1/25/21 $\boxed{\text{PROBABILITY}}$ The basis of stat. Mech.

Probability in its simplest case is just a fraction. (5)

Consider 1 fair coin, we will measure the probability of getting a number of heads

Experiment: 1) throw the coin once

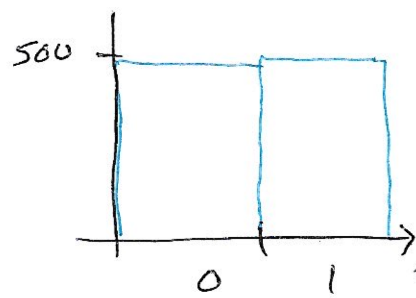
minimum number of heads: 0

maximum " " " : 1

2) repeat step 1) ~~to~~ one thousand times

3) plot the histogram (probability density function)

probability space: H, T



If # of throws is small, there will be variation, but it becomes relatively smaller as the number of throws increases. In stat. mech. we deal with $\sim 10^{23}$ particles.

Now consider 2 fair coins

Exp. 1) Throw both coins once

2) repeat 1000 times

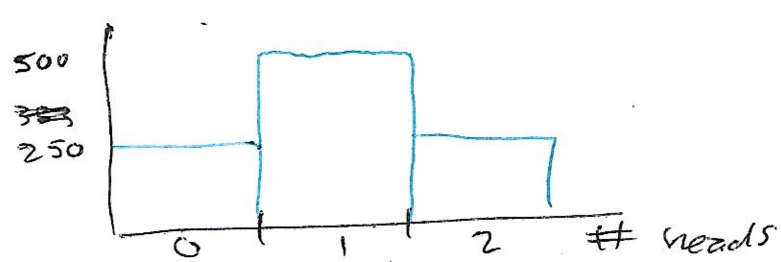
3) plot

min # heads: 0

max # heads: 2

Probability space

HH, HT, TH, TT



For 3 coins

	# heads	
HHH	3	1/8
HHT	2	3/8
HTH	2	
THH	2	
HTT	1	3/8
THT	1	
TTH	1	
TTT	0	1/8

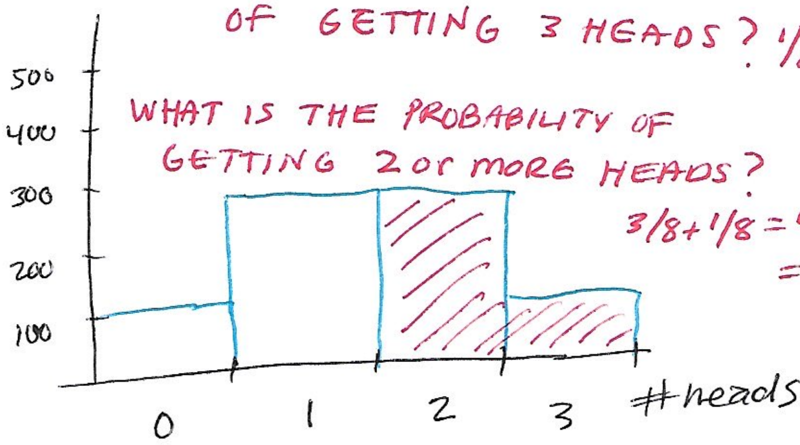
8 possibilities

Repeat 800 times

WHAT IS THE PROBABILITY OF GETTING 3 HEADS? 1/8

WHAT IS THE PROBABILITY OF GETTING 2 or MORE HEADS?

$3/8 + 1/8 = 4/8 = 1/2$



IF NORMALIZED, THE PROBABILITY IS THE AREA UNDER THE CURVE.

☆ WHAT DO YOU EXPECT THE BEHAVIOR OF THE HISTOGRAM TO BE AS THE NUMBER OF COINS INCREASES??

☆ STAT. MECH. IS ALL ABOUT COUNTING, BUT AS THE NUMBER OF PARTICLES INCREASES, IT IS EASIER TO USE PROBABILITY DENSITY FUNCTIONS (PDFs) RATHER THAN HISTOGRAMS.

For 2 dice

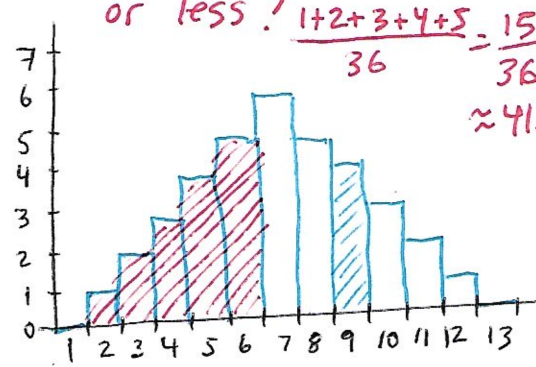
sum

11	21	31	41	51	61
12	22	32	42	52	62
13	23	33	43	53	63
14	24	34	44	54	64
15	25	35	45	55	65
16	26	36	46	56	66

2 3 4 5 6 7 8 9 10 11 12

Sum	Prob.
1	0
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

Prob. of getting 6 points or less? $\frac{1+2+3+4+5}{36} = \frac{15}{36} \approx 41.7\%$



Prob. of getting 9 points? $4/36 \approx 11.1\%$

The fundamental assumption of Stat. Mech.

$$\boxed{\text{A closed system is equally likely to be in any of the quantum states accesible to it. All accessible quantum states are assumed to be equally probable}}$$

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"Quantum" just means that the states are discreete

★ WHAT IS THE "EXPECTATION VALUE" OF THE SUM OF TWO DICE?

MEAN or WEIGHTED AVERAGE

$$\sum p \cdot \text{sum} = (1/36)(2) + (2/36)(3) + (3/36)(4) + (4/36)(5) + (5/36)(6) + (6/36)(7) + (5/36)(8) + (4/36)(9) + (3/36)(10) + (2/36)(11) + (1/36)(12)$$

$$= \frac{2}{36} + \frac{6}{36} + \frac{12}{36} + \frac{20}{36} + \frac{30}{36} + \frac{42}{36} + \frac{40}{36} + \frac{36}{36} + \frac{30}{36} + \frac{22}{36} + \frac{12}{36} = \frac{252}{36} = \underline{\underline{7}}$$

★ WHAT IS THE "EXPECTATION VALUE" OF THE SQUARE OF THE SUM OF TWO DICE?

$$\sum p \cdot \text{sum}^2 = (1/36)(2^2) + (2/36)(3^2) + (3/36)(4^2) + (4/36)(5^2) + (5/36)(6^2) + (6/36)(7^2) + (5/36)(8^2) + (4/36)(9^2) + (3/36)(10^2) + (2/36)(11^2) + (1/36)(12^2)$$

$$\sum p \cdot sum^2 = \frac{4}{36} + \frac{18}{36} + \frac{48}{36} + \frac{100}{36} + \frac{180}{36} + \frac{474}{36} + \frac{320}{36}$$

$$+ \frac{324}{36} + \frac{300}{36} + \frac{242}{36} + \frac{144}{36} = \frac{2154}{36} = 59.83$$

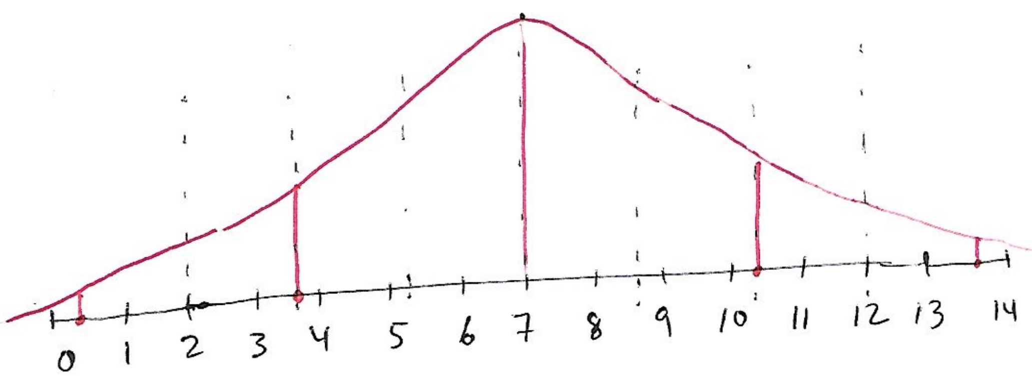
★ WHY DOES THIS MATTER?

$$\text{Variance} = (\text{std. dev})^2 = E[x^2] - E[x]^2$$

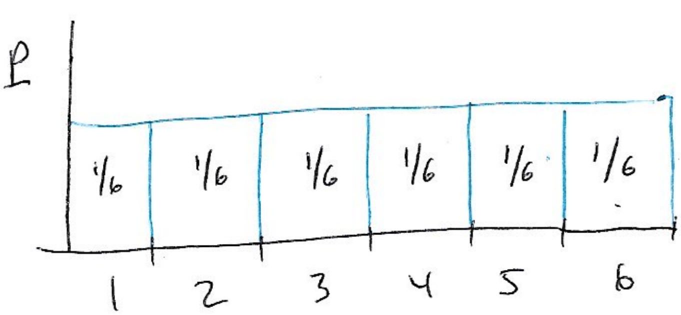
For the 2 dice, the variance is $59.83 - 7^2 = \underline{\underline{10.8}}$

so std. dev $= 3.3$

if you wanted to represent it as a Gaussian, then



★ NOTICE THAT THE PROBABILITY DISTRIBUTIONS OF THE DICE ARE UNIFORM



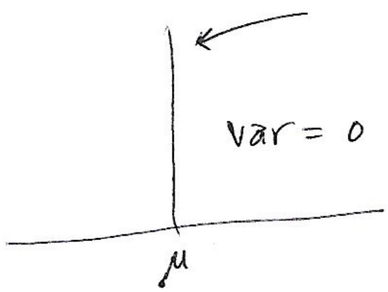
★ WHY DOES THE SUM STARTS TO RESEMBLE A GAUSSIAN??

ANSWER: CENTRAL LIMIT THEOREM

Central limit theorem: when independent random variables are added, their properly normalized sum tends towards a normal distribution even if the original variables themselves are not normally distributed.

As the number of samples n increases, $n \to \infty$, $var = \sigma^2/n$

So in the limit $n \to \infty$, the distribution is a delta function



Remember: Stat. Mech. is a mathematical framework that does not assume any natural laws. It assumes that all states are equally likely (probability) and the CLT.

BINARY MODEL SYSTEMS Ch. 1 pg. 10 KK

$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$
1	2	3	4	5	6	7	8	9	10

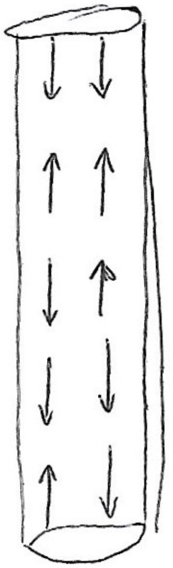
Number of site

$\uparrow = 4$ $\downarrow = 6$ created using a coin

Fig. 1.3

these are magnetic moments $(+m, -m)$ spin up/spin down, but they can also represent on/off, blue/red, etc.

we could say that it is a magnet. There is no



interaction between the spins. Let M be the magnetization, the sum of spin ups and downs. The possible values of the magnetization are:

Eq 1.7 KK

$$M = Nm,\ (N-2)m,\ (N-4)m,\ \ldots,\ -Nm$$

where N is the number of spins and m the scalar value of the magnetic moment of a spin.

In example above, $N = 10$, $M = 6(-m) + 4(m) = -2m = (N-12)m$

if we flip a $\downarrow$ to $\uparrow$, then $M = 5(-m) + 5(m) = 0 = (N-10)m$

Magnetization changes by $\pm 2m$ when a spin flips.

There is only 1 state of the system that has $M = Nm$, $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$ Eq. 1.8 KK

There are N ways to form a state with one magnet down

one such state $\downarrow\uparrow\uparrow\uparrow\cdots\uparrow\uparrow\uparrow\uparrow$ Eq. 1.9

another such state: $\uparrow\downarrow\uparrow\uparrow\cdots\uparrow\uparrow\uparrow\uparrow$ Eq. 1.10

etc.

Is there a better way to enumerate the states??

Assume N is even, then $N_\uparrow = \frac{1}{2}N + s$ and $N_\downarrow = \frac{1}{2}N - s$

where s is an integer. The difference

$$N_\uparrow - N_\downarrow = \frac{1}{2}N + s - \frac{1}{2}N + s = 2s \qquad \text{Eq 1.11}$$

is called the spin excess.

The magnetization $\underline{M}$ is a macroscopic variable

$$M = (N_\uparrow - N_\downarrow)m = (2s)m$$

To find out how many states are there, we use the binomial theorem. If there are

two spins $(\uparrow + \downarrow)^2 = \uparrow\uparrow + 2\uparrow\downarrow + \downarrow\downarrow$ (4)

$2s = 2$, $2s = 0$, $2s = -2$

three spins $(\uparrow + \downarrow)^3 = \uparrow\uparrow\uparrow + 3\uparrow\uparrow\downarrow + 3\uparrow\downarrow\downarrow + \downarrow\downarrow\downarrow$ (8)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

these are the cases we saw before

$2s = -4$
$2s = -2$
$2s = 0$
$2s = 2$
$2s = 4$

$$(\uparrow + \downarrow)^N = 2^N \text{ states}$$

The <u>multiplicity</u> $g(N,s)$ is the number of states (sequences) having the same value s.