SHARPNESS

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

n choose r

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\binom{N}{N_\uparrow} = \frac{N!}{N_\uparrow!\,(N-N_\uparrow)!} = \frac{N!}{N_\uparrow!\,N_\downarrow!}$$

so $$g(N,s) = \frac{N!}{\left(\frac{1}{2}N+s\right)!\left(\frac{1}{2}N-s\right)!}$$ ← Eq. 1.15 remember

Sharpness of the multiplicity function

Let's look at the case $s \ll N$ (small fluctuation)

★ WHAT IS THE LARGEST NUMBER YOU CAN GET THE FACTORIAL OF USING YOUR CALCULATOR?

First, introduce Stirling's approximation

$$\ln N! = \ln\left[N(N-1)(N-2)\cdots(1)\right] = \ln \prod_{m=1}^{N} m = \sum_{m=1}^{N} \ln m$$

$$\sum_{m=1}^{N} \ln m \approx \int_1^N \ln m \, dm = m\ln m - m \Big|_1^N \approx N\ln N - N$$

$$N! = e^{\ln N!} = e^{N\ln N - N} = \frac{e^{N\ln N}}{e^N} = \frac{(e^{\ln N})^N}{e^N} = \frac{N^N}{e^N} = \left(\frac{N}{e}\right)^N$$

A more precise approx. gives $N! = \sqrt{2\pi N}\,(N/e)^N$, see Appendix A in KK

$$g(N,s) = \frac{N!}{\left(\frac{1}{2}N+s\right)!\left(\frac{1}{2}N-s\right)!}$$

use Sterling's

$$g(N,s) = \frac{(2\pi N)^{1/2} N^N e^{-N}}{\left[2\pi\left(\frac{1}{2}N+s\right)\right]^{1/2}\left[\frac{1}{2}N+s\right]^{\frac{1}{2}N+s} e^{-\left(\frac{1}{2}N+s\right)}\left[2\pi\left(\frac{1}{2}N-s\right)\right]^{1/2}\left[\frac{1}{2}N-s\right]^{\frac{1}{2}N-s} e^{-\left(\frac{1}{2}N-s\right)}}$$

with

$$\frac{1}{2}N+s = \frac{1}{2}N\left(1+\frac{2s}{N}\right)$$

$$\frac{1}{2}N-s = \frac{1}{2}N\left(1-\frac{2s}{N}\right),$$

$$g(N,s) = (\cancel{2}\pi N)^{1/2} N^N e^{-N} \cdot \left[\frac{2\pi N}{\cancel{2}}\left(1+\frac{2s}{N}\right)\right]^{-1/2}\left[\frac{1}{2}\cancel{N}\left(1+\frac{2s}{N}\right)\right]^{-\frac{1}{2}N\left(1+\frac{2s}{N}\right)} \cancel{e^{\frac{1}{2}N\left(1+\frac{2s}{N}\right)}}$$

$$\cdot \left[\frac{2\pi N}{\cancel{2}}\left(1-\frac{2s}{N}\right)\right]^{-1/2}\left[\frac{1}{2}\cancel{N}\left(1-\frac{2s}{N}\right)\right]^{-\frac{1}{2}N\left(1-\frac{2s}{N}\right)} \cancel{e^{\frac{1}{2}N\left(1-\frac{2s}{N}\right)}}$$

$$N^{-\frac{1}{2}N\left(1+\frac{2s}{N}\right)-\frac{1}{2}N\left(1-\frac{2s}{N}\right)} \qquad e^{\frac{1}{2}N\left(1+\frac{2s}{N}\right)+\frac{1}{2}N\left(1-\frac{2s}{N}\right)}$$

$$\left(\frac{1}{2}\right)^{-1/2}\left(\frac{1}{2}\right)^{-1/2} = 2^{1/2}2^{1/2} = 2 \qquad N^{-\frac{N}{2}-\cancel{s}-\frac{1}{2}N+\cancel{s}} = \cancel{N^{-N}} \qquad e^{\frac{N}{2}+\cancel{s}+\frac{N}{2}-\cancel{s}} = e^N$$

$$g(N,s) = \cancel{N^{1/2}}\, 2\,(2\pi)^{-1/2}\left[\cancel{N}\left(1+\frac{2s}{N}\right)\right]^{-1/2}\left[\cancel{\frac{1}{2}}\left(1+\frac{2s}{N}\right)\right]^{-\frac{1}{2}N\left(1+\frac{2s}{N}\right)}\left[N\left(1-\frac{2s}{N}\right)\right]^{-1/2}\left[\cancel{\frac{1}{2}}\left(1-\frac{2s}{N}\right)\right]^{-\frac{1}{2}N\left(1-\frac{2s}{N}\right)}$$

$$\left(\frac{1}{2}\right)^{-\frac{1}{2}N\left(1+\frac{2s}{N}\right)} + \left(\frac{1}{2}\right)^{-\frac{1}{2}N\left(1-\frac{2s}{N}\right)}$$

$$\left(\frac{1}{2}\right)^{-\frac{N}{2}-\cancel{s}-\frac{N}{2}+\cancel{s}} = \left(\frac{1}{2}\right)^{-N} = 2^N$$

$$g(N,s) = 2\,(2\pi)^{-1/2}\,N^{-1/2}\,2^{N}\left[1+\frac{2s}{N}\right]^{-1/2}\left[1-\frac{2s}{N}\right]^{-1/2}\left[1+\frac{2s}{N}\right]^{-\frac{1}{2}N\left(1+\frac{2s}{N}\right)} \cdot \left[1-\frac{2s}{N}\right]^{-\frac{1}{2}N\left(1-\frac{2s}{N}\right)} \qquad (14)$$

$$g(N,s) = 2\,(2\pi)^{-1/2}\,N^{-1/2}\,2^{N}\left(1+\frac{2s}{N}\right)^{-\frac{1}{2}N\left(1+\frac{2s}{N}\right)}\left(1-\frac{2s}{N}\right)^{-\frac{1}{2}N\left(1-\frac{2s}{N}\right)}\cancel{\left(1-\frac{4s^2}{N^2}\right)^{-1/2}}\xrightarrow{} 1$$

goes to 1 since $s \ll N$,

$$\ln\left[\left(1+\frac{2s}{N}\right)^{-\frac{1}{2}N\left(1+\frac{2s}{N}\right)}\right] = -\frac{1}{2}N\left(1+\frac{2s}{N}\right)\ln\left(1+\frac{2s}{N}\right)$$

The power expansion of $\ln(1+x)$ for x small is $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$

$$= -\frac{1}{2}N\left(1+\frac{2s}{N}\right)\left(\frac{2s}{N} - \frac{4s^2}{2N^2}\cancel{+\frac{8s^3}{3N^3}} - \ldots\right)$$

$$= -\frac{1}{2}N\left(\frac{2s}{N} - \frac{2s^2}{N^2} + \frac{4s^2}{N^2} - \frac{4s^3}{N^3} + \right)$$

similarly,

$$\ln\left[\left(1-\frac{2s}{N}\right)^{-\frac{1}{2}N\left(1-\frac{2s}{N}\right)}\right] = -\frac{1}{2}N\left(1-\frac{2s}{N}\right)\ln\left(1-\frac{2s}{N}\right)$$

$$= -\frac{1}{2}N\left(1-\frac{2s}{N}\right)\left(-\frac{2s}{N} - \frac{4s^2}{2N^2}\right)$$

$$= -\frac{1}{2}N\left(-\frac{2s}{N} - \frac{2s^2}{N^2} + \frac{4s^2}{N^2} + \frac{4s^3}{N^3}\right)$$

$$= -\frac{1}{2}N\left[\cancel{\frac{2s}{N}} - \cancel{\frac{2s}{N}} + \frac{2s^2}{N^2} + \frac{2s^2}{N^2} - \cancel{\frac{4s^3}{N^3}} + \cancel{\frac{4s^3}{N^3}}\right] = -\frac{4s^2}{2N} = -\frac{2s^2}{N}$$

$$g(N,s) \approx \frac{2}{(2\pi N)^{1/2}} 2^N \exp\left(-\frac{2s^2}{N}\right)$$

$$g(N,s) \approx \left(\frac{2}{\pi N}\right)^{1/2} 2^N \exp\left(-\frac{2s^2}{N}\right)$$

Let $g(N,0) = \left(\frac{2}{\pi N}\right)^{1/2} 2^N$ Eq. 1.15

Then

$$g(N,s) = g(N,0) \exp\left(-\frac{2s^2}{N}\right)$$ Eq. 1.35 KK

This is a Gaussian centered at 0

$$e^{-x^2/2\sigma} = e^{-2s^2/N}$$

$$\frac{-x^2}{2\sigma^2} = -\frac{2s^2}{N}$$

$\boxed{\text{variance is } \frac{N}{4}}$

$$\frac{1}{2\sigma^2} = \frac{2}{N} \Rightarrow N = 4\sigma^2 \qquad \sigma = \left(\frac{N}{4}\right)^{1/2} = \frac{N^{1/2}}{2}$$

if $N = 1\times10^{22}$, then $\sigma = 1\times10^{11}$

The fluctuation is $\frac{10^{11}}{10^{22}} = 10^{-11}$, 10 parts in 1 trillion we know that the exact value of

$$g(N,0) = \frac{N!}{\left(\frac{1}{2}N+0\right)!\left(\frac{1}{2}N-0\right)!}$$

and the approximate is $\left(\frac{2}{\pi N}\right)^{1/2} 2^N \approx g(N,0)$

Let $N = 50$

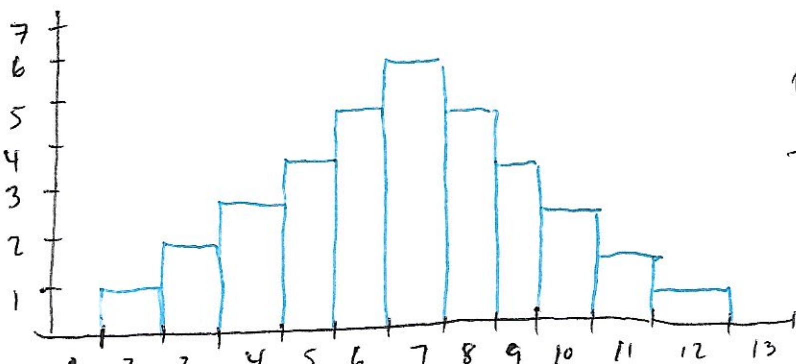
$$g(50,0) = \frac{50!}{25!\,25!} = \frac{3.04\times10^{64}}{1.55\times10^{25}\cdot1.55\times10^{25}} = \frac{3.04\times10^{64}}{2.40\times10^{50}}$$

$g(50,0) = 1.52\times10^{14}$ $\left(1.264\times10^{14}\text{ according to KK}\right)$

perhaps my calculator is not precise enough

$$g(50,0) \approx \left(\frac{2}{50\pi}\right)^{1/2} 2^{50} = 0.1128\cdot1.125\times10^{15} = 1.270\times10^{14}$$

Average values



In the example with 2 dice, we multiplied the sum of the two numbers times the probability of appearance and added all the possible values.

In short form, $\langle s \rangle = \sum_s s\,P(s)$. We also had $\langle s^2 \rangle = \sum_s s^2\,P(s)$. In fact, the mean value of a function $f(s)$ taken over a probability distribution $P(s)$ is $\langle f \rangle = \sum_s f(s)P(s)$

Eq. 1.39 KK

total probability $f(s) = 1$ zeroth moment

In the case of the mean, $f(s) = s$ (where the center is) first moment (17)

the variance $f(s) = s^2$ (how spread apart) second moment

the skewness $f(s) = s^3$ (heavier on left or right side) third moment

the kurtosis $f(s) = s^4$ (tailedness) fourth moment

⋮ these are called moments, you can use more moments of a distribution for more accurate descriptions

Trivia: The arithmetic mean, usually called the 'average' is the sum of a collection of numbers divided by the number of elements, so

$$\frac{1}{N}\sum_s s = \sum_s s \frac{1}{N}$$, so equal to $\langle s \rangle$ iff $P(s) = \frac{1}{N}$, the probability is constant

In physics, a moment is an expression involving the product of a distance and a physical quantity.

Consider mass

zeroth moment $\sum_i r_i^0 m_i = \sum_i m_i$ (total mass)

first moment $\sum_i r_i^1 m_i = \sum_i r_i m_i$ (center of mass) (weighted average of the mass)

second moment $\sum_i r_i^2 m_i$ (moment of inertia) (the greater the "variance" of the mass, the more distributed the mass is)

Let's look at our magnet again, which follows the binomial distribution, (18)

Eq. 1.41 KK

Zeroth moment $\sum_s s^0 P(N,s) = \sum_s P(N,s) = \boxed{\sum_s g(N,s) = 2^N} = 1$

$$\text{so} \quad P(N,s) = \frac{g(N,s)}{2^N} = \left(\frac{2}{\pi N}\right)^{1/2} \frac{\cancel{2^N}}{\cancel{2^N}} \exp\left(-\frac{2s^2}{N}\right)$$

~~First moment~~ $\cancel{\sum_s s P(N,s)}$ The distribution function should be normalized to 1,

$$\text{so} \quad \sum_s P(s) = 1 \qquad \text{Eq. 1.40 KK}$$

Determine normalization factor, $A \sum_s P(s) = 1$

$$A \sum_{2s=-N}^{2s=N} \left(\frac{2}{\pi N}\right)^{1/2} \exp\left(-\frac{2s^2}{N}\right) = 1$$

Approximate summation with integral, since $s \ll N$, change limits to $-\infty, +\infty$

$$A\left(\frac{2}{\pi N}\right)^{1/2} \int_{-\infty}^{\infty} e^{-2s^2/N}\, ds = 1$$

$$\text{Let } x^2 = \frac{2s^2}{N} \Rightarrow x = \sqrt{\frac{2}{N}}\, s$$

$$dx = \sqrt{\frac{2}{N}}\, ds \Rightarrow ds = \sqrt{\frac{N}{2}}\, dx$$

This is equation A.4 in ~~Appendix A~~ of KK, equals $\sqrt{\pi}$

$$\text{rewrite} \quad A\left(\frac{\cancel{2}}{\pi \cancel{N}}\right)^{1/2}\left(\frac{\cancel{N}}{\cancel{2}}\right)^{1/2} \int_{-\infty}^{\infty} e^{-x^2}\, dx = 1$$

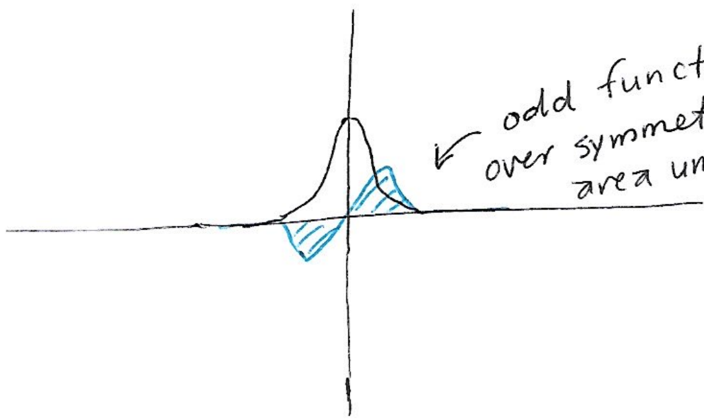
$$A \frac{\cancel{\sqrt{\pi}}}{\cancel{\sqrt{\pi}}} = 1 \Rightarrow A = 1,$$ so $P(N,s)$ is already normalized!

First moment $$\sum_s s P(N,s) = \sum_{2s=-N}^{2s=N} s \left(\frac{2}{\pi N}\right)^{1/2} \exp\left(-\frac{2s^2}{N}\right)$$

Approximate with integral, change limits

$$\langle s \rangle = \left(\frac{2}{\pi N}\right)^{1/2} \int_{-\infty}^{\infty} s\, e^{-2s^2/N}\, ds$$

$$\langle s \rangle = \left(\frac{2}{\pi N}\right)^{1/2} \left(\frac{N}{2}\right)^{2/2} \int_{-\infty}^{\infty} x e^{-x^2}\, dx = 0$$



odd function integrated over symmetric interval, so area under the curve cancels out

Eq. 1.43 KK

$$\boxed{\text{Hence, } \langle s^2 \rangle = \left(\frac{1}{\pi}\right)^{1/2} \left(\frac{N}{2}\right) \frac{\sqrt{\pi}}{2} = \frac{N}{4} \text{ as determined before}}$$

Second moment $$\langle s^2 \rangle = \left(\frac{2}{\pi N}\right)^{1/2} \int_{-\infty}^{\infty} s^2 e^{-2s^2/N}\, ds = \left(\frac{2}{\pi N}\right)^{1/2} \left(\frac{N}{2}\right)^{3/2} \int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx$$

In Eq. A.5 and A.6, $2\int_0^{\infty} x^m e^{-x^2}\, dx = \Gamma(n+1)$, where

$n = (m-1)/2$ and $\Gamma(n)$ is the Gamma function of n. For $m=2$,

even function

$$n = \frac{2-1}{2} = \frac{1}{2}, \text{ so } \int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx = 2\int_0^{\infty} x^2 e^{-x^2}\, dx = \Gamma\left(\tfrac{1}{2}+1\right) = \Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

From table