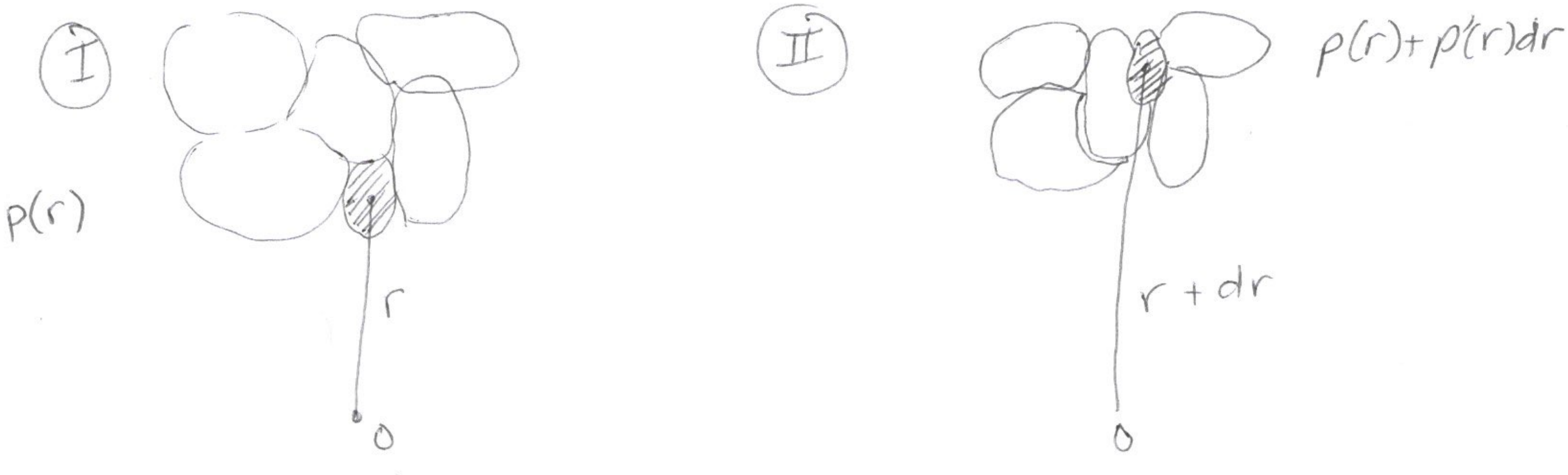
CONVECTION

Consider a small element of stellar fluid that moves upward from r to $r+dr$



Forces are always balanced, so the pressure in the element changes from $p(r)$ to $p(r)+p'(r)\,dr$.

This "small" element is only small compared with the rest of the star. On the sun's photosphere they are about the size of Texas, with a diameter of 1.5×10^{6} m and last 8 to 20 minutes. They are called granules and at any given time the sun is covered by about 4 million of them.

Underneath, the supergranules are 3×10^{7} m (bigger than the earth) and live for 24 hours or so.

Because conduction is really slow, the temperature and the density will change only due to the change in pressure and in general it will be different than the environment.

The process is <u>adiabatic</u> (no heat flow).

The new density is

$$\rho(r) + \left.\frac{\partial \rho(p)}{\partial p}\right|_{p=p(r)} p'(r)\,dr \leftarrow \text{change in radius}$$

original density

rate of change of the density with respect to pressure

rate of change of the pressure with respect to radius

★ WHAT HAPPENS IF THE DENSITY IS GREATER THAN THE AMBIENT DENSITY AT THE NEW POS?

$\rho(r) + \rho'(r)\,dr$?

Sinks back down, Archimides

★ WHAT IF DENSITY IS LOWER?

will continue moving up

So the condition for stability against moving up is:

$$\underbrace{\cancel{\rho(r)} + \left.\frac{\partial \rho(p)}{\partial p}\right|_{p=p(r)} p'(r)\,dr}_{\mathrm{I}} > \underbrace{\cancel{\rho(r)} + \rho'(r)\,dr}_{\mathrm{II}}$$

if they are equal, then blob can move at constant velocity or not move.

I The previous equation is usually given in terms of pressure and temperature, so for $\rho(p,T)$

$$\frac{\partial \rho(p,T)}{\partial p} = \frac{\partial \rho}{\partial p} \cdot \cancel{\frac{\partial p}{\partial p}} + \frac{\partial \rho}{\partial T} \cdot \frac{\partial T}{\partial p} \qquad \text{Chain rule}$$

For an ideal gas, $pV = nk_B T$

$$\Rightarrow V = \frac{nk_B T}{p}$$

$$\rho = \frac{M}{V} = \frac{\cancel{n} m}{\cancel{n} k_B T / p} = \frac{mp}{k_B T}$$

where m is the mass of a gas particle

$$m = \frac{k_B T \rho}{p}$$

$$\frac{\partial \rho}{\partial p} = \frac{m}{k_B T} = \frac{\cancel{k_B T}\rho/p}{\cancel{k_B T}} = \frac{\rho}{p}$$

$$\frac{\partial \rho}{\partial T} = -\frac{mp}{k_B T^2} = -\frac{\;\frac{k_B T \rho \cancel{p}}{\cancel{p}}\;}{\frac{k_B T^2}{1}} = -\frac{\cancel{k_B T}\rho}{\cancel{k_B} T^{\cancel{2}}} = -\frac{\rho}{T}$$

so

$$\frac{\partial \rho}{\partial p} = \frac{\rho}{p} - \frac{\rho}{T}\frac{\partial T}{\partial p}$$

Remember that $\frac{d}{dx}\ln(x) = \frac{1}{x}$, so

$$d\ln(x) = \frac{dx}{x},$$

we can rewrite $\frac{\partial T}{T} = \partial \ln T$

$$\frac{\partial P}{P} = \partial \ln P$$

$$\frac{\partial \rho}{\partial P} = \frac{\rho}{P} - \frac{\partial \ln T}{\partial \ln P}$$

II We can do the same for $\rho'(r) = \frac{d\rho(r)}{dr}$

$$\frac{d\rho(P)}{dr} = \frac{d\rho(P)}{dP} \cdot \frac{dP}{dr} = \frac{\rho}{P}\frac{dP}{dr} = \frac{\rho P'}{P}$$

$$\frac{d\rho(T)}{dr} = \frac{d\rho(T)}{\partial T} \cdot \frac{dT}{dr} = \frac{\rho}{T}\frac{dT}{dr} = \frac{\rho T'}{T}$$

AND GUESS WHAT...

$$P' = \frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}, \quad T' = \frac{dT}{dr} = -\frac{3\kappa(r)\rho(r)L(r)}{4caT^3(r)4\pi r^2}$$

Let $$\nabla_{ad}(r) \equiv \left.\frac{\partial \ln T(p)}{\partial \ln p}\right|_{p=p(r)}$$

$$\nabla(r) \equiv \frac{T'(r)/T(r)}{p'(r)/p(r)}$$

Stable against moving up

$$\left.\frac{\partial \rho(p)}{\partial p}\right|_{p=p(r)} p'(r)\,\cancel{dr} - \rho'\,\cancel{dr} > 0$$

$$\left[\frac{\rho}{p} - \nabla_{ad}\right] p'(r) - \nabla\left[\frac{\rho T p'^2}{T' p^2} - \frac{\rho p'}{p}\right] > 0$$

$$\frac{\rho p'}{p} - p'\nabla_{ad} -$$

Can be written as

$$-\frac{p'(r)\rho(r)}{p(r)}\left[\nabla_{ad}(r) - \nabla(r)\right] > 0$$

$\rho(r)$ and $p(r)$ are positive for all r

$p'(r)$ is negative for all r, so the term is negative times negative = positive

The stability condition is then $\nabla_{ad}(r) > \nabla(r)$

$$\nabla(r) = \left(-\frac{3\kappa(r)\cancel{\rho(r)}L(r)}{4caT^3(r)\,4\pi \cancel{r^2}}\right)\left(\frac{1}{T(r)}\right)\left(-\frac{\cancel{r^2}}{GM(r)\cancel{\rho(r)}}\right)\left(\frac{P(r)}{1}\right)$$

$$\nabla(r) = \frac{3\kappa(r)L(r)P(r)}{16\pi c a T^4(r)\,GM(r)}$$

$$\Rightarrow \frac{16\pi c a T^4(r)\,GM(r)\,\nabla_{ad}(r)}{3\kappa(r)P(r)} > L(r)$$

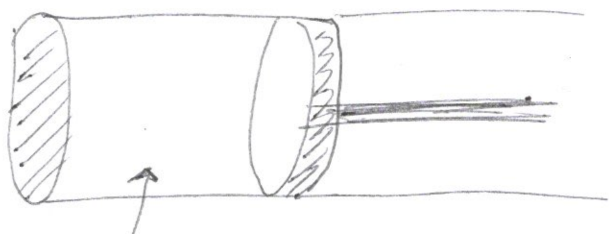
with $P_{rad}(r) = \dfrac{aT^4(r)}{3}$

$$\frac{16\pi c\,P_{rad}(r)\,GM(r)\,\nabla_{ad}(r)}{\kappa(r)P(r)} > L(r)$$

with $L_{Edd} = \dfrac{4\pi c\,GM(r)}{\kappa(r)}$ Eddington limit

$$\left(\frac{4P_{rad}(r)\nabla_{ad}(r)}{P(r)}\right)\left(L_{Edd}(r)\right) > L(r)$$

In order for radiation not to destroy the star, $L(r) < L_{Edd}(r)$ anyways, so consistent.



Consider 1 kg of stellar fluid

The thermal energy density is $\mathcal{E}$ (J/m^3)

so the thermal energy per kilogram is $\frac{\mathcal{E}}{\rho}$ $\left(\frac{J/m^3}{kg/m^3} = J/kg\right)$

when the volume per kilogram $\frac{1}{\rho}$ (m^3/kg) changes,

the work is $P\,\delta V = P\,\delta\left(\frac{1}{\rho}\right)$ Done on the environment

This must come from the thermal energy, so decreases temperature

$$P\,\delta\left(\frac{1}{\rho}\right) = -\,\delta\left(\frac{\mathcal{E}}{\rho}\right)$$

Remember that $\mathcal{E} = \frac{P}{\Gamma - 1}$, so $\quad$ $\mathcal{E} = nP$ with $n = \frac{1}{\Gamma-1}$

$$P\,\delta\left(\frac{1}{\rho}\right) = -\,\delta\left(\frac{P}{\rho\Gamma - \rho}\right)$$

$$P\,\delta\left(\frac{1}{\rho}\right) + \delta\left(\frac{nP}{\rho}\right) = 0 \qquad P\,\delta\left(\frac{1}{\rho}\right) + \delta\left(\frac{P}{(\Gamma-1)\rho}\right) = 0$$

$$p(\Gamma-1)\,\delta\left(\frac{1}{\rho}\right) + \delta\left(\frac{p}{\rho}\right) = 0$$

$$\delta\left(\frac{p}{\rho^{\Gamma}}\right) = 0$$

so $$\left.\frac{\partial \rho(r)}{\partial p}\right|_{p=p(r)} = \frac{\rho}{\Gamma p}$$

and stability achieved if $$\frac{\rho(r)\,p'(r)}{\Gamma p(r)} > \rho'(r)$$

$$\frac{p'(r)}{p(r)} - \frac{\Gamma \rho'(r)}{\rho(r)} > 0$$ Schwarzschild discriminant

$$\underline{\underline{\nabla_{ad} = 1 - 1/\Gamma}}$$