{a_1} \otimes \dots \otimes x_s^{a_s} \otimes y_1^{b_1} \otimes \dots \otimes y_t^{b_t}.$$

One checks that the obvious map defined by the universal property is an isomorphism. \square

Corollary 2.49. Suppose

$$\begin{aligned}
 B &= k[x_1, \dots, x_s]/I, \\
 C &= k[y_1, \dots, y_t]/J.
 \end{aligned}$$

Then

$$B \otimes_k C \cong k[x_1, \dots, x_s, y_1, \dots, y_t]/(I + J).$$

In differential geometry, the fiber of a map $f: X \rightarrow Y$ over a point is not typically a manifold.

In contrast, in the Spec-world, we have the following.

Definition 2.50. Let $f: A \rightarrow B$ be a homomorphism, $\mathfrak{m} \subseteq A$ a maximal ideal. Then the fiber of f over \mathfrak{m} is

$$B/\mathfrak{m}B.$$

Proposition 2.51. In the above setting,

$$\text{Spec}(B/\mathfrak{m}B) \cong \{\mathfrak{p} \subseteq B \mid f^*(\mathfrak{p}) = \{\mathfrak{m}\}\}.$$

Note that

$$\text{Spec}(A_{\mathfrak{p}}) = \{\mathfrak{q} \subseteq \mathfrak{p}\}, \quad \text{Spec}(A/\mathfrak{p}) = \{\mathfrak{q} \supseteq \mathfrak{p}\}.$$

Definition 2.52. More generally, if $f: A \rightarrow B$ is a homomorphism, $\mathfrak{p} \subseteq B$ a prime ideal, then the fiber of f over \mathfrak{p} is given by

$$B \otimes_A \kappa(\mathfrak{p}),$$

where

$$\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}.$$

Proposition 2.53. For any prime ideal $\mathfrak{p} \subseteq B$, $\kappa(\mathfrak{p})$ is a field.

Note 2.54. If \mathfrak{m} is maximal, then

$$\kappa(\mathfrak{m}) \cong A/\mathfrak{m}.$$

Proposition 2.55. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longleftarrow & \mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p})) \\ \downarrow f^* & & \downarrow \\ \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(\kappa(\mathfrak{p})) \end{array}$$

2.7 Noetherian modules

Proposition 2.56. Let $M \in \text{mod } A$. Then the following conditions are equivalent:

1. every submodule of M is finitely generated,
2. every sequence of submodules

$$M_1 \subseteq M_2 \subseteq \dots \subseteq M$$

stabilises, that is,

$$\exists n_0 \in \mathbb{N} \ \forall n \geq n_0 \ M_n = M_{n_0},$$

3. every family of submodules of M has a maximal element with respect to inclusion.

Proof. (Proposition 2.56): The implication “2 \implies 3” follows from Kuratowski-Zorn.

For “3 \implies 2”, take $\{M_i\}_{i \in \mathbb{N}}$ as the family in statement of 3; then the maximal element is also necessarily the one on which the sequence stabilises.

“2 \implies 1”: choose $N \subseteq M$ a submodule. Further, take

$$n_1 \in N \setminus \{0\}, \ n_2 \in N \setminus A_{n_1}, \ n_3 \in N \setminus A_{n_2} \oplus A_{n_3}, \ \dots$$

Either one can do this for all $k \in \mathbb{N}$ or not. In the latter case, N is necessarily finitely generated. In the former, we get the sequence

$$A_{n_1} \subseteq A_{n_1} + A_{n_2} \subseteq A_{n_1} + A_{n_2} + A_{n_3} \subseteq \dots$$

that does not stabilize.

“1 \implies 2”: take a sequence

$$M_1 \subseteq M_2 \subseteq \dots;$$

we want to stabilise it. To that end, consider

$$N = \bigcup_{n \in \mathbb{N}} M_n.$$

Since the sequence is increasing, this is a submodule of M .

By 1, this is finitely generated, and by a finite set of elements from N . Then this very set is already contained in one of the M_k , and then for $n > k$, $M_n = M_k$. \square

Definition 2.57. An A -module M is called Noetherian if it satisfies the equivalent conditions of Proposition 2.56.

2 Modules

Definition 2.58. A ring A is Noetherian if and only if the A -module A is Noetherian.

Proposition 2.59. This happens exactly when every ideal of A is finitely generated, and exactly when every prime ideal of A is finitely generated.

Proposition 2.60. Let $N \subseteq M$ be A -modules. Then the following conditions are equivalent:

1. M is Noetherian,
2. N and M/N are Noetherian.

Proof. We start with “1 \implies 2”. First, we show that N is Noetherian; indeed, if

$$N_1 \subseteq N_2 \subseteq \dots \subseteq N \text{ subseteq } M$$

is a sequence of submodules of N , it is also a sequence of submodules of M and the claim follows.

To show that M/N is Noetherian, consider a sequence of submodules

$$P_1 \subseteq P_2 \subseteq M/N.$$

If then $\pi: M \rightarrow M/N$ denotes the quotient map, then the sequence

$$\pi^{-1}(P_1) \subseteq \pi^{-1}(P_2) \subseteq \dots \subseteq M$$

stabilizes, and since π is surjective, we have

$$\pi(\pi^{-1}(P_n)) = P_n,$$

ending the proof of “1 \implies 2”.

For “2 \implies 1”, pick a sequence of submodules

$$M_1 \subseteq M_2 \subseteq \dots \subseteq M$$

and define

$$N_k = M_k \cap N, \quad P_k = \pi(M_k).$$

The sequences defined by the N_k and P_k stabilize, say at common n_0 .

We will show that

$$\forall n > n_0 \quad M_n = M_{n_0}.$$

First, show $M_n \subseteq M_{n_0}$. Take $m \in M_n$, then

$$\pi(m) \in P_n = P_{n_0} = M_{n_0}/\ker \pi \cap M_{n_0}$$

Pick $\tilde{m} \in M_{n_0}$ such that $\pi(m) = \pi(\tilde{m})$. Then we have

$$m - \tilde{m} \in \ker(\pi) \cap M_n = N_n = N_{n_0} \subseteq M_{n_0}.$$

Since $m - \tilde{m} \in M_{n_0}$, we have that in fact

$$m = \tilde{m} + m - \tilde{m} \in M_{n_0}$$

and the claim follows since m was arbitrary in M_n . □

Note that this translates to the claim that in the short exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$$

the middle term is Noetherian if and only if the other ones are.

Corollary 2.61. If A is a Noetherian ring, then all finitely generated A -modules are Noetherian.

Proof. First, we show that all finitely generated free modules are Noetherian. This follows easily by induction since we get short exact sequences

$$0 \rightarrow A \rightarrow A^{\oplus n} \rightarrow A^{\oplus(n-1)} \rightarrow 0.$$

If then K is a finitely generated A -module, say generated by k elements, we get a short exact sequence

$$0 \rightarrow \ker \alpha \rightarrow A^{\oplus k} \xrightarrow{\alpha} K \rightarrow 0,$$

and the claim follows from Proposition 2.60 again. \square

Example 2.62. Some Noetherian rings include:

1. fields,
2. principal ideal domains (e.g. \mathbb{Z} , $k[x]$, $\mathbb{Z}[x]$).

Theorem 2.63 (Hilbert basis theorem). If A is Noetherian, then

$$A[x_1, \dots, x_n]$$

is also Noetherian.

Corollary 2.64. If A is Noetherian and B is a finitely generated A -algebra, then B is also Noetherian.

Proof. Immediate, since

$$B = A[x_1, \dots, x_n]/I.$$

\square

Question 2.65. Are tensor products of Noetherian modules also Noetherian?

Answer. No, since

$$\bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$$

is not Noetherian. In fact,

$$\text{Spec}(\bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}) \simeq \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

is a natural isomorphism of sets (and not of groups).

Lemma 2.66. If the ring A is Noetherian, then so is A/I for all ideals $I \subseteq A$.

Lemma 2.67. If A is Noetherian, then for any multiplicative subset $S \subseteq A$, the localization $S^{-1}A$ is also Noetherian.

2 Modules

Proof. Consider $I \subseteq S^{-1}A$, $J = i^{-1}(I) \subseteq A$. Write

$$I = \left\{ \frac{j}{s} \mid j \in J, s \in S \right\}.$$

The ideal $J \subseteq A$ is finitely generated, say

$$J = A(j_1, \dots, j_r).$$

Then

$$\forall j \in J \quad \exists a_1, \dots, a_r \quad j = \sum_{k=1}^r a_k j_k.$$

We then have

$$\frac{j}{s} = \sum_{k=1}^r \frac{a_k}{s} \cdot \frac{j_k}{1},$$

and so I is generated by

$$\frac{j_1}{1}, \dots, \frac{j_r}{1}.$$

□

Note that we have used the previous characterization of Proposition 2.56; indeed, as the submodules of a ring are exactly the ideals.

Proof. (Hilbert basis theorem) Let $I \subseteq A[x]$. Write

$$J = \{a \in A \mid \exists f \in I \quad f = ax^n + a_1x^{n-1} + \dots\},$$

that is, J is the set of polynomials with leading term ax^n .

One claims that J is an ideal: indeed, let $j_1, j_2 \in J$. Put

$$r = j_1 - j_2.$$

If $r = 0$, we are done; in the other case, proceed. Take $f_1, f_2 \in I$ with leading term $f_i = j_i x^{n_i}$. Then the leading term of

$$x^{n_2} f_1 - x^{n_1} f_2$$

is

$$(j_1 - j_2)x^{n_1+n_2}.$$

Hence, J is an abelian subgroup.

If $j \in J, a \in A$, and so J is an ideal.

Since $J \subseteq A$ is an ideal of a Noetherian ring, it is finitely generated, say by elements

$$j_1, \dots, j_r.$$

One then has

$$\exists f_1, \dots, f_r \in I$$

with leading terms of f_i equal to $j_i x^{n_i}$.

If we let $n = \max(n_i)$, one can modify the f_i to get leading term $f_i = j_i x^n$.

Let

$$SP = I \cap A[x]_{<n}.$$

This is not an ideal, but an A -module; it is isomorphic to $A^{\oplus n}$, and so finitely generated, we write

$$SP = Ag_1 + \dots + Ag_t.$$

We claim that

$$I = (f_1, \dots, f_r, g_1, \dots, g_t)$$

as an ideal in $A[x]$.

Clearly,

$$I \supseteq (f_1, \dots, f_r, g_1, \dots, g_t).$$

For the other inclusion, we pick $h \in I$ and proceed by induction on degree.

Write the leading term of h as

$$bx^{\deg(h)}.$$

Now, if $\deg(h) < n$, h must necessarily be an element of SP .

In the other case,

$$\deg(h) \geq n$$

and we can divide with remainder. Write

$$b = \sum_{s=1}^r a_s j_s$$

and consider

$$h' = h - \sum a_s f_s x^{\deg(h)-n}.$$

Then

$$\deg(h') < \deg(h) \implies h' \in (f_1, \dots, f_r, g_1, \dots, g_t)$$

and so also

$$h \in (f_1, \dots, f_r, g_1, \dots, g_t).$$

This ends the proof. □

2.8 Finite and integral ring extensions

Consider B an A -algebra.

Definition 2.68. An element $b \in B$ is *integral* over A if

$$\exists a_{n-1}, \dots, a_0 \in A \quad b^n + a_{n-1}b^{n-1} + \dots + a_0 = 0.$$

B is integral over A if all its elements are integral over A .

Definition 2.69. An A -algebra B is *finite* over A if it is finitely generated as an A -module, that is,

$$\exists b_1, \dots, b_k \quad B = Ab_1 + \dots + Ab_k.$$

Example 2.70. The following hold:

1. $\mathbb{Z} \rightarrow \mathbb{Z}[i]$ is finite,
2. $\mathbb{Z} \rightarrow \mathbb{Z}/n$ is finite,
3. more generally, $A \rightarrow A/I$ is finite,

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4. $\mathbb{Z} \rightarrow \mathbb{Z}[x]$ is not finite.

Lemma 2.71. If $A \rightarrow B$ is finite, then it is integral.

Proof. Let $b \in B$ and consider the multiplication map

$$\phi: B \rightarrow B, \quad \phi(a) = a \cdot b.$$

This is an A -module homomorphism. By Cayley-Hamilton with $I = (1)$, one has

$$\exists a_{n-1}, \dots, a_0 \in A \quad b^n + a_{n-1}b^{n-1} + \dots + a_0 = 0.$$

□

Lemma 2.72. If $A \rightarrow B$ is integral and B is a finitely generated A -algebra, then $A \rightarrow B$ is finite.

Proof. Let

$$B = A[x_1, \dots, x_n]/I.$$

Then every $x_i \in B$ is integral over A and so

$$\exists n \quad \forall i \quad \exists a_0^{(i)} \quad x_i^n + a_{n-1}^{(i)} + \dots + a_0^{(i)} = 0.$$

We check that B is generated as an A -module by the finitely many monomials

$$\{x_1^{c_1}, \dots, x_n^{c_n}, 0 \leq c_i \leq n\}.$$

□

Example 2.73. The extension

$$\mathbb{Q} \rightarrow \bar{\mathbb{Q}}$$

is integral, but not finite.

Definition 2.74. For B and A -algebra, $b \in B$ an element, $A[b]$ will denote the smallest A -algebra contained in B and containing $b \in B$.

Proposition 2.75. Let $A \rightarrow B$ be an A -algebra, $b \in B$. The following conditions are equivalent:

1. v is integral over A ,
2. $A[b]$ is a finitely generated A -module,
3. $A[b]$ is contained in a finitely generated A -module

Proof. “1 \implies 2 \implies 3” is formal: the first one uses the same trick as in Lemma 2.72.

For “3 \implies 1, fix a finitely generated A -module $A[b] \subseteq C$. Then the multiplication map

$$\phi_b: C \rightarrow C$$

is an A -module homomorphism, so Cayley-Hamilton implies

$$\exists n \exists a_{n-1}, \dots, a_0 \in A \quad b^n + a_{n-1}b^{n-1} + \dots + a_0 = 0.$$

□

Definition 2.76. Let B be an A -algebra. Then the *integral closure* of A in B is

$$\bar{A} = \{b \in B \mid b \text{ is integral in } A\}.$$

The *normalization* of a domain A is its integral closure in the field of fractions $\text{Frac}(A)$.

Corollary 2.77. \bar{A} is an A -algebra.

Proof. Consider $x, y \in \bar{A}$. By Proposition 2.75, $A[x]$ and $A[y]$ are finitely generated A -modules. In fact, by the proof of Lemma 2.72, $A[x, y]$ is a finitely generated A -module.

One has

$$A[x - y] \subseteq A[x, y], \quad A[xy] \subseteq A[x, y],$$

so by Proposition 2.75, point 3,

$$x - y, xy \in \bar{A}.$$

Hence, \bar{A} is a ring. One has also the map

$$A \rightarrow \bar{A}$$

which makes \bar{A} an A -algebra; indeed, an A -subalgebra of B . □

Example 2.78. Let $\mathbb{Z} \rightarrow \mathbb{Q} \rightarrow K$ with $\mathbb{Q} \rightarrow K$ finite. Then one has also

$$O_K = \bar{\mathbb{Z}} \rightarrow K$$

- the ring of algebraic integers in K . This is Noetherian.

Theorem 2.79 (Nagata). If A is finitely generated over k , then the normalization of A is as well. Moreover,

$$A \rightarrow \bar{A}$$

is finite.