① PHYS 3359 Astrophysics

Fall 2020

8/25/20

☆ Hydrostatic equilibrium of a sphere of fluid

Are stars spherical? Most, to a good approximation

Are they in equilibrium? Most are not collapsing or exploding, so yes

We know gravity is always present. Consider a thin spherical shell of mass with radius r and thickness dr

$$\rho = \frac{M_s}{V_s} \Rightarrow M_s = \rho V_s = \rho A_s\, dr = \rho 4\pi r^2 dr$$

This shell is gravitationally attracted to the solid sphere of mass it encloses

Density in general is a function of r

$M(r)$

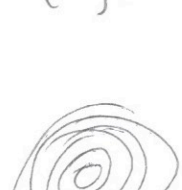
Mass enclosed also a function of r

$$F_{grav} = -\frac{G M_s M_c}{r^2} = -\frac{\cancel{r^2}\, 4\pi G \rho(r) M(r)\, dr}{\cancel{r^2}}$$

Towards the center

with $M(r) = \int_0^r 4\pi r'^2 \rho(r')\, dr'$

A bit like an onion with infinitesimally thin layers



important for derivations / Conservation of mass

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \; ; \; M(0) = 0$$

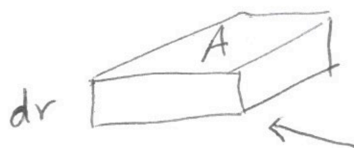
Since the body is in equilibrium, then (2)
the net force at each point should be zero.

There should be another force, or combination of forces, that perfectly cancel out the force of gravity.

For now, the nature of these forces is irrelevant.

In general, $F_{buoyant}$

$$\Sigma F = -F_g + F_b = 0 \Rightarrow F_b = F_g$$



$F_b(r) > F_b(r+dr)$ in order to be able to push up

pressure

$$p = \frac{F}{A} \Rightarrow F = p \cdot A$$

but if the force of gravity depends on the radius, the pressure must depend on it too.

$$F_{buoyant} = F_b(r) - F_b(r+dr) = A\left[p(r) - p(r+dr)\right]$$

$$= 4\pi r^2\left[p(r) - p(r+dr)\right] = -4\pi r^2 p'(r)\,dr$$

(in the negative direction)

From free-body diagram,

$$\sum F_r = -4\pi G\rho(r) M(r)\, dr \cancel{+} - 4\pi r^2 \frac{dp(r)}{dr}\, dr = 0$$

$$\cancel{4\pi} r^2 \frac{dp(r)}{dr} \cancel{dr} = -\cancel{4\pi} G\rho(r) M(r) \cancel{dr}$$

$$\boxed{\frac{dp(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}} \qquad \underline{W1.1.4}$$

Fundamental equation of hydrostatic equilibrium.

How general is W1.1.4? Quite, because the buoyant force can be produced ~~provided~~ by several phenomena: nuclear reaction, Pauli exclusion principle, fluids, etc.

Let's analize W1.1.4. Let $\rho(r) = \rho =$ constant

Then $M(r) = 4\pi\rho \int_0^r r'^2 dr' = \frac{4\pi\rho r^3}{3} = V \cdot \frac{M}{V} = M$

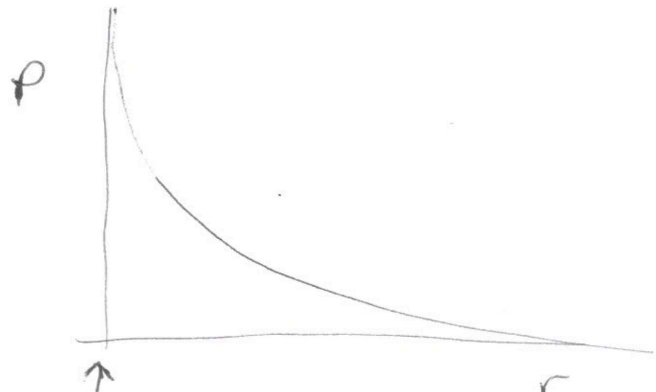
$$\frac{dp(r)}{dr} = -\frac{GM\rho}{r^2}$$

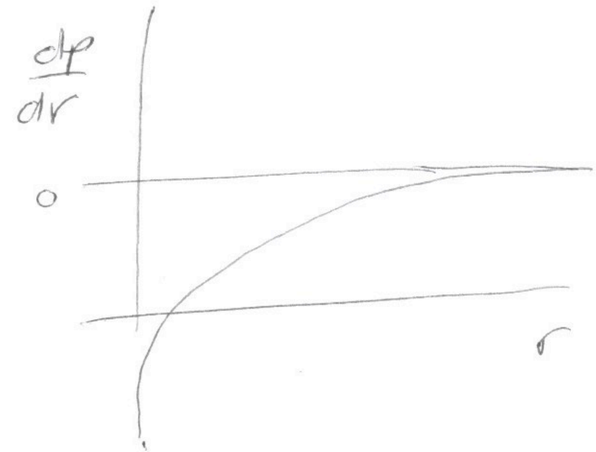
Since $F = ma = \cancel{m}g = \frac{GM\cancel{m}}{r^2}$, $\frac{dp(r)}{dr} = -g\rho$

g is the local acceleration due to gravity at r

From W1.1.4 we see that it is not pressure that stabilizes stars (and other astronomical objects), it is a pressure gradient.

$$p = \int dp = -GM\rho \int_0^r r'^{-2} dr' = \frac{2GM\rho}{r}$$





Let's combine $\frac{dp(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$ and $\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$

carefully callibrated

$$\frac{d}{dr}\left[p(r) + \frac{GM^2(r)}{8\pi r^4}\right] = \frac{dp(r)}{dr} + \frac{d}{dr}\left[r^{-4} \cdot M^2(r)\right]\frac{G}{8\pi}$$

Let $u = r^{-4}$ $v = M^2(r)$

with ~~than~~ $\frac{d(uv)}{dr} = u\frac{dv}{dr} + v\frac{du}{dr}$, $r^{-4}\frac{dM^2(r)}{dr} + M^2(r)\frac{dr^{-4}}{dr}$

$$\frac{1}{r^4}\frac{d}{dr}\left[M(r) \cdot M(r)\right] - \frac{4M^2(r)}{r^5}$$

Let $u = M(r)$ $v = M(r)$

with $\dfrac{d(uv)}{dr} = u\dfrac{dv}{dr} + v\dfrac{du}{dr}$; $\dfrac{d}{dr}\left[M^2(r)\right] = 2M(r)\dfrac{dM(r)}{dr}$

so $\dfrac{d}{dr}\left[r^{-4}M^2(r)\right]\dfrac{G}{8\pi} = \dfrac{G}{\cancel{8}\pi}\left[-{}_{2}\dfrac{\cancel{4}M^2(r)}{r^5} + {}_{4}\dfrac{\cancel{2}M(r)M'(r)}{r^4}\right]$

and $\dfrac{d}{dr}\left[p(r) + \dfrac{GM^2(r)}{8\pi r^4}\right] = -\dfrac{GM(r)\rho(r)}{r^2} - \dfrac{GM^2(r)}{2\pi r^5} + \dfrac{GM(r)M'(r)}{4\pi r^4}$

but $\dfrac{GM(r)M'(r)}{4\pi r^4} = \dfrac{GM(r)\,\cancel{4\pi r^2}\,\rho(r)}{\cancel{4\pi}\, r^{\cancel{4}\,2}} = \dfrac{GM(r)\rho(r)}{r^2}$

so the first and third terms cancel out

$$\frac{d}{dr}\left[p(r) + \overset{A(r)}{\frac{GM^2(r)}{8\pi r^4}}\right] = -\frac{GM^2(r)}{2\pi r^5} \le 0$$

Let ρ = constant and nonzero close to the center

$\dfrac{dM(r)}{dr} = 4\pi r^2\rho(r) \Rightarrow \int dM(r) = \int 4\pi r^2 \rho\, dr = \dfrac{4\pi\rho r^3}{3}$

Close to the center, as $r \to 0$, $M \propto r^3$

so $\dfrac{M^2(r)}{r^4} \propto \dfrac{r^6}{r^4} = r^2 \to 0$, so $\cancel{A(0)=0}$ $GM^2(r)/8\pi r^4$ vanishes

The pressure vanishes at some radius R, the edge of the star or astronomical body, so

$$\frac{d}{dr} p(R) = 0 \quad \text{and} \quad A(R) = \frac{GM^2(R)}{8\pi R^4}$$

and of course $M(R) = M$, the total mass, so

$$A(R) = \frac{GM^2}{8\pi R^4}$$

Since the pressure at $r=0$ is greater than at $r=R$,

$$p(0) \geq \frac{GM^2}{8\pi R^4}$$

$$\frac{G}{8\pi} = \frac{6.674 \times 10^{-11} \ m^3/kg\, s^2}{8\pi} = 2.655 \times 10^{-12} \ m^3/kg\, s^2$$

	Sun	Earth	Neutron Star
M	1.98×10^{30} kg	5.97×10^{24} kg	$1.4 M_\odot$
R	6.95×10^{8} m	6.37×10^{6} m	10×10^{3} m
$p(0)$	$\geq 4.4 \times 10^{4}$ GPa	39.8 GPa	2.0×10^{24} GPa
BKV	2.5×10^{7} GPa	390 GPa	1.6×10^{25} GPa