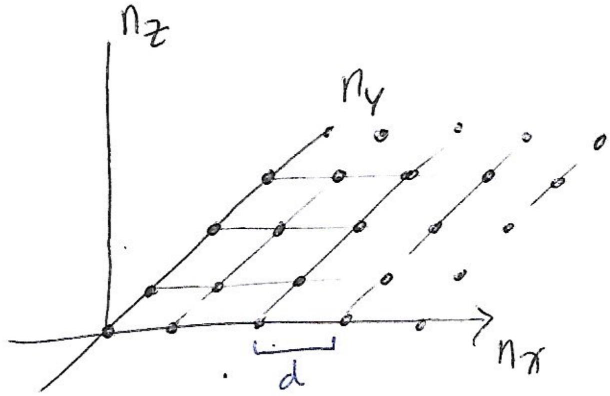
This is really a triple sum

$$U = 2 \sum_{n_x} \sum_{n_y} \sum_{n_z} \frac{\hbar \omega_n}{\exp(\hbar\omega_n/\tau) - 1}$$

$$\omega_n = \frac{c\pi}{L} \underbrace{\left(n_x^2 + n_y^2 + n_z^2\right)^{1/2}}$$

This is a vector that can take discrete values



using $k_n = \omega_n / c$,

$$k_n = \frac{\pi}{L}\left[k_x \hat{i} + k_y \hat{j} + k_z \hat{k}\right]$$

As $L \to \infty$, $\frac{d}{L} \to 0$, so we can replace the sums by integrals. This approach is used in quantum physics and we will use several times in this class.

$$\frac{L}{\pi} k_n = k_x \hat{i} + k_y \hat{j} + k_z \hat{k}$$

$$\Rightarrow \frac{L}{\pi} dk_n = dk_x \hat{i} + dk_y \hat{j} + dk_z \hat{k}$$

$$U = 2 \int_0^\infty \left(\frac{L}{\pi}\right) dk_x \int_0^\infty \left(\frac{L}{\pi}\right) dk_y \int_0^\infty \left(\frac{L}{\pi}\right) dk_z \quad \langle \varepsilon_n \rangle$$

$$U = 2 \left(\frac{L}{\pi}\right)^3 \left(\frac{1}{2}\right) \int_{-\infty}^{\infty} dk_x \left(\frac{1}{2}\right) \int_{-\infty}^{\infty} dk_y \left(\frac{1}{2}\right) \int_{-\infty}^{\infty} dk_z \quad \langle \varepsilon_n \rangle$$

$$U = 2\left(\frac{L}{\pi}\right)^3\left(\frac{1}{8}\right) \underbrace{\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z}\ \langle \varepsilon_n \rangle$$

This is an integral over all space in cartesian coordinates. We can write it in spherical coordinates

$$U = 2\left(\frac{L}{\pi}\right)^3\left(\frac{1}{8}\right) \int_0^{\infty} k^2 dk \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi \quad \langle \varepsilon_n \rangle$$

$$-\cos\theta \Big|_0^{\pi} \qquad \varphi \Big|_0^{2\pi}$$

$$-(-1)+(+1) \rightarrow 2 \cdot 2\pi = 4\pi$$

$$U = \cancel{2}\left(\frac{L}{\pi}\right)^3\left(\frac{1}{\cancel{8}}\right) \cancel{4}\pi \int_0^{\infty} k^2 dk \ \langle \varepsilon_n \rangle$$ ≈ KK Eq. 4.17

$$U = \frac{L^3}{\pi^2} \int_0^{\infty} dk\, k^2 \frac{\hbar c \overset{\omega_n}{k}}{\exp\left(\frac{\hbar c k}{\tau}\right) - 1}$$ ≈ KK Eq. 4.18

Let $x \equiv \hbar c k / \tau$, then $k = \dfrac{\tau x}{\hbar c}$

$$dx = dk\,\frac{\hbar c}{\tau} \Rightarrow dk = \frac{\tau\, dx}{\hbar c}$$

$$U = \frac{L^3}{\pi^2} \int_0^{\infty} \frac{\tau}{\hbar c} dx \frac{\tau^2 x^2}{(\hbar c)^2} \frac{\cancel{\hbar c}\, \frac{\tau x}{\cancel{\hbar c}}}{e^x - 1}$$

$$U = \frac{L^3}{\pi^2} \frac{\tau^4}{(\hbar c)^3} \int_0^\infty dx \, \frac{x^3}{e^x - 1}$$ KK Eq. 4.19

The Riemann zeta function can be expressed as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \quad \text{for} \quad \mathrm{Re}(s) > 1, \text{ so}$$

$$\int_0^\infty \frac{x^3}{e^x - 1} dx \Rightarrow s = 4, \text{ so } \int_0^\infty \frac{x^3}{e^x - 1} dx = \zeta(4)\Gamma(4),$$

where Γ is the gamma function.

$$\zeta(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = \frac{\pi^4}{90}$$

$$\Gamma(4) = 3! = 6, \text{ so } \int_0^\infty dx \, \frac{x^3}{e^x - 1} = \frac{\pi^4 \, 6}{90} = \frac{\pi^4}{15}$$

$$U = \frac{L^3}{\cancel{\pi^2}} \frac{\tau^4}{(\hbar c)^3} \frac{\pi^{\cancel{4}\,2}}{15} = \frac{V \pi^2 \tau^4}{15 (\hbar c)^3}$$

$$\Rightarrow \boxed{\frac{U}{V} = \frac{\pi^2}{15 \hbar^3 c^3} \tau^4}$$ KK Eq. 4.20 !

Stefan–Boltzmann law of radiation

Radiant energy density is proportional to the fourth power of the temperature.

$$U = \frac{L^3}{\pi^2}\int_0^\infty dk\, k^2 \frac{\hbar c k}{\exp\left(\frac{\hbar c k}{\tau}\right)-1}$$ KK Eq. 4.18

Since $k = \frac{\omega}{c}$, $dk = \frac{d\omega}{c}$

$$U = \frac{L^3}{\pi^2}\int_0^\infty \frac{d\omega}{c}\frac{\omega^2}{c^2}\frac{\hbar \cancel{c}\, \omega/\cancel{c}}{\exp\left(\frac{\hbar \cancel{c}\, \omega/\cancel{c}}{\tau}\right)-1}$$

KK Eq. 4.21

$$\frac{U}{L^3} = \frac{U}{V} = \frac{\hbar}{\pi^2 c^3}\int_0^\infty d\omega \frac{\omega^3}{\exp(\hbar\omega/\tau)-1}$$

If we write it as a distribution $\frac{U}{V} = \int d\omega\, u_\omega$,

then $$\boxed{u_\omega = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{\exp(\hbar\omega/\tau)-1}}$$

KK Eq. 4.22

! Planck radiation law

"Quantum theory began here." - Kittel & Kroemer pg. 95

Let's look at the limiting cases of this distribution

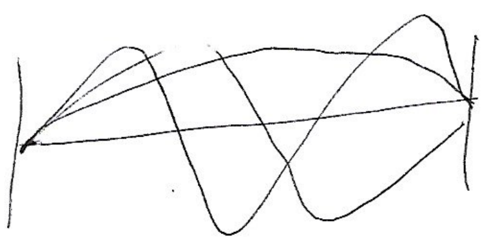
Let $\omega \to 0$, $\hbar\omega \ll \tau$

In this case, like before, $e^x \approx 1 + x$, so $e^{\hbar\omega/\tau} \approx 1 + \frac{\hbar\omega}{\tau}$

$$u_\omega = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{1+\frac{\hbar\omega}{\tau}-1} = \frac{\cancel{\hbar}}{\pi^2 c^3}\frac{\omega^{\cancel{3}\,2}\tau}{\cancel{\hbar\omega}} = \frac{\omega^2\tau}{\pi^2 c^3}$$

This is the Rayleigh-Jeans law

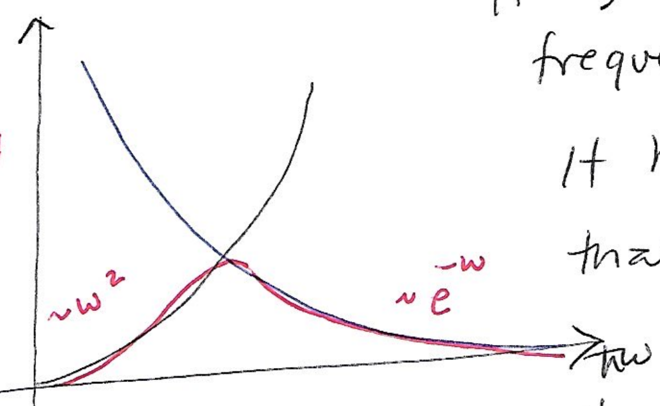
The Rayleigh-Jeans law can be derived from classical physics also using electromagnetic standing waves. These waves are not quantized and according to the equipartition theorem, each standing wave has the same energy.
every



Since there are more short-wavelength standing waves, there is more energy in the high frequencies

It agrees with experiment at low frequency, but not at high frequency.

KK
Fig. 4.4



It has the unfortunate characteristic that $\displaystyle\int_0^\infty d\omega \, \frac{\omega^2 \tau}{\pi^2 c^3} = \infty = \frac{U}{V}$

It predicts that the energy density of a black body is infinite, which is obviously wrong.

Planck, trying to solve this problem, quantized the energies of the standing waves, which allowed him to then apply the Boltzmann distribution (what we just did). So at $\omega \to \infty$, $\hbar\omega >> \tau$,

$\sigma_B \equiv \dfrac{\pi^2 k_B^4}{60\hbar^3 c^2}$
Stefan-Boltzmann constant

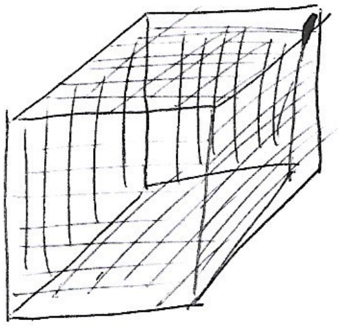
$$u_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{\exp(\hbar\omega/\tau) - 1} \approx \frac{\hbar}{\pi^2 c^3} \frac{1}{\exp(\hbar\omega/\tau)}$$

One last thing: $J_u = \frac{1}{4} c \, U(\tau) \frac{1}{V} = \dfrac{\pi^2 \tau^4}{60\hbar^3 c^2} = \sigma_B T^4$ flux

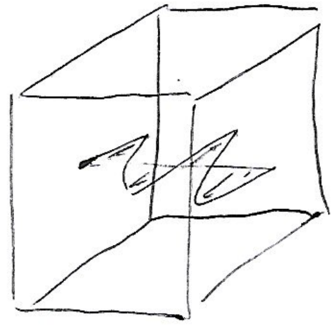
Also called phonon gas model

Phonons in solids: Debye theory

pg. 102 KK



solid with quantized vibrations



Transversly polarized electromagnetic field. (photon)

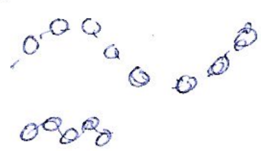
Both have the same distribution function, the Planck distribution

$$\langle s \rangle = \frac{1}{\exp(\hbar\omega/\tau)-1}$$

KK 4.35

but as mentioned before their origin is different. One comes from the fact that we can have many photons of energy $\hbar\omega$ in the same state, the other from the energy levels of the quantum harmonic oscillator that are equally spaced. There are some differences, though.

	Photons	Phonons
Dispersion relation	$\omega^2 = c^2 \vec{k}^2$ speed of light	Not quite $\omega^2 = v^2 \vec{k}^2$ but we will assume it. speed of sound
Maximum frequency	Infinity	Given by the interatomic distance
Number of modes	2 transverse	2 transverse 1 longitudinal



For electromagnetic waves, we derived

$$U = \underbrace{2\left(\frac{L}{\pi}\right)^3\left(\frac{1}{8}\right)4\pi\int_0^\infty k^2\,dk}_{\text{Number}}\;\underbrace{\langle \varepsilon_n\rangle}_{\text{Energy}}$$

For phonons, we can make the substitutions $2 \longrightarrow 3$

$\infty \rightarrow k_{max}$ (k_D)

maximum number of modes $\longrightarrow 3N$

(each atom is a node and there are 3 degrees of freedom per node/atom)

$$3\left(\frac{L}{\pi}\right)^3\left(\frac{1}{8}\right)4\pi\int_0^{k_D} k^2\,dk = 3N \qquad \approx \text{KK } 4.37$$

$$\cancel{3}\left(\frac{L}{\pi}\right)^3\left(\frac{1}{\cancel{8}\,2}\right)\cancel{4\pi}\,\frac{k^3}{\cancel{3}}\Bigg|_0^{k_D} = 3N$$

$$\frac{L^3}{2\pi^2}k_D^3 = 3N \qquad k_D = \left(\frac{6N\pi^2}{V}\right)^{1/3} = \left(6\rho\pi^2\right)^{1/3}$$

most energetic phonon

depends on the density of the solid

which is the interatomic spacing related

Since $k_n = n\dfrac{\pi}{L}$, KK Eq. 4.38

$$\frac{\cancel{L^3}}{2\cancel{\pi^2}}\,n_D^3\,\frac{\pi^{\cancel{3}}}{\cancel{L^3}} = 3N = \frac{1}{2}\pi n_D^3 \Rightarrow n_D = \left(6N/\pi\right)^{1/3}$$

Also, since $k = \dfrac{2\pi}{\lambda}$,

$$\frac{L^3}{2\pi^2}\frac{(2\pi)^3}{\lambda^3} = 3N \Rightarrow \lambda^3 = \frac{L^3\,\cancel{8}^{\,4}\pi^{3}}{\cancel{2}\pi^2\,3N} \Rightarrow \lambda_{min} = L\left(\frac{4\pi}{3N}\right)^{1/3}$$