Nuclear energy generation

~~Ed~~ Sir Arthur Eddington derived the previous equation without knowing any of the details about opacity or energy generation, in 1920.

In fact fusion, thermonuclear energy and the fact that stars are mainly made of hydrogen WAS NOT KNOWN !

But there was a big problem with the assumption that the energy came from gravitational energy only.

$$\Omega \sim \frac{3}{5}\frac{GM^2}{R} = \frac{\frac{3}{5}\left(6.67\times10^{-11}\ \frac{m^3}{Kg\,s^2}\right)\left(2\times10^{30}\ Kg\right)^2}{7\times10^{8}\ m}$$

$$= 2.3\times10^{41}\ \frac{m^{\cancel{3}}\ Kg^{\cancel{2}}}{\cancel{Kg}\ \cancel{m}\ s^2} = J$$

The luminosity of the sun is $L_\odot = 3.8\times10^{26}\ W$ J/s

$$P = \frac{W}{t} \quad L = \frac{E}{t} \Rightarrow t = \frac{E}{L}$$

If gravity alone is the source of energy, all the energy would be radiated away in $\sim \frac{2\times10^{41}\ J}{4\times10^{26}\ J/s} = 0.5\times10^{15}\ s$

$5\times10^{14}\ s$

$$5\times10^{14}\,\cancel{s}\underbrace{\left(\frac{1\,\cancel{h}}{3600\,\cancel{s}}\right)\left(\frac{1\,\cancel{d}}{24\,\cancel{h}}\right)\left(\frac{1\,yr}{365\,\cancel{d}}\right)}_{3.15\times10^{7}} = 1.6\times10^{7}\,yr$$

so 16 Myr

This was not horrible for estimates of the age of the earth before the discovery of radioactivity, but based on fossils and evolution, biologists thought that at least several ~~ra~~ hundreds of millions of years were necessary to get to current species.

Radioactivity was discovered by Henri ~~Becquerel~~ Becquerel in 1896 in uranium salts. Marie and Pierre Curie discovered Polonium and Radium in 1898. This introduced a few problems. The earth would take more than ~ 20 Myr to cool down to its current temperature if radioactivity (heat) ~~was~~ is considered, and radioactive dating put the oldest rocks at a few billion years old (measurements from about 1915)

So Eddington brought together a few ideas

1. If stars are rotating, their angular velocity should increase as they contract to conserve angular momentum, but this was not really had been observed.

2. $E = mc^2$

3. Francis Aston developed a mass spectrograph with enough resolution to find out that a Helium atom is only 99.3% as massive as 4 Hydrogen atoms

4. You don't need to much hydrogen to get to reasonable values for the lifetimes of stars (he used 5%.)

$$\text{So} \left(\frac{5}{100} M_{\odot}\right)\left(\overset{\%\Delta m}{0.007}\right)\left(3\times10^{8}\,\frac{m}{s}\right)^2$$

$$\frac{5}{100}\left(2\times10^{30}\,kg\right)\left(0.007\right)\left(3\times10^{8}\,\frac{m}{s}\right)^2 \approx 6.3\times10^{43}\,J$$

$$\frac{6\times10^{43}\,J}{4\times10^{26}\,\frac{J}{s}} = 1.5\times10^{17}\,s$$

$$\left(1.5\times10^{17}\,s\right)\left(\frac{1h}{3600s}\right)\left(\frac{1d}{24\,h}\right)\left(\frac{1yr}{365\,d}\right) = 5\times10^{9}\,yr$$

5 Gyr

Now we know that a star like the sun will burn about 10% of its hydrogen, so its lifetime is ~ 10 Gyr.

Consider two nuclei with charges Z_A and Z_B and masses m_A and m_B. They interact through the Coulomb potential $\dfrac{Z_A Z_B e^2}{4\pi\varepsilon_0 r}$ which is repulsive

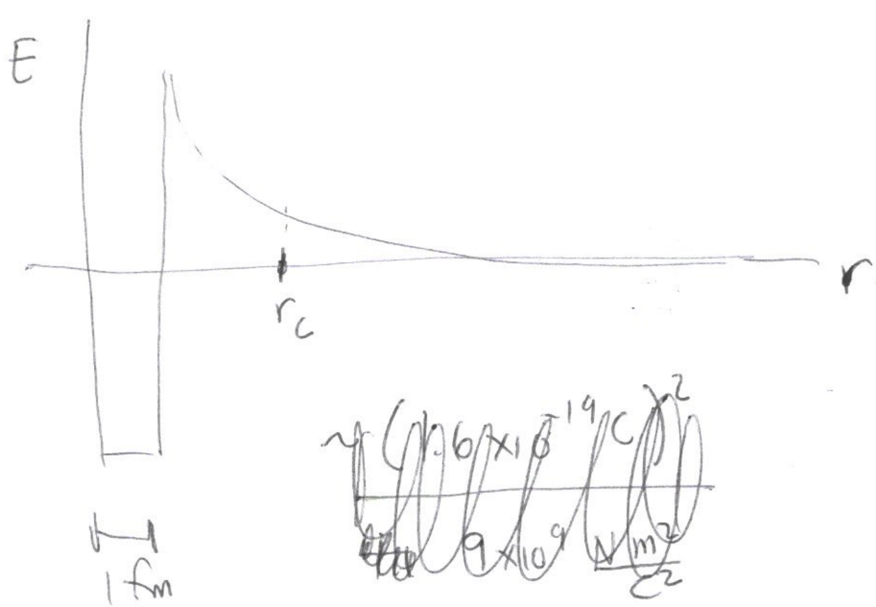
The closest approach r_c depends on the kinetic energy $E = \dfrac{Z_A Z_B e^2}{4\pi\varepsilon_0 r_c}$ $\qquad \dfrac{1}{4\pi\varepsilon_0} = 9\times10^9$

$r_c = \dfrac{Z_A Z_B e^2}{4\pi\varepsilon_0 E}$

At distances $\sim 1\times10^{-15}$ m (a femtometer, sometimes also called a fermi)

the strong nuclear force kicks in and it is attractive. So it looks like this

($100\ \text{fm}^2 = 1\times10^{-28}\ \text{m}^2$ is called a "barn")



The energy required to overcome Coulomb repulsion and achieve fusion is for 2 protons

$\cancel{\sim \dfrac{(1.6\times10^{-19}\,C)^2}{9\times10^{9}\ N m^2/C^2}}$

$$\frac{(1.6\times10^{-19}\ C)^2\left(9\times10^{9}\ Nm^2/C^2\right)}{\left(1\times10^{-15}\ m\right)} = 2.3\times10^{-13}\ J = 1.44\times10^{6}\ eV$$

$1.44\ MeV$

$1\ eV = 1.6\times10^{-19}\ J$

~~so~~ energy required is

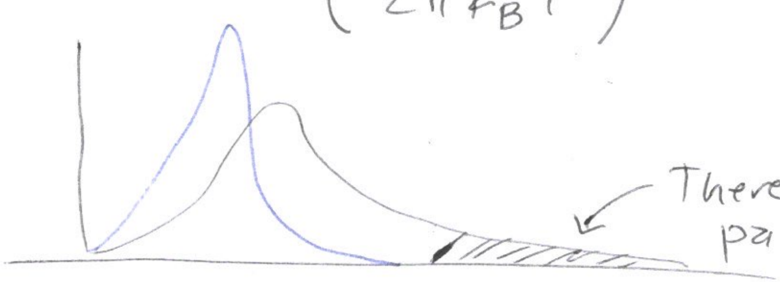
$$E = \frac{3}{2}k_B T \Rightarrow T = \frac{2}{3}E\frac{1}{k_B} = \frac{2}{3}\frac{1.44\times10^{6}\ \cancel{eV}}{8.62\times10^{-5}\ \cancel{eV}/K}$$

$$T = 1.7\times10^{10}\ K$$

Nevertheless, fusion occurs in stars with much lower temperatures $\sim 10^7\ K$. Why? ☆

Also, remember that at any given temperature, the distribution of velocities of particles in thermodynamic equilibrium is given by ~~a~~ Maxwell-Boltzmann

$$f(v)\,d^3v = \left(\frac{m}{2\pi k_B T}\right) e^{-\frac{mv^2}{2k_BT}}\,d^3v$$

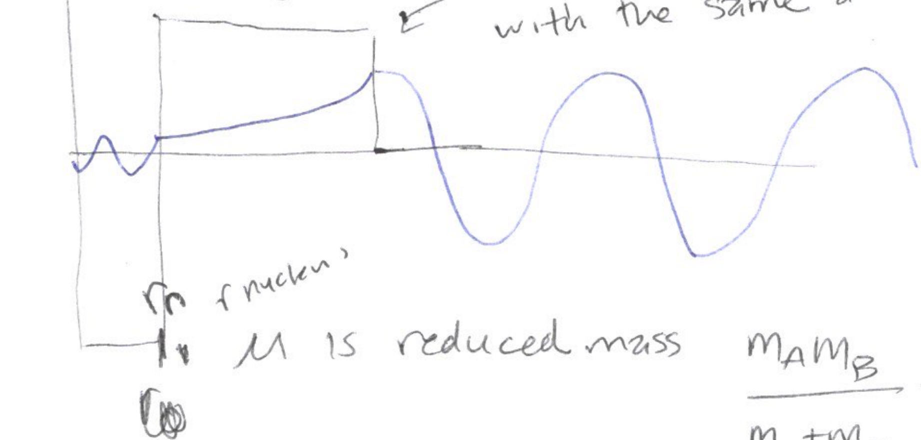


There might a fraction of particles that have enough energy

$$\langle V(r)\rangle = \frac{\int_0^{r_c} 4\pi r^2 V(r)\,dr}{\int_0^{r_c} 4\pi r^2\,dr}$$ spherically symmetric potential

energy barrier with the same area as Coulomb potential up to r_c

$$\left[\frac{-\hbar^2\nabla^2}{2\mu} + V(r)\right]\psi(r) = E\psi(r)$$



μ is reduced mass $\quad \frac{m_A m_B}{m_A + m_B} = \mu$

The solution is $\quad \psi = \frac{A e^{\beta r}}{r} \quad$ with $\beta = \frac{\sqrt{\mu E}}{\hbar}$

The wave function squared is proportional to the probability density for the particle being at position r.

Probability of tunneling:

$$\frac{|\psi(r_n)|^2 \cancel{4\pi} r_n^2 \cancel{dr}}{|\psi(r_c)|^2 \cancel{4\pi} r_c^2 \cancel{dr}} = \frac{|\psi(r_n)|^2 r_n^2}{|\psi(r_c)|^2 r_c^2} = \frac{\cancel{A^2} e^{2\beta r_n}}{\cancel{A^2} e^{2\beta r_c}} \frac{\cancel{r_n^2}\,\cancel{r_c^2}}{\cancel{r_n^2}\,\cancel{r_c^2}}$$

Let $r_n << r_c$, then $e^{2\beta r_n} \to 1$ and

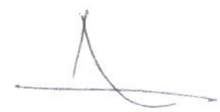
$$\text{Probability of tunneling} \simeq e^{-2\beta r_c} = \exp\left[-\frac{2\sqrt{\mu E}}{\hbar}\frac{Z_A Z_B e^2}{4\pi\varepsilon_0 E}\right]$$

$$= \exp\left[-\frac{2\sqrt{\mu}}{\hbar} Z_A Z_B \frac{e^2}{4\pi\varepsilon_0}\frac{1}{\sqrt{E}}\right]$$

The strenght of the electromagnetic interaction is given by the fine structure constant $\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c}$

$$\frac{e^2}{4\pi\varepsilon_0} = \alpha \hbar c$$

if solving actual potential



so

$$\exp\left[-\frac{\pi}{\sqrt{2}}\,\frac{2\sqrt{\mu}}{\hbar}\, Z_A Z_B\, \alpha \hbar c\, \frac{1}{\sqrt{E}}\right]$$

Let $\pi\sqrt{2\mu}\, Z_A Z_B \alpha c = \sqrt{E_G}$

$E_G = (\pi \alpha Z_A Z_B)^2\, 2\mu c^2$ Gamow Energy

then $g(E) = e^{-\sqrt{E_G/E}}$ Gamow factor

For 2 protons, $E_G = \left(\pi \frac{1}{137}\right)^2 m_p c^2 = 4.94 \times 10^5$ eV

The mass of a proton $m_p = 0.94\ \frac{\mathrm{GeV}}{c^2}$

what is $g(E)$?