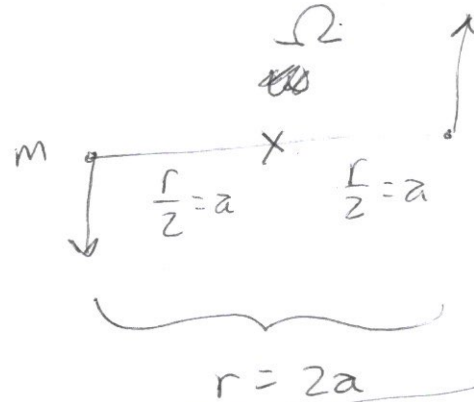
GRAVITATIONAL WAVES

An alternative scenario for a supernova Ia is that two white dwarfs ~~close~~ that are gravitationally bound lose energy due to gravitational waves until they merge, exceeding the Chandrasekar limit an exploding.

First we show that the orbital kinetic energy is:



Consider equal masses m separated by distance r and with angular frequency Ω, the energy is

$$\cancel{KE = 2\left[\frac{1}{2} m v^2\right] = m\Omega^2 \left(\frac{r}{2}\right)^2}$$

$$\cancel{\text{with } \Omega^2 = \frac{MG}{\left(\frac{r}{2}\right)^3} = \frac{2mG}{\frac{r^3}{8}}}$$

$$\cancel{KE = 2\left[\frac{1}{2} m v^2\right] = m\Omega^2 a^2}$$

$$\cancel{\Omega^2 = \frac{MG}{(2a)^3} = \frac{2mG}{8a^3}}$$

$$\cancel{KE = \frac{2m^2 G r^2}{4r^3} =}$$

$$KE = \cancel{2}\left[\frac{1}{\cancel{2}} m v^2\right] = m\Omega^2 \left(\frac{r}{2}\right)^2$$

$$\text{with } \Omega^2 = \frac{MG}{r^3} = \frac{2mG}{r^3}\text{; } KE = \frac{\cancel{2} m^2 G}{r^{\cancel{3}}}\frac{\cancel{r^2}}{\cancel{4}\,2} = \underline{\underline{\frac{Gm^2}{2r}}}$$

The potential energy is $-\frac{Gm^2}{r}$, so the total energy is $\frac{Gm^2}{2r} - \frac{Gm^2}{r} = \frac{Gm^2}{r}\left(\frac{1}{2} - \frac{1}{1}\right) = -\frac{Gm^2}{2r}$

The following Eq. is W2.3.7 for the time derivative of the energy, it comes from the mass tensor, but we will not derive it, just state it

$$-\left\langle \frac{dE}{dt} \right\rangle = \langle P \rangle = \frac{32 G \mu^2}{5c^5}\left(\frac{MG}{r^3}\right)^3 r^4 f(e)$$

where $f(e)$ is a function of the ellipticity/eccentricity, with $f(0) = 1$. Since $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m^2}{2m} = \frac{m}{2}$; $\mu^2 = \frac{m^2}{4}$

$$\langle P \rangle = -\frac{m^2}{4} \cdot \frac{32 G}{5c^5} \frac{(2m)^3 G^3}{r^{9}} r^4 = -\frac{m^5 2^5 G^4 2}{2^2 \cdot 5c^5 r^5} = -\frac{2}{5G}\left(\frac{2Gm}{c^2 r}\right)^5$$

$$\langle P \rangle = -\frac{2c^5}{5G}\left(\frac{2Gm}{c^2 r}\right)^5 = -\frac{2c^5}{5G}\left(\frac{2Gm}{c^2 r}\right)^5$$

This is a nice way to write it.

$$\frac{Gm}{c^2 r} \Rightarrow \frac{\frac{m^3}{kg\, s^2} \cdot kg}{\frac{m^2}{s^2} \cdot m} = \frac{m^3\, kg\, s^2}{kg\, s^2\, m^3} = \text{unitless}$$

$$\frac{c^5}{G} = \frac{m^5/s^5}{m^3/kg\,s^2} = \frac{m^{\cancel{5}\,2}\,kg\,\cancel{s^2}}{\cancel{m^3}\,s\,\cancel{s}^3} = \frac{kg\,m^2}{s^3} = J/s$$

so $\frac{dE_{gw}}{dt} = -\frac{2c^5}{5G}\left(\frac{2Gm}{c^2 r}\right)^5$ and $\frac{dE}{dt} = -\frac{d}{dt}\left(\frac{Gm^2}{2r}\right)$ before

If ~~all~~ the energy can only be lost due to gravitational waves, then

$$\frac{dE_{gw}}{dt} = \frac{d}{dt}E \Rightarrow \frac{d}{dt}\left(\frac{Gm^2}{2r}\right) = \frac{Gm^2}{2}\frac{d}{dt}(1/r)$$

$$= \frac{2c^5}{5G}\left(\frac{2Gm}{c^2 r}\right)^5 = \frac{2c^5}{5G}\left(\frac{2Gm}{c^2}\right)^5\frac{1}{r^5}$$

Can we find the separation between the whitedwarfs as a function of time? $r(t)$

Let $u = 1/r$, then

$$\frac{Gm^2}{2}\frac{du}{dt} = \frac{2c^5}{5G}\left(\frac{2Gm}{c^2}\right)^5 u^5$$

$$\int\frac{du}{u^5} = \frac{2}{\cancel{Gm^2}}\,\frac{2\cancel{c^5}}{5\cancel{G}}\,\frac{2^5 G^{\cancel{5}\,3} m^{\cancel{5}\,3}}{c^{\cancel{10}\,5}}\int dt = \frac{2^7 G^3 m^3}{5c^5}\int dt$$

$$-\frac{1}{4u^4} = \frac{2^7 G^3 m^3 t}{5c^5} + C$$

$$r = \left\{ -\frac{2^9 G^3 m^3 t}{5c^5} + C \right\}^{1/4}$$

we know that $r(0) = r_0$, so $r_0 = C^{1/4}$

$$r_0^4 = C$$

$$r^4(t) = -\frac{2^9 G^3 m^3 t}{5c^5} + r_0^4$$

Let's assume that the white dwarfs, of $1 M_\odot$ each, collide after 10 Gyr. What was their initial separation?

$$r(t) = \left[r_0^4 - \frac{2^9 G^3 M_\odot^3 t}{5c^5} \right]^{1/4}$$

$$10 \times 10^9 \text{ years} \left(\frac{3.15 \times 10^7 \text{ s}}{1 \text{ year}} \right) = 3.15 \times 10^{17} \text{ s}$$

$$r(3.15 \times 10^{16} \text{ s})^4 = 0^4 = r_0^4 - \frac{2^9 G^3 M_\odot^3 (3.15 \times 10^{17} \text{ s})}{5c^5}$$

$$r_0^4 = \frac{2^9 G^3 M_\odot^3 t}{5c^5} = \frac{512 \cdot (6.67\times10^{-11} m^3/kg s^2)^3 (2\times10^{30} kg)^3 (3.15\times10^{17} s)}{5\,(3\times10^{8} \frac{m}{s})^5}$$

$$r_0^4 = \frac{512 \cdot 2.96\times10^{-31} \frac{m^{9}}{\cancel{kg^3 s^6}} \cdot 8\times10^{90} \cancel{kg^3}\ 3.15\times10^{17} \cancel{s}}{5 \cdot 2.43\times10^{42} \cancel{m^5}/\cancel{s^5}}$$

$$r_0^4 = \frac{3.83\times10^{80} m^4}{1.22\times10^{43}} = 3.15\times10^{37} m^4$$

$$\Rightarrow r_0 = 2.37\times10^{9} m \ \frac{1\ A.U.}{1.5\times10^{11} m} = 0.016\ A.U.$$

This is about 6 times the distance from the earth to the moon.

* IS THE EARTH MOON PRODUCING GRAVITATIONAL RADIATION?

* IF YES, ARE THEY GOING TO COLLIDE? WHY NOT?

Consider the Maxwell Equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

They tell you what is the relationship between the electric and magnetic fields and the distribution of charges and currents ~~that produce them~~.

The Einstein field Equations do an analogous thing: they tell you what is the relationship between spacetime and ~~matter~~ the distribution of matter

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad \text{or even} \quad G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

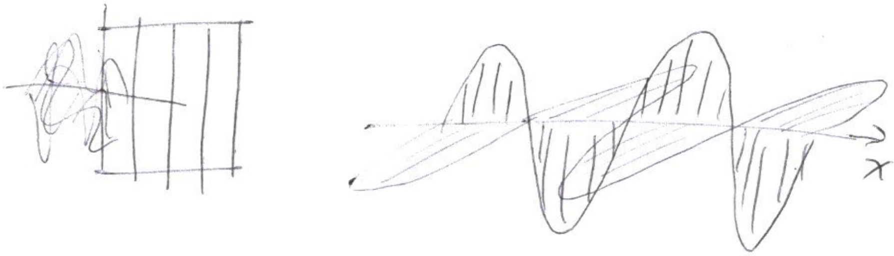
For Maxwell, $\nabla\cdot$ and $\nabla\times$ are shorthand first order vector derivatives

For Einstein, $G_{\mu\nu}$ are first and second order tensor derivatives

it actually consists of 60 equations

So mathematically, General Relativity is much more complicated than E&M, but conceptually is not that much different.

consider light getting polarized



E-field moving in a plane propagating throug space, described by a polarization vector

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \vec{E} = \begin{bmatrix} E_x \\ 0 \\ 0 \end{bmatrix}$$

HOW WOULD THIS LOOK LIKE FOR A GRAVITATIONAL WAVE?

$g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ ← perturbation added to space

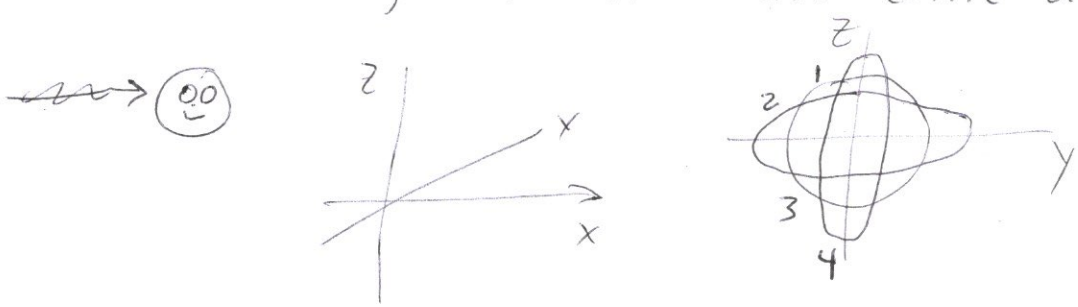
$$h_{\mu\nu} = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x & y & z & t \end{matrix} \end{array}$$

This is known as the polarization tensor

h_+ is a stretch/compress mode

$h_\times$ is a rotation mode

if you look at the gravitational wave come directly at you



The rotation modes can be observed if at an angle.

The amplitude of the gravitational wave is

$$\bar{h} \approx \frac{MG\bar{v}^2}{dc^4} \qquad \underline{W\,2.4.1} \qquad \text{unitless}$$

where d is the distance from the source to the observer.

LIGO can detect currently $\bar{h} \simeq 8\times10^{-24}$

For the white dwarf system we analized before, can we detect it with LIGO? Depends on d

$1\ ly = 9.46\times10^{15}\ m$

$$M = 2M_\odot$$

$$v^2 = \Omega^2\left(\frac{r}{2}\right)^2 = \frac{2mG}{r^3}\frac{r^2}{4} = \frac{M_\odot G}{r}$$

$$\bar{h} = 2M_\odot G\frac{M_\odot G}{r}\frac{1}{dc^4} = \frac{2M_\odot^2G^2}{dc^4r}$$

$$d\bar{h} = \frac{2(2\times10^{30}\,kg)^2(6.67\times10^{-11}\,m^3/kg\,s^2)^2}{(2.37\times10^9\,m)(3\times10^8\,m/s)^4} = \frac{(8\times10^{60}\,kg^2)(4.44\times10^{-21}\,m^6/kg^2s^4)}{(2.37\times10^9\,m)(8.1\times10^{33}\,m^4/s^4)}$$

$$d\bar{h} = \frac{3.56\times10^{40}\,m}{1.91\times10^{43}} = 1.85\times10^{-3}\,m$$

$$h = 1\times10^{-23} = \frac{1.85\times10^{-3}\,m}{d} \Rightarrow d = \frac{1.85\times10^{-3}\,m}{1\times10^{-23}} = 1.85\times10^{20}\,m$$

$$\approx 20{,}000\ ly$$