Consider a system of n noninteracting identical particles. The wave function of the system

$$\Psi(x_1, x_2, \ldots, x_n) = \psi(x_1)\,\psi(x_2)\ldots\psi(x_n)$$

where $\psi(x_n)$ is the wavefunction of particle n.

Does it matter if we exchange some of the particles?

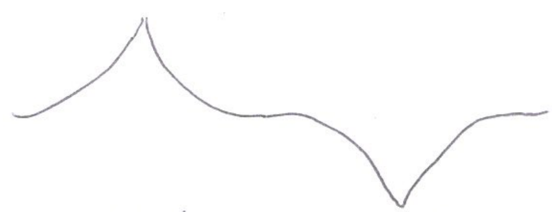
$$|\Psi(x_1, x_2)|^2 = |\Psi(x_2, x_1)|^2$$

And what we can measure is the square of the wave function. This means that we have two options

$$\Psi(x_2, x_1) = \Psi(x_1, x_2) \qquad \Psi(x_2, x_1) = -\Psi(x_1, x_2)$$

E.g.

Symmetric



Antisymmetric

This holds true in the case of n particles: if the sign of the wavefunction changes upon exchange of particles, then it is antisymmetric.

Consider two identical particles 1 and 2 that can occupy two different states A and B.

If particle 1 is in state A and particle 2 is in state B, the wavefunction of the system is given by:

$$\Psi_I = \Psi_A(x_1)\,\Psi_B(x_2)$$

vice versa

$$\Psi_{II} = \Psi_A(x_2)\,\Psi_B(x_1)$$

In this system there is nothing that makes I or II more likely, so according to Schrödinger, the true wave function is a superposition of Ψ_I and Ψ_{II}

$$\cancel{|\Psi|^2 = \left[|\Psi_I|^2 + |\Psi_{II}|^2 \right]^2 \frac{1}{2}}$$

(Probability must be normalized)

$$\Psi_S = \frac{1}{\sqrt{2}}\left[\Psi_A(x_1)\,\Psi_B(x_2) + \Psi_A(x_2)\,\Psi_B(x_1) \right]$$

but also

$$\Psi_{AS} = \frac{1}{\sqrt{2}}\left[\Psi_A(x_1)\,\Psi_B(x_2) - \Psi_A(x_2)\,\Psi_B(x_1) \right]$$

If we exchange x_1 and x_2

$$\psi_{S}(x_2, x_1) = \frac{1}{\sqrt{2}}\left[\psi_A(x_2)\psi_B(x_1) + \psi_A(x_1)\psi(x_2)\right] = \psi_{S}(x_1, x_2)$$

So it is symmetric under exchange

$$\psi_{AS}(x_2, x_1) = \frac{1}{\sqrt{2}}\left[\psi_A(x_2)\psi_B(x_1) - \psi_A(x_1)\psi_B(x_2)\right] = -\psi_{AS}(x_1, x_2)$$

so it is anti symmetric under exchange.

Let's put both particles in state A

$$\psi_{S}(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_A(x_1)\psi_A(x_2) + \psi_A(x_1)\psi_A(x_2)\right]$$

$$= \frac{2}{\sqrt{2}}\left[\psi_A(x_1)\psi_A(x_2)\right]$$

$$\psi_{AS}(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_A(x_1)\psi_A(x_2) - \psi_A(x_1)\psi_A(x_2)\right]$$

$$= 0$$

This is the Pauli exclusion principle (PEP)

If you think about it, if the particles are distinguishable you would have very different physics. But because they are indistinguishable and we can only measure the square of the wavefunction, their wavefunction can be either symmetric or antisymmetric. If the wavefunctions are symmetric, then the probability of finding particles in the same state is greater than zero, but if they are antisymmetric, the probability is exactly zero

★ Let's assume that A is a lower energy state than B. What would particles with symmetric wave functions do at 0K (no kinetic energy)?

THEY ALL FALL TO A, THE GROUND STATE

For antisymmetric THE GROUND STATE IS one particle in A and one in B

Particles with antisymmetric wave functions are called FERMIONS They follow Fermi-Dirac Statistics
" " symmetric " " " "
BOSONS They Follow Bose-Einstein Statistics

$$\langle N \rangle_T(\varepsilon) = \frac{1}{e^{(E-\mu)/k_BT}+1}$$

Fermi dirac

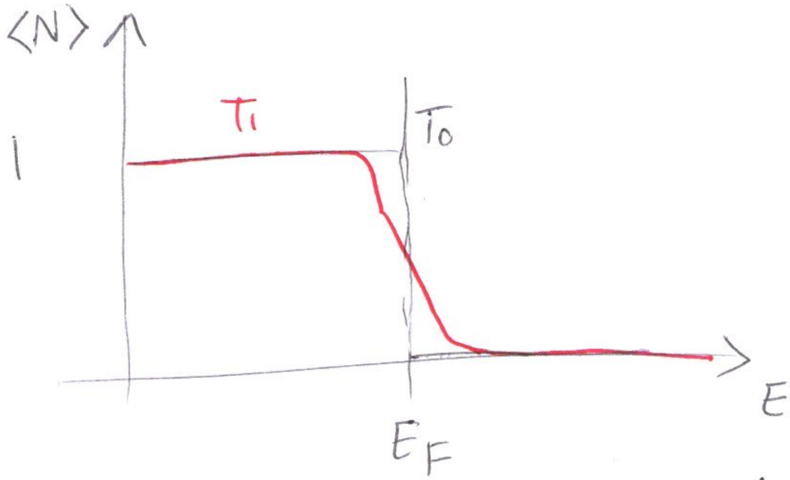
Consider $T \to 0$, then $e^{(E-\mu)/k_BT} \to e^{+/0}$ for $E>\mu$

then

$$f(E) = \frac{1}{e^{+\infty}+1} = \frac{1}{\infty} = 0 \quad \text{for } E > \cancel{\mu}\, E_F$$

otherwise

$$f(E) = \frac{1}{e^{-\infty}+1} = \frac{1}{1} = 1 \quad \text{for } E < \cancel{\mu}\, E_F$$



So even in ~~the gro~~ at 0K, fermions have kinetic energy

Volume is due to F-D

Examples of fermions (fractional spin, e.g. 1/2)

Leptons
- charged
  - electron — positron
  - muon — anti muon
  - tau — anti tau
- neutral
  - electron neutrino
  - muon "
  - tau "

Quarks
- up down
- charm strange
- top bottom

anti

If you combine 3 quarks you form baryons

most typical ones: proton – 2up 1down
neutron – 1up 2down

Baryonic matter includes all nuclei but not electrons, neutrinos, dark matter

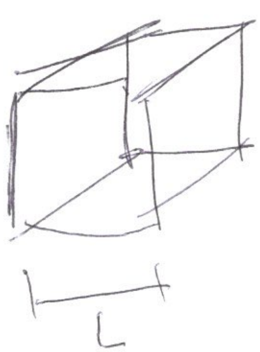
We will not look at them in detail, but bosons can all fall to the same state at 0K and share the same wave function. This makes for very intersting physics.

Deuterium is a boson $\left(\overset{1/2}{\text{proton}} + \overset{1/2}{\text{neutron}}\right)$

Helium-4 is a boson (2protons + 2neutrons)

superfluidity Bose Einstein condensates

Cooper pairs 2 fermions joined by a phonon
Super conductivity



How many states do we have below E_F in a box? Sum becomes an integral if L is large. Grid of points in k-space

$$\left(\frac{L}{2\pi}\right)^3 \int d^3k$$

2 electrons per state $N = 2\left(\frac{L}{2\pi}\right)^3 \int_0^{k_F} 4\pi k^2 dk$ 70

k_F is the Fermi wavevector, the wavevector of the most energetic ~~par~~ electron.

$$N = \frac{\cancel{8} V \cancel{\pi}}{\cancel{8}\pi^{\cancel{3}2}} \int k^2 dk = \frac{V}{\pi}\frac{k_F^3}{3} \Rightarrow k_F^3 = 3\pi^2 \frac{N}{V} = 3\pi^2 n$$

Remember that $\lambda = 2\pi/k$

So typical separation between electrons

$$\lambda = \frac{2\pi}{k_F} = \frac{2\pi}{(3\pi^2 n)^{1/3}}$$

If the particles are non-relativistic and not interacting

$$E = \frac{p^2}{2m} = \frac{(\hbar k)^2}{2m} \Rightarrow E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

To get the ground state energy we put the energy inside the density of states integral

$$U_0 = 2\left(\frac{L}{2\pi}\right)^3 \int_0^{k_F} 4\pi k^2 dk \cdot \frac{(\hbar k)^2}{2m} = \frac{\cancel{2}\cdot\cancel{4\pi}\cdot\hbar^2\cdot V}{\cancel{2}\cancel{8}\pi^{\cancel{3}}\cancel{2}m}\int_0^{k_F} k^4 dk$$

$$= \frac{V\hbar^2}{2\pi^2 m}\frac{k_F^5}{5} = \frac{V\hbar^2}{2\pi^2 m}\frac{1}{5}\left(3\pi^2\frac{N}{V}\right)k_F^2 = \frac{3}{5}N\frac{(\hbar k_F)^2}{2m}$$

$$\boxed{U_0 = \frac{3}{5}NE_F}$$