Fermi-Dirac and Bose-Einstein

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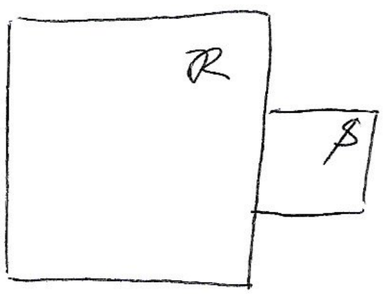


Consider a system $\mathcal{S}$ composed of a single orbital that may be occupied by a <u>fermion</u>. (zero or one)
In this case the reservoir $\mathcal{R}$ consists of N_0 Fermions and enough orbitals.

Fig 6.1

If $\mathcal{S}$ has 0 fermions
0 energy

The $\mathcal{R}$ has N_0 fermions
U_0 energy
$g(N_0, U_0)$ mult.
$\sigma(N_0, U_0)$ entr.

If $\mathcal{S}$ has 1 fermion
ε energy

The $\mathcal{R}$ has $N_0 - 1$ fermions
$U_0 - \varepsilon$ energy
$g(N_0-1, U_0-\varepsilon)$ mult.

$$\sigma(N_0-1, U_0-\varepsilon) = \sigma(N_0, U_0) - \underbrace{\left(\frac{\partial\sigma}{\partial N}\right)_U}_{-\mu/\tau} - \underbrace{\left(\frac{\partial\sigma}{\partial U}\right)_N}_{1/\tau}\varepsilon$$

$$\sigma(N_0-1, U_0-\varepsilon) \doteq \sigma(N_0, U_0) + \frac{\mu - \varepsilon}{\tau}$$

The Gibbs sum, from the definition

KK Eq. 5.61 KK Eq. 5.60

$$\mathcal{Z} = \sum_{ASN} \lambda^N \exp(-\varepsilon_s/\tau) \quad \text{with} \quad \lambda \equiv \exp(\mu/\tau)$$

KK Eq. 6.1

$$\mathcal{Z} = \lambda^0 \exp(0/\tau) + \lambda^1 \exp(-\varepsilon/\tau) = 1 + \lambda \exp(-\varepsilon/\tau)$$

Before we had shown that $\langle N \rangle = \lambda \dfrac{d \ln \mathcal{Z}}{d\lambda}$ KK 5.62

$$\langle N \rangle = \frac{\lambda}{\mathfrak{z}} \frac{d}{d\lambda} \mathfrak{z} = \frac{\lambda}{\mathfrak{z}} \frac{d}{d\lambda} \left[1 + \lambda \exp(-\varepsilon/\tau) \right]$$

$$= \frac{\lambda}{\mathfrak{z}} \left[0 + \exp(-\varepsilon/\tau) \right]$$

Eq. KK 5.69

$$\langle N \rangle = \frac{\lambda \exp(-\varepsilon/\tau) / \lambda \exp(-\varepsilon/\tau)}{1 + \lambda \exp(-\varepsilon/\tau) / \lambda \exp(-\varepsilon/\tau)} = \frac{1}{\lambda^{-1} e^{\varepsilon/\tau} + 1}$$

Since $\lambda \equiv e^{\mu/\tau}$

convention

$$\langle N \rangle = \frac{1}{\dfrac{e^{\varepsilon/\tau}}{e^{\mu/\tau}} + 1} = \frac{1}{e^{\varepsilon/\tau} e^{-\mu/\tau} + 1} = f(\varepsilon)$$

$$\boxed{f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/\tau} + 1}}$$

Fermi-Dirac distribution function

KK Eq. 6.4

In solid state physics, the chemical potential, which usually depends on the temperature $\mu(\tau)$ is called the Fermi level. The value of $\mu(\tau=0)$ is called the

Fermi energy ε_F:

There is another way to write the F-D distribution which sometimes is more useful (with derivatives, etc.). It is in terms of the hyperbolic tangent:

$$\tanh x = \frac{e^{2x}-1}{e^{2x}+1}, \quad \text{so} \quad 1-\tanh x = \frac{1}{1} - \frac{(e^{2x}-1)}{(e^{2x}+1)}$$

$$1-\tanh x = \frac{\cancel{e^{2x}}+1-\cancel{e^{2x}}+1}{e^{2x}+1} = \frac{2}{e^{2x}+1}$$

$$\frac{1}{2}\left(1-\tanh x\right) = \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{e^{2x}+1} = \frac{1}{e^{2x}+1}$$

With $2x = (\varepsilon-\mu)/\tau$, $\Rightarrow$ $x = \dfrac{\varepsilon-\mu}{2\tau}$

$$f(\varepsilon) = \frac{1}{2}\left[1-\tanh\left(\frac{\varepsilon-\mu}{2\tau}\right)\right]$$

Now let's look at the behavior of the F-D distribution

if $\varepsilon = \mu(t)$, $f(\varepsilon=\mu) = \dfrac{1}{e^{0/\tau}+1} = \dfrac{1}{1+1} = \dfrac{1}{2}$

if there is an orbital at the Fermi level, the occupancy of that orbital is exactly one half (thermal average) and this is independent of the temperature.

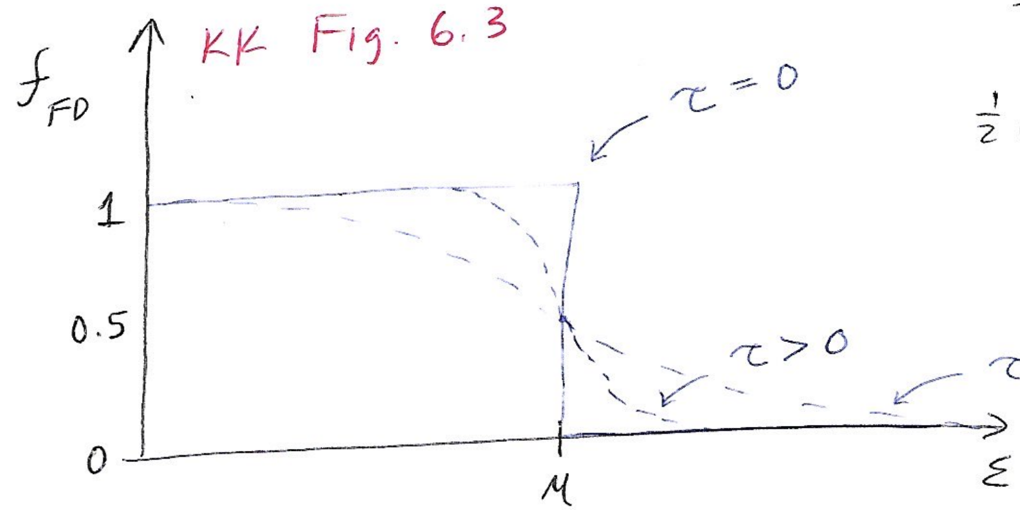
if $\varepsilon < \mu$, $f(\varepsilon<\mu) = \dfrac{1}{e^{-x/\tau}+1} = \dfrac{1}{\frac{1}{e^{x/\tau}}+1}$

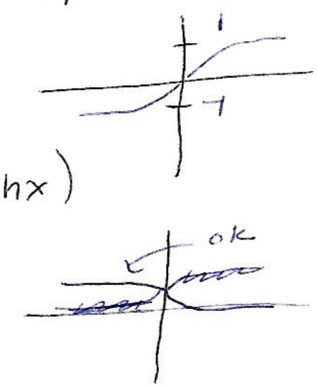
In the low temperature limit $\tau \to 0$ $\quad \dfrac{1}{\frac{1}{e^{\infty}}+1} = 1$

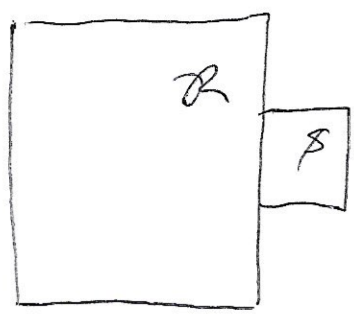
At low temperature, the occupancy below the Fermi ~~level~~ energy is equal to 1 independent of the energy if $\varepsilon > \mu$, $f(\varepsilon > \mu) = \dfrac{1}{e^{x/\tau} + 1}$

In the low temperature limit $\tau \to 0$, $\dfrac{1}{e^{\infty} + 1} = 0$

At low temperature, the occupancy below the Fermi energy is equal to zero independent of the energy







Now consider a system $\mathcal{S}$ composed of a single orbital that may be occupied by a boson. (zero or more)

In this case the reservoir $\mathcal{R}$ consists of N_0 bosons and enough orbitals.

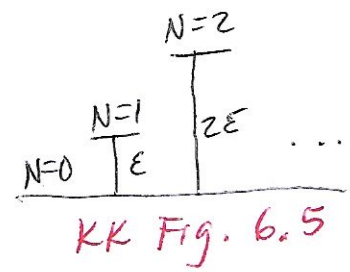
If $\mathcal{S}$ has 1 boson, then energy is ε

2 bosons, 2ε

3 bosons, 3ε

$\vdots$

N bosons, $N\varepsilon$



The Gibbs sum, from the definition

$$\mathcal{Z} = \sum_{ASN} \lambda^N \exp(-\varepsilon_s/\tau)$$

KK Eq. 6.7

since there is only one state, $\mathcal{Z} = \sum_{N=0}^{\infty} \lambda^N \exp(-N\varepsilon/\tau)$
per particle number

$$\mathcal{Z} = \sum_{N=0}^{\infty} \lambda^N e^{N(-\varepsilon/\tau)} = \sum_{N=0}^{\infty} \left[\lambda e^{-\varepsilon/\tau}\right]^N$$ KK Eq. 6.7

We have seen equations like this before, several times.
Look at pg. 60 of my notes.

Let $x = \lambda e^{-\varepsilon/\tau}$, then $\mathcal{Z} = \sum_{N=0}^{\infty} x^N = \dfrac{1}{1-x}$ iff $x<1$

which is a well-know geometric series

$$\mathcal{Z} = \frac{1}{1-\lambda\exp(-\varepsilon/\tau)}$$ KK Eq. 6.8

(but need $\lambda\exp(-\varepsilon/\tau) < 1$)

$$e^{-\varepsilon/\tau} < \lambda^{-1} = e^{-\mu/\tau}$$

$$e^{\mu/\tau} < e^{\varepsilon/\tau}$$

$$\frac{\mu}{\tau} < \frac{\varepsilon}{\tau}$$

$$\mu < \varepsilon$$

the energy of the orbital must be higher than the chemical potential

we know $\langle N \rangle = \lambda \dfrac{d}{d\lambda} \ln \mathcal{Z}$

$$\langle N \rangle = \frac{\lambda}{\mathcal{Z}} \frac{d}{d\lambda} \mathcal{Z}$$

$$\langle N \rangle = \frac{\lambda}{\mathcal{Z}} \frac{d}{d\lambda} \left[\frac{1}{1-\lambda e^{-\varepsilon/\tau}}\right]$$

$$\langle N\rangle = \frac{\lambda}{\mathcal{Z}}\frac{d}{d\lambda}\left(1-\lambda e^{-\varepsilon/\tau}\right)^{-1} = -\left(1-\lambda e^{-\varepsilon/\tau}\right)^{-2}\left(-e^{-\varepsilon/\tau}\right)\frac{\lambda}{\mathcal{Z}}$$

$$\langle N\rangle = \frac{\lambda e^{-\varepsilon/\tau}}{\left(1-\lambda e^{-\varepsilon/\tau}\right)^{\cancel{2}}}\cdot\frac{\cancel{\left(1-\lambda e^{-\varepsilon/\tau}\right)}}{1} = \frac{\lambda e^{-\varepsilon/\tau}/\lambda e^{-\varepsilon/\tau}}{\left(1-\lambda e^{-\varepsilon/\tau}\right)/\lambda e^{-\varepsilon/\tau}}$$

$$\langle N\rangle = \frac{1}{\lambda^{-1}e^{\varepsilon/\tau}-1}$$ KK Eq. 6.9

since $\lambda \equiv e^{\mu/\tau}$

$$\langle N\rangle = \frac{1}{\frac{e^{\varepsilon/\tau}}{e^{\mu/\tau}}-1} = \frac{1}{e^{\varepsilon/\tau}e^{-\mu/\tau}-1} = f(\varepsilon)$$ ← convention

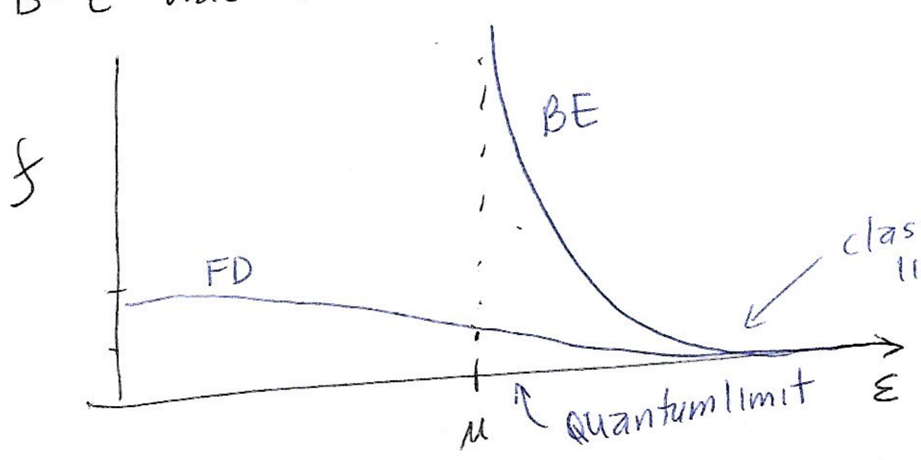
$$\boxed{f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/\tau}-1}}$$

Bose-Einstein distribution function

KK Eq. 6.10

★ The difference~~s~~ between the F-D and B-E distributions is that the F-D has a +1 in the denominator while the B-E has a -1 in the denominator.

KK Fig. 6.6



Number of particles per orbital is close to zero

In the "classical" regime, $f_{FD} \approx f_{BE}$. This is true if

$$e^{(\varepsilon-\mu)/\tau} >> 1 \Rightarrow \frac{\varepsilon-\mu}{\tau} >> 0 \Rightarrow \varepsilon >> \mu$$ KK Eq. 6.12

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/\tau}} = e^{(\mu-\varepsilon)/\tau} = \lambda e^{-\varepsilon/\tau}$$

$$\boxed{f(\varepsilon) = \lambda e^{-\varepsilon/\tau}}$$ Classical distribution function

KK Eq. 6.13 (although this is a quantum result)

Let's study some of the thermal properties of an ideal gas using the classical limit of the F-D and B-E distributions.

Chemical potential

$$N = \sum_s f(\varepsilon_s) = \sum_s \lambda e^{-\varepsilon/\tau} = \lambda \sum_s e^{-\varepsilon/\tau} = \lambda Z_1$$ KK Eq. 6.15

where Z_1 is the partition function for a single free atom

$$Z_1 = n_Q V \quad \text{with } n_Q \equiv \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2}$$ is the quantum concentration

Hence,

$$N = \lambda Z_1 = \lambda n_Q V \Rightarrow \lambda = \frac{N}{n_Q V} = \frac{n}{n_Q}$$ KK Eq. 6.16

$$e^{\mu/\tau} = n/n_Q \Rightarrow \frac{\mu}{\tau} = \ln(n/n_Q) \Rightarrow \boxed{\mu = \tau \ln n/n_Q}$$ KK Eq. 6.1[illegible] KK Eq 5.12a

Before, we got the free energy of the ideal gas, divided by $N!$ to correct for the indistinguishability, and used $\mu = (\partial F/\partial N)_{\tau,V}$. Here it is direct.

Free energy

We know $\left(\frac{\partial F}{\partial N}\right)_{\tau,V} = \mu$, KK Eq. 6.21

so $\int dF(N,\tau,V) = \int_0^N \mu(N,\tau,V)\, dN$

$$F = \int \mu \, dN = \tau\left[\int \ln N \, dN - \int \ln(V n_Q)\, dN\right]$$

$$= \tau\left[N \ln N - N - N\ln(V n_Q)\right]$$

$$= N\tau\left[\ln\left(\frac{N}{V}\frac{1}{n_Q}\right) - 1\right]$$

$$\boxed{F = N\tau\left[\ln(n/n_Q) - 1\right]}$$

KK Eq. 6.24

Pressure

KK Eq. 3.49 and KK Eq. 6.28

we know $p = -\left(\frac{\partial F}{\partial V}\right)_{\tau,N}$

since $F = N\tau\left[\ln N - 1 - \ln V - \ln n_Q\right]$

$$-\left(\frac{\partial F}{\partial V}\right)_{\tau,N} = +N\tau \frac{\partial}{\partial V}\ln V = \frac{N\tau}{V} = p \Rightarrow \boxed{pV = N\tau}$$

KK Eq. 6.29