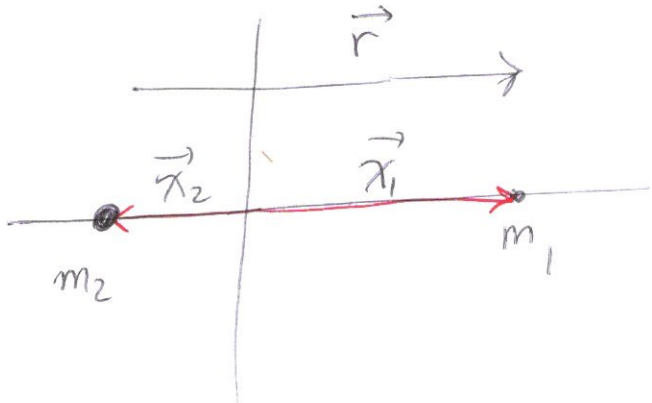
BINARIES



m_1, m_2 masses of stars 1 & 2

$M = m_1 + m_2$ total mass of the system

$$m_1 \vec{x}_1 + m_2 \vec{x}_2 = 0$$

origin at the center of mass

$$\vec{r} \equiv \vec{x}_1 - \vec{x}_2$$

$$\vec{x}_1 = \vec{r} + \vec{x}_2$$

$$\vec{x}_1 = \vec{r} + \left(-\frac{m_1}{m_2}\vec{x}_1\right)$$

$$\left(1 + m_1/m_2\right)\vec{x}_1 = \vec{r}$$

$$\left(\frac{m_1+m_2}{m_2}\right)\vec{x}_1 = \vec{r}$$

$$\frac{M}{m_2}\vec{x}_1 = \vec{r}$$

$$\vec{x}_1 = \left(\frac{m_2}{M}\right)\vec{r}$$

$$-\vec{x}_2 = \vec{r} - \vec{x}_1$$

$$\vec{x}_2 = \vec{x}_1 - \vec{r}$$

$$\vec{x}_2 = -\frac{m_2}{m_1}\vec{x}_2 - \vec{r}$$

$$\left(1 + m_2/m_1\right)\vec{x}_2 = -\vec{r}$$

$$\left(\frac{m_1+m_2}{m_1}\right)\vec{x}_2 = -\vec{r}$$

$$\frac{M}{m_1}\vec{x}_2 = -\vec{r}$$

$$\vec{x}_2 = \left(\frac{m_1}{M}\right)\vec{r} \qquad \underline{\underline{\text{W2.1.1}}}$$

For stars moving much slower than the speed of light,

$$F = m \frac{d^2x}{dt^2}, \text{ so}$$

$$\left| \cancel{m_1} \frac{d^2\vec{x}_1}{dt^2} \right| = \left| \frac{G \cancel{m_1} m_2}{r^2} \right| \qquad \left| \cancel{m_2} \frac{d^2\vec{x}_2}{dt^2} \right| = \left| \frac{G m_1 \cancel{m_2}}{r^2} \right|$$

Since $\vec{r}$ is parallel to $\hat{\imath}$,

$$\frac{d^2x_1}{dt^2} = -Gm_2\vec{r}/r^3 \qquad \frac{d^2x_2}{dt^2} = +Gm_1\vec{r}/r^3$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2}{dt^2}(\vec{x}_1 - \vec{x}_2) = \left[-Gm_2\vec{r}/r^3 - Gm_1\vec{r}/r^3\right]$$

$$\frac{d^2\vec{r}}{dt^2} = -G(m_1+m_2)\vec{r}/r^3 = -GM\vec{r}/r^3 \qquad \text{W2.1.2}$$

☆ So we reduce a 2-body problem into a 1-body problem.

This innocent-looking differential equation is actually pretty difficult to solve analytically as you need elliptic integrals. It is called an ORBIT EQUATION.

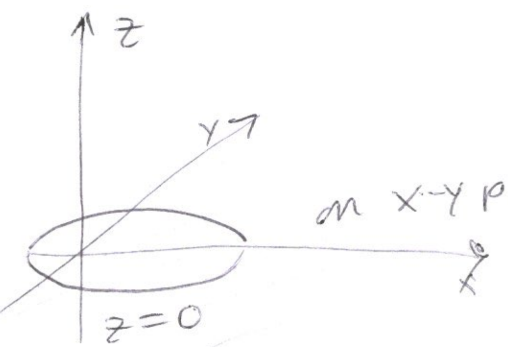
Luckily, we know the solution thanks to Euler and others.

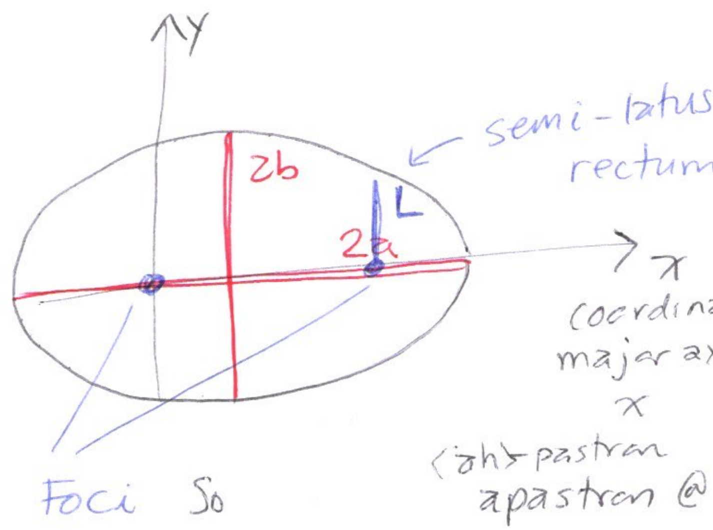
$$\vec{r} = r\,(\cos\varphi,\ \sin\varphi,\ 0)$$

where $r = \dfrac{L}{1 - e\cos\varphi}$ $\qquad \dfrac{d\varphi}{dt} = \sqrt{\dfrac{GM}{L^3}}\,(1 - e\cos\varphi)^2$

e is the eccentricity and L the semi-latus rectum

For an ellipse with major axis $\underline{\underline{2a}}$ and minor axis $\underline{\underline{2b}}$, $\quad e = \sqrt{1 - \dfrac{b^2}{a^2}} \qquad L = a(1-e^2)$





coordinate system major axis along x

~~(ah)~~ pastron
apastron @ $\varphi = 0$
(furthest point)

$r_{max} = a(1+e)$ aphelion in the case of the earth sun

peri astron @ $\varphi = \pi$
closest point

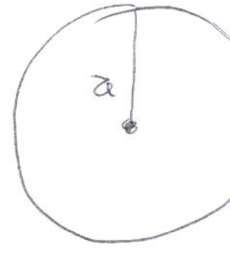
$r_{min} = a(1-e)$

perihelium for the earth sun

For low eccentricity ($e \approx 0$), orbit looks circular

$$\sqrt{1 - b^2/a^2} \approx 0 \Rightarrow b^2/a^2 \approx 1 \Longrightarrow a \approx b$$

$$L = a(1-e^2) \approx a$$



$r_{max} \approx a \quad , \quad r_{min} \approx a$

For high eccentricity ~~[illegible]~~ $(e \lesssim 1)$

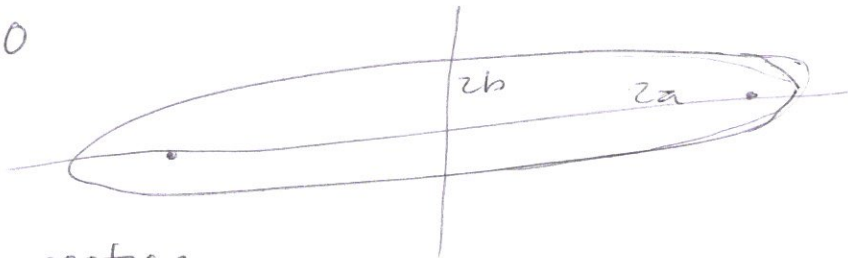
orbit is elongated

$$\sqrt{1-\frac{b^2}{a^2}} \lesssim 1 \Rightarrow \frac{b^2}{a^2} \lesssim 0 \Rightarrow b \lesssim 0$$

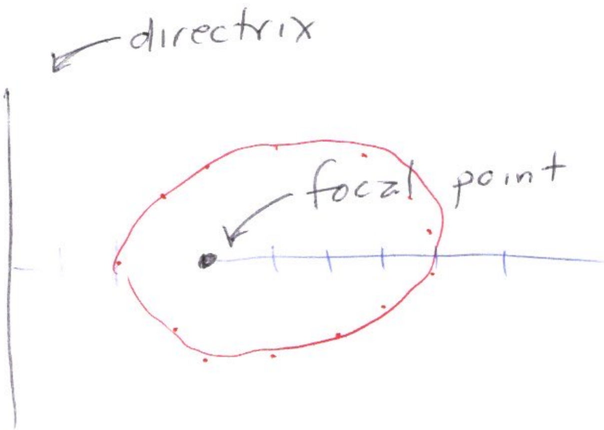
or $a \lesssim \infty$

$$L = a(1-e^2) \gtrsim 0$$

$$\Rightarrow a \simeq \infty$$



Definition of ~~ellipse~~ conic section – Locus of set of points P whose distance ~~from p to~~ to focus is a constant e of the distance from p to directrix



$e = 0$ for circle

$0 < e < 1$ for ellipse

$e = 1$ for parabola

$e > 1$ for hyperbola

Notice that $\dfrac{d\varphi}{dt} = \sqrt{\dfrac{GM}{L^3}}\,(1 - e\cos\varphi)^2$

For periastron (closest point) $\varphi = \pi \Rightarrow \cos\varphi = 0$

$$\left.\frac{d\varphi}{dt}\right|_{\varphi=\pi} = \sqrt{\frac{GM}{L^3}}$$

For apastron (furthest point) $\varphi = 0 \Rightarrow \cos\varphi = 1$

$$\left.\frac{d\varphi}{dt}\right|_{\varphi=0} = \sqrt{\frac{GM}{L^3}}\,(1-e)^2 < \left.\frac{d\varphi}{dt}\right|_{\varphi=\pi}$$

☆ DOES IT MAKES SENSE FROM NEWTONIAN GRAVITATION?

Total energy

$$E = \frac{m_1 \dot{\vec{x}}_1^{\,2}}{2} + \frac{m_2 \dot{\vec{x}}_2^{\,2}}{2} - \frac{G m_1 m_2}{r}$$

kinetic potential

Angular momentum $J = \vec{r} \times \vec{p} = r \times m v$

$$\cancel{\Sigma m_1 x_1} \quad J = m_1 \vec{x}_1 \times \dot{\vec{x}}_1 + m_2 \vec{x}_2 \times \dot{\vec{x}}_2$$

$$|J| = \frac{m_1 m_2}{m_1 + m_2} \sqrt{GML} \quad , \quad E = -\frac{m_1 m_2}{m_1 m_2} \frac{MG}{2a}$$

negative because it is bound gravitationally

Binaries ~~Opti~~

- Visual binary – stars can be resolved independently
  Far enough apart from each other
  and close enough to earth

- Eclipsing binary – Orbital planes oriented along
  earth line of sight and
  periodically pass in front of each
  other. Change in luminosity.

- Spectrum binary – Two spectrums superimposed

If some component of the velocity is in the line of sight, then the spectra will be shifted (Dopler shift, $\nu_{obs} = \nu_{rest}\sqrt{\frac{1-v/c}{1+v/c}}$)

- Spectroscopic binary – if r is small, then the period will be short (s

$$\frac{\Delta \nu_1}{\nu_1} = \frac{m_2}{M}\sqrt{\frac{MG}{Lc^2}}\,\sin i\,[$$

i angle between line of sight and orbit