# Galaxies — Collisionless Dynamics

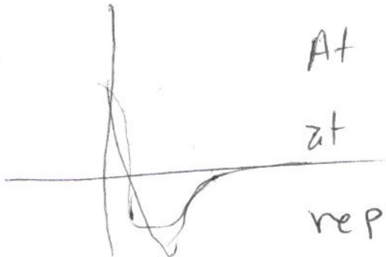
## Coming full circle

Collisionless dynamics is the study of the motion of a system of particles that orbit under their self-gravity. Examples:

| | | |
|---|---|---|
| Star clusters | $10^4 - 10^6$ | stars |
| Galaxies | $10^6 - 10^{11}$ | stars |
| Galaxy clusters | $10^2 - 10^3$ | galaxies |
| Cold Dark Matter Haloes | $\gg 10^{50}$ | haloes |

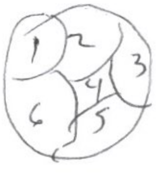
Particles interact gravitationally only, so in priciple we can use discrete Newtonian mechanics to study the systems, but the most powerful computers can only simulate up to $\sim 10^6$ particles, so in practice we need continuous mass density $\rho(\vec{x})$ and gravitational potential $\phi(\vec{x})$ functions.

This is one <u>strange</u> fluid. In a regular fluid, for example a gas, the particles collide with each other and interact via van der Waals forces (dipoles).



At large distance the forces are attractive, but at short distances the electron cloud is strongly repulsive.

In a "gravitational gas" the forces are always attractive, but the density is so low that the particles never collide (hence the term "collisionless").

| Fluid | Gravitational system |
|---|---|
| mean free path << size of the system | mean free path >> size of the system |
| particles collide frequently, this produces a pressure that balances out gravity, so hydrostatic equilibrium | particles don't collide, but they exchange kinetic and potential energy, so they are in virial equilibrium |
| Energy is an extensive variable, total energy is the sum of energies of subsystems (because no strong long-range interaction  1+2+3+4+5+6 | Energy is non-extensive variable. Because of longrange interactions, adding new subsystem modifies the energies of the other ones. |
| Equation of state: well-defined relationship between pressure and density | No equation of state, although there is a self-consistency relationship between mass density, gravitational potential, and orbits. |

Virial theorem ~~[illegible]~~ $2T + \Omega = 0$

since $E = T + \Omega$, $\quad 2T + (E - T) = T + E = 0$

$$\Rightarrow \underline{\underline{T = -E}}$$

$$2(E - \Omega) + \Omega = 2E - 2\Omega + \Omega = 2E - \Omega =$$

$$\Rightarrow \underline{\underline{\Omega = 2E}}$$

The virial velocity is given by:

$$T = \frac{1}{2} M v_v^2 \Rightarrow v_v^2 = \frac{2T}{M} = \frac{2|E|}{M}$$

where M is the total mass of the system

The virial radius is given by:

$$\Omega = \frac{GM^2}{R_v} \Rightarrow R_v = \frac{GM^2}{\Omega} = \frac{GM^2}{2E}$$

The virial time is given by:

$$v_v = \frac{R_v}{t_v} \Rightarrow t_v = \frac{R_v}{v_v} = \frac{GM^2}{2E}\left(\frac{M^{1/2}}{2^{1/2}|E|^{1/2}}\right)$$

$$t_v^2 = \frac{G^2 M^4}{4E^2} \frac{M}{2|E|} = \frac{G^2 M^5}{8|E|^3}$$

$$\Rightarrow t_v = G\left(\frac{M^5}{8|E|^3}\right)^{1/2}$$

Since both the total mass and the total energy are conserved, the virial time is constant even out of equilibrium

Crossing time.



$t_v$ approximates the time scale over which the system reaches equilibrium (relaxes)

★ WE CAN USE $t_v$ TO ESTIMATE THE TIME IT WILL TAKE THE ANDROMEDA-MILKYWAY COMBINED GALAXY TO RELAX TO EQUILIBRIUM. (MILKOMEDA)

★ WHAT WILL HAPPEN TO THE SUPERMASSIVE BLACK HOLES AT THE CENTERS OF THE GALAXIES?

A: THEY WILL MERGE EMITTING GRAVITATIONAL WAVES, ~~THEN~~ WE WILL HAVE AN ACTIVE GALACTIC NUCLEUS OR QUASAR

★ WHAT WILL HAPPEN TO THE SUN?

A: UNCLEAR AT THIS POINT. SOME SIMULATIONS PREDICT THAT IT WILL GET EJECTED FROM MILKOMEDA, OTHERS PREDICT THAT IT WILL BE SWALLOWED BY THE SMBH.

---

Another expression for $t_v$

$$2T + \Omega = 0$$

$$\cancel{2}\frac{1}{\cancel{2}} M v_v^2 = \frac{GM^{\cancel{2}}}{R_v}$$

$$v_v^2 = \frac{R_v^2}{t_v^2} \Rightarrow t_v^2 = \frac{R_v^2/1}{GM/R_v} = \frac{R_v^3}{GM}$$

with $\rho = \dfrac{M}{\frac{4}{3}\pi R^3} = \dfrac{3M}{4\pi R^3}$

$$\Rightarrow \frac{R^3}{M} = \frac{3}{4\pi\rho}, \quad \text{so} \quad t_V = \left(\frac{3}{4\pi G\rho}\right)^{1/2}$$

A typical value for $t_V$ for galaxies is $10^8$ years (hundreds of millions of years, pretty quick).

In order to simulate via brute force, $6N$ equations are needed, where $N$ is the number of stars. This is a huge parameter space.

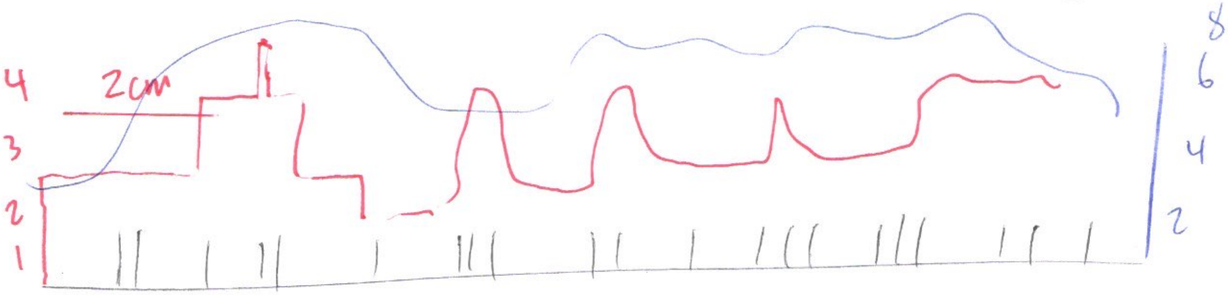
INSTEAD, consider that each star moves in the gravitational field of the galaxy as a whole.

The number of stars in an element $d^3x$ of space volume at position $\vec{x}$ an in an element $d^3v$ in velocity space at velocity $\vec{v}$ and at time $t$ is $f(\vec{x}, \vec{v}, t)\, d^3x\, d^3v$, where

$$f(\vec{x}, \vec{v}, t) = \overline{\sum_N \delta^3(\vec{x} - x_N(t))\, \delta^3(\vec{v} - \vec{v}_N(t))} \qquad \underline{\text{W4.1.1}}$$

The bar indicates that it is a time average, but for time $\ll t_V$, so almost instantaneous. The $\delta^3$ are Dirac delta functions in position and velocity space

Rigurously speaking, the deltas would result in zero everywhere except for where you have actual stars. In 1-D this might look like:

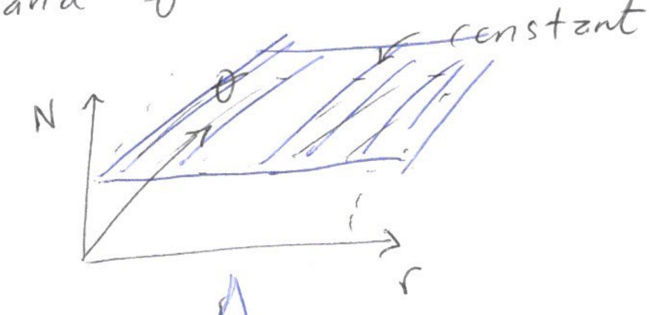


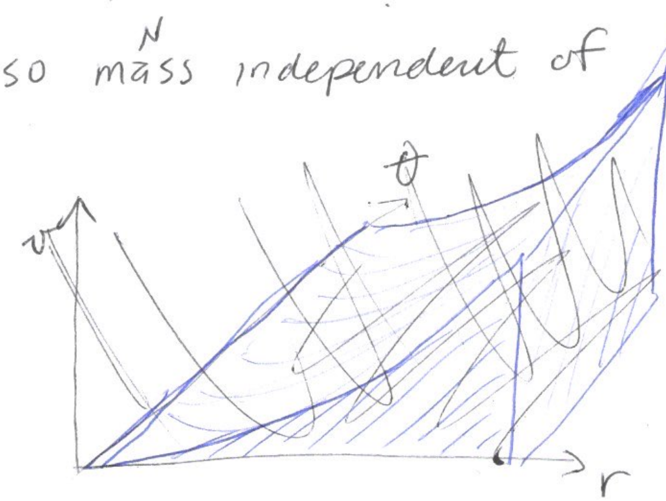

3cm  But you can consider larger regions and this will result in a smoother function

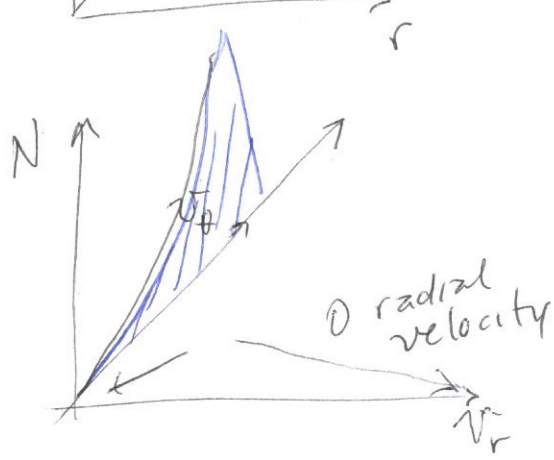
$x_N(t)$ and $v_N(t)$ are the position and velocity of star N at time t.

How does f look for a 2-D uniform rotating rigid disk?

~~Cstar~~ Cylindrical symmetry, so $\overset{N}{\text{mass}}$ independent of r and $\theta$

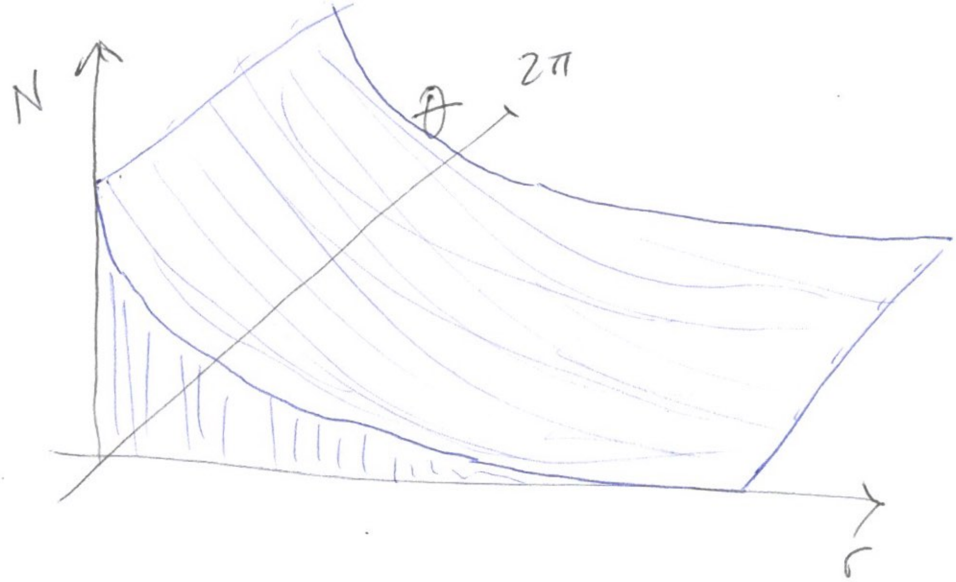


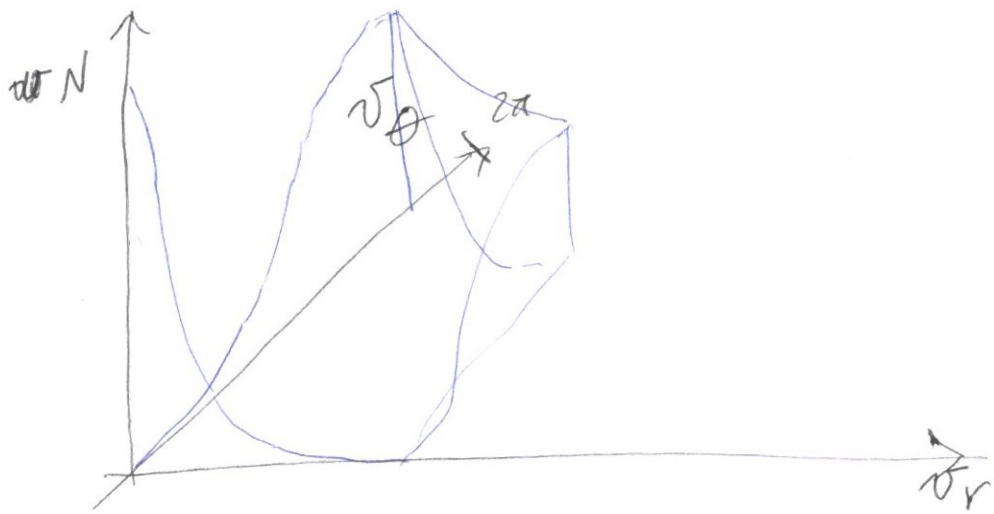



What about an accretion disk, Keplerian?

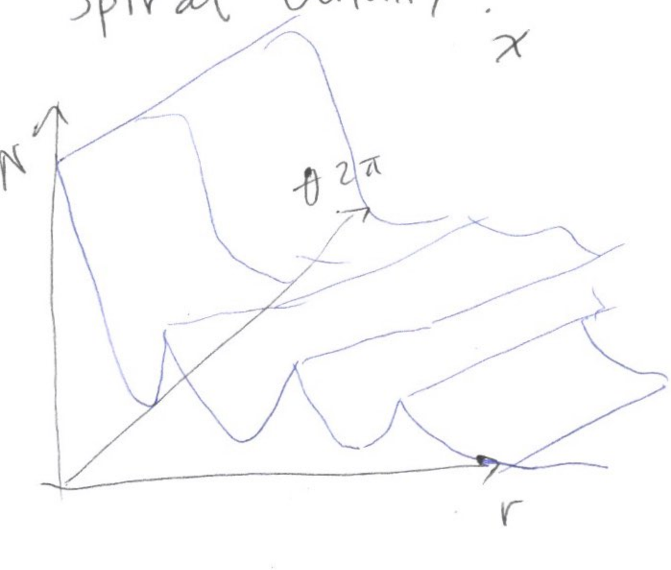




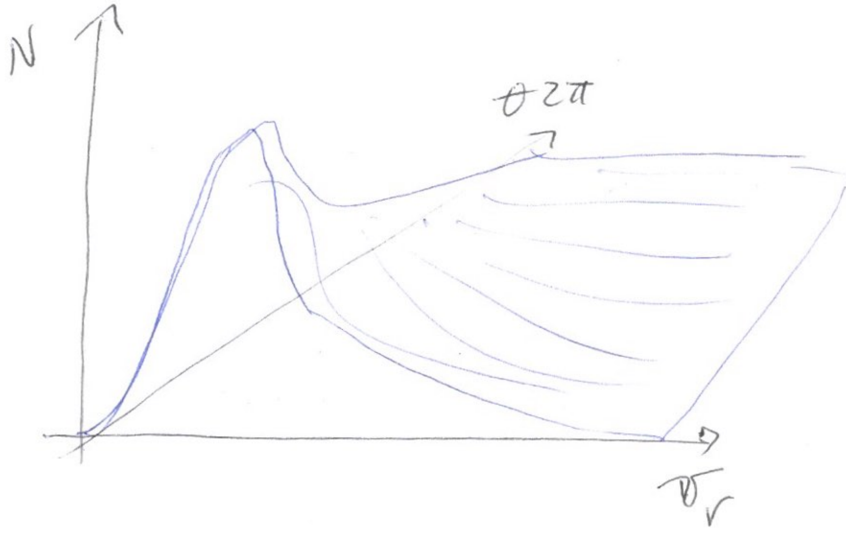




Spiral Galaxy?

x




v



The stars positions and velocities obey the equations of motion

$$\cancel{\ast} \quad \dot{\vec{x}}_N(t) = \vec{v}_N(t) \qquad \dot{\vec{v}}_N(t) = -\nabla\phi(\vec{x}_N(t), t)$$

where $\phi(\vec{x}, t)$ is the gravitational potential of the galaxy due to dark ~~gas~~ matter, gas, and stars, planets.

The time derivative is the <u>Collisionless Boltzmann Equation</u>

$$\frac{\partial f(\vec{x}, \vec{v}, t)}{\partial t} = -\vec{v}\cdot\nabla_x f(\vec{x}, \vec{v}, t) + \nabla_x \phi(\vec{x}, t)\cdot\nabla_v f(\vec{x}, \vec{v}, t)$$

W4.1.3

The number density is $n(\vec{x}, t) \equiv \int d^3v\, f(\vec{x}, \vec{v}, t)$

The mean velocity is $n(\vec{x}, t)\bar{\vec{v}}(\vec{x}, t) \equiv \int d^3v\, \vec{v}\, f(\vec{x}, \vec{v}, t)$

There is a continuity equation from 4.1.3

$$\frac{\partial}{\partial t} n(\vec{x}, t) + \nabla\cdot\left(n(\vec{x}, t)\bar{\vec{v}}(\vec{x}, t)\right) = 0$$

But interestingly, this is in phase space

Conservation of phase space density