Topics of global warming, solar energy, and academia can be further explored by investigating the quantification of the leaf mass of trees and its relation to energy absorption and efficiency. We present a model which showcases these properties in a theoretical manner. Using convex, continuous, differentiable curves in $\mathbb{R}^3$, we model trees as 3-dimensional surfaces produced by rotating said curves about the z-axis. In our model, trees grow in the positive z direction and lie directly underneath the path of the Sun. At midday, the Sun will be precisely above the center of the tree.

We frequently use the term "leaf density" to denote the number of leaves per unit volume. It is important to distinguish this from "leaf mass density" which is the usual mass per unit volume. It is our assumption that the maximum leaf density for a given height, z_0, is directly proportional to the daily energy observed along the tree's profile at the same height z_0. We claim that knowing the coefficient of proportionality, α, of these two quantities and γ, the leaf mass density, we can estimate the total leaf mass of a given tree.

In the process of determining leaf mass, we provide an expression for leaf density, which we call ρ. Based on this ρ, we define a probability density for leaf stem locations as a function of distance from the tree's trunk. We compare the relative efficiency of leaf shapes by running a probabilistic simulation which plots leaves on a branch according to the aforementioned distribution.

Our results include a symbolic expression for leaf mass with respect to the tree's profile. Further, we show that leaves with natural, leaf-like (folium) shapes perform much better than non-traditional leaf shapes such as squares and circles. It is interesting to note that circles and squares are conventionally optimal shapes (such as in coverage and packing problems), but are suboptimal in the case of leaves. Performance was based on the percentage of overlapping leaf surface area across a branch.

We provide many theoretical relationships between model variables which are strengthened by experimental data produced by others. We developed relationships rather than specific calculations, providing a more conceptual solution to the problem.

Tree Leaf Allometry

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Abstract

We present a model for estimating tree allometry on leaf mass and leaf mass distributions for a given tree. Certain more explicit formulations of the question such as which type of tree or which type of environment contain more or less leaf mass content are essentially trivial to answer and are briefly addressed. We focus on the sun distribution of energy on the surface of the tree for a given day as the heart of our model. We argue that the leaf mass distribution is strongly correlated to the energy distribution around the shell of the tree. In addition, we compare surface area overlap of different leaf shapes distributed across a branch at arbitrary height. The qualitative results of our model agree with that of trees found in southern Moravia. [EERMAK, 1998]

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1 Introduction

Modeling and quantifying leaf mass on an arbitrary tree has applications in global warming, solar energy, and academia. The global warming applications are obvious, a greater/smaller leaf mass implies greater/less carbon sequestration by photosynthesis producing a warmer /cooler planet. A less-obvious application is for design of a solar panel and cell configurations based on leaf distributions to allow for a more optimal scheme. An even more abstract application is the concept of looking at a conventional coverage problem as a evolutionary programming problem. We present a model based on sun light collection on a convex tree shape for predicting growth density. Using this model, we show how this density affects leaf overlap and how a leaf shape reduces said overlap more effectively over conventionally optimal shapes such as circles or squares. These results are directly applicable to other fields such as those mentioned previously.

1.1 Model Overview

We model a tree as a rotated curve in $\mathbb{R}^3$ with the x-y axis parallel to the Earth's surface and z being the direction in which the tree grows. The Sun travels in an arc above the tree illuminating its profile.

Throughout the paper, we use the term "leaf density" to denote the number of leaves per unit volume. In most cases we talk about the leaf density in relation to the lateral distance from the trunk. Note that this is different than "leaf mass-density" which is the mass per unit volume of a leaf.

We build a framework using both daily energy observed along the profile of the tree, and leaf density to build a comprehensive model for estimating the total leaf mass of the tree. Our model also allows us to explore the efficiency of different leaf shapes.

1.2 Assumptions

- A tree's profile can be defined by a convex, continuous, differentiable curve rotated about the z axis. In other words, the tree is symmetrical around its trunk. While not a perfect fit for natural trees, it has been shown that most are symmetrical to some degree [Schneider and Sagan, 2005].
- The intensity of light throughout the day is constant at approximately 580W/m^2, as a result of only 43% of incident light being at a usable wavelength[Hall and Roe, 1999]. Considered trees lie directly underneath the Sun's path through the sky.
- Horizontal leaf density follows a logistic curve which increases with distance from the trunk. Logistic curves occur frequently in nature, and this type of distribution has been seen in tree growth patterns [Hofmeyer et al., 2010], [Setiyono et al., 2008].
- Maximum leaf density at a given height is proportional to the daily energy observed along the profile at that height. This is a natural assumption which loosely means that trees will grow more leaves where more sunlight is present. This behavior is also noted in [EERMAK, 1998].
- Leaf mass-density, γ, is constant across all leaves on a tree [Roderick and Cochrane, 2002].

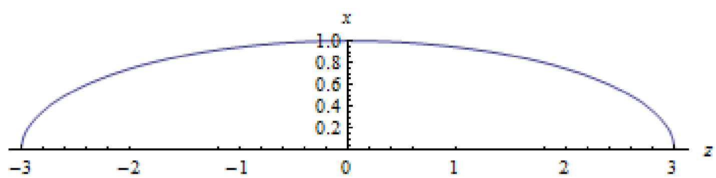
1.3 Problem Definition

Suppose $x = l(z)$ is a convex, continuous, and differentiable curve which when rotated about the z axis gives the surface of a tree. Using only γ, the mass-density of leaves, and α, the proportionality constant between the daily energy and maximum leaf density at a given height, determine the leaf mass of the tree. Also discuss the efficiency of leaf shape in terms of overlapping area.

2 Model and Methods

We model a tree by taking some function $x = l(z)$ and rotating it about the z axis. The resulting three-dimensional surface is the profile of some tree L. We require $l(z)$ to be convex, continuous, and differentiable for $z_0 \leq z \leq z_1$ where z_0 and z_1 are the lower and upper bounds for the tree.

Figure 1: Hypothetical Tree: $l(z) = \sqrt{1 - (\frac{z}{2})^2}$



For example, take $l(z) = \sqrt{1 - (\frac{z}{2})^2}$, rotate it around the z axis giving a surface which represents the profile of a tree (see **Figures 1 and 2**). Describing tree profiles in this fashion is not only convenient, but fairly representative of trees in nature [Schneider and Sagan, 2005].

Having a model of trees in $\mathbb{R}^3$, we now wish to represent incoming sunlight in relation to our tree L. To simplify this relationship, we assume the Sun's path coincides with $y = 0$; in other words, the Sun travels directly over the tree along the x axis. We set the x-y plane parallel to the Earth's surface, and let θ_t denote the angle rays from the Sun make with the positive x axis at time t. Letting t range from 0 to 1 we have $\theta_t = \theta_{min} + t(\theta_{max} - \theta_{min})$ where θ_{min} and θ_{max} are the minimum and maximum angles for which Sun rays will reach L respectively.

We let $\vec{T_z}$ denote the tangent vector at a point $P = (l(z), 0, z)$ on L. The intensity vector $\vec{I_\theta}$ represents a Sun

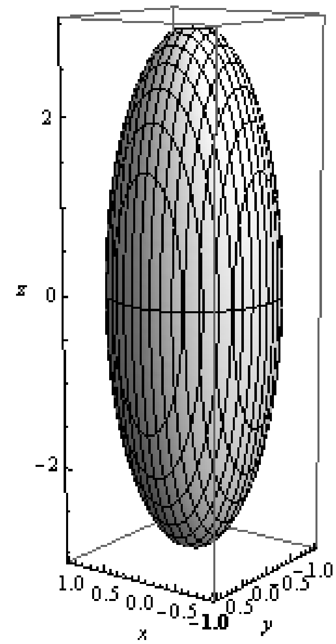


Figure 2: Tree Surface L

ray that makes an angle θ with the x-y plane. $|\vec{I_\theta}| = 580$ for all θ [Kopp and Lean, 2011]. **Figure 3** shows the previously defined vectors and angles.

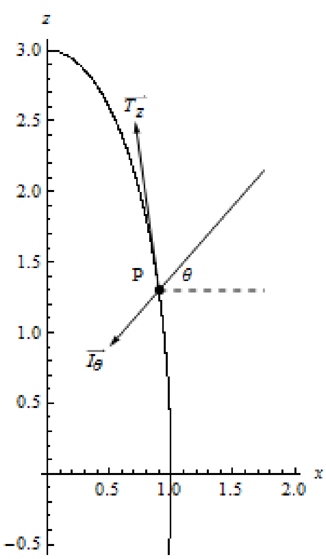


Figure 3: Angles and vectors on $l(z)$

2.1 Daily Energy Along the Tree Profile

We now wish to determine the total energy a point receives over one period (a full day). First, we examine the instantaneous intensity at point $P = (l(z), 0, z)$ using [Jones, 2010]:

$$|I_z| = |I_\theta|\ \cos\phi \tag{2.1.1}$$

Here, ϕ is the angle between $\vec{I_\theta}^{\perp}$ and $\vec{T_z}$, notice this is just the projection of $\vec{I_\theta}$ on to $\vec{T_z}$ as seen in **Figure 3**. Solving for ϕ using the definition of the dot product yields:

$$\phi = \cos^{-1}\left(\frac{\vec{I_0}^{\perp}\cdot\vec{T_z}}{|\vec{I_0}^{\perp}||\vec{T_z}|}\right)$$
$$I(z,\theta) = |I_\theta|\left(\frac{-\sin\theta\ l'(z) + \cos\theta}{\sqrt{1+(l'(z))^2}}\right) \tag{2.1.2}$$

Equation (2.1.2) defines the instantaneous intensity at a point $(l(z), 0, z)$ for a given θ. Having an expression for intensity allows us to determine the total energy a point receives over the course of one full day. To calculate total energy, we must integrate intensity over a full period, $0 \leq t \leq 1$ [Stine and Geyer, 1987].

$$\begin{aligned} E(z) &= \int_0^1 I(z,\theta_t)\mathrm{d}t \\ &= \int_0^1 |I_\theta|\left(\frac{-\sin\theta\ l'(z)+\cos\theta}{\sqrt{1+(l'(z))^2}}\right)\mathrm{d}t \end{aligned} \tag{2.1.3}$$

Continuing with our example of a tree with profile $l(z) = \sqrt{1-(\frac{z}{2})^2}$, we graph the energy observed per day for each z for which our tree is defined ($-3 \leq z \leq 3$). This graph (**Figure 4**) shows exactly what you might expect: very small energy at the base of the tree, fairly average energy in the middle, and very high energy near the top.

2.2 Estimating Leaf Mass

Our approach to estimating the leaf mass requires knowledge of the leaf density, ρ, as a function of height and lateral distance from the trunk. It is our assumption that the tree is fully symmetrical about the z axis, thus we simply work in the cross section at $y = 0$.

We believe that the leaf density is logistic with respect to x, the lateral distance from the trunk. For small x near the trunk, there will be few leaves, but approaching the boundary of the tree profile, the leaf density must grow rapidly. Not only is this an intuitive model, but is mentioned in [Setiyono et al., 2008]. We suppose that the leaf density function is of the approximate form (**Figure 5**) with $\rho_{\max}(z)$ being the maximum leaf density for a given height:

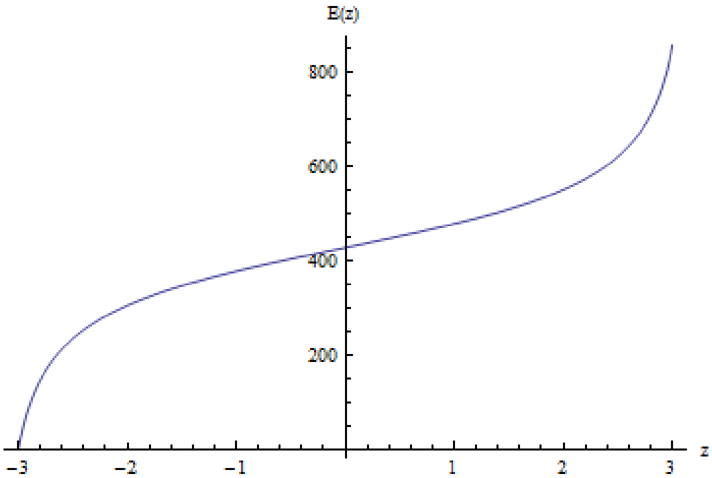


Figure 4: Energy over a full day with respect to z

$$\rho(z,x) = \frac{\rho_{\max}(z)}{1+e^{-6(\frac{2x}{l(z)}-1)}} \tag{2.2.1}$$

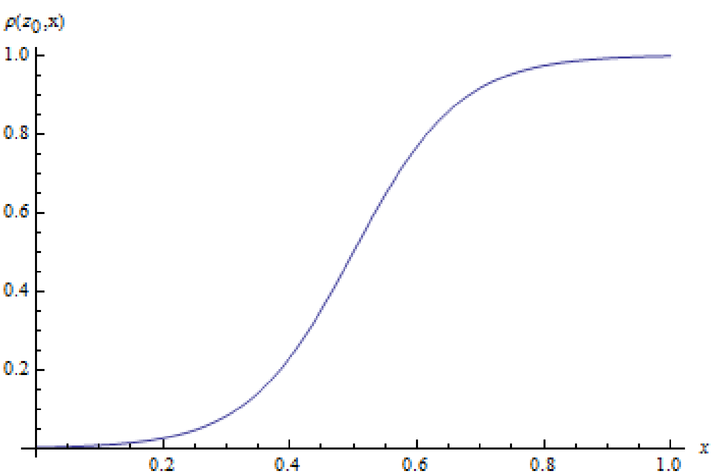


Figure 5: Shape of $\rho(z,x)$ for fixed z

Integrating equation (2.2.1) across all values of z and then going around the z axis with θ going from 0 to 2π, we find the leaf mass proprotional to the following expression:

$$\begin{aligned} m &= \int_0^{2\pi}\int_0^h\int_0^{l(z)} \gamma\rho(z,x)\,\mathrm{d}x\mathrm{d}z\mathrm{d}\theta \\ &= 2\pi\gamma\int_0^h\left(\int_0^{l(z)}\frac{\rho_{\max}(z)}{1+e^{-6(\frac{2x}{l(z)}-1)}}\mathrm{d}x\right)\mathrm{d}z \\ &= 2\pi\gamma\int_0^h \rho_{\max}(z)\left[\frac{l(z)}{12}log\left(e^{\frac{12x}{l(z)}}+e^6\right)\right]_0^{l(z)}\mathrm{d}z \end{aligned} \tag{2.2.2}$$

We have now defined a leaf mass function which depends on the parameters:

1. $l(z)$ The profile function of a tree.
2. $\rho_{\max}(z)$ The maximum leaf density at a height z.
3. h The height of the tree.
4. γ The leaf mass-density of the tree.

2.3 Relationship Between Maximum Leaf Density and Energy

Our final discussion brings together our expression for the leaf density with energy. We do this by noting that since it has been shown that in certain cases the maximum leaf density at a given height is directly proportional to the daily energy observed at that point [EERMAK, 1998]. Using notation described thus far, we make the claim that $\rho_{\max}(z) \propto E(z)$.

Using this relationship, we now rewrite our expression for the leaf mass using the substitution $\alpha E(z) = \rho_{\max}(z)$. We start by rewriting equation (2.2.1) as:

$$\begin{aligned} \rho(z,x) &= \frac{\rho_{\max}(z)}{1+e^{-6(\frac{2x}{l(z)}-1)}} \\ &= \frac{\alpha E(z)}{1+e^{-6(\frac{2x}{l(z)}-1)}} \end{aligned} \tag{2.3.1}$$

Finally, bringing everything together, we find an expression for the leaf mass of a tree using the new version of $\rho(z,x)$. This equation is exciting because it lets us approximate the leaf mass of a tree needing knowledge only of $l(z)$ and α, the proportionality constant for $\rho_{\max}(z)$ and $E(z)$.

$$\begin{aligned} m &= 2\pi\alpha\int_0^h E(z)\left[\frac{l(z)}{12}log\left(e^{\frac{12x}{l(z)}}+e^6\right)\right]_0^{l(z)}\mathrm{d}z \\ &= 2\pi\alpha\int_0^h\left[\int_0^1 I(z,\theta_t)dt\right]\left[\frac{l(z)}{12}log\left(e^{\frac{12x}{l(z)}}+e^6\right)\right]_0^{l(z)}\mathrm{d}z \\ &= 2\pi\alpha|I_\theta|\int_0^h\left(\int_0^1\frac{-\sin\theta_t\ l'(z)+\cos\theta_t}{\sqrt{1+(l'(z))^2}}dt\right. \\ &\qquad\qquad \left.\left[\frac{l(z)}{12}log\left(e^{\frac{12x}{l(z)}}+e^6\right)\right]_0^{l(z)}\right)\mathrm{d}z \end{aligned} \tag{2.3.2}$$

2.4 Leaf Shape Overlap Comparison

We now explore the relationship between a leaf's shape and the ratio of overlapping surface area of leaves on a branch at a given height on a tree. To do this, we model a branch as a horizontal line in the x-y plane at some fixed z_0. Supposing there are n leaves on this branch, we would expect them to be distributed across the branch according to a distribution based on $\rho(z_0, x)$.

Appealing to probability theory, we need to express this distribution as a probability density function. Using equation (2.2.1), we define the following function f as a PDF describing the distribution of stem locations on the branch at height z_0. Note that $f(x) = 0$ for $x < 0$ and $x > l(z)$.

$$\begin{aligned} \rho_{\text{mass}} &= \int_0^{l(z_0)} \rho(z_0, x)\mathrm{d}x \\ f(x) &= \frac{\rho(z_0, x)}{\rho_{\text{mass}}} \end{aligned} \tag{2.4.1}$$

Next, we determine the percentage of overlapping area on this z_0 branch with leaves of three different shapes: circle, square, and folium (which we describe shortly). Fixing the area of each leaf to 1 cm^2, we place n leaves along the x axis according to the distribution f (2.4.1).

To determine the location of the stems, we use inverse transformation sampling to choose values of x according to distribution f. Essentially, we evaluate the inverse cumulative distribution of f at uniformly random values between 0 and 1. This method is outlined in [Lam, 2009]. We calculate the CDF as:

$$\begin{aligned} F(y) &= \int_0^y f(x)\mathrm{d}x \\ &= \frac{1}{\rho_{\text{mass}}} \int_0^y \rho(z_0, x)\mathrm{d}x \end{aligned} \tag{2.4.2}$$

At this point, keeping track of an analytic function had become unwieldly. We need the inverse of F which we will be using to sample x values at our z_0. Using MATLAB's built-in interpolation functionality, we were able to approximate F^{-1} to a high degree of accuracy.

2.4.1 Quick Discussion About Shapes

Before we continue, we would like to mention our justification of the shapes we analyzed. First, the folium shape which literally means "leaf" or "petal". We believe it to be a good representation of an average leaf, see **Figure 6**. Unfortunately, we were constrained to convex shapes due to the geometry libraries we had available. Circles and squares seem to be good candidates for convex shapes at other ends of the spectrum.

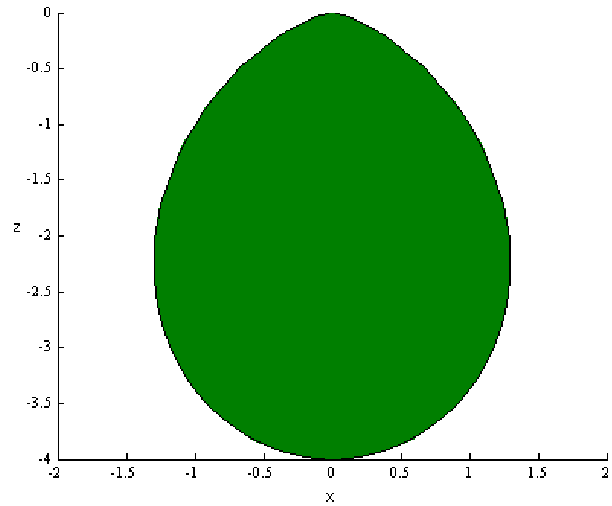


Figure 6: Folium shape described by:
$r = -\mathrm{b}\cos\theta + 4\mathrm{a}\cos\theta\sin 2\theta$ [JOC/EFR/BS, 1997]

3 Results and Discussion

We started with a very basic model of a tree and have developed sophisticated solutions to energy, mass, and shape considerations. As a summary of our results, we list a few key equations:

$$E(z) = \int_0^1 |I_\theta| \left(\frac{-\sin\theta\ l'(z) + \cos\theta}{\sqrt{1 + (l'(z))^2}} \right) \mathrm{d}t$$

$$\rho(z, x) = \frac{\alpha E(z)}{1 + e^{-6(\frac{2x}{l(z)} - 1)}}$$

$$m = 2\pi\alpha \int_0^h E(z) \left[\frac{l(z)}{12} log \left(e^{\frac{12x}{l(z)}} + e^6 \right) \right]_0^{l(z)} \mathrm{d}z$$

Based on these relationships we were able to set up a very detailed simulation modeling the effect leaf shapes have on surface area efficiency. Using MATLAB, and the MPT library [Kvasnica et al., 2004] we represented individual leaves as convex polytopes. Using the inverse cumulative distribution for sampling lateral distances (as discussed above) we were able to plot a series of leaves across an imaginary branch at any given height. Leaves were distributed according to $\rho(z, x)$, the leaf density giving an accurate picture of a typical branch.

We used the same distances for each leaf shape as to provide a foundation for fairly judging performance. Upon calculating the distances, we placed a leaf at each point and the calculated the overlapping surface area. Our measure was the average percent of area overlapped for each leaf. **Figure 7** shows the plot created for each shape. Green outlines indicate a leaf, and red filled area represents overlapping area.

Table 1 shows the average overlap for each leaf shape as well as the parameters to the simulation. There is quite a large disparity between the folium shape and the two non-traditional leaf shapes. As we would expect, the natural, leaf-like shapes perform much better.

3.1 Further Research

For a closer examination of leaf mass with respect to tree profile, we would suggest investigating different $l(z)$ curves. Throughout this paper, we use a simple ellipsoid tree profile as our key example, but there are many other natural tree shapes.

During our research, we did not uncover any experimental data that shed light on the relationship between tree profile and leaf mass. It would be revealing to pit our model's prediction against hard data.

Finally, it would be interesting to extend our model of light intensity from the Sun to include attenuation as the radiation travels through the atmosphere.

Leaf Shape	Branch Length	Leaves	Leaf Size	% Overlap
Folium	40cm	30	1cm^2	61.446%
Square	40cm	30	1cm^2	75.805%
Circular	40cm	30	1cm^2	69.460%
Folium	60cm	30	1cm^2	46.678%
Square	60cm	30	1cm^2	56.677%
Circular	60cm	30	1cm^2	54.628%

Table 1: Overlap comparison for circle, square and folium shaped leaves.

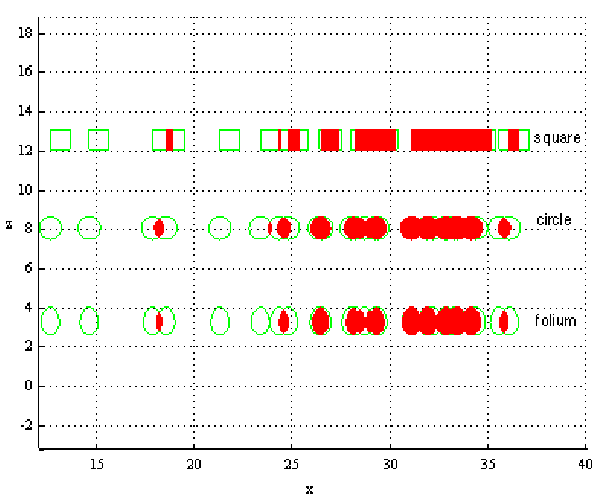


Figure 7: Branch visualization for each leaf shape.

4 Conclusion

We have built a powerful yet comprehensible framework modeling many properties of a tree based on a few important characteristics. Given height, profile, and α as defined above, we can calculate:

1. Total leaf mass.
2. Leaf density.
3. Estimate for total number of leaves.
4. Estimate maximum energy absorbed per day.

By keeping our model generic, we retained the ability to keep separate the notions of a tree profile and a leaf shape. We can extend our model to encompass most any tree that fits within our simple assumptions. Finally, an encouraging result of our model is that leaves of a natural folium shape are more efficient in terms of overlapping surface area.

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