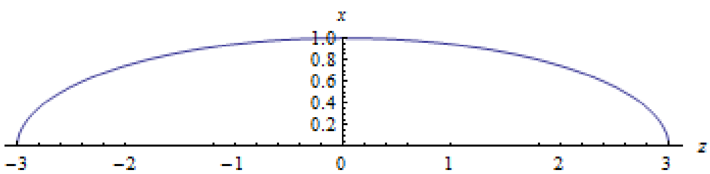
1 Model

We model a tree by taking some function $x = l(z)$ and rotating it about the z axis. The resulting three-dimensional surface is the profile of some tree L. We require $l(z)$ to be both convex and continuous for $z_0 \leq z \leq z_1$ where z_0 and z_1 are the lower and upper bounds for the tree.

Figure 1: $l(z) = \sqrt{1 - (\frac{z}{2})^2}$



For example, we take $l(z) = \sqrt{1 - (\frac{z}{2})^2}$, rotate it around the z axis giving a surface which represents the profile of a tree (see **figures 1 and 2**). Describing tree profiles in this fashion is not only convenient, but fairly representative of trees in nature [CITATION ABOUT TREE SYMMETRY].

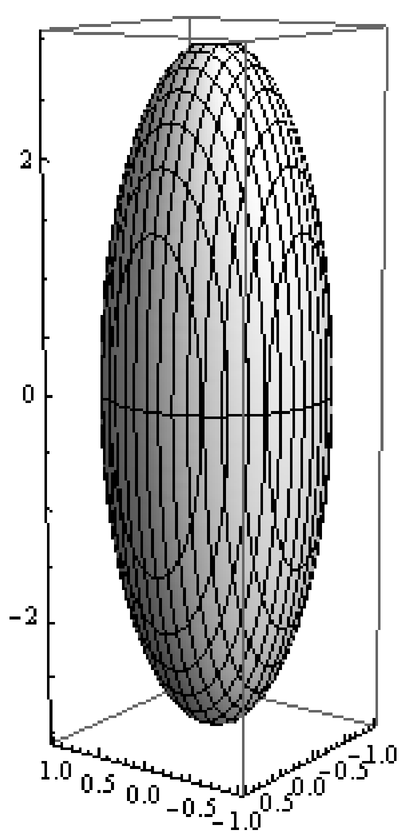
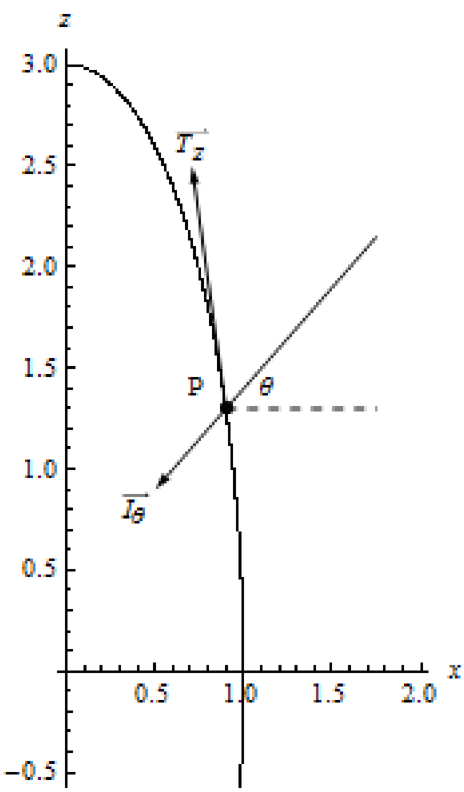


Figure 2: Surface L

Having a model of trees in $\mathbb{R}^3$, we now wish to represent incoming sunlight in relation to our tree L. To simplify this relationship, we assume the Sun's path coincides with $y = 0$; in other words, the Sun travels directly over the tree along the x axis. We choose the x-y plane as being parallel to the Earth's surface, and let θ_t denote the angle rays from the Sun make with the positive x axis at time t. Letting t range from 0 to 1 we have $\theta_t = \theta_{min} + t(\theta_{max} - \theta_{min})$ where θ_{min} and θ_{max} are the minimum and maximum angles for which Sun rays will reach L respectively.

We let $\vec{T_z}$ denote the tangent vector at a point $P = (l(z), 0, z)$ on L. The intensity vector $\vec{I_\theta}$ represents a Sun ray that makes an angle θ with the x-y plane. $|\vec{I_\theta}| = 1367$ for all θ [CITATION FOR SUN INTENSITY]. **Figure 3** shows the previously defined vectors and angles.

Figure 3: Angles and vectors on $l(z)$



1.1 Finding Energy at a Point

We now wish to determine the total energy a point receives over one period (a full day). First, we examine the instantaneous intensity at point $P = (l(z), 0, z)$ using [CITATION FOR EQN]:

$$|I_z| = |I_\theta| \; cos\phi \tag{1}$$

Here, ϕ is the angle between $\vec{I_\theta}^\perp$ and $\vec{T_z}$, notice this is just the projection of $\vec{I_\theta}$ on to $\vec{T_z}$. Solving for ϕ using the definition of the dot product yields:

$$\phi = cos^{-1}\left(\frac{\vec{I_0}^\perp \cdot \vec{T_z}}{|\vec{I_0}^\perp||\vec{T_z}|}\right) \tag{2}$$

$$I(z,\theta) = |I_\theta|\left(\frac{-sin\theta\ l'(z) + cos\theta}{\sqrt{1+(l'(z))^2}}\right) \tag{3}$$

Equation (3) defines the instantaneous intensity at a point ($l(z)$, 0, z) for a given θ. Having an expression for intensity allows us to determine the total energy a point receives over the course of one full day. To calculate total energy, we must integrate intensity over a full period, $0 \leq t \leq 1$ [CITATION].

$$E(z) = \int_0^1 I(z,\theta_t)dt \tag{4}$$

Continuing with our hypothetical tree with profile $l(z) = \sqrt{1-(\frac{z}{2})^2}$, we graph the energy observed per day for each z for which our tree is defined ($-3 \leq z \leq 3$). This graph (**figure 4**) shows exactly what you might expect: very small energy at the base of the tree, fairly average energy in the middle, and very high energy near the top.

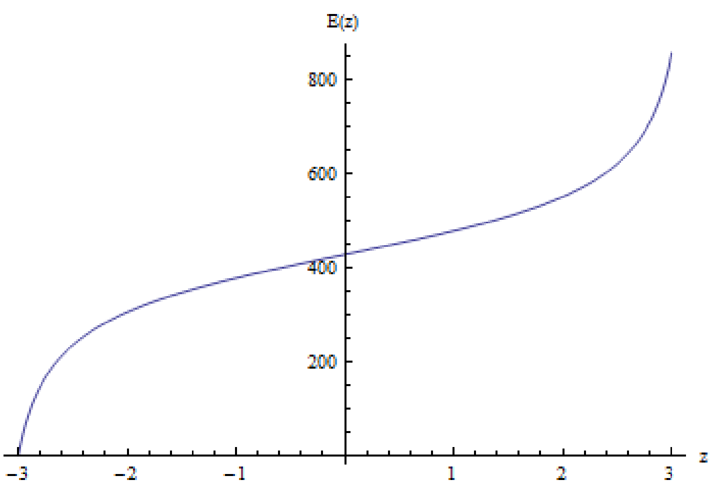


Figure 4: Energy over a full day with respect to z

1.2 Estimating Leaf Mass

Our approach to estimating the leaf mass requires knowledge of the leaf density, ρ, as a function of height and lateral distance from the trunk. It is our assumption that the tree is fully symmetrical about the z axis, thus we can simply work in the [[slice]] $y = 0$.

We believe that the leaf density is logistic with respect to x, the lateral distance from the trunk. For small x near the trunk, there will be few leaves, but approaching the boundary of the tree profile, the leaf density much grow very rapidly. Not only is this an intuitive model, but is mentioned in [CITATION??]. We suppose that the leaf density function is of the approximate form (**figure 5**) with $\rho_0(z)$ being the maximum leaf density for a given height.:

$$\rho(z,x) = \frac{\rho_0(z)}{1+e^{-6(\frac{2x}{l(z)}-1)}} \tag{5}$$

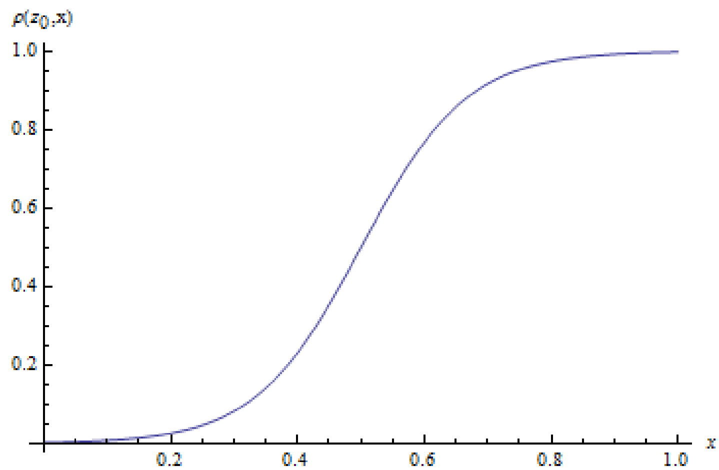


Figure 5: Shape of $\rho(z,x)$ for fixed z

Integrating equation (5) across all values of z and then doing around the z axis with θ going from 0 to 2π, we find the leaf mass with the following expression:

$$\begin{aligned} m &= \int_0^{2\pi}\int_0^h\int_0^{l(z)} \rho(z,x)\ dxdzd\theta \\ &= 2\pi\int_0^h\left(\int_0^{l(z)}\frac{\rho_0(z)}{1+e^{-6(\frac{2x}{l(z)}-1)}}dx\right)dz \\ &= 2\pi\int_0^h \rho_0(z)\left[\frac{l(z)}{12}log\left(e^{\frac{12x}{l(z)}}+e^6\right)\right]_0^{l(z)} dz \end{aligned} \tag{6}$$

We have now defined a leaf mass function which depends on the parameters:

1. $l(z)$ The profile function of a tree.
2. $\rho_0(z)$ The maximum leaf density at a height z.
3. h The height of the tree.

1.3 Relationship Between Maximum Leaf Density and Energy

It has been found experimentally that in some cases the maximum leaf density for a given height z is directly proportional to the daily energy observed at that point [CITATION MOTHAFUCKA]. Using notation described in this paper, we are claiming that $\rho_0(z) \propto E(z)$.

Using this relationship, we can now rewrite our expression for the leaf mass of a tree substituting $\alpha E(z) = \rho_0(z)$:

$$
\begin{aligned}
m &= 2\pi\alpha \int_0^h E(z) \left[\frac{l(z)}{12} log \left(e^{\frac{12x}{l(z)}} + e^6 \right) \right]_0^{l(z)} dz \\
&= 2\pi\alpha \int_0^h \left[\int_0^1 I(z, \theta_t) dt \right] \left[\frac{l(z)}{12} log \left(e^{\frac{12x}{l(z)}} + e^6 \right) \right]_0^{l(z)} dz \\
&= 2\pi\alpha |I_\theta| \int_0^h \left[\int_0^1 \frac{-sin\theta \; l'(z) + cos\theta}{\sqrt{1 + (l'(z))^2}} dt \right] \left[\frac{l(z)}{12} log \left(e^{\frac{12x}{l(z)}} + e^6 \right) \right]_0^{l(z)} dz
\end{aligned}
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