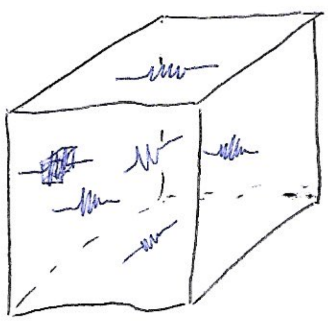
Chapter 7: Fermi and Bose Gases

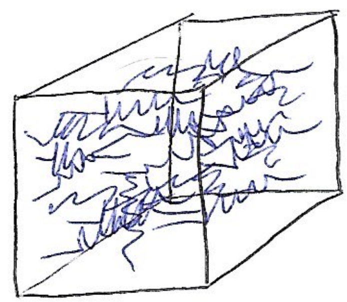
It is a fundamental result of quantum physics (symmetric and anti-symmetric wavefunctions) that all particles are either fermions or bosons. They behave alike in the classical regime, but not in the quantum regime.

The gas is in the quantum regime when its concentration is greater than its quantum concentration, $n \geq n_Q$. When this is the case, it is called a <u>quantum gas.</u>



classical

concentration is low so the features of the wavefunctions are irrelevant, quantum interactions can be safely ignored



$$n \geq n_Q \equiv \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2}$$

quantum

concentration is so high that wave functions start to overlap with each other, so quantum interactions can't be ignored.

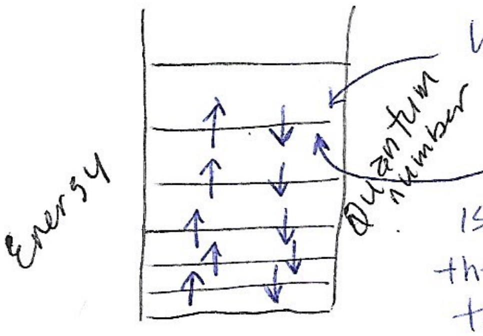
A <u>Fermi gas</u> has high kinetic energy and low heat capacity because particles occupy different orbitals that are dense. A <u>Bose gas</u> has high concentration of particles in the ground orbital because they can all occupy the same orbital.

For many systems, the concentration is fixed and the temperature is the important parameter $\tau_0 \equiv \left(\frac{2\pi\hbar^2}{M}\right) n^{2/3}$. A gas in the quantum regime with $\tau << \tau_0$ is called a <u>degenerate gas</u>.

04/05/21

Fermi gas

Since fermions can't occupy the same quantum state, the lowest energy orbitals will be occupied in the ground state (at 0 kelvin), but this means the kinetic energy of the ground state is high



high kinetic energy

Fermi energy is the energy of the highest state that is occupied

When $\tau \ll \varepsilon_F$, the states below ε_F are almost entirely occupied and states above are almost entirely unoccupied

★ Ground state of Fermi gas in 3-D

Consider the wavefunctions of the particle in a box

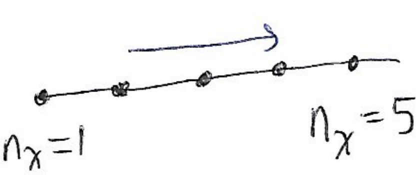
KK Eq. 3.58

$$\psi(x,y,z) = A\sin(n_x \pi x/L)\sin(n_y \pi y/L)\sin(n_z \pi z/L)$$

which has energy values KK Eq. 3.59

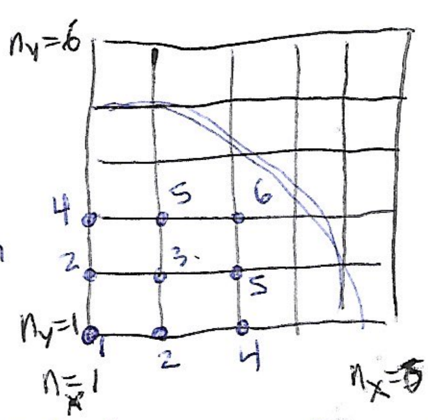
$$\varepsilon_n = \frac{\hbar^2}{2M}\left(\frac{\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2)$$

In 1-D



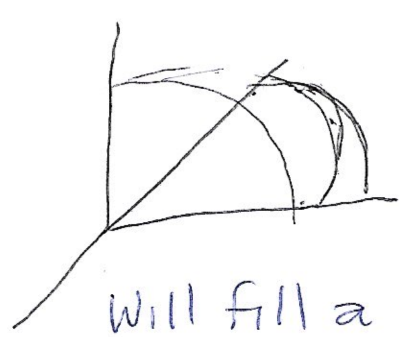
Fills in this direction

In 2-D



Fills the area of a circle

In 3-D



Will fill a sphere

In 3-D, the Fermi energy is $\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi n_F}{L}\right)^2$ where n_F is the radius of the sphere. KK Eq. 7.5

Since the system is in its ground state, we can calculate the radius of the Fermi sphere for a given number of particles N: KK Eq. 7.6

$$\left(\frac{4}{3}\pi n_F^3\right)\underbrace{\left(\frac{1}{8}\right)}_{\text{1 octant}}\underbrace{(2)}_{\text{spin up/spin down}} = \frac{\pi}{3}n_F^3 = N \Rightarrow n_F = \left(\frac{3N}{\pi}\right)^{1/3}$$

so
$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2\left(\frac{3N}{\pi}\right)^{2/3} = \frac{\hbar^2}{2m}\left(\frac{\pi^{2\cdot 3/2}\cdot 3N}{L^{2\cdot 3/2}\,\pi}\right)^{2/3}$$

$$= \frac{\hbar^2}{2m}\left(\frac{\pi^{\cancel{3}\,2}\,3N}{\underbrace{L^3}_{V}\,\cancel{\pi}}\right)^{2/3} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

$$\varepsilon_F = \frac{\hbar^2}{2m}(3\pi^2 \underbrace{n}_{\text{concentration}})^{2/3} \equiv \underbrace{\tau_F}_{\text{Fermi temperature}}$$ KK Eq. 7.7

We can also calculate the ground state energy

$$U_0 = 2\underbrace{\sum_{n\le n_F}\varepsilon_n}_{} = 2\left(\frac{1}{8}\right)4\pi\int_0^{n_F} dn\, n^2\,\varepsilon_n$$

sum of the energies of the states below the Fermi radius

4π comes from converting to spherical coordinates and integrating θ and ϕ, we have done it before

The notation is confusing here, n is the radius of the sphere, not the concentration

$$U_0 = \pi \int_0^{n_F} dn\, n^2 \frac{\hbar^2}{2m}\left(\frac{\pi n}{L}\right)^2$$

$$U_0 = \frac{\pi^3}{2m}\left(\frac{\hbar}{L}\right)^2 \int_0^{n_F} dn\, n^4$$

KK Eq. 7.8

$$\frac{3N}{\pi} = n_F^3$$

$$U_0 = \frac{\pi^3}{2m}\left(\frac{\hbar}{L}\right)^2 \frac{n_F^5}{5} = \frac{\pi^3}{10m}\left(\frac{\hbar}{L}\right)^2 n_F^2\, n_F^3$$

$$U_0 = \frac{\pi^{\cancel{3}\,2}}{10m}\left(\frac{\hbar n_F}{L}\right)^2 \frac{3N}{\cancel{\pi}} = \frac{3\hbar^2}{10m}\left(\frac{\pi n_F}{L}\right)^2 N$$

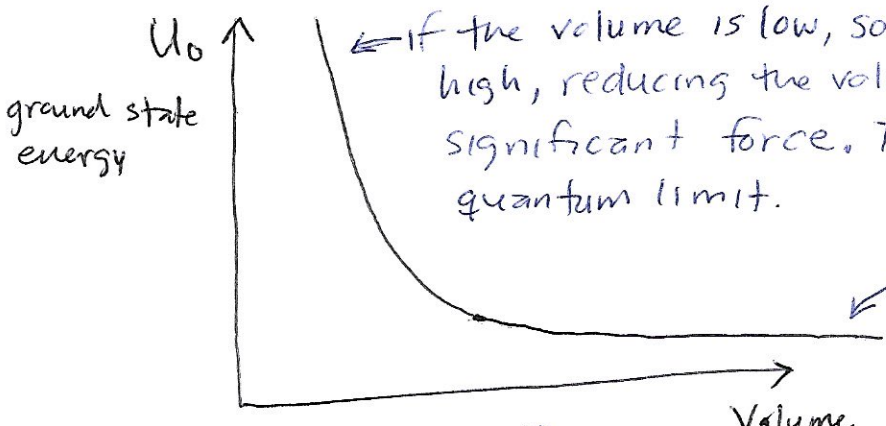
$$U_0 = \frac{3\hbar^2}{5\cdot 2m}\left(\frac{\pi}{L}\right)^2\left(\frac{3N}{\pi}\right)^{2/3} N = \frac{3}{5}\,\frac{\hbar^2}{2m}\left(\frac{\pi^{\cancel{3}\,2}\, 3N}{L^3 \cancel{\pi}}\right)^{2/3} N$$

$$U_0 = \frac{3}{5} N \underbrace{\frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}}_{\text{Fermi energy}} = \frac{3}{5} N \varepsilon_F$$

KK Eq. 7.9

Notice that if N is constant, then $U_0(V) \propto V^{-2/3}$

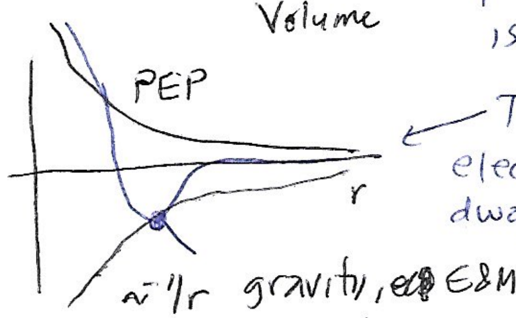
$$F = -\frac{dU}{dx}$$



← if the volume is low, so concentration is high, reducing the volume produces a significant force. This is the quantum limit.

← if the volume is large, so concentration is low, reducing the volume produces almost no force. This is the classical limit.

KK Fig. 7.2



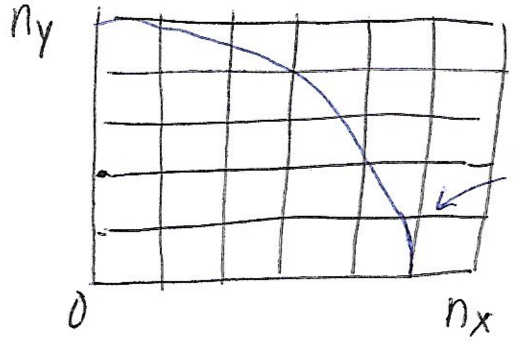
← The volume occupied by the electrons in a metal or white dwarf minimizes the sum of the Fermionic repulsion and attraction

Density of states

KK Eq. 7.11

Thermal averages have the form $\langle X \rangle = \sum_n f(\varepsilon_n, \tau, \mu) X_n$,

but the sum is somewhat cumbersome. In 2-D



states with the same energy will have the same value for the desired ~~many states might have the~~ property, but there might be more than one state with the same energy (orbital)

If we want to compute the thermal average ~~as~~ using an integral over the orbital (state) energy ε, then we need an additional function to tell us how many states or orbital exist at that energy. This function is called the Density of States $\mathcal{D}(\varepsilon)$. This gives us the rule

$$\sum_n (\cdots) \longrightarrow \int d\varepsilon\, \mathcal{D}(\varepsilon)(\cdots) \quad , \text{ so}$$

$$\langle X \rangle = \int d\varepsilon\, \mathcal{D}(\varepsilon) f(\varepsilon, \tau, \mu)\, X(\varepsilon) \qquad \text{KK Eq. 7.12}$$

We just saw that for the ideal degenerate Fermi gas in 3-D

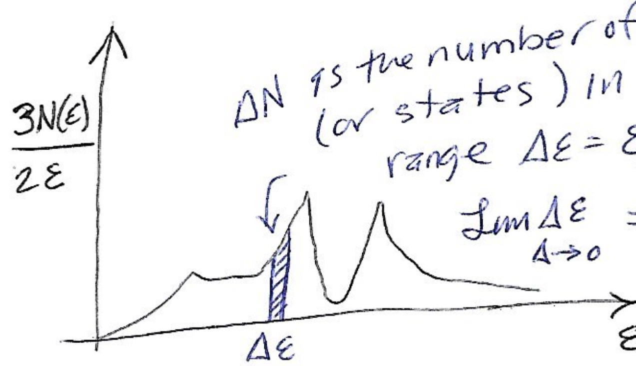
$$\varepsilon = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \Rightarrow \varepsilon^{3/2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{V}{3\pi^2} = N \qquad \text{KK Eq. 7.14}$$

$$\ln N = \ln \varepsilon^{3/2} + \ln\left[\left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{V}{3\pi^2}\right] = \frac{3}{2}\ln \varepsilon + \text{constant} \qquad \text{KK Eq. 7.15}$$

$$\cancel{\frac{dN}{N}} \quad d\ln N = \frac{1}{N}\,dN \ ; \quad d\ln\varepsilon = \frac{1}{\varepsilon}\,d\varepsilon$$

so $$\frac{dN}{N} = \frac{3}{2}\frac{d\varepsilon}{\varepsilon} \Rightarrow \frac{dN}{d\varepsilon} = \frac{3N(\varepsilon)}{2\varepsilon} = \mathcal{D}(\varepsilon)$$ KK Eq. 7.17

$$dN \approx \Delta N = \frac{3N(\varepsilon)}{2\varepsilon}\,\Delta\varepsilon$$



For the Fermi gas,

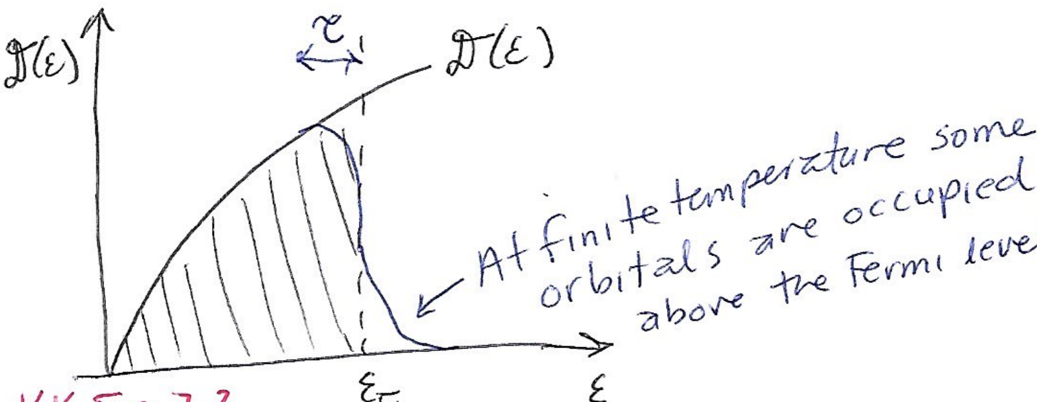
$$\mathcal{D}(\varepsilon) = \frac{\cancel{3}}{2}\,\frac{1}{\cancel{\varepsilon}}\,\varepsilon^{\cancel{3/2}\,1/2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{V}{\cancel{3}\pi^2}$$

$$\mathcal{D}(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}$$ KK Eq. 7.19

Now we can express the number of particles and kinetic energy as integrals over the energy instead of Fermi radius

$$N = \int_0^\infty d\varepsilon\,\mathcal{D}(\varepsilon)\,f(\varepsilon,\tau,\mu) = \int_0^{\varepsilon_F} d\varepsilon\,\mathcal{D}(\varepsilon)$$ KK Eq. 7.22

$$U_0 = \int_0^\infty d\varepsilon\,\varepsilon\,\mathcal{D}(\varepsilon)\,f(\varepsilon,\tau,\mu) = \int_0^{\varepsilon_F} d\varepsilon\,\varepsilon\,\mathcal{D}(\varepsilon)$$ KK Eq. 7.23



KK Fig. 7.3