Chapter 6: Ideal Gas

The ideal gas is a gas of non-interacting atoms in the limit of low concentrations. whether the gas is in this limit or not depends on its distribution function $f(\varepsilon, \tau, \mu)$, where ε is the energy of the orbital.

An orbital is a state of the Schrödinger equation for only 1 particle. We are not going to solve the S.E., we will just assume that these states exist. In an ideal gas, by definition, the particles are non interacting, so if allowed, we can put several particles in an orbital. We will assume that the solution to the S.E. of a system with N particles is the same as N 1-particle solutions.

When we looked at the ideal gas before, we did not make a distinction regarding the type of particle. In fact, there are two kinds of particles in the Universe: bosons and fermions.

- Bosons have integral spin (photons, Higgs boson)
- Fermions have half-integral spin (electrons, quarks)

Consider a system of N non-interacting identical particles. The wave function of the system is

$$\Psi(x_1, x_2, \ldots, x_N) = \psi(x_1)\psi(x_2)\ldots\psi(x_N)$$

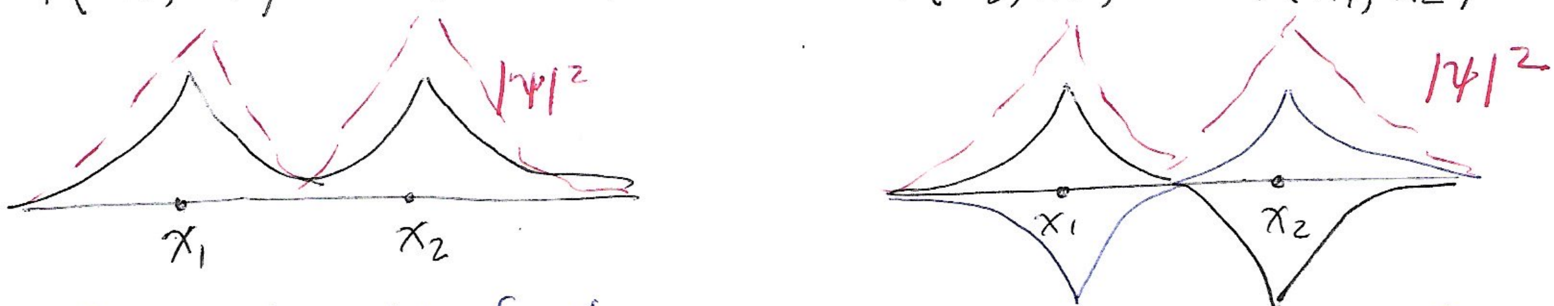
where $\psi(x_N)$ is the wave function of particle N.

Remember that particles are <u>indistinguishable</u>, that means that if we <u>exchange</u> a pair of them, then we should not be able to notice it. What we actually measure is the square of the wavefunction, so

$$|\Psi(x_1, x_2)|^2 = |\Psi(x_2, x_1)|^2$$

But this gives us two options:

$$\Psi(x_2, x_1) = \Psi(x_1, x_2) \qquad\qquad \Psi(x_2, x_1) = -\Psi(x_1, x_2)$$



Symmetric Wavefunction   Anti-symmetric wavefunction

This holds in the general case with N particles when we exchange any pair of indistinguishable particles.

Consider two identical particles 1 and 2 that can occupy two different orbitals A and B.

If particle 1 is in state A and particle 2 is in state B, the wavefunction is given by:

$$\psi_{I} = \psi_A(x_1)\psi_B(x_2)$$

If particle 2 is in state A and particle 1 is in state B, the wavefunction is given by:

$$\psi_{II} = \psi_A(x_2)\psi_B(x_1)$$

There is nothing that makes I or II more likely, so the wavefunction is a superposition of I and II, normalized. So it could be either

$$\psi_{S} = \frac{1}{\sqrt{2}}\left[\psi_A(x_1)\psi_B(x_2) + \psi_A(x_2)\psi_B(x_1)\right]$$

or

$$\psi_{AS} = \frac{1}{\sqrt{2}}\left[\psi_A(x_1)\psi_B(x_2) - \psi_A(x_2)\psi_B(x_1)\right]$$

If we exchange x_1 and x_2,

$$\psi_{S}(x_2, x_1) = \frac{1}{\sqrt{2}}\left[\psi_A(x_2)\psi_B(x_1) + \psi_A(x_1)\psi_B(x_2)\right] = \psi_S(x_1, x_2)$$

Since $\psi_S(x_2, x_1) = \psi(x_1, x_2)$, it is symmetric under exchange

$$\psi_{AS}(x_2,x_1) = \frac{1}{\sqrt{2}}\left[\psi_A(x_2)\psi_B(x_1) - \psi_A(x_1)\psi_B(x_2)\right]$$

$$= -\frac{1}{\sqrt{2}}\left[\psi_A(x_1)\psi_B(x_2) - \psi_A(x_2)\psi_B(x_1)\right] = -\psi_{AS}(x_1,x_2)$$

since $\psi_{AS}(x_2,x_1) = -\psi_{AS}(x_1,x_2)$, it is anti-symmetric under exchange

So this holds even when particles are in different orbitals.

Now let's put both particles in orbital A.

$$\psi_{S} = \frac{1}{\sqrt{2}}\left[\psi_A(x_1)\psi_A(x_2) + \psi_A(x_2)\psi_A(x_1)\right] = 2\psi_{S}(x_1,x_2)$$

$$\psi_{AS} = \frac{1}{\sqrt{2}}\left[\psi_A(x_1)\psi_A(x_2) - \psi_A(x_2)\psi(x_1)\right] = 0$$

In the anti-symmetric case, you destroy the particles if you put them in the same orbital, which is not allowed. This is called the <u>Pauli exclusion principle</u>; discovered by Wolfgang Pauli in 1924. The expectation value of the distance between particles with a wave function that is symmetric under exchange is smaller than if particles were distinguishable, and larger in the case of the anti-symmetric wavefunction. This is called the <u>exchange interaction</u>, was discovered independently by Werner Heisenberg and Paul Dirac in 1926 and is responsible for the <u>volume</u> of matter.

The results of quantum theory as applied to the orbital model of non-interacting particles appear as occupancy rules:

1. An orbital can be occupied by any integral number of <u>bosons</u> of the same species, including zero. (symmetric)
2. An orbital can be occupied by 0 or 1 <u>fermion</u> of the same species. (anti-symmetric) KK pg. 152

Some fermions and bosons with no known substructure

Spin

1/2 Leptons
- charged
  - electron — antielectron
  - muon — anti muon
  - tau — anti tau
- neutral
  - electron neutrino
  - muon neutrino
  - tau neutrino

1/2 Quarks
- up
  - up — anti up
  - charm — anti charm
  - top — anti top
- down
  - down — anti down
  - strange — anti strange
  - bottom — anti bottom

1 Bosons
- photon — carrier of EM force
- gluon — carrier of strong force
- W, Z — carrier of weak force

0 Higgs boson — interaction with Higgs field gives leptons mass

Some fermions with know substructure

Quarks are strongly bound by the strong force. Often they come in triplets, these are called baryons. The most common quarks are "up" and "down."

up quark has spin $1/2$ and charge $2/3\, e^-$

down quark has spin $1/2$ and charge $-1/3\, e^-$

$$\left.\begin{array}{ll} 2\,(\text{up}) + 1\,(\text{down}) \text{ has spin} & 1/2 - 1/2 + 1/2 = 1/2 \\ \text{charge} & 2/3 + 2/3 - 1/3 = 1e^- \end{array}\right\} \text{proton}$$

$$\left.\begin{array}{ll} 1(\text{up}) + 2(\text{down}) \text{ has spin} & 1/2 + 1/2 - 1/2 = 1/2 \\ \text{charge} & 2/3 - 1/3 - 1/3 = 0 \end{array}\right\} \text{neutron}$$

Some bosons with known substructure

Nucleus of deuterium, called deuteron, is 1 proton + 1 neutron

uud udd has spin $1/2 - 1/2 + 1/2 + 1/2 + 1/2 - 1/2 = 1$

Atom of ^4He, 2 protons and 2 neutrons, alpha particle

superfluid at low T.

uud udd
uud udd spin = 0