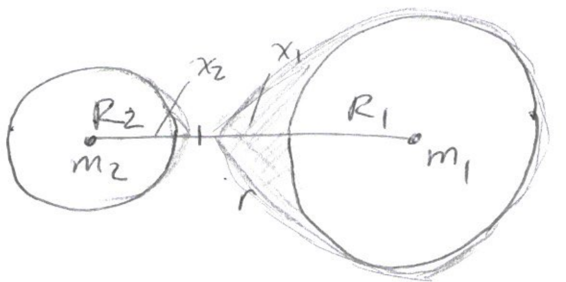
Close binaries

We have seen the orbits of binaries that can be represented by point masses. Is there a distance at which this paradigm breaks down?

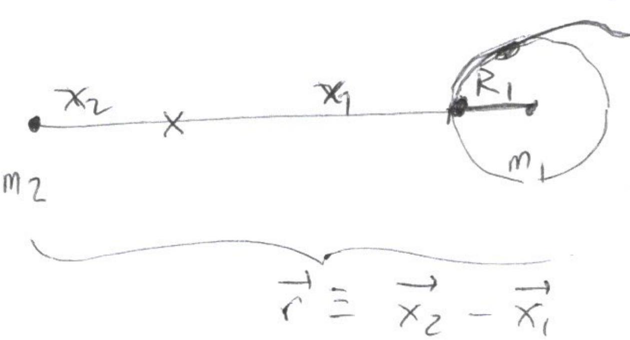


If $r \sim R_1$, tidal effects will distort the stars, so they will look more like drops of liquid than spheres

If the mass distribution becomes extremely distorted, then you need numerical methods to analize the orbits, but if most of the mass remains in a sphere and only the outer layer is distorted, you can study the system analitically

★HOW FAR AWAY MUST THE STARS BE FOR THE TIDAL DISTORTION TO BE NEGLIGIBLE?

Circular frequency Ω (if the orbits are small, they will be very close to circular)



δm on the surface of star 1 at the point closest to star 2

Then $F_{grav} = \dfrac{G m_1 \delta m}{R_1^{\,2}} - \dfrac{G m_2 \delta m}{(r-R_1)^2}$

The frame of reference is rotating, so there is a centrifugal force (an inertial force)

$$F_{cent} = \delta m \, a_{cent} = \delta m \, \Omega^2 (x_1 - R_1)$$

since $x_1 = \left(\frac{m_2}{M}\right) r$,

$$F_{cent} = \Omega^2 \left[\frac{m_2 r}{M} - R_1\right] \delta m$$

For circular orbits, $a = r$ (semimajor axis)

Also, $\Omega = \frac{2\pi}{T}$ Angular frequency is 2π/period.

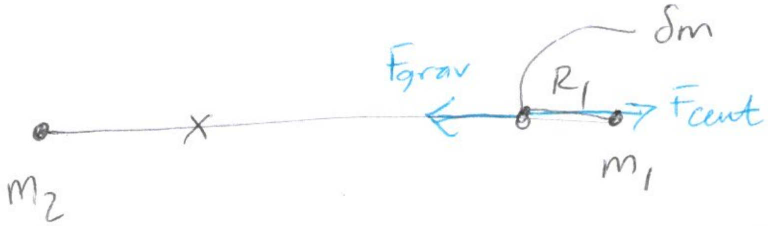
Kepler's third law for circular orbits $\Omega^2 = \frac{MG}{r^3}$

$$F_{cent} = \frac{MG}{r^3}\left[\frac{m_2 r}{M} - \frac{R_1}{1}\right] \delta m \; \cancel{= \left[\frac{MGm_2 r}{r^3 M} - \frac{MGR_1}{r^3}\right] \delta m}$$

$$F_{cent} = \frac{MG}{r^3}\left[\frac{m_2 r - MR_1}{M}\right] \delta m = \frac{G}{r^3}\left[m_2 r - (m_1 R_1 + m_2 R_1)\right] \delta m$$

$$F_{cent} = \frac{G}{r^3}\left[\cancel{m_2 R_1}\; m_2 (r - R_1)^* - m_1 R_1\right] \delta m$$

$$F_{cent} = -\frac{G m_1 R_1 \delta m}{r^3} + \frac{G m_2 (r - R_1)^* \delta m}{r^3}$$



Goes away

Compare $$F_{grav} = \boxed{\frac{G m_1 \delta m}{R_1^{2}}} - \frac{G m_2 \delta m}{\boxed{(r - R_1)^2}}$$ — Goes to r^2

Goes away

$$F_{cent} = -\boxed{\frac{G m_1 R_1 \delta m}{r^3}} + \frac{G m_2 (r - R_1) \delta m}{r^3}$$ — Goes to r^2

If $m_1 = 0$ and $R_1 \to 0$ the gravitational and centrifugal forces balance each other (as expected)

if $m_1 = 0$ and $R \neq 0$, then $(r-R_1)^2$ term in the denominator of Fgrav becomes smaller and $|F_{grav}|$ increases as R_1 increases (closer to m_2), but $(r-R_1)$ term in the nominator of F_{cent} also becomes smaller so F_{cent} ~~becomes~~ decreases. δm flies off surface of m_1 unless m_1 is non-zero and large enough.

☆ THE CONDITION FOR δm ON THE SURFACE OF ~~R1~~ m_1 NOT TO FLY OFF IS

positive direction

$$F_{grav} + F_{cent} > 0$$

$$\frac{G m_1 \cancel{\delta m}}{R_1^{2}} - \frac{G m_2 \cancel{\delta m}}{(r-R_1)^2} + \Omega^2 \left[\frac{m_2 r}{M} - R_1 \right] \cancel{\delta m} > 0$$

$$\frac{Gm_1}{R_1^2} - \frac{Gm_2}{(r-R_1)^2} > -\Omega^2\left[\frac{m_2 r}{M} - R_1\right]$$

$$\frac{\cancel{G}m_1}{R_1^2} - \frac{\cancel{G}m_2}{(r-R_1)^2} > \frac{\cancel{M}\cancel{G}}{r^3}\left[\frac{MR_1 - m_2 r}{\cancel{M}}\right]$$

$$\frac{m_1}{R_1^2} - \frac{m_2}{(r-R_1)^2} > \frac{MR_1 - m_2 r}{r^3}$$

If this does not hold, then there will be a tidal distortion. ★WHAT DOES THIS TELL US FOR THE EARTH-MOON SYSTEM?

Let $m_2 = M_\oplus = 6\times10^{24}$ kg

$m_1 = M_{moon} = 7.3\times10^{22}$ kg

$R_1 = r_{moon} = 1.7\times10^{6}$ m

$r = d_{earth\text{-}moon} = 3.8\times10^{8}$ m

$$\frac{7.3\times10^{22}\text{ kg}}{(1.7\times10^{6}\text{ m})^2} - \frac{6\times10^{24}\text{ kg}}{(3.8\times10^{8} - 1.7\times10^{6}\text{ m})^2} > \frac{(6\times10^{24}\text{ kg} + 7.3\times10^{22}\text{ kg})(1.7\times10^{6}\text{ m})}{(3.8\times10^{8}\text{ m})^3}$$

$$- \frac{(6\times10^{24}\text{ kg})\cancel{(3.8\times10^{8}\text{ m})}}{(3.8\times10^{8}\text{ m})^{\cancel{3}\,2}}$$

$$2.5 \times 10^{10}\ kg/m^2 - 4.2 \times 10^7\ kg/m^2 > 1.9 \times 10^5\ \frac{kg}{m^2} - 4.2 \times 10^7\ \frac{kg}{m^3}$$

$$\approx 2.5 \times 10^{10}\ kg/m^2 > -4.2 \times 10^7\ kg/m^2$$

No tidal distortion for earth-moon system.

★IS THERE A TIDAL DISTORTION ON THE MOON DUE TO EARTH'S GRAVITY?

To be sure, the conditions are

$$\frac{m_1}{R_1^2} - \frac{m_2}{(r-R_1)^2} >> \frac{MR_1 - m_2 r}{r^3}$$ <u>W 2.2.3</u> Side closer to m_1

$$\frac{m_1}{R_1^2} + \frac{m_2}{(r+R_1)^2} >> \frac{MR_1 + m_2 r}{r^3}$$ <u>W 2.2.4</u> Side furthest from m_1

Often $r >> R_1$, so

$$\frac{m_1}{R_1^2} \mp \frac{m_2}{r^2} >> \mp \frac{m_2}{r^2}$$

$$r >> R_1 \left(1 + \frac{3m_2}{m_1}\right)^{1/3}$$

$$\frac{m_1}{R_1^2} >> \mp \frac{m_2}{r^2} \pm \frac{m_2}{r^2} = 0$$

<u>Roche limit</u> <u>W2.2.5</u>

$$\frac{m_1}{R_1^2} >> 0$$

so either m_1 is very large or R_1^2 is very small (compact object)

or both!

Consider Sirius A and Sirius B

$m_1 = m_B = 0.98\,M_\odot \qquad R_1 = R_B = 0.008\,R_\odot$

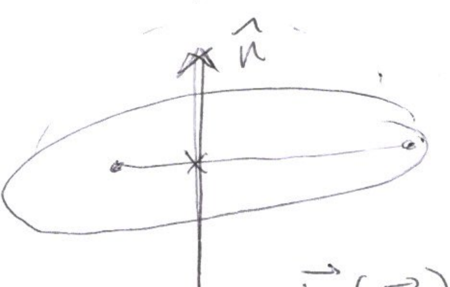
$m_2 = m_A = 2.28\,M_\odot \qquad R = R_A = 1.71\,R_\odot \qquad r = 1.7\times10^3 R_\odot$

$$0.008R_\odot\left(1+\frac{3(2.28M_\odot)}{0.98M_\odot}\right)^{1/3} = 0.008R_\odot\left(1+7\right)^{1/3} = \cancel{0.064}\ 0.016\,R_\odot$$

$1.7\times10^3\,R_\odot \gg 1.6\times10^{-2}\,R_\odot$ ✓

The more interesting case is when the condition is NOT met. What happens then? ☆

Consider the direction $\hat{n}$ out of the orbital plane and through the center of mass



The force at position $\vec{x}$ (arbitrary) felt by a test body of mass δm is

zero if over the center of mass

$$\vec{F}(\vec{x}) = \delta m\left(-\nabla\phi(\vec{x}) + \Omega^2\left[\vec{x} - \hat{n}(\hat{n}\cdot\vec{x})\right]\right)$$

zero if on the orbital plane

☆ $\phi(\vec{x})$ is the total gravitational potential

so $-\nabla\phi(\vec{x})$ is the total gravitational force

$F = -\frac{dU}{dx}$, so $-\int dU = \int F\,dx$

$$-U = \Omega^2\int\left[x - \hat{n}(\hat{n}\cdot x)\right]dx = \frac{\Omega^2}{2}\left[x^2 - (\hat{n}\cdot\vec{x})^2\right]$$

So we can rewrite the force in terms of an effective potential,

$$\vec{F}(\vec{x}) = -\delta m \, \nabla \Phi(\vec{x})$$

where $\Phi(\vec{x}) = \phi(\vec{x}) - \frac{\Omega^2}{2}\left[\vec{x}^2 - (\hat{n}\cdot\vec{x})^2\right]$

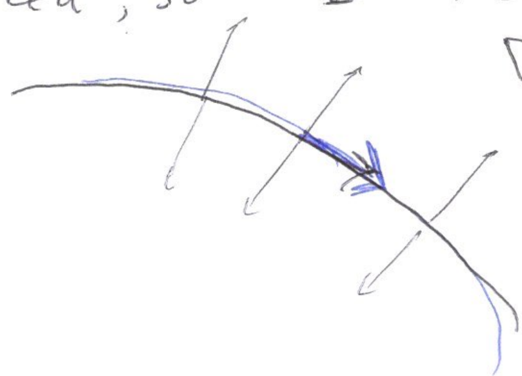
Inside of the sphere/star, the force will be balanced by pressure (radiation, ideal gas, etc.), also at the surface.

At the very surface, though, the tangential components are not balanced, so the tangential part of

$$\nabla_{\perp}\Phi(\vec{x}) = 0$$

$$\Rightarrow \Phi(\vec{x}) = \Phi$$

constant



Notice that $\Phi(\vec{x})$ is general for any mass distribution, but unless the tidal effects are small (so m_1 and m_2 are still close to spherical) you need numerical methods.

Also notice that the condition holds outside of the star, so you get equipotential surfaces.

so with $\phi(\vec{x}) = -\dfrac{m_1 G}{|\vec{x}-\vec{x}_1|} - \dfrac{m_2 G}{|\vec{x}-\vec{x}_2|}$

$m_2 \quad \vec{x}_2 \quad \times \quad \vec{x}_1 \quad m_1$

and $\Omega^2 = \dfrac{MG}{r^3}$

$$-\Phi = +\frac{m_1 G}{|\vec{x}-\vec{x}_1|} + \frac{m_2 G}{|\vec{x}-\vec{x}_2|} + \frac{MG}{2r^3}\left[\vec{x}^2 - (\hat{n}\cdot\vec{x})^2\right]$$

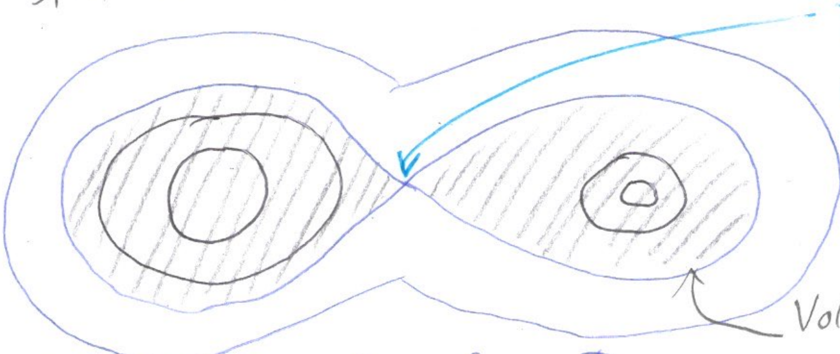
Since $\vec{x}_1 = \left(\dfrac{m_2}{M}\right)\vec{r}$ and $\vec{x}_2 = \left(\dfrac{m_1}{M}\right)\vec{r}$

with $\vec{x}_1 = \left(\dfrac{m_2 r}{M}, 0, 0\right)$; $\vec{x}_2 = \left(-\dfrac{m_1 r}{M}, 0, 0\right)$; $\hat{n} = (0,0,1)$

for position $x = (x, y, z)$,

$$-\Phi = \frac{m_1 G}{\left[\left(x - \frac{m_2 r}{M}\right)^2 + y^2 + z^2\right]^{1/2}} + \frac{m_2 G}{\left[\left(x + \frac{m_1 r}{M}\right)^2 + y^2 + z^2\right]^{1/2}} + \frac{MG}{2r^3}(x^2+y^2)$$

For large values of $-\Phi$ (which will be closer to the masses) Spherical and disconnected



The point where equipotentials touch at critical value of $-\Phi$ is called the L_1 (first) Lagrangian point, the force is zero.

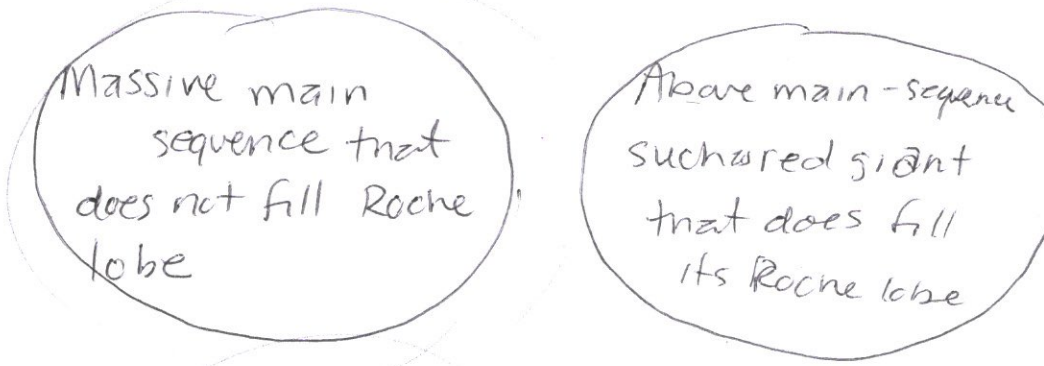
Volume enclosed is called a Roche lobe

For smaller values of $-\Phi$, equipotential connect and are distorted by tidal effects

The cool thing about these equipotential surfaces is that you can start filling them with mass. Matter will want to minimize its energy, so it configures itself is a sphere initially, but if more mass is added with low density, the matter might "spill" onto the other star.

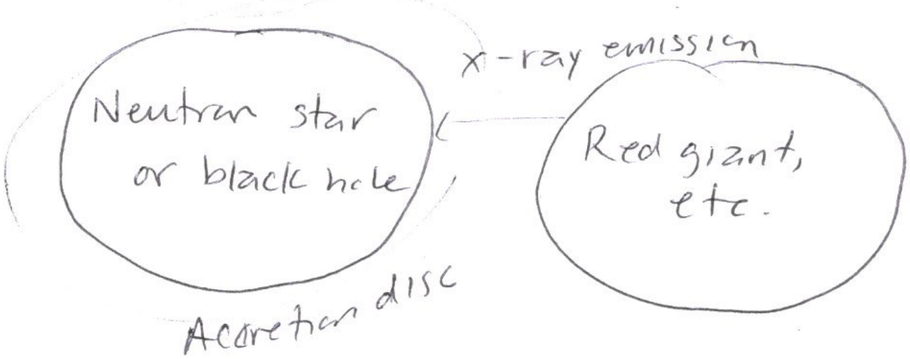
- Detached binary — Neither star fills its Roche lobe
- Semi-detached — One star fills its Roche lobe, so it transfers mass to the other.

  + Algol-type



  + Type I supernova

  + X-ray binary



- Contact binary — Both stars fill their Roche lobes, the more massive transfers mass and luminosity to second one, so same temperature.