8/27/2020

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The fundamental equation of hydrostatic equilibrium is a (fundamental, so no parameters) relationship between the phenomenon of gravity and the pressure gradient.

$$\boxed{\frac{dp(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}} \quad \text{W1.1.4}$$

Let's use it to get the potential energy as a functional of the pressure. Let $\Omega$ be the gravitational potential energy of the star, so $-\Omega$ is the energy required to remove the mass layer by layer

pick layer of thickness dr, move to infinity

At $r_*$, it feels the force of gravity of the rest of the mass and much energy is required to move the layer, but the force decreases with distance and it is <u>zero</u> at infinity.

The force $F_{gravitational} = \underbrace{\frac{GM(r)}{r'^2}}_{\text{rest of the star}} \cdot \underbrace{4\pi r^2 \rho(r)\, dr}_{\text{layer}}$

The energy required to move all the mass in the layer is $F \cdot dr'$, so $GM(r)\, 4\pi r^2 \rho(r)\, dr \int_r^\infty \frac{dr'}{r'^2}$

$$\int_r^\infty r'^{-2} dr' = -r'^{-1}\Big|_r^\infty = -\cancel{\frac{1}{\infty}}^{\,0} + \frac{1}{r} = \frac{1}{r}$$

$$\therefore \Omega_{layer} = GM(r)\, 4\pi r \rho(r) dr$$

to get the total energy, we integrate over all the layers, so 

$$-\Omega = 4\pi \int_0^R G M(r) r \rho(r)\, dr$$

with $-GM(r)\rho(r) = r^2 \dfrac{dp(r)}{dr}$

$$+\Omega = +4\pi \int_0^R r^2 \frac{dp(r)}{dr} r\, dr = +4\pi \int_0^R \frac{dp(r)}{dr} r^3 dr$$

Let $u = r^3 \quad dv = dp(r)$

$du = 3r^2 dr \quad v = p(r)$

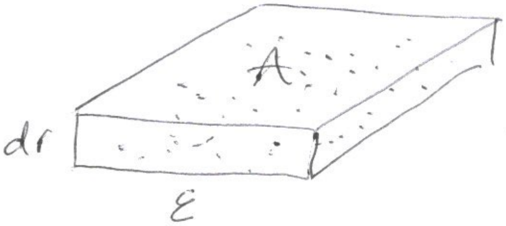
Integrating by parts, $\int u\,dv = uv - \int v\,du$

$$\frac{\Omega}{4\pi} = r^3 p(r) - 3\int_0^R p(r) r^2\, dr$$

The first term on the right vanishes because

$r^3 = 0$ when $r = 0$, and $p(r) = 0$ when $r = R$

$$\Omega = -3\int_0^R p(r)\, 4\pi r^2\, dr$$




Now let's look at the thermal energy, which is associated with motion. If the energy density at a distance $r$ from the center of the sphere is $\mathcal{E}(r)$, then the total thermal energy is $\Upsilon \equiv \int_0^R \mathcal{E}(r)\, 4\pi r^2\, dr$

The total energy is $E = \Upsilon + \Omega = 4\pi \int_0^R \left[\mathcal{E}(r) - 3p(r)\right] r^2\, dr$

In order for the star or object to be stable, $E < 0$

This implies $\mathcal{E}(r) < 3p(r)$

$$p = \frac{F}{A} \Rightarrow F = p \cdot A$$

thin layer

Work done against a pressure is $F \cdot dx = p \cdot A \cdot dx = p \cdot V$

In general $p = -\frac{\partial U}{\partial V}$

$$\frac{W}{V} = p \propto \mathcal{E}$$

Let $\mathcal{E} = \frac{p}{\Gamma - 1}$ where $\frac{1}{\Gamma - 1}$ is the proportionality constant

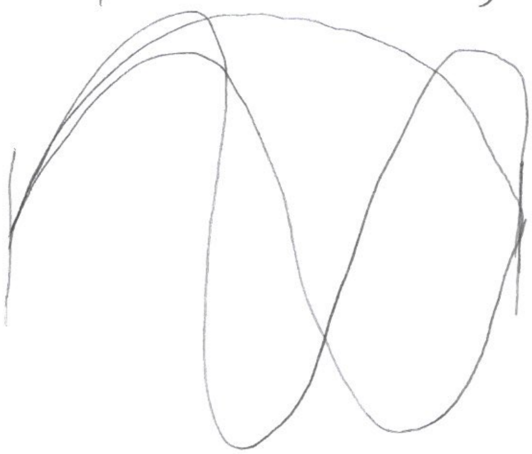
The actual value of $\Gamma$ depends on the phenomenon creating the thermal energy.

Consider the ideal gas, $pV = nRT$ and $U = \frac{3}{2} nRT$

$$p = \frac{nRT}{V} \quad \text{and} \quad \mathcal{E} = \frac{U}{V} = \frac{3}{2}\frac{nRT}{V} \Rightarrow \frac{nRT}{V} = \frac{2}{3}\mathcal{E}$$

so $p = \frac{2}{3}\mathcal{E}$ and $\frac{1}{\Gamma - 1} = \frac{3}{2} \Rightarrow 2 = 3\Gamma - 3 \Rightarrow \Gamma = 5/3$

For pure radiation, $P = \frac{U}{3V} = \frac{\varepsilon}{3}$



etc.

$$U = \sum_j s_j \hbar \omega_j$$

the frequency depends on the wavelenght, given by the box

$$P = -\frac{\partial U}{\partial V} = -\sum_j s_j \hbar \frac{d\omega_j}{dV}$$

$$\frac{d\omega_j}{dV} = -\frac{1}{3}\frac{\omega_j}{V}$$

In this case, $\frac{1}{\Gamma - 1} = \frac{3}{1} \Rightarrow 1 = 3\Gamma - 3 \Rightarrow \boxed{\Gamma = \frac{4}{3}}$

keep it in mind.

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In general, $\varepsilon = \frac{P}{\Gamma - 1}$, so $\Upsilon = \frac{1}{\Gamma - 1}\int_0^R P(r)\, 4\pi r^2 dr$

$-\frac{\Omega}{3} = \int_0^R P(r)\, 4\pi r^2 dr$, so $\Upsilon = \left(\frac{1}{\Gamma-1}\right)\left(-\frac{\Omega}{3}\right) = -\frac{\Omega}{3\Gamma - 3}$

$-\Upsilon = \frac{\Omega}{3\Gamma - 3} \Rightarrow \Omega = -(3\Gamma - 3)\Upsilon$

do some algebra

$E = \Omega + \Upsilon \Rightarrow \Omega = E - \Upsilon = -(3\Gamma - 3)\Upsilon$

$$\frac{E}{3\Gamma-3} - \frac{\Upsilon}{3\Gamma-3} = -\underline{\Upsilon}$$

$$\frac{E}{3\Gamma-3} = \frac{\Upsilon}{3\Gamma-3} - \frac{\Upsilon}{1} = \frac{\Upsilon - \Upsilon(3\Gamma-3)}{\cancel{3\Gamma-3}}$$

$$\Rightarrow E = \Upsilon(1-3\Gamma+3) = \Upsilon(4-3\Gamma)$$

$$\Rightarrow \underline{\Upsilon} = \frac{E}{4-3\Gamma} = -\frac{E}{3\Gamma-4}$$

Also, $\Omega = E - \Upsilon = E + \dfrac{E}{3\Gamma-4} = \dfrac{E(3\Gamma-4)+E}{3\Gamma-4}$

$$\Omega = \frac{E(3\Gamma-4+1)}{3\Gamma-4} = \frac{E(3\Gamma-3)}{3\Gamma-4} = \frac{3E(\Gamma-1)}{3\Gamma-4}$$

$$\Omega = \frac{\frac{1}{\cancel{3}}}{\frac{1}{3}} \cdot \frac{\frac{\cancel{3}E(\Gamma-1)}{1}}{\frac{3\Gamma-4}{1}} = \frac{E(\Gamma-1)}{\Gamma-4/3}$$

For an ideal gas, $\Gamma = 5/3$, so

$$\Upsilon = -\frac{E}{3\Gamma - 4} = -\frac{E}{5-4} = -E$$

$$\Omega = \frac{(\Gamma - 1)E}{\Gamma - 4/3} = \frac{2/3\, E}{1/3} = 2E \qquad E = \frac{\Omega}{2}$$

$$\Upsilon = -\frac{1}{2}\Omega \qquad \text{(virial theorem)}$$

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For pure radiation, $\Gamma = 4/3$, so

$$\Upsilon = -\frac{E}{3\Gamma - 4} = -\frac{E}{4-4} = -\infty$$

$$\Omega = \frac{(\Gamma - 1)E}{\Gamma - 4/3} = \frac{1/3\, E}{1/3(0)} = \infty$$

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In order for the body to be stable, $\Gamma > 4/3$

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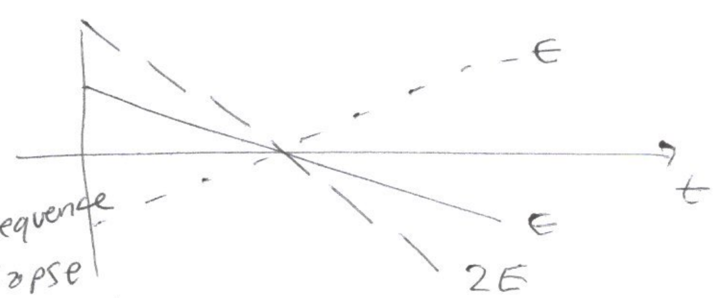
Stability requires that $E < 0 \Rightarrow \Omega < 0$

cloud of gas. If $T > 0\,K$ (which is always the case) then it is losing energy due to thermal radiation




At some point in time, gravity wins and the cloud starts collapsing into stars, planets, etc. You are a consequence of this collapse