Final Homework

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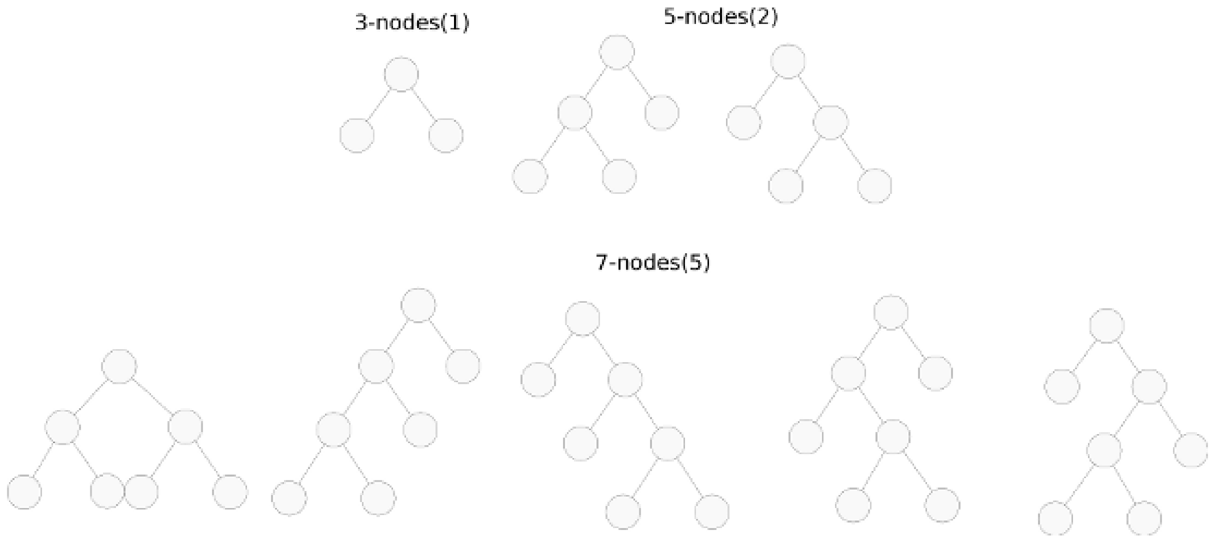
Problem 1

The cost of going from exit j to k is $C_j + C_{j+1} + C_{j+2} + \cdots + C_{k-1}$. I propose the data structure H such that $H_i = C_1 + C_2 + \cdots + C_{i-1}$. To calculate the cost exit j to k using H, it would simply be $H_k - H_j$. This expression expands to $(C_1 + C_2 + \cdots + C_{j-1}) - (C_1 + C_2 + \cdots + C_{k-1})$, which simplifies to $C_j + C_{j+1} + \cdots + C_{k-1}$. Showing that $H_k - H_j$ is equivalent to the cost we calculated for exit j to k. Given that H is already calculated, this computation is a simple subtraction, $O(1)$.

Generating this data structure is very easy and would take $O(n)$ time and holds n elements. Each element H_i is equal to $C_i + H_{i-1}$ which lends itself easily to an accumlating loop from 1 to n.

```
H[1] = C[1]
for i = 2...n:
  H[i] = C[i] + H[i-1]
```

Problem 2



a) $B_3 = 1, B_5 = 2, B_7 = 5$.

b) You can't construct a full binary tree with an even number of nodes. Every node always has zero or two child nodes, meaning everytime the tree grows, it must grow by a multiple of two nodes. So starting with the root, and growing n times, the total number of nodes will always be of the form $1 + 2n$, which is odd.

c) Before determining an upper bound for the number of full binary trees of some size, it is necessary to know how many leaves such a tree has. All full binary trees of size n can be built by simply adding two nodes as the left and right child to any leaf node on a tree of size $n - 2$. For this reason, all full trees with the same number of nodes have the same number of leaves.

Starting from the most simple full tree of size 3, it can be seen that every time the tree grows (by two nodes), the number of leaves increases by one. This is because to grow the tree, two nodes must fill a leaf's children, which results in one more leaf than before. So the number of leaves on a full tree of size k is just the number of times two leaves were added to the tree of three nodes plus the original two on that tree. So as a function of k, the number of leaves of a full tree, L_k, is $L_k = 2 + \frac{k-3}{2}$. The first term is the original two leaves on the base tree, and $\frac{k-3}{2}$ is the number of times two nodes were added to the base tree.

With this expression for the number of leaves in a full tree, we can now discuss the number of full trees of size n. All of these trees of size n had to be constructed by growing some smaller tree with $n - 2$ nodes. Assuming that we know B_{n-2}, then $B_n \leq L_{n-2}B_{n-2}$. This is just the number of trees multiplied by the number of leaves on each tree, which constructs every possible tree of size n.

Expanding this expression we get:

$$\begin{aligned} B_n &= L_{n-2}B_{n-2} \\ &= (2 + \frac{(n-2)-3}{2})B_{n-2} \\ &= 2B_{n-2} + (\frac{n-5}{2})B_{n-2} \end{aligned}$$

Problem 3

```
structure weirdqueue {
   pushstack (stack pointer)
   popstack (stack pointer)
}

def enqueue(Q, elt):
   Q.pushstack.push(elt)
```

```
def dequeue(Q):
   if Q.pushstack and Q.popstack are empty:
      error underflow
   if Q.popstack is empty:
      while Q.pushstack is not empty:
         Q.popstack.push( Q.pushstack.pop() )
      swap Q.popstack pointer with Q.pushstack
   return Q.popstack.pop()
```

a) Under the assumption that the 'popstack' is empty, we would have to pop each element off the 'pushstack' and then push that on to the 'popstack.' By doing this we are reversing the order, guaranteeing that we get the first queued item, but means we are also doing work proportional to the size of the structure, $O(n)$.

b) In practice, we could not possibly have to do this mass popping and pushing to reorient the structure every dequeue. This means that we will have a much faster amortized analysis of the running time. If we follow the lifetime of one element in the structure, there are only about 4 operations associated with it. We initially push it on to the 'pushstack' and then at some later time we will transfer it to the 'popstack' and finally pop it one more time when it is removed. So there can never be more than 4 operations per element lifecycle. Using amortized analysis, we can see that n insertions could never be worse than about $4n$ stack operations. Therefore, the average cost per enqueue/dequeue operation is $\frac{4n}{n} = 4$, which is $O(1)$.

Problem 4

a) I'm not quite sure how to go about analyzing this algorithm. There really seems to be a lot of dependencies, and I'm not yet skilled in the art of probability. My best attempt was to try partition the bets in to two groups: the first $\frac{n}{10}$ and the rest. If we call the largest bid in the first partition M, and let the list of bets, L, be all the bets in the second partition that are greater than M, we can find the probability. Assuming that we can find this list L, or even the number of elements in it then the probability of picking the highest bet is equal to the probability that the highest bet is the first element of L.

So, the probabiliy of taking the highest bet is $\dfrac{1}{\text{length}(L)}$, where L is the list described above (number of bets in the partition that are greater than maximum of first).

Now we have some way of determining the probability, albeit one that assumes a lot. I really don't know how to go about determining the length of L formally, but we can analyze the smallest and largest values it can take. First of all, the worst case is that the $\frac{n}{10}$ smallest elements are the first bets in the sequence. In that case all the bets in the second partition are greater than the maximum of the first. Therefore, the probability of picking the highest bid would be $\frac{1}{\frac{9n}{10}} = \frac{10}{9n}$ in this scenario.

On the other hand, if the 2^{nd} largest bet was one of the first $\frac{n}{10}$ bets seen, then we are guaranteed to accept the very highest bet. So in this case, the probability is 1.