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2. (Pb. 10.19 in Carroll and Ostlie) Derive an expression for the total mass of a $\Gamma = 6/5$ polytrope and show that although $\xi_1 \to \infty$, the mass is finite.

In general, $M = \int_0^R 4\pi r^2 \rho(r)\, dr$

We will rewrite $\rho(r)$ in terms of the variables in the Lane-Emden equation (page 61 of Weinberg and also pg. 61 of my notes).

$$\Theta \equiv \left(\frac{\rho(r)}{\rho(0)}\right)^{\Gamma-1} \Rightarrow \rho(r)^{\Gamma-1/\Gamma-1} = \left[\Theta \rho(0)^{\Gamma-1}\right]^{1/\Gamma-1} \qquad \text{W 1.8.3}$$

$$\Rightarrow \rho(r) = \Theta^{1/\Gamma-1} \rho(0)$$

$$r^2 = \xi^2 \frac{K\Gamma}{4\pi G(\Gamma-1)\rho(0)^{(2-\Gamma)}} \qquad \text{W 1.8.4}$$

$$r = \frac{\xi}{\left(\frac{4\pi G(\Gamma-1)}{K\Gamma}\right)^{1/2} \rho(0)^{(2-\Gamma)/2}} = \frac{(K\Gamma)^{1/2}\, \xi}{\left[4\pi G(\Gamma-1)\right]^{1/2} \rho(0)^{(2-\Gamma)/2}}$$

$$dr = \frac{d\xi}{\left(\frac{4\pi G(\Gamma-1)}{K\Gamma}\right)^{1/2} \rho(0)^{(2-\Gamma)/2}} = \frac{(K\Gamma)^{1/2}\, d\xi}{\left[4\pi G(\Gamma-1)\right]^{1/2} \rho(0)^{(2-\Gamma)/2}}$$

Putting $r^2\rho(r)dr$ together,

$$4\pi r^2\rho(r)dr = 4\pi\left(\frac{K\Gamma}{4\pi G(\Gamma-1)}\right)^{3/2} \xi^2 \frac{\Theta^{1/\Gamma-1}\rho(0)}{\rho(0)^{3(2-\Gamma)/2}} d\xi$$

Then

$$M = 4\pi\rho(0)^{(3\Gamma-4)/2}\left(\frac{K\Gamma}{4\pi G(\Gamma-1)}\right)^{3/2}\int_0^{\xi_1} \xi^2\,\Theta^{1/\Gamma-1}(\xi)\,d\xi$$

ξ_1 is the first root of the Lane-Emden Equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}\Theta(\xi)\right) + \Theta(\xi)^{1/\Gamma-1} = 0$$

Notice that

$$\Theta(\xi)^{1/\Gamma-1} = -\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}\Theta(\xi)\right)$$

so the integral then becomes

$$-\int_0^{\xi_1}\cancel{\xi^2}\frac{1}{\cancel{\xi^2}}\frac{\cancel{d}}{\cancel{d\xi}}\left(\xi^2\frac{d}{d\xi}\Theta(\xi)\right)\cancel{d\xi} = -\xi^2\frac{d}{d\xi}\Theta(\xi)\Big|_0^{\xi_1} = -\xi_1^2\Theta'(\xi_1)$$

and

$$M = 4\pi\rho(0)^{(3\Gamma-4)/2}\left(\frac{K\Gamma}{4\pi G(\Gamma-1)}\right)^{3/2}\xi_1^2\,|\Theta'(\xi_1)|$$

Notice that $\Theta'(\xi)$ is negative or zero for all ξ, so $-\Theta'(\xi_1) = |\Theta'(\xi_1)|$

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$$\rho(r) = m_1 \mu \frac{N}{V}$$

where m_1 is the nuclear mass for unit atomic weight

$m_1 = 931.49 \ \text{MeV}/c^2$
1.66×10^{-27} kg

(a Dalton defined as $\frac{1}{12}$ the mass of an unbound neutral atom of ^{12}C in its nuclear and electronic ground state and at rest.

$\mu \equiv \frac{A}{Z}$ atomic weight per electron

For Fe, $m_1 \mu = \frac{55.847}{26}$ C would be $\frac{12.011}{6}$

which is a weighted average of the isotopes

$$\frac{5.85}{100} \, {}^{54}Fe + \frac{91.75}{100} \, {}^{56}Fe + \frac{2.12}{100} \, {}^{57}Fe + \frac{0.28}{100} \, {}^{58}Fe$$

Remember that $N = 2\left(\frac{L}{2\pi}\right)^3 \int_0^{k_F} 4\pi k^2 \, dk$

and momentum $p \equiv \hbar \vec{k} = 2\pi h \vec{k}$

* Weinberg uses k for the momentum, and I will follow the notation. Do not get confused it just changes by a constant $\frac{L}{2\pi} \longrightarrow \frac{L}{2\pi\hbar}$, ~~so~~

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$$N = 2\left(\frac{L}{2\pi\hbar}\right)^3 \int_0^{k_F} 4\pi k^2\, dk$$

$$\left(\frac{3h^3\rho}{8\pi m_1 \mu}\right)^{1/3} = k_F$$

$$N = \frac{8\pi V}{8\pi^3 \hbar^3} \int_0^{k_F} k^2\, dk$$

since $\hbar = \frac{h}{2\pi} \Rightarrow \hbar^3 = \frac{h^3}{8\pi^3}$ so

$$N = \frac{8\pi V}{\cancel{8\pi^3} \frac{h^3}{\cancel{8\pi^3}}} \int_0^{k_F} k^2\, dk = \frac{8\pi V}{h^3} \int_0^{k_F} k^2\, dk$$

W 1.10.1

$$\rho(r) = m_1 \mu \frac{N}{V} = \frac{\cancel{V}}{\cancel{V}} \frac{8\pi m_1 \mu}{h^3} \int_0^{k_F(r)} k^2 dk = \frac{8\pi m_1 \mu k_F^3(r)}{3h^3}$$

Last time we got the internal energy in terms of the Fermi wavevector (Fermi momentum). More generally, with the energy-momentum (relativistic dispersion) relation

$$E^2 = \cancel{(pc)^2}\ (kc)^2 + (m_e c^2)^2$$

$$\frac{U}{V} = \mathcal{E}(r) = \frac{V}{V} \frac{8\pi}{h^3} \int_0^{k_F} k^2 \left[(k^2c^2 + m_e^2 c^4)^{1/2} - m_e c^2 \right] dk$$

Energy density

excluding rest mass

Last time we derived, so

$$U_0 = \frac{V\hbar^2}{2\pi^2 m}\,\frac{k_F^5}{5} = \frac{\hbar^2 (3\pi^2 N)^{5/3}}{10\pi^2 m}\, V^{-2/3}$$

$$dU_0 = -\frac{2}{3}\,\frac{\hbar^2 (3\pi^2 N)^{5/3}}{10\pi^2 m}\, V^{-5/3}\, dV$$

$$dU_0 = -\frac{2}{3}\, U_0\, \frac{dV}{V}$$

Let $\overset{-dW}{dU_0} = p\, dV$, then

$$p = +\frac{2}{3}\,\frac{U_0}{V}$$

with $p = \frac{2}{3}\,\frac{k^2}{2m}\,\frac{1}{V}$

$$\rho(r) = \frac{8\pi c^2}{3h^3}\int_0^{k_F} \frac{k^4}{k^2c^2 + m_e^2 c^4}\, dk$$

W1.10-3

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What is the critical density at which the Fermi momentum becomes $m_e c$

$$\rho_c = \frac{8\pi m_1 \mu k_F^3}{3h^3} = \frac{8\pi m_1 \mu m_e^3 c^3}{3h^3}$$

$$\rho_c = \frac{8\pi \cdot 1.66\times10^{-27}\,\text{kg} \cdot (9.11\times10^{-31}\,\text{kg})^3 \cdot \left(3\times10^8 \frac{m}{s}\right)^3}{3\cdot(6.626\times10^{-34}\,\text{J}\cdot\text{s})^3}\,\mu$$

composition

$$\rho_c = \frac{8.51\times10^{-91}\ \frac{kg^4\cancel{m^3}}{\cancel{s^3}}}{8.72\times10^{-100}\ \frac{kg^3 m^{6}}{\cancel{s^3}}} = \left(9.75\times10^{8}\ kg/m^3\right)\mu$$

non-relativistic case:

For $\rho \ll \rho_c$, $k_F \ll m_e c$, $\sqrt{k^2c^2 + m_e^2 c^4} \approx m_e c^2$

$\sqrt{k^2c^2 + m_e^2c^4} - m_e c^2 \approx k c$

$$p = \frac{8\pi \cancel{c^2}}{3h^3 m_e \cancel{c^2}} \int_0^{k_F} k^4\,dk = \frac{8\pi k_F^5}{3\cdot 5\, m_e h^3} = \frac{8\pi}{15 m_e h^3}\left(\frac{3h^3\rho}{8\pi m_1 \mu}\right)^{\frac{5}{3}}$$

Just like before,

$p = \frac{2}{3}\varepsilon$

This is a polytrope $\mathcal{E} = \frac{3}{2} P$

$$\frac{1}{\Gamma - 1} = \frac{3}{2} \Rightarrow 3\Gamma - 3 = 2 \Rightarrow \boxed{\Gamma = 5/3}$$

$P = K\rho^{\Gamma}$ will give $K = \frac{8\pi}{15 m_e h^3}\left(\frac{3h^3}{8\pi m_1 \mu}\right)^{5/3}$

$P = K\rho^{5/3}$

only variable, depends on composition.

$$R = \left(\frac{4\pi G(\Gamma - 1)}{K\Gamma}\right)^{-1/2} \rho(0)^{-(2-\Gamma)/2} \xi_1$$

central density

$$M = 4\pi \rho(0)^{(3\Gamma - 4)/2} \left(\frac{K\Gamma}{4\pi G(\Gamma - 1)}\right)^{3/2} \xi_1^{\,2} \left|\Theta'(\xi_1)\right|$$

For $\Gamma = 5/3$, $\xi_1 = 3.65375$ $\quad \xi_1^{\,2}\left|\Theta'(\xi)\right| = 2.71406$

$$\mu = \frac{3h^3 \rho_c}{8\pi m_1 m_e^3 c^3}$$

$$K = \frac{8\pi}{15 m_e h^3}\left(\frac{\cancel{3h^3} \cdot \cancel{8\pi m_1} m_e^3 c^3}{\cancel{8\pi m_1} \cdot \cancel{3h^3} \rho_c}\right)^{5/3} = \frac{8\pi}{15 m_e h^3} m_e^{\cancel{5}\,4} c^5 \rho_c^{-5/3}$$

$$K = \frac{8\pi m_e^4 c^5}{15 h^3} \rho_c^{-5/3}$$

in relativistic case $\Gamma = 4/3$

$$R = 2 \times 10^4 \, \mu^{-1} \left(\frac{\rho(0)}{\rho_c} \right)^{-1/6} \text{ km} \qquad \Gamma = 5/3$$

$$M = 2.79 \, \mu^{-2} \left(\frac{\rho(0)}{\rho_c} \right)^{1/2} M_\odot$$

$$R = 5.3 \times 10^4 \, \mu^{-1} \left(\frac{\rho(0)}{\rho_c} \right)^{-1/3} \qquad \Gamma = 4/3$$

$$M = 5.87 \, \mu^{-2} \, M_\odot \longleftarrow$$ Chandrasekar limit

W 1.10.13

R decreases with increasing central density whereas mass increases