Lecture notes for

Commutative Algebra

Jakub Paliga

October 26, 2019

1 Preface

The following is a set of notes that have been taken during the lecture given by dr Joachim Jelisiejew as part of a WS2019/2020 course in commutative algebra (*Algebra przemienna*) at the faculty of Mathematics, Informatics and Mechanics, University of Warsaw.

The text of these notes was compiled by Jakub Paliga, who does not guarantee their correctness and disclaims any warranties when applicable. Indeed, he asserts any deficiency within, of which there is a nonzero number, to have been introduced by him in the process.

Contents

1 Basics

1.1 Conventions

Rings will be understood to be commutative, associative, unitary; the ring "0=1" is considered a ring. k will denote a field, $\bar{k}$ its algebraic closure. Usually, $k = \mathbb{C}$ - but not always.

Prime ideals will be denoted by $\mathfrak{p}$, $\mathfrak{q}$; maximal ideals, by $\mathfrak{m}$, $\mathfrak{n}$.

1.2 First definitions

Definition 1.1. Let A be a ring. Then an A-algebra is a ring B together with a fixed homomorphism $A \to B$. This homomorphism is called the structural homomorphism.

The homomorphism is presumed to preserve 1.

Definition 1.2. Le A be a ring. A homomorphism of A-algebras is a ring homomorphism commuting with the structural maps, that is, if $\phi : A \to B$, $\psi : A \to C$ are A-algebras, then a ring homomorphism $f : B \to C$ is an algebra homomorphism if $\psi = f \circ \phi$.

Example 1.3. $f : \mathbb{C} \to \mathbb{C}$ is given by $z \mapsto \bar{z}$ (the complex conjugation). Here:

- f is a ring homomorphism

- f is a real algebra homomorphism

- f is not a complex algebra homomorphism

The intuition for this is: we wish to see no "dirty tricks" with respect to A.

Example 1.4. In the case
$$B = \mathbb{C}[x]/I, \;\; C = \mathbb{C}[y]/J,$$
the quotient map $B \to C$ is a complex algebra homomorphism.

Definition 1.5.

1. A ring A is a domain if
$$\forall a, b \in A \;\; (ab = 0 \implies a = 0 \vee b = 0.)$$

2. An ideal I in a ring B is prime if B/I is a domain.

3. A ring A is a field if
$$\forall 0 \neq a \in A \;\; \exists b \in A \;\; ab = 1,$$
that is, every nonzero element has a multiplicative inverse.

4. An ideal I in a ring B is maximal if B/I is a field.

1.3 Motivations

Lemma 1.6. Let $f : A \to B$ be a ring homomorphism. If $\mathfrak{p} \subset B$ is prime, then $f^{-1}(\mathfrak{p}) \subset A$ is prime.

Proof. Consider the following diagram.

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A/f^{-1}(\mathfrak{p}) & \longrightarrow & B/\mathfrak{p}
\end{array}
$$

$A/f^{-1}(\mathfrak{p}) \subseteq B/\mathfrak{p}$ is a subring of a domain. Hence, $A/f^{-1}(\mathfrak{p})$ is a domain, and so $f^{-1}(\mathfrak{p}) \subseteq A$ is prime. $\qquad\square$

Example 1.7. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ has $f^{-1}(0) = 0$ is not maximal, even though $0 \in \mathbb{Q}$ is maximal. Ergo, Lemma 1.6 does not hold with "prime" replaced by "max".

Example 1.8. Consider $A = C(S^1, \mathbb{R})$, $S^1 = \{z \in \mathbb{C} | |z| = 1\}$. Then there is a bijection between S^1 and maximal ideals in A, given by every maximal ideal being some $\mathfrak{m}_x$. Moreover, we can recover the topology of S^1 from $A = C(S^1, \mathbb{R})$. How do we do it? For $f \in A$, consider the vanishing set $V(f) = \{\mathfrak{m} \text{ maximal ideal in } A | f \in m\}$. The topology is then generated by closed subsets with subbase $= \{V(f) | f \in A\}$.

1.4 The Spec() functor

We aim to revisit previous considerations for an arbitrary ring.

Definition 1.9. Let A be a ring. We let $\mathrm{Spec}(A) = \{\mathfrak{p} \text{ a prime ideal in} A\}$.

Example 1.10. $\mathrm{Spec}(\mathbb{C}[x]) = \{0\} \cup \{(x - a) | a \in \mathbb{C}\}$.

For a ring homomorphism $f \colon A \to B$ we obtain a map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ given by

$$(\mathfrak{q} \subseteq B) \mapsto (f^{-1}(\mathfrak{q}) \subseteq A).$$

This is a good start; however, we wish to upgrade this set $\mathrm{Spec}(A)$ to a topological space. In this, we follow the previous motivation.

Definition 1.11. For $E \subseteq A$ any subset, we define $V(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) | f \in \mathfrak{p}\}$ and call $V(f)$ the vanishing locus of f.

Proposition 1.12. The sets $V(E)$, $E \subseteq A$, are the closed subsets of a topology. This topology is called the Zariski topology.

Proof. $V(1) = \emptyset$, $V(0) = \mathrm{Spec}(A)$. It is clear that $\cap_{i \in I} V(E_i) = V(\cup_{i \in I} E_i)$. Further, we need to show that
$$V(E_1) \cup V(E_2) = V(E_1 \cdot E_2),$$
where $E_1 \cdot E_2 = \{e_1 \cdot e_2 \mid e_1 \in E_1, e_2 \in E_2\}$.

The nontrivial part is the left-way inclusion in the last statement: that $\mathfrak{p}$ is prime is used here.

Suppose $\mathfrak{p} \not\supseteq E_1$ and $\mathfrak{p} \not\supseteq E_2$. Then there exist $e_i \in E_i \setminus \mathfrak{p}$, as well as $e_1 \cdot e_2 \in E_1 \cdot E_2 \setminus \mathfrak{p}$ because $\mathfrak{p}$ is prime. $\qquad\square$

Question 1.13. What is the Zariski topology for $A = \mathbb{C}[x]$?

We note that

$$
\begin{aligned}
V(f_1, \ldots, f_r) &= \{\mathfrak{p} \text{ prime in } A \mid f_1, \ldots, f_r \in \mathfrak{p}\} \\
&= \{(x - a) \text{ maximal in } A \mid f_1, \ldots, f_r \in (x - a)\} \\
&= \{(x - a) \mid f_1(a) = \ldots = f_r(a) = 0\}.
\end{aligned}
$$

If any f_i is nonzero, then $0 \notin V(f_1, \ldots, f_r)$.

Now,

$$
V(f_1, \ldots, f_r) = \text{ set of common roots of } f_1, \ldots, f_r.
$$

Hence, $V(f_1, \ldots, f_r)$ is finite.

Note 1.14. All finite subsets of the set of maximal ideals are of the form $V(f_1, \ldots, f_r)$.

After some deliberation, this yields the cofinite topology augmented by $\{0\}$.

Note 1.15. The closure of $\{0\} \subseteq \operatorname{Spec}(A)$ is $\operatorname{Spec}(A)$ itself, since all other closed subsets do not contain 0; hence, all open subsets contain 0. In effect, the resulting space is not T_1, let alone Hausdorff.

Proposition 1.16. Let $f \colon A \to B$ be a ring homomorphism. Then the induced map

$$
f^* \colon \operatorname{Spec}(B) \to \operatorname{Spec}(A) \quad \text{with} \quad \mathfrak{q} \mapsto f^{-1}(\mathfrak{q})
$$

is continuous.

Proof. $E \subseteq A \implies V(E) \subseteq \operatorname{Spec}(A)$. We claim further that

$$
(f^*)^{-1}(V(E)) = V(f(E)),
$$

which ends the proof. $\qquad\square$

The upshot is: when $k = \bar{k}$, $\operatorname{Spec}_{max}(k[x_1, \ldots, x_n])$ is "nice". We will see later, in the Nullstellensatz, tha all maximal ideals in $k[x_1, \ldots, x_n]$ are of the form $(x_1 - a_1, \ldots x_n - a_n)$.

In fact, $\operatorname{Spec}_{max}(k[x_1, \ldots, x_n]) \cong k^n$ under $(x_1 - a_1, \ldots, x_n - a_n) \mapsto (a_1, \ldots, a_n)$.

The intuition here is: $k[x_1, \ldots, x_n]$ is the ring of regular functions on k^n.

1.5 Pictures of spectra

For any I, we get

$$
V(I) = \operatorname{Spec}(A/I) \hookrightarrow \operatorname{Spec}(A)
$$

with a bijection:

$$
\{\bar{\mathfrak{p}} \subseteq A/I\} \simeq \{\mathfrak{q} \subseteq A \mid \mathfrak{q} \supseteq I\}.
$$

From

$$
\pi : A \to A/I
$$

we get π^*, which is injective with image $V(I)$. Hence, we identify

$$
\operatorname{Spec}(A/I) \quad \text{with} \quad V(I) \subseteq \operatorname{Spec}(A).
$$

Example 1.17. Consider

$$\mathrm{Spec}(k[x,y]/(xy-1)) \hookrightarrow \mathrm{Spec}(k[x,y]) \supseteq \mathrm{Spec}_{\mathrm{max}}(k[x,y]) = k^2.$$

A point $(a,b) \in k^2$ is seen as the maximal ideal $(x-a, y-b)$; that comes from $\mathrm{Spec}(k[x,y]/(xy-1))$ if and only if $ab - 1 = 0$.

Example 1.18. $\mathrm{Spec}(k[x,y]/xy)$ gives the "cross" $\{ab = 0\}$.

Example 1.19. In the case $\mathrm{Spec}(k[x,y]/(x^2+y^2+1)) \hookleftarrow \mathrm{Spec}(k[x,y])$ we do not know how to draw.

Note 1.20. In this case we do not lose anything by thinking about $\mathrm{Spec}_{\mathrm{max}}$ instead of Spec.

1.6 Localization

Definition 1.21. Let A be a ring. $S \subseteq A$ is called a multiplicative subset if the both conditions hold:

1. $1 \in S$,

2. $\forall s, t \in S \ \ st \in S$.

We wish to obtain a ring $S^{-1}A \xleftarrow{i} A$ such that the $i(s)$ are invertible; a pseudo-

$$\{\frac{a}{s} \mid a \in A, s \in S\}.$$

Construction 1.22 (localization of a ring at a multiplicative subset)**.**
  Step 1. Let
$$I = \{a \in A \mid \exists s \in S \ \exists sa = 0\}.$$

  Note that if $a, b \in I$, then

$$\exists s \in S \ \ sa = 0, \quad \exists t \in S \ \ tb = 0.$$

Then $(st)(a \pm b) = 0$.
  Upshot: I is an ideal. We let $A \xrightarrow{\pi} A/I =: A'$.
  Step 2. $\pi(S) \subseteq A'$ consists of non-zerodivisors.
  Indeed, suppose that $s \in S$, $\pi(s)$ is a zero divisor, that is

$$\begin{aligned}
\exists 0 \neq a \in A \ \ &\pi(s)\pi(a) = 0 \\
&\implies \pi(sa) = 0 \\
&\implies sa \in I \\
&\implies \exists t \in S \ tsa = 0 \\
&\implies (ts)a = 0.
\end{aligned}$$

But
$$ts \in S \implies a \in I \implies \pi(a) = 0.$$

Which means that $\pi(s)$ was not really a zero divisor.

Define $S' := \pi(S)$. We want

$$(S')^{-1}A' = \{\frac{a'}{s'} \mid a' \in A', s' \in S'\}.$$

Consider pairs

$$\{(a', s') \mid a' \in A', s' \in S'\}.$$

Define a relation

$$(a'_1, s'_1) \sim (a'_2, s'_2) \iff a'_1 s'_2 = a'_2 s'_1.$$

To prove that this is an equivalence relation, we show transitivity (exercise; use that the S' are non-zerodivisors).

Step 3.

$$(S')^{-1}A' := \{(a', s') \mid a' \in A', s' \in S'\}/\sim$$

is a well-defined set. The ring operations are given by thinking about this as

$$\{\frac{a'}{s'} \mid a' \in A', s' \in S'\}.$$

This turns the set into an associative commutative ring with unity.

Lemma 1.23. Let $a, b \in A$, $s, t \in S$. Then

$$a/s = b/t \iff \exists u \in S \ u(at - bs) = 0.$$

Proof. "($\Longleftarrow$)":

$$u(at - bs) = 0$$
$$\Longrightarrow uta = usb$$
$$\Longrightarrow uta/1 = usb/1$$
$$\Longrightarrow a/s = uta/stu = usb/stu = b/t$$

"($\Longrightarrow$)": Observe: the map $\phi : A/I = A' \to S'^{-1}A^{-1}$ is injective. Note that

$$\forall \bar{a}, \bar{b} \in A' \ \bar{a}/1 = \bar{b}/1 \iff \bar{a} = \bar{b} \iff 1 \cdot \bar{a} = 1 \cdot \bar{b}$$

$\square$

Corollary 1.24. We could define $S^{-1}A$ as the set of points

$$\{(a, s) \mid a \in A, s \in S\}/\sim$$

under

$$(a, s) \sim (b, t) \iff \exists s \in S \ u(at - bs) = 0.$$

Example 1.25. For $f \in A$, let $S = \{1, f, f^2, \ldots\}$. We denote $S^{-1}A$ by A_f.

Example 1.26. $\mathfrak{p} \in \text{Spec}(A)$, $S := A \setminus \mathfrak{p}$. Then we denote $S^{-1}A$ by $A_\mathfrak{p}$.

Note 1.27. $S^{-1}A$ does not determine S, for instance

$$\mathbb{C}[x]_x = \mathbb{C}[x]_{x^{2019}}$$

Proposition 1.28 (universal property of localization)**.** Let $f : A \to B$ be such that $f(S)$ consists of invertible elements.

$$
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \\
& \searrow i \qquad \nearrow S^{-1}f & \\
& S^{-1}A &
\end{array}
$$

Then $\exists ! S^{-1} \colon S^{-1}A \to B$ such that $f = (S^{-1}f) \circ i$.

Proof. We write $\tilde{f} := S^{-1}f$. Suppose that $\tilde{f}$ exists. Then

$$\forall a \in A \ \tilde{f}(a/1) = f(a)$$

and thus

$$\forall s \in S \tilde{f}(\frac{a}{s} \cdot s) = f(\frac{a}{1}) = f(a)$$

and the formula $\tilde{f}(a/s) = f(a)/f(s)$ is recovered. One needs only to check that such a function is a homomorphism. $\qquad\square$

Lemma 1.29. $A_f \cong A[x]/(fx - 1)$.

Proof. Note that f is invertible in $A[x]/(fx - 1)$. Hence by universal property

$$\tilde{\phi}(a/f) = \phi(a)/\phi(f) = \bar{a}\bar{x}.$$

Now take $\psi \colon A[x] \to A_f$ a homomorphism of A-algebras defined by $\psi(x) = 1/f$. Then

$$\psi(fx - 1) = f \cdot 1/f - 1 = 0$$

and so

$$\exists \tilde{\psi} \colon A[x]/(xf - 1) \to A_f$$

such that

$$\forall a \in A, n \in \mathbb{N} \ \tilde{\psi}(ax^n) = a/f^n.$$

The maps $\tilde{\phi}$ and $\tilde{\psi}$ are then mutual inverses. $\qquad\square$

Corollary 1.30. If A is a finitely generated algebra, then A_f is also a finitely generated algebra.

Lemma 1.31. For A an algebra and S a multiplicative subset:

1. $S^{-1}A = 0 \iff 0 \in S$

2. $A \xrightarrow{i} S^{-1}A$ is injective if and only if all elements of S are non-zerodivisors in A

3. $A \to S^{-1}A$ is an isomorphism if and only if all elements of S are invertible in A

Proof. 1. By Lemma 1.29:

$$\frac{1}{1} = \frac{0}{1} \iff \exists u \in S \ u(1 \cdot 1 - 0 \cdot 1) = 0 \iff \exists u \in S \, u = 0.$$

2. i is injective if and only if

$$(\{a \in A \mid \exists s \in S \ sa = 0\}).$$

This happens if and only if s is a non-zerodivisor.

3. If $A \to S^{-1}A$ is iso, then $\forall s \in S$ $s/1$ is invertible. Hence, so is s.

   Conversely, if all elements of S are invertible in A, then they are non-zerodivisors. Hence $A \to S^{-1}A$ is injective by the previous point. Moreover,

$$i(as^{-1}) = \frac{as^{-1}}{a} = \frac{a}{s},$$

   so i is "onto".

$\square$

Intuitively, we think of A_f as the ring of functions on X_f, where $X = \operatorname{Spec}(A)$. We would expect $\operatorname{Spec}(A_f)$ to be the same as X_f. Indeed, the following holds.

Proposition 1.32. Consider the maps

$$i \colon A \to S^{-1}A$$
$$i^* \colon \operatorname{Spec}(S^{-1}A) \to \operatorname{Spec}(A).$$

Then i^* is injective and

$$\operatorname{im} i^* = \{\mathfrak{p} \mid \mathfrak{p} \cap S = \emptyset\}.$$

Before presenting the proof, for which we will require some additional facts, we note the following corollaries.

Corollary 1.33. If $S = \{1, f, f^2, \ldots\}$, then

$$\operatorname{im} i^* = \{\mathfrak{p} \mid \mathfrak{p} \not\ni f\} = (\operatorname{Spec}(A))_f.$$

Corollary 1.34. For $\mathfrak{q} \in \operatorname{Spec}(A)$, $S = A \setminus q$,

$$\operatorname{im} i^* = \{\mathfrak{p} \mid \mathfrak{p} \subseteq \mathfrak{q}\} = \operatorname{Spec}(A_p).$$

Lemma 1.35. Let $I \subseteq S^{-1}A$ be an ideal. If we let

$$J = \{a \in A \mid \frac{a}{1} \in I\},$$

then

$$I = \{\frac{j}{s} \mid j \in J, s \in S\}.$$

Proof. "$\supseteq$":

$$\forall j \in J \frac{j}{i} \in I \implies \frac{j}{s} = \frac{j}{1} \cdot \frac{1}{s} \in I$$

"$\subseteq$":

$$\frac{a}{s} \in I \implies \frac{a}{1} = \frac{a}{s} \cdot \frac{s}{i} \in I \implies a \in J \implies \frac{a}{s} \in \{\frac{j}{s} \mid j \in J, s \in S\}$$

$\square$

Proposition 1.32. Let $\mathfrak{q} \in \operatorname{Spec}(S^{-1}A)$. We let

$$\mathfrak{p} = i^*(\mathfrak{q}) = i^{-1}(\mathfrak{q}) = \{a \in A \mid f \in \mathfrak{q}\}.$$

By Lemma 1.35 we can recover $\mathfrak{q}$ from $\mathfrak{p}$ alone, and so i^* is injective.

Let $\mathfrak{p} = i^*(\mathfrak{q})$. Suppose $\mathfrak{p} \cap S \neq \emptyset$. This means that

$$\frac{s}{1} \in \mathfrak{q} \text{ and } \frac{s}{t} \text{ is invertible} \implies \mathfrak{q} = (1).$$

Suppose $\mathfrak{p} \in \operatorname{Spec}(A)$, $\mathfrak{p} \cap S = \emptyset$. We wish to find $\mathfrak{q}$. We guess that

$$q := \{\frac{p}{s} \mid p \in \mathfrak{p}, s \in S\} \subseteq S^{-1}A$$

does the job. This $\mathfrak{q}$ is an ideal, we want to see that it is prime.

Suppose that

$$\frac{a}{s} \cdot \frac{b}{t} \in \mathfrak{q}.$$

Then $ab/1 \in \mathfrak{q}$ and so $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 1.36. For $f \in A$, the following are equivalent:

1. f is nilpotent (that is, $\exists n > 0 \; f^n = 0$),

2. $f \in \cap\{\mathfrak{p} \mid p \in \operatorname{Spec}(A)\}$,

3. $V(f = \operatorname{Spec}(A)$.

Proof. The equivalence between 2 and 3 is checked formally.

$1 \implies 2$: $\forall \mathfrak{p} \; f^n = 0 \in \mathfrak{p} \implies f \in \mathfrak{p}$.

Negation of 1 implies negation of 2: let $0 \notin \{1, f, f^2, \ldots\}$ so $A_f \neq 0$ by Lemma 1.35. Hence, $\operatorname{Spec}(A_f) \neq \emptyset$, so by Proposition 1.32

$$\operatorname{Spec}(A_f) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid f \notin \mathfrak{p}\} \neq \emptyset$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 1.37. The nilradical of a ring A is defined as

$$\operatorname{nil}(A) = \cap\{\mathfrak{p} \mid p \in \operatorname{Spec}(A)\} = \{f \in A \mid \exists n > 0 \; f^n = 0\}.$$

2 Modules

2.1 Modules

Definition 2.1. Let A be a ring. An A-module M is an abelian group M together with a ring homomorphism $A \to \operatorname{End}_{\mathbb{Z}}(M)$.

Equivalently, M is an abelian group together with a map

$$A \times M \to M, \quad (a, m) \mapsto am,$$

such that:

1. $\forall a \in A, \ m_1, m_2 \in M \ \ a(m_1 + m_2) = am_1 + am_2$

2. $(a_1 + a_2)m = a_1 m + a_2 m$

3. $1 \cdot m = m$

4. $a_1(a_2 m) = (a_1 a_2)m$

Definition 2.2. A homomorphism of A-modules $\phi \colon M \to N$ is a homomorphism of abelian groups such that

$$\forall a \in A, \ m \in M \ \ a\phi(m) = \phi(am).$$

Example 2.3. For A a ring, A is an A-module; in fact, any ideal $I \subseteq A$ is an A-module.

Example 2.4. For any A-algebra B, B is an A-module. In particular, A/I is an A-module for any ideal $I \subseteq A$.

Definition 2.5. If M is an A-module and $m_1, \ldots, m_k \in M$ its elements, then the submodule generated by $m_1, \ldots, m_k$ is

$$Am_1 + Am_2 + \ldots + Am_k = \{\sum_{i=1}^{k} a_i m_i \mid a_i \in A\}.$$

Note 2.6. The set

$$\hom_A(M, N) = \{\phi : M \to N \text{ an } A\text{-module homomorphism}\}$$

is an A-module under

$$(a.\phi)(m) := \phi(am) = a\phi(m).$$

Definition 2.7. An A-module is finitely generated if $\exists k \in \mathbb{N}, \ m_1, \ldots, m_k \in M$ such that

$$M = A_1 + \ldots + A_{m_k}.$$

Definition 2.8. An A-module is free if it is isomorphic to an A-module of the form

$$\bigoplus_{i \in I} A.$$

Example 2.9. All k-vector spaces are free k-modules. Not all abelian groups are free.

Lemma 2.10. An A-module M is finitely generated if and only if

$$\exists \phi : \bigoplus_{i=1}^{k} A \to M \text{ surjective.}$$

Proof. □

Theorem 2.11 (Cayley-Hamilton)**.** Let M be an A-module. Let $\phi : M \to M$ be a homomorphism of A-modules such that

$$\phi(m) = IM = \{\sum i_k m_k \mid i_k \in I, m_k \in M\}.$$

Then there exist elements

$$a_{n-1} \in I, a_{n-2} \in I^2, \ldots, a_0 \in I^n$$

such that

$$\phi^n + a_{n-1}\phi^{n-1} + \ldots + a_0 = 0 \text{ (the zero morphism).}$$

2.2 More on modules

Note 2.12*.* When we write $I \cdot M$, we mean the linear span of elements of the form $i \cdot m$.

Lemma 2.13 (adjugate matrix)**.** Let $X \in M_{n \times n}(A)$. Then

$$\exists Y \in M_{n \times n}(A) \quad \text{such that} \quad Y \cdot X = \text{diag}(d), \ d = \det(X).$$

Concretely, Y is given by the formula

$$Y := [(-1)^{i+j} \det X_{ji}]_{ij},$$

where X_{ij} is the submatrix of X formed by excluding the i-th row and the j-th column.

Proof. Let

$$\pi : A^{\oplus n} \to M \quad \text{via} \quad e_i \mapsto m_i.$$
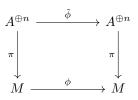
Then

$$\forall i \ \exists u_{ij} \in I \ \phi(m_i) = \sum_j u_{ij} m_j$$

in face of the assumptions. Define the lift of ϕ to a self-map $\tilde{\phi}$ of the free module $A^{\oplus n}$ by

$$\tilde{\phi} : A^{\oplus n} \to A^{\oplus n} \quad \text{via} \quad \tilde{\phi}(e_i) = \sum_j u_{ij} e_j.$$

This can be fitted inside a commutative diagram

$$
\begin{array}{ccc}
A^{\oplus n} & \xrightarrow{\ \tilde{\phi}\ } & A^{\oplus n} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
M & \xrightarrow{\ \phi\ } & M
\end{array}
$$

Let $B := A[x]$ and make $A^{\oplus n}$ into a B-module by

$$\forall f \in A^{\oplus n} \quad x.f := \tilde{\phi}(f).$$

Now,

$$X := [u_{ij}]_{ij} - \operatorname{diag}(x) \in M_{n \times n}(B).$$

View X as an endomorphism of $A^{\oplus n}$. Then

$$X.e_i = \sum_j u_{ij}e_j - x \cdot e_i = \sum_j u_{ij}e_j - \tilde{\phi}(e_j) = 0.$$

Hence X acts on $A^{\oplus n}$ as the zero endomorphism.

Now, use Lemma 2.13 to see that

$$\exists Y \ \ Y \cdot X = \operatorname{diag}(d).$$

Hence

$$\forall i \ \ 0 = Y \cdot X \cdot e_i = \operatorname{diag}de_i = d \cdot e_i.$$

Expand the determinant by columns:

$$d = \det(X) = x^n + a_1 x^{n-1} + \ldots + a_n, \quad a_k \in I^k.$$

So as an endomorphism of $A^{\oplus n}$,

$$\tilde{\phi}^n + a_1\tilde{\phi}^{n-1} + \ldots + a_n = 0.$$

But

$$\phi \circ \pi = \pi \circ \tilde{\phi}.$$

So for all i

$$(\phi^n + a_1\phi^{n-1} + \ldots + a_n)(m_i) = \pi((\tilde{\phi}^n + a_1\tilde{\phi}^{n-1} + \ldots + a_n)(e_i))$$
$$= \pi(0)$$
$$= 0.$$

$\square$

Lemma 2.14 (Nakayama)**.** Let M be an A-module, $I \cdot M = M$. If M is finitely generated, then

$$\exists i \in I \ \ (1 - i) \cdot M = 0.$$

Proof. Consider $\operatorname{id}_M \colon M \xrightarrow{\simeq} M$. By Section 2.2,

$$\exists a_i \ \ \operatorname{id}^n + a_1\operatorname{id}^{n-1} + \ldots + a_n = 0$$

and then

$$\forall m \in M (1 + a_1 + \ldots + a_n) \cdot m = 0.$$

We now let

$$-i := a_1 + \ldots + a_n.$$

$\square$

Corollary 2.15 (local Nakayama)**.** Let A be a local ring, $\mathfrak{m}$ the maximal ideal of A, M an A-module, M finitely generated, $M = \mathfrak{m} \cdot M$. Then

$$M = 0.$$

Proof. Apply Lemma 2.14:

$$\exists a \in \mathfrak{m} \ (1-a) \cdot M = 0, \quad (1-a) \notin \mathfrak{m}.$$

But $(1-a)$ is invertible; let its inverse be b. Then

$$M = 1 \cdot M = b(1-a)M = b \cdot 0 = 0.$$

$\square$

Corollary 2.16. Let $N \subseteq M$, M finitely generated, A local, $\mathfrak{m}$ maximal. Let

$$M = N + \mathfrak{m} \cdot M.$$

Then $M = N$.

Proof. $(M/N) = (M/N)$. By Corollary 2.15 $M/N = 0$. $\square$

Corollary 2.17. Let M be a finitely generated A-module, $(A, \mathfrak{m})$ a local ring. Suppose

$$m_1, \ldots, m_k \in M$$

be such that their images in $M/\mathfrak{m} \cdot M$ generate $M/\mathfrak{m} \cdot M$. Then $m_1, \ldots, m_k$ generate M.

Proof. Let

$$N := \sum_{i=1}^{k} A \cdot m_i \subseteq M.$$

Then the map

$$N \to M/\mathfrak{m} \cdot M$$

is surjective by assumption, so by Corollary 2.15:

$$M = \mathfrak{m} \cdot M + N \implies M = N = \sum_{i=1}^{k} A \cdot m_i.$$

Note that

$$A/\mathfrak{m}$$

is a field, so

$$M/\mathfrak{m} \cdot M$$

is a vector space. This finishes the proof. $\square$

Example 2.18. Consider $\mathbb{Q} \in \mathbb{Z}_{\mathrm{mod}}$. Consider $I = (2019) \subseteq \mathbb{Z}$. Certainly $\mathbb{Q} = I \cdot \mathbb{Q}$, but there does not exist an iI such that

$$(1-i) \cdot \mathbb{Q} = 0,$$

so $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module.

Example 2.19. Another counterexample (this time local) is given by $\mathbb{Z}_{(2)}$.

Note 2.20. Localizing is a typical way of finding non-finitely generated modules.

Example 2.21. Let $X = S^1$, $A = C(S^1, \mathbb{R})$. Consider

$$\mathfrak{m}_x = \{f \mid f(x) = 0\}.$$

Then $\mathfrak{m}_x^2 = \mathfrak{m}_x$, hence $\mathfrak{m}_x$ is not a finitely generated ideal.

2.3 Tensor product

Fix A-modules M, N. We will construct another A-module, denoted by

$$M \otimes_A N,$$

and study its properties.

Definition 2.22. Let P be an A-module. A function (not a homomorphism!)

$$f \colon M \times N \to P$$

is called bilinear if:

1. $\forall a \in A, m \in M, n \in N \ \ af(m,n) = f(am,n) = f(m,an)$,

2. $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$,

3. $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$.

The set of such functions will be denoted by

$$\mathrm{Bilin}_{M \times N}(P).$$

Intuitively, we wil think about f as a "multiplication".

Example 2.23. For a homomorphism

$$\phi \colon P \to R,$$

where R is an A-module, we get

$$\phi^* \colon \mathrm{Bilin}_{M \times N}(P) \to \mathrm{Bilin}_{M \times N}(R) \quad \text{via} \quad f \mapsto \phi \circ f.$$

Theorem 2.24. In Example 2.23, there exists

$$T \in A_{\mathrm{mod}}, g \in \mathrm{Bilin}_{M \times N}(T) \quad \text{such that} \quad \forall P \in A_{\mathrm{mod}} \quad \forall f \in \mathrm{Bilin}_{M \times N}(P) \ \exists! \ f' \colon T \to P$$

such that

$$f = f' \circ g.$$

Moreover, this T is unique.

Note 2.25. One can rephrase the theorem as the existence of a natural isomorphism

$$\hom(T, -) \simeq \mathrm{Bilin}_{M \times N}(-).$$

Definition 2.26. The A-module T of Theorem 2.24 is denoted by

$$M \otimes_A N$$

and called a tensor product of M and N. The map g is written as

$$m \otimes n \coloneqq g(m, n).$$

Theorem 2.24. Let

$$T_0 = A^{M \times N}.$$

Then

$$\forall p \ \hom(T_0, P) \simeq \mathrm{Set}(M \times N, P) \quad \text{via} \quad f \mapsto \sum a_i f(e_i),$$

that is, the free module functor is adjoint to the forgetful functor to Set. Denote the generator of T_0 corresponding to (m, n) by $e(m, n)$. Now define

$$T = T_0/\sim,$$

where $\sim$ describes the relations given in Definition 2.22. We now claim that T together with the quotient of e satisfies the universal property.

Now, for uniqueness: adjoints are unique by the faithfulness of the Yoneda embedding.

Alternatively, apply standard abstract nonsense for universal properties. $\square$

Question 2.27. Do all representable functors have left adjoints?

Answer 2.28. Yes, provided the category considered has all coproducts, as then for functors into Set the two conditions are in fact equivalent; the same holds in the case of corepresentability and having a right adjoint.

Concretely, the left adjoint to a functor represented by X is given by

$$Y \mapsto \bigsqcup_Y X.$$

Definition 2.29. Let k be a ring, $k \to A$ a k-algebra. For $P \in A_{\mathrm{mod}}$ we define

$$\mathrm{Der}_k(A, p) = \{d \colon A \to P \mid d(a + b) = d(a) + d(b), d(ab) = ad(b) + d(a)b, d(k) = \{0\}\}.$$

For $\phi \colon P \to R$ we get pullback

$$\mathrm{Der}_k(A, P) \to \mathrm{Der}_k(A, R) \quad \text{via} \quad d \mapsto \phi \circ d$$

Theorem 2.30. There exists an A-module

$$\Omega_{A/k}, \ d \colon A \to \Omega_{A/k}$$

such that

$$\forall P \in A_{\mathrm{mod}} \forall f \in \mathrm{Der}_k(A, P) \exists! f' \colon \Omega_{A/k} \to P \quad \text{such that} \quad f = f' \circ d.$$

One can rephrase this as

$$\hom_{\Omega_{A/k}} \simeq \mathrm{Der}_k(A, -).$$

Proof. Define Ω as a quotient of A^A by the obvious relations, denote

$$A^A =: \{e(a) \mid a \in A\}.$$

Give $d \colon A \to \Omega$ by $d(a) = e(a)$. $\square$

Example 2.31. Let $A = k[x_1, \ldots, x_n]$. Then $\Omega_{A/k}$ is a free module on n generators

$$d(x_1), \ldots, d(x_n).$$

Indeed, as $\Omega_{A/k}$ is generated by

$$\{\mathrm{d}f \mid f \in A\}.$$

We claim that

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i.$$

Then $\Omega_{A/k}$ is indeed generated by

$$\mathrm{d}x_1, \ldots, \mathrm{d}x_n.$$

Consider the k-linear derivation gradient

$$\nabla \colon A \to A^{\oplus n} \quad \text{via} \quad f \mapsto (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}).$$

By the universal property of localization we obtain

$$\pi \colon \Omega_{A/k} \text{ such that } \forall 1 \le i \le r \ \pi(\mathrm{d}x_i) = e_i.$$

If we let

$$\phi \colon S^{\oplus n} \to \Omega_{A/k} \quad \text{via} \quad e_i \mapsto \mathrm{d}x_i.$$

Then ϕ is onto and since

$$\pi \circ \phi = \mathrm{id}_{S^{\oplus n}},$$

ϕ is a bijection (and hence, an isomorphism).

2.4 Localization of modules

Definition 2.32. The rank of a free module F is

$$\mathrm{rk}(F) := \dim(F/\mathfrak{m}F) \text{ over } A/\mathfrak{m}.$$

This does not depend on the choice of $\mathfrak{m}$.

Example 2.33. $\mathrm{rk}(A^{\oplus I}) = |I|$.

Example 2.34. $\Omega_{A/k}$ is free of rank n.

Definition 2.35. Let $S \subseteq A$ be a multiplicative subset, $M \in \mod A$. The one forms an $S^{-1}A$-module $S^{-1}M$ analogously to $S^{-1}A$ itself. That is,

$$S^{-1}M := \{m/s \mid m \in M, s \in S\}$$

with

$$m/s = m'/s' \iff \exists t \in S \ t(s'm - sm') = 0.$$

Example 2.36. If
$$S = \{1, f, f^2, \dots\},$$
then we denote
$$M_f = S^{-1}M.$$
Analogously, for
$$S = A \setminus \mathfrak{p},$$
write
$$M_{\mathfrak{p}} = S^{-1}M.$$

Definition 2.37. A finitely generated k-algebra A is smooth if
$$\forall p \in \mathrm{Spec}(A) \ (\Omega_{A/k})_{\mathfrak{p}} \text{ is free as an } A_{\mathfrak{p}}\text{-module.}$$

Example 2.38. $k[x_1, \dots, x_n]$ is smooth.

Lemma 2.39. Let $S \subseteq A$ be a multiplicative subset, M an A-module. Then
$$S^{-1}A \otimes_A M \cong S^{-1}M$$
under the map
$$(a/s) \otimes m \mapsto am/s.$$

Proof. The given map
$$f \colon S^{-1}A \otimes M \to S^{-1}M$$
is linear by the universal property of the tensor product. We show that it is an isomorphism.

1. f is "onto":
$$f(1/s \otimes m) = m/s,$$

2. f is "into": take an element
$$n \in S^{-1}A \otimes_A M.$$

    We write
    $$n = \sum_{k=1}^{r} a_k/s_k \otimes m_k = \sum_{k=1}^{r} \tilde{a}_k/s \otimes \tilde{m}_k.$$

    If we let
    $$s = \prod_{k=1}^{r} s_k,$$

    we can write
    $$\sum_{k=1}^{r} 1/s \otimes \tilde{a}_k m_k = 1/s \otimes \left(\sum_{k=1}^{r} \tilde{a}_k m_k\right) =: 1/s \otimes m.$$

    Since any n is of the form, we may proceed as follows. Assume $0 = f(n)$; we will show $n = 0$. Have:
    $$0 = f(n) = m/s \iff \exists t \in S \ \ tm = 0.$$

    Then
    $$n = 1/s \otimes m = 1/(st) \otimes tm = 1/(st) \otimes 0 = 0.$$

    $\square$

Corollary 2.40.
$$S^{-1}(\oplus_{i \in I} M_i) = \oplus_{i \in I} S^{-1}M_i.$$

Proof. $\square$

2.5 Exactness

Definition 2.41. Let M_i be A-modules, $i = 1, 2, 3$.

1. The sequence
$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$$
   is exact if
$$\operatorname{im} f_1 = \ker f_2.$$

2. The sequence
$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$
   is exact if and only if
$$M_1 \to M_2 \to M_3$$
   is exact and f_2 is surjective.

3. The sequence
$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$
   is a short exact sequence if:
   - f_1 is injective,
   - $\operatorname{im} f_1 = \ker f_2$,
   - f_2 is surjective.

Proposition 2.42 (right-exactness of the tensor product)**.** For all exact sequences
$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

and for all $N \in \mod A$, the induced sequence
$$M_1 \otimes_A N \xrightarrow{f_1 \otimes_A \mathrm{id}} M_2 \otimes_A N \xrightarrow{f_2 \otimes \mathrm{id}} M_3 \otimes N \to 0$$

is exact.

Proof. First, we need to check that
$$M_2 \otimes_A N \to M_3 \otimes_A N$$

is "onto". This is checked on elements: $M_3 \otimes_A N$ is generated by elements of the form
$$m_3 \otimes n \quad \text{such that} \quad m_3 \in M_3, n \in N.$$

The claim then follows by surjectivity of f_2 itself.

Exactness in the middle means that
$$M_2 \otimes N / (f_1 \otimes \mathrm{id})(M_1 \otimes N) \to M_3 \otimes N$$

is an isomorphism. We already checked that it is "onto"; now we need to know that there exists a section, that is, a one-sided inverse.

Have
$$M_2/f_1(M_1) \cong M_3 \times N \to M_2 \otimes_A N / (f_1 \otimes \mathrm{id})(M_1 \otimes_A N) \text{ via } (m_3, n) \mapsto \overline{m_2 \otimes n}.$$

This is bilinear if it is well-defined. Hence, it yields a map

$$s \colon M_3 \otimes_A N \to M_2 \otimes N$$

and indeed, the equality

$$s \circ \pi = \mathrm{id}_{M_3 \otimes N}$$

holds, and so π is injective and further, an isomorphism. $\qquad\square$

Note 2.43. We have shown that
$$M_3 \cong M_2/f_1(M_1).$$

Hence, we get
$$M_2/(f_1(M_1) \otimes N \cong M_2 \otimes_A N/(f_1(M_1) \otimes_A N).$$

Example 2.44. $A/I \otimes_A N \cong N/IN$.

And in fact, quotients and localizations of modules can be computed by means of the tensor product.

Note 2.45. The notions of right-exactness as defined above in Proposition 2.42 coincide with the preserving of colimits. The definition of Proposition 2.42 is valid in the case of preadditive categories.

2.6 Why do we care?

First of all, as noted before, we have the folowing isomorphisms:

$$M \otimes_A A/I \cong M/IM,$$

$$M \otimes_A S^{-1}A \cong S^{-1}M.$$

Moreover:

Proposition 2.46. Tensoring with an A-algebra B is left adjoint to the forgetful functor

$$B_{\mathrm{mod}} \to A_{\mathrm{mod}}.$$

Proposition 2.47. Suppose A-algebras B, C are given. Then

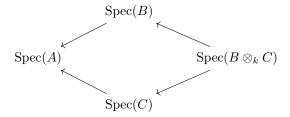$$B \otimes_A C$$

is an A-algebra with multiplication

$$(b_1 \otimes c_1) \cdot (b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2).$$

We claim that

$$B \otimes_A C$$

is in fact the coproduct of B and C.

Note 2.48. In the same setting, one has the diagram

$$\text{Spec}(B)$$

$$\text{Spec}(A) \qquad \text{Spec}(B \otimes_k C)$$

$$\text{Spec}(C)$$

Note that then

$$\text{Spec}(B \otimes_k C) \neq \text{Spec}(B) \times \text{Spec}(C);$$

we only get the arrows demanded that make the diagram commute.

We note the following special cases of Proposition 2.47.

Corollary 2.49. Let $A = k$ a field,

$$B = k[x_1, \ldots, x_s], \quad C = k[y_1, \ldots, y_k].$$

Then

$$B \otimes_k C \cong k[x_1, \ldots, x_s, y_1, \ldots, y_t].$$

Proof. Note that B and C are free k-modules, so $B \otimes_k C$ is a free k-module with basis

$$x_1^{a_1} \otimes \ldots \otimes x_s^{a_s} \otimes y_1^{b_1} \otimes \ldots \otimes y_t^{b_t}.$$

One checks that the obvious map defined by the universal property is an isomorphism. $\square$

Corollary 2.50. Suppose

$$B = k[x_1, \ldots, x_s]/I,$$
$$C = k[y_1, \ldots, y_t]/J.$$

Then

$$B \otimes_k C \cong k[x_1, \ldots, x_s, y_1, \ldots, y_t]/(I + J).$$

In differential geometry, the fiber of a map $f \colon X \to Y$ over a point is not typically a manifold. In contrast, in the Spec-world, we have the following.

Definition 2.51. Let $f \colon A \to B$ be a homomorphism, $\mathfrak{m} \subseteq A$ a maximal ideal. Then the fiber of f over $\mathfrak{m}$ is

$$B/\mathfrak{m}B.$$

Proposition 2.52. In the above setting,

$$\text{Spec}(B/\mathfrak{m}B) \cong \{\mathfrak{p} \subseteq B \mid f^*(\mathfrak{p}) = \{\mathfrak{m}\}\}.$$

Note that

$$\text{Spec}(A_{\mathfrak{p}}) = \{\mathfrak{q} \subseteq \mathfrak{p}\}, \quad \text{Spec}(A/\mathfrak{p}) = \{\mathfrak{q} \supseteq \mathfrak{o}\}.$$

Definition 2.53. More generally, if $f \colon A \to B$ is a homomorphism, $\mathfrak{p} \subseteq B$ a prime ideal, then the fiber of f over $\mathfrak{p}$ is given by

$$B \otimes_A \kappa(\mathfrak{p}),$$

where

$$\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}.$$

Proposition 2.54. For any prime ideal $\mathfrak{p} \subseteq B$, $\kappa(\mathfrak{p})$ is a field.

Note 2.55. If $\mathfrak{m}$ is maximal, then

$$\kappa(\mathfrak{m}) \cong A/\mathfrak{m}.$$

Proposition 2.56. There is a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(B) & \longleftarrow & \mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p})) \\
\downarrow{\scriptstyle f^*} & & \downarrow \\
\mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(\kappa(\mathfrak{p}))
\end{array}
$$