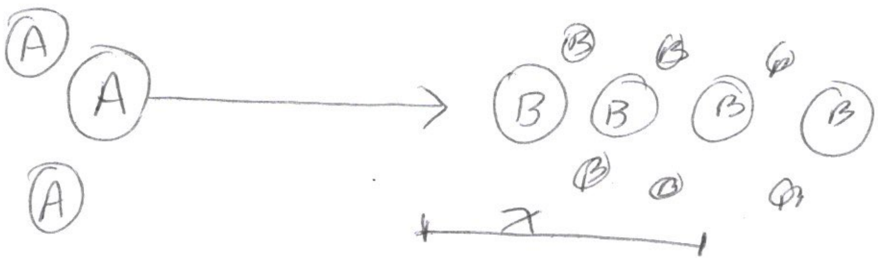
# Nuclear reaction rates

The cross-section for a nuclear reaction involving nuclei A and B is

$$\sigma_{AB}(E) = \frac{S_0}{E} e^{-\sqrt{E_G/E}}$$

where $S_0$ can be measured experimentally



$n_B \sigma_{AB} dx = dN_A$

Number of reactions depends on $\cancel{\rho_B \sigma_{AB} dx = dN_A}$

$$n_B \sigma_{AB} dx/dt = n_B \sigma_{AB} v_{AB}$$

$$\frac{dN_A}{dt} = \cancel{\rho_B \sigma_{AB} \frac{dx}{dt}} = \rho_B \sigma_{AB} v_{AB}$$

per unit time gives us a rate

reaction rate

$$\Gamma_{AB} = n_A n_B \sigma_{AB} v_{AB}$$

$$\left(\frac{1}{m^3}\right)\left(\frac{1}{\cancel{m^3}}\right)\left(\frac{\cancel{m^2}}{1}\right)\left(\frac{\cancel{m}}{s}\right)$$

number of reactions per unit time and unit volume

Let the reaction AB release energy Q

$Q_{AB}\Gamma_{AB}$ is Energy per unit time unit volume, so power per unit volume

$\epsilon = \dfrac{Q_{AB}\Gamma_{AB}}{\rho}$ is power per unit mass $\epsilon = n_A n_B \sigma_{AB} v_{AB} Q/\rho$

Let $X_A + X_B = 1$ where $X_A, X_B$ are the relative abundances of A and B, then

$$n_A = \frac{\rho X_A}{A_A m_H} \quad , \quad n_B = \frac{\rho X_B}{A_B m_H}$$

where A is the atomic mass, then

$$\epsilon = \frac{\rho^{\cancel{2}} X_A X_B}{m_H^2 A_A A_B} \sigma_{AB} v_{AB} Q \frac{1}{\cancel{\rho}}$$

The velocity $v_{AB}$ is a distribution, and $\sigma_{AB}(E)$ depends on the actual velocity of the particles (through $E$) so,

$$\langle \epsilon \rangle = \frac{\rho X_A X_B}{m_H^2 A_A A_B} \langle \sigma_{AB} v_{AB} \rangle Q$$

with

weighted average.
what is the probability distribution??

$$\langle \sigma_{AB} v_{AB} \rangle = \int_0^\infty \sigma_{AB} v_{AB} P(v_{AB}) dv_{AB}$$

Maxwell – Boltzman.

This is not the distribution of both A and B, instead it is the relative velocity between A and B atoms, will also use M-B but with the reduced mass,

$$P(v_{AB}) = 4\pi \left(\frac{\mu}{2\pi k_B T}\right)^{3/2} v_{AB}^2 \exp\left(-\frac{\mu v_{AB}^2}{2 k_B T}\right) dv_{AB}$$

with $E = \frac{1}{2}\mu v_{AB}^2 \Rightarrow v_{AB}^2 = 2E/\mu$

$$dE = \frac{1}{\cancel{2}}\mu \cdot \cancel{2} v_{AB}\, dv_{AB} = \mu v_{AB}\, dv_{AB}$$

$$\Rightarrow dv_{AB} = dE/\mu v_{AB}$$

$$P(E) = 4\pi\left(\frac{\mu}{2\pi k_B T}\right)^{3/2} \frac{2E}{\mu} \exp\left(-\frac{\cancel{\mu}\, \cancel{2}E/\cancel{\mu}}{\cancel{2} k_B T}\right) \frac{1}{\mu\sqrt{\frac{2E}{\mu}}}\, dE$$

$$P(E) = \frac{4\pi}{(2\pi)^{3/2}(k_B T)^{3/2}} \frac{\cancel{\mu^{3/2}}\,\cancel{\mu^{1/2}}}{\cancel{\mu^2}} \frac{2E}{\sqrt{2E}} \exp\left(-\frac{E}{k_B T}\right) dE$$

$$P(E) = \frac{1}{(k_B T)^{3/2}}\sqrt{\frac{2}{\pi}}\sqrt{2E} \exp\left(-\frac{E}{k_B T}\right) dE = \sqrt{\frac{4E}{\pi}} \exp\left(-\frac{E}{k_B T}\right)$$

Then

$$\langle \sigma_{AB} v_{AB} \rangle = \int_0^\infty \frac{1}{(k_B T)^{3/2}} \frac{S_0}{E} \exp\left(-\sqrt{\frac{E_G}{E}}\right) \sqrt{\frac{2E}{\mu}} \sqrt{\frac{4E}{\pi}} \exp\left(-\frac{E}{k_B T}\right)$$

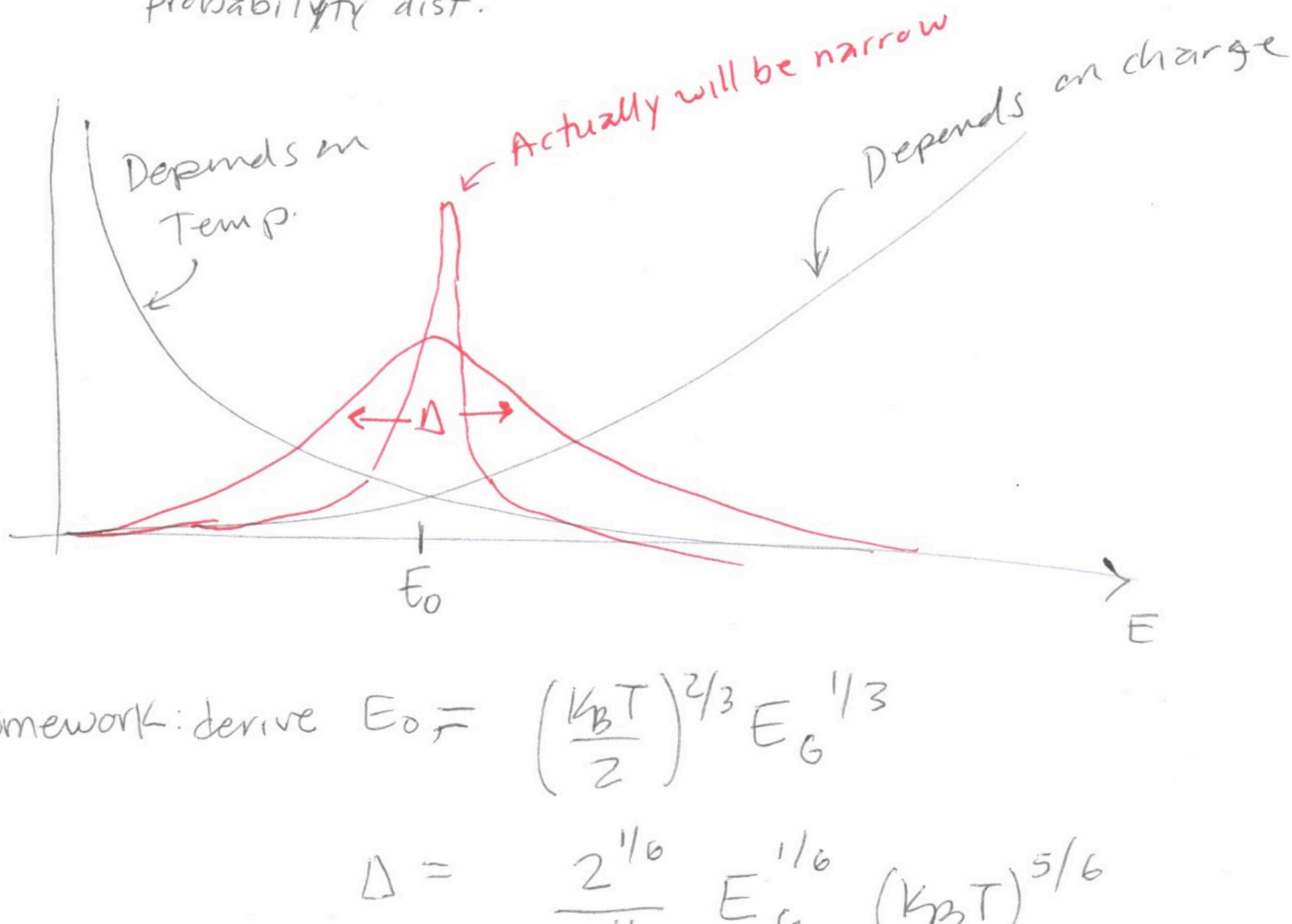
$$\langle \sigma_{AB} v_{AB} \rangle = \int_0^\infty \frac{S_0}{E} \frac{\sqrt{8}\, E}{\sqrt{\mu \pi}} \frac{1}{(kT)^{3/2}} e^{-\sqrt{E_G/E}} e^{-E/k_BT} \, dE$$

$$= \overbrace{\left(\frac{8}{\mu \pi}\right)^{1/2} \frac{S_0}{(k_BT)^{3/2}}}^{K} \int_0^\infty e^{-\sqrt{E_G/E}} e^{-E/k_BT}$$

Let $f(E) = e^{-E/k_BT} e^{-\sqrt{E_G/E}}$

Boltzmann probability dist.

Gamow factor




Homework: derive $E_0 = \left(\frac{k_BT}{2}\right)^{2/3} E_G^{1/3}$

$$\Delta = \frac{2^{1/6}}{3^{1/2}} E_G^{1/6} (k_BT)^{5/6}$$

by approximating $f(E)$ as a Gaussian centered at $E_0$ with standard deviation $\Delta$

$f(E) \sim E_0$

Integral of arbitrary Gaussian $\int_{-\infty}^{\infty} a\, e^{-(x-b)^2/2c^2}\, dx = \sqrt{2}\, ac\sqrt{\pi}$

let be $f(E_0)$

$$f(E) = \frac{1}{\Delta\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{E-E_0}{\Delta}\right)^2}$$

Let $E = x$

$\Delta = c$

$b = E_0$

$a = \frac{1}{\Delta\sqrt{2\pi}} \rightarrow f(E_0)$

so $I(E) \sim \sqrt{2\pi} \cdot \frac{\Delta}{\Delta\sqrt{2\pi}} = 1$ if $a = \frac{1}{\Delta\sqrt{2\pi}}$

$I(E) \sim \sqrt{2\pi}\, f(E_0)\, \Delta$ in general

Then

$$\langle \sigma_{AB} v_{AB} \rangle = \sqrt{2\pi}\, K\, e^{-E_0/k_B T}\, e^{-\sqrt{E_G/E_0}}\, \frac{2^{1/6}}{3^{1/2}} E_G^{1/6} (k_B T)^{5/6}$$

with $\frac{E_0}{k_B T} = \left(\frac{k_B T}{2}\right)^{2/3} \frac{E_G^{1/3}}{(k_B T)} = \frac{1}{2^{2/3}}\left(\frac{E_G}{k_B T}\right)^{1/3}$

$$\left(\frac{E_G}{E_0}\right)^{1/2} = \left[\frac{E_G}{\left(\frac{k_B T}{2}\right)^{2/3}} \frac{1}{E_G^{1/3}}\right]^{1/2} = \left[\frac{2^{2/3} E_G^{2/3}}{(k_B T)^{2/3}}\right]^{1/2}$$

$$\left(\frac{E_G}{E_0}\right)^{1/2} = \left(\frac{2 E_G}{k_B T}\right)^{1/3}$$

$$\left(\frac{E_G}{4 k_B T}\right)^{1/3} + \left(\frac{2 E_G}{k_B T}\right)^{1/3}$$

$$\langle \sigma_{AB} v_{AB} \rangle = K \sqrt{2\pi}\, \frac{2^{1/6}}{3^{1/2}} E_G^{1/6} (k_B T)^{5/6} \exp\left[-\frac{1}{2^{2/3}}\left(\frac{E_G}{k_B T}\right)^{1/3} - \left(\frac{2 E_G}{k_B T}\right)^{1/3}\right]$$

$$\cancel{\left(\frac{E_G}{k_B T}\right)^{1/3}\left(\frac{1}{2^{2/3}} + \frac{2^{1/3}}{4}\right)}$$

Focus on exponent

$$\exp\left[-\left(\frac{E_G}{4 k_B T}\right)^{1/3} - \left(\frac{2 E_G}{k_B T}\right)^{1/3}\right] = \exp\left[-\left(\frac{E_G}{4 k_B T}\right)^{1/3} - \overset{2}{8^{1/3}}\left(\frac{E_G}{4 k_B T}\right)^{1/3}\right]$$

$$= \exp\left[\left(\frac{E_G}{4 k_B T}\right)^{1/3}(-1-2)\right] = \exp\left[-3\left(\frac{E_G}{4 k_B T}\right)^{1/3}\right]$$

so

$$\langle \sigma_{AB} v_{AB} \rangle = K \sqrt{2\pi}\, \frac{2^{1/6}}{3^{1/2}} E_G^{1/6} (k_B T)^{5/6} \exp\left[-3\left(\frac{E_G}{4 k_B T}\right)^{1/3}\right]$$

Recall $\langle \epsilon \rangle = \frac{\rho X_A X_B}{m_H^2 A_A A_B} \langle \sigma_{AB} v_{AB} \rangle Q$ , so

$$\langle \epsilon \rangle = \left( \frac{\overset{2^{3/2}}{8}}{\cancel{\pi}\mu} \right)^{1/2} \frac{S_0}{(k_B T)^{3/2}} (2\pi)^{1/2} \frac{2^{1/6}}{3^{1/2}} E_G^{1/6} (k_B T)^{5/6} \frac{\rho X_A X_B Q}{m_H^2 A_A A_B} \exp[$$

$$\langle \epsilon \rangle = \frac{2^{3/2 + 1/2 + 1/6}}{3^{1/2}} \frac{\rho X_A X_B}{m_H^2 A_A A_B \sqrt{\mu}} Q S_0 \frac{E_G^{1/6}}{(k_B T)^{2/3}} \exp\left[ -3 \left( \frac{E_G}{4 k_B T} \right)^{1/3} \right]$$

$\frac{5}{6} - \frac{3}{2} = \frac{10-18}{12} = \frac{-\cancel{8}}{\cancel{12}} = -\frac{2}{3}$

Since it is narrow, $\epsilon \approx \langle \epsilon \rangle$

WHAT DOES IT ALL MEAN?

$E_G = (\pi \alpha Z_A Z_B)^2 2 \mu c^2$ — charge ~ number of protons

$\mu = \frac{m_H^2 A_A A_B}{A_A + A_B}$ — mass ~ number of nucleons

$S_0$ how often reaction occurs

$E_G$ parameterizes the nuclear reaction, e.g. $p + p \rightarrow d + e^+ + \nu_e$

$Q$ is the energy relesed by the reaction $\quad p + {}^{12}C \rightarrow {}^{13}N + \gamma$

$X_A, X_B$ composition, percentage of atoms A and B

depends on the chemical composition of the star

In the exponent, $\epsilon \sim e^{-E_G}$

more energy can be extracted from reactions with small Gamow energy, so lighter elements

Also $\epsilon \sim e^{-1/T}$

so higher temperatures can make reactions with heavier atoms feasible

The relationship between $\epsilon$ and $\rho$ is interesting as they self-regulate: if $\rho$ increases, $\epsilon$ increases, and the luminosity ~~the~~ L increases, but because of the opacity H, some radiation is trapped and this increases the pressure, so the density decreases and this decreases $\epsilon$.

This is why the temperature of the core of stars that burn the same atoms are very similar even if their masses are very different.