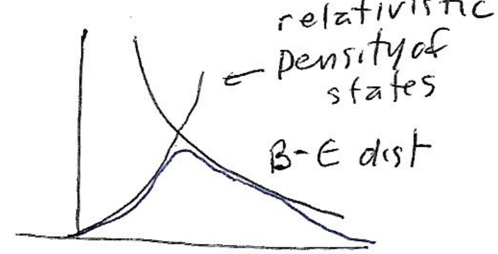
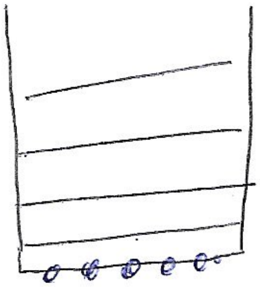
Boson gas and Bose-Einstein Condensation

Satyendra Nath Bose was an Indian mathematician and physicist. He was a polymath and a polyglot. In 1924, at age 30, he derived the Planck radiation law without classical mechanics using a novel, at the time, way of counting indistinguishable particles. His paper was not accepted for publication, so he sent it to A. Einstein.

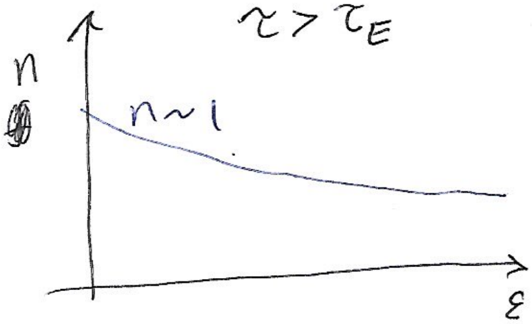
Einstein understood the importance of the result. He personally translated the paper to German and sent it, on Bose's behalf, to the leading physics journal of the time. In a subsequent paper, Einstein used Bose's ideas (now called Bose-Einstein statistics) to predict a new state of matter (now called a Bose-Einstein condensate). This paper was published in 1925 and it was Einstein's last major scientific discovery, age 46.

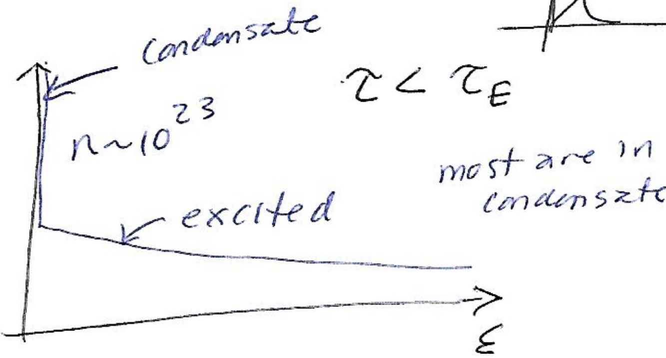
Bose was not awarded a Nobel prize, although 7 Nobel prizes related to Bosons and Bose-Einstein statistics have been awarded, including in 2001 for the observation of Bose-Einstein condensates. Fermi named spin integer particles "bosons" in honor of Bose and the name stuck.





In realistic systems, the energy difference between the ground state (lowest energy state) and first state is about 1×10^{-14} K, but condensates appear at ~ 1 K, so 14 orders of magnitude higher temperature. This is due to the behavior of the chemical potential.





Consider the BE distribution $f(\varepsilon,\mu,\tau) = \dfrac{1}{e^{(\varepsilon-\mu)/\tau}-1}$

Let the energy of the ground orbital be $\varepsilon=0$, $f(0,\tau,\mu) = \dfrac{1}{e^{-\mu/\tau}-1}$

as $\tau\to0$,

with $e^{-x} = 1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \dots$ and $x \ll 1$

$$f(0,\tau\to0,\mu) = \frac{1}{1-\frac{\mu}{\tau}-1} = \frac{1}{-\frac{\mu}{\tau}} = -\tau/\mu = N$$

$$\boxed{N = -\tau/\mu \implies \mu = -\tau/N}$$ KK Eq. 7.55

Another way to look at it: the number of particles in the ground energy level is much greater than the number of particles in excited states, so

$$f(0,\tau,\mu) = \frac{1}{e^{-\mu/\tau}-1} \gg 1$$ (Number of particles in the ground orbital)

$$\Rightarrow e^{-\mu/\tau}-1 \approx 0$$

Chemical potential very close to zero

$$\Rightarrow e^{-\mu/\tau} \approx 1 \Rightarrow -\mu/\tau \approx 0 \Rightarrow \mu \approx 0$$

since $\mu \approx 0$, $e^{-\mu/\tau} \approx 1 - \frac{\mu}{\tau} - 1 = -\mu/\tau$

KK Eq. 7.56

$\lambda \equiv e^{\mu/\tau}$. Since $\mu \approx 0$, $\lambda \approx 1 + \mu/\tau = 1 - \frac{1}{N}$

In a BEC ($\lambda \approx 1$), the number of excited particles, $N_e(\tau)$, function of temperature, divided by the total number of particles is less than 1,

$$\frac{N_e(\tau)}{N} < 1$$

with $N = N_0 + N_e$, $\frac{N_e(\tau)}{N_0(\tau)+N_e(\tau)} < 1$

since $\frac{N_e(\tau)}{N} + \frac{N_0(\tau)}{N} = 1 \Rightarrow \frac{N_0(\tau)}{N} = 1 - \frac{N_e(\tau)}{N}$

Remember that the density of states for an ideal gas of non-interacting particles is $\mathcal{D}(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$ as shown on pg. 127 of my notes. KK Eq. 7.19, 7.65

Also, $N = \int_0^\infty d\varepsilon\, \mathcal{D}(\varepsilon) f(\varepsilon, \tau, \mu)$ KK Eq. 7.20

spinless, or no spin degeneracy

$$N = N_0(\tau) + N_e(\tau) = \int_0^\infty d\varepsilon\, \mathcal{D}(0) f(0, \tau) + \int_0^\infty d\varepsilon\, \mathcal{D}(\varepsilon) f(\varepsilon, \tau)$$

$$N = N_0(\tau) + \int_0^\infty d\varepsilon\, \mathcal{D}(\varepsilon) f(\varepsilon, \tau)$$ KK Eq. 7.66

$$N_e(\tau) = \int_0^\infty d\varepsilon\, \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2} \frac{1}{\lambda^{-1} e^{\varepsilon/\tau} - 1}$$ KK Eq. 7.68

Let $x \equiv \varepsilon/\tau$; $dx = d\varepsilon/\tau$; $d\varepsilon = \tau dx$ $\quad \varepsilon = \tau x$

$$N_e(\tau) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \tau\, dx\, \tau^{1/2} x^{1/2} \frac{1}{\lambda^{-1} e^{x} - 1}$$

$$N_e(\tau) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \tau^{3/2} \int_0^\infty dx\, \frac{x^{1/2}}{\lambda^{-1} e^{x} - 1}$$

Recall (my notes page 70) that the Riemann Zeta function $\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$ for $Re(s) > 1$, so if $\lambda \approx 1$,

$\int_0^\infty \frac{x^{1/2}}{\lambda^{-1} e^{x} - 1} dx \Rightarrow s = 3/2$, so $\int_0^\infty \frac{x^{1/2}}{\lambda^{-1} e^{x} - 1} dx = \zeta(3/2)\Gamma(3/2)$

where Γ is the Gamma function. From wikipedia,

$$\zeta(3/2) = 2.612\ 375\ 348\ \ldots$$

$$\Gamma(3/2) = \sqrt{\pi}/2$$

so

$$N_e(\tau) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \tau^{3/2}\ \frac{2.612\sqrt{\pi}}{2}$$

$$n_Q = \left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2}$$

$$N_e(\tau) = \frac{V}{2^2 \cdot 2\ \pi^{3/2}}\left(\frac{2m\tau}{\hbar^2}\right)^{3/2} \cdot 2.612$$

our friend the quantum concentration

$$N_e(\tau) = V\left(\frac{\cancel{2}m\tau}{2^{\cancel{2}}\pi\hbar^2}\right) 2.612 = 2.612\, n_Q V$$ KK Eq. 7.70

The fraction of particles in excited orbitals is

$$\frac{N_e(\tau)}{N} \simeq 2.612\frac{n_Q V}{N} = 2.612 n_Q/n$$ KK Eq. 7.71

If all the particles are in an excited state (so not a BEC), then $N_e(\tau_E) = N$. Define τ_E as the temperature for which all the particles are in an excited state. This temperature is

$$N = 2.612\left(\frac{m\tau_E}{2\pi\hbar^2}\right)^{3/2} V \Rightarrow \left(\frac{N}{2.612V}\right)^{2/3} = \frac{m\tau_E}{2\pi\hbar^2}$$

$$\boxed{\tau_E \equiv \frac{2\pi\hbar^2}{m}\left(\frac{N}{2.612V}\right)^{2/3}}$$ KK Eq. 7.72

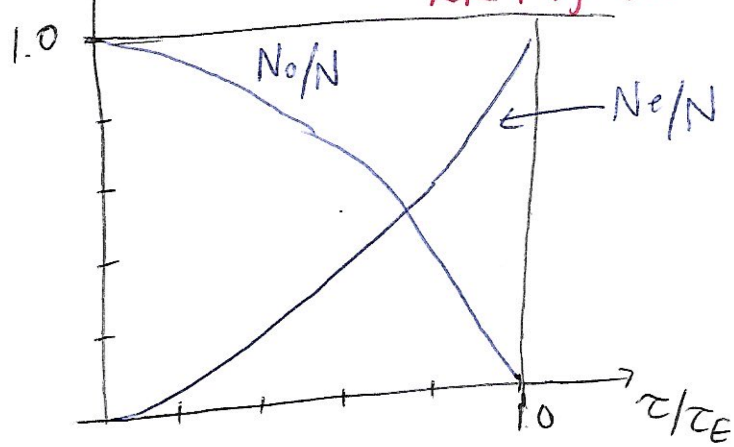
so matter will form a BEC if the temperature $\tau < \tau_E$

$$\frac{N_e(\tau)}{N} = \frac{N_e(\tau)}{N_e(\tau_E)} = \left(\tau/\tau_E\right)^{3/2}$$ KK Eq. 7.73

KK Eq. 7.74

$$\frac{N_0}{N} = \frac{N-N_e}{N} = 1-\left(\tau/\tau_E\right)^{3/2} \Rightarrow N_0 = N\left[1-\left(\frac{\tau}{\tau_E}\right)^{3/2}\right]$$

KK Fig. 7.11



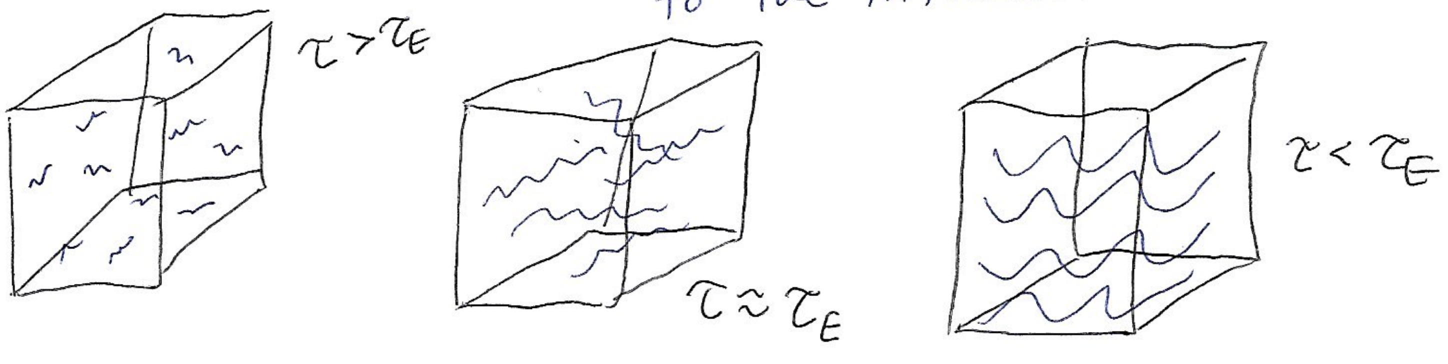
Remember that $\varepsilon = \frac{\hbar^2 k^2}{2m}$, where k is the wave vector

$$\tau_E = \varepsilon_E = \frac{\hbar^2 k_E^2}{2m} \Rightarrow k_E^2 = \frac{2m}{\hbar^2}\tau_E = \frac{2\cancel{m}}{\cancel{\hbar^2}}\cdot\frac{2\pi\cancel{\hbar^2}}{\cancel{m}}\left(\frac{N}{2.612V}\right)^{2/3}$$

$$k_E^{2/2} = (4\pi)^{1/2}\left(\frac{N}{2.612V}\right)^{2/3\cdot 2} \Rightarrow k_E = \frac{\cancel{2}\pi}{\lambda_E} = \cancel{2}\pi^{1/2}\left(\frac{N}{2.612V}\right)^{1/3}$$

$$\lambda_E = \frac{\pi}{\pi^{1/2}}\left(\frac{2.612}{N}\right)^{1/3} L$$

BEC occur when the thermal de Broglie wavelenght is comparable to the interatomic distance



In order to obtain a high τ_E, a small mass is desirable. Consider the lightest atomic boson, ^4He, with a molar volume $V = 27.6 \times 10^{-6}\ m^3/mol$ and a mass of 2 protons + 2 neutrons $= 2(1.672 \times 10^{-27}\ kg) + 2(1.674 \times 10^{-27}\ kg)$ 140

$m = 6.69 \times 10^{-27}\ kg$

$$\tau_E = \frac{2\pi\left(1.054\times10^{-34}\ J\cdot s\right)^2}{6.69\times10^{-27}\ kg}\left(\frac{1\ \cancel{mol}\ \left(6.022\times10^{23}/\cancel{mol}\right)}{\left(27.6\times10^{-6}\ m^3/\cancel{mol}\right)\left(2.612\right)}\right)^{2/3}$$

$$\tau_E = \frac{6.98\times10^{-68}\ \frac{kg^2\, m^4}{s^2}}{6.69\times10^{-27}\ \cancel{kg}}\left(\frac{8.35\times10^{27}}{m^3}\right)^{2/3}$$

$$\tau_E = \left(1.04\times10^{-41}\ J\right)\left(4.12\times10^{18}\right) = 4.29\times10^{-23}\ J$$

$$\tau_E = k_B T_E \Rightarrow T_E = \frac{\tau_E}{k_B} = \frac{4.29\times10^{-23}\ J}{1.38\times10^{-23}\ J/K} = 3.11\ K$$

The measured phase transition is at $T = 2.17\,K$, it is lower because atoms do have some (weak) interactions.