POLYTROPES

9/24/2020 (57)

Consider the equations of hydrostatic equilibrium

$$\frac{dp(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2} \qquad \frac{dM(r)}{dr} = -4\pi r^2 \rho(r)$$

$$\frac{r^2}{\rho(r)} \frac{dp(r)}{dr} = -GM(r)$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)} \frac{dp(r)}{dr}\right) = -G\frac{dM(r)}{dr} = -4\pi r^2 \rho(r) G$$

$$\Rightarrow -\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)} \frac{dp(r)}{dr}\right) = 4\pi G\rho(r)$$ Poisson's Equation

★ WOULDN'T IT BE NICE TO HAVE AN EQUATION RELATING $p(r)$ to $\rho(r)$?

WE WOULDN'T NEED $\frac{dT(r)}{dr}$ which makes things difficult!

Consider the first law of thermodynamics

$$dU = dQ - p\,dV$$

$$dU = \left.\frac{\partial Q}{\partial T}\right|_{p,n} dT - p \left.\frac{\partial V}{\partial T}\right|_{p,n} dT$$

$$C_p \equiv \left.\frac{\partial Q}{\partial T}\right|_{p,n}$$ heat capacity at constant pressure

For an ideal gas: $pV = nk_B T$

$$\Rightarrow p\,dV + V\,dp = nk_B\,dT + Tk_B\,dn$$

For constant pressure and constant number of particles $dp = dn = 0$

$$\Rightarrow p\,dV = nk_B\,dT$$

$$p\frac{\partial V}{\partial T} = nk_B$$

$$\frac{dU}{dT} = \frac{C_p \cancel{dT}}{\cancel{dT}} - \frac{nk_B \cancel{dT}}{\cancel{dT}} \equiv C_V$$

$$\Rightarrow C_p - C_V = nk_B$$

☆ DOES IT MAKE SENSE THAT C_p and C_V ARE NOT EQUAL?

$$\frac{C_p - C_V}{C_V} = \frac{C_p}{C_V} - 1 = \frac{nk_B}{C_V}$$

$$\Rightarrow C_p/C_V = \frac{nk_B}{C_V} + 1 = \Gamma$$ Adiabatic index

Consider the first law of thermodynamics

for an ideal gas expanding adiabatically, so $dQ = 0$ ~~at constant pressure~~ ✓

$$dU = -p\,dV$$

$$p\,dV + V\,dp = nk_B\,dT + Tk_B\,dn$$

for constant pressure and constant number of particles,

$$p\,dV = nk_B\,dT$$

~~so $dU = -nk_B\,dT$~~

from the definition of $C_V = \frac{dU}{dT} \Rightarrow dT = \frac{dU}{C_V}$

$$dT = \frac{-p\,dV}{C_V}$$

$$\Rightarrow p\,dV + V\,dp = -\frac{nk_B\,p\,dV}{C_V}$$

$$V\,dp = -p\,dV\left[\frac{nk_B}{C_V} + 1\right]$$

$$\int \frac{dp}{p} = -\Gamma \int \frac{dV}{V}$$

$$\ln p = -\Gamma \ln V + K$$

$$\ln p + \Gamma \ln V = K$$

$$\ln p + \ln V^{\Gamma} = K$$

$$\ln(pV^{\Gamma}) = K$$

$$pV^{\Gamma} = e^{K} = K$$

$$\boxed{pV^{\Gamma} = K}$$

Adiabatic gas law

$$P = K\left(\frac{1}{V}\right)^{\Gamma} = K\rho^{\Gamma}$$

W 1.8.1

Poisson's Equation

$$-\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{d}{dr}K\rho^{\Gamma}(r)\right) = 4\pi G\rho(r)$$

Remember $$\frac{d\rho^{\Gamma}}{dr} = \Gamma\rho^{\Gamma-1}\frac{d\rho}{dr} + \cancel{\rho^{\Gamma}\ln\rho\frac{d\Gamma}{dr}}$$ c

so

$$-\frac{K}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\Gamma\rho^{\Gamma-1}(r)\frac{d\rho}{dr}\right) = 4\pi G\rho(r)$$

$$\frac{K\Gamma}{r^2}\frac{d}{dr}\left(r^2\rho^{\Gamma-2}(r)\frac{d\rho}{dr}\right) = -4\pi G\rho(r)$$

Second-order differential equation can be solved uniquely with two boundary conditions.

① $\rho(0) =$ some value

② $$\frac{dP(r)}{dr} = K\frac{d\rho^{\Gamma}}{dr} = K\Gamma\rho^{\Gamma-1}\frac{d\rho}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\Rightarrow \left.\frac{d\rho}{dr}\right|_{r=0} = -\frac{GM(r)\rho^{2-\Gamma}(r)}{K\Gamma r^2} = 0 \quad \text{(L'Hôpital)}$$

Let $\Theta \equiv \left(\frac{\rho(r)}{\rho(0)}\right)^{\Gamma-1}$, then

Capital omega

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\Theta\right) = -\frac{4\pi G(\Gamma-1)}{K\Gamma}\rho(0)^{(2-\Gamma)}\,\Theta^{1/(\Gamma-1)}$$

Let $\xi \equiv \left[\frac{4\pi G(\Gamma-1)}{K\Gamma}\right]^{1/2}\rho(0)^{(2-\Gamma)/2}\, r$

xi

Thus poisson's becomes

Lane-Emden Equation

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}\Theta(\xi)\right) + \Theta(\xi)^{1/(\Gamma-1)} = 0} \quad \underline{W1.8.5}$$

with boundary conditions

$\Theta(0) = 1 \qquad \Theta'(0) = 0$

K free parameter (depends on $\rho(c)$, $\rho(0)$, etc

Anoter way to write Lane-Emden is $\Gamma = \frac{n+1}{n}$

so $\frac{1}{\Gamma-1} = \frac{1}{\frac{n+1}{n}-1} = \frac{1}{\frac{n+1-n}{n}} = n$ where n is called the index of the polytrope

☆WHY IS LANE-EMDEN USEFUL?

For ideal gas $\Gamma = 5/3$, $\quad 5n = 3n+3 \Rightarrow 2n = 3 \Rightarrow \underline{n = 1.5}$

For pure radiation $\Gamma = 4/3$, $\quad 4n = 3n+3 \Rightarrow \underline{n = 3}$

idealgas: light white dwarfs, Gas giants, Convective stars

pure radiation: heavy white dwarfs, Supermassive stars

There are only <u>three</u> values of Γ for which Lane-Emden has an analytic solution: ∞, 2, 6/5

$\Gamma = \infty \Rightarrow n = 0$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2 \frac{d}{d\xi}\Theta(\xi)\right] + \cancel{\Theta(\xi)^{\frac{1}{\infty}}}^{\,1} = 0$$

$$\int \frac{d}{d\xi}\left[\xi^2 \frac{d}{d\xi}\Theta(\xi)\right] = -\int \xi^2\, d\xi$$

$$\frac{d}{d\xi}\Theta(\xi) = -\frac{\xi}{3} + \frac{C}{\xi^2}$$

since $\Theta'(0) = 0 \qquad C = 0$

$$\int \frac{d}{d\xi}\Theta(\xi) = \int -\frac{\xi}{3}\, d\xi$$

$$\Theta(\xi) = -\frac{1}{3}\frac{\xi^2}{2} + D = -\frac{\xi^2}{6} + D$$

since $\Theta(0) = 1 \qquad D = 1$

so $\Theta(\xi) = -\dfrac{\xi^2}{6} + 1$

The root is at $\Theta(\xi) = -\dfrac{\xi^2}{6} + 1 = 0 \Rightarrow \xi_1 = \sqrt{6}$

Weinberg pg. 62

	Γ	ξ_1	$\xi^2 \lvert \Theta'(\xi_1) \rvert$	n
*A	$\frac{6}{5}$	∞	$\sqrt{3}$	5
N	4/3	6.89	2.01	3
N	3/2	4.35	2.41	2
N	5/3	3.65	2.71	3/2
*A	2	π	π	1
*A	∞	$\sqrt{6}$	$2\sqrt{6}$	0

Important because $$R = \left(\frac{4\pi G(\Gamma - 1)}{K\Gamma} \right)^{-1/2} \rho(0)^{-(2-\Gamma)/2} \xi_1$$

$$M = 4\pi \rho(0)^{(3\Gamma-4)/2} \left(\frac{K\Gamma}{4\pi G(\Gamma-1)} \right)^{3/2} \xi_1^{2} \lvert \Theta'(\xi_1) \rvert$$

From integrating the solution.