INTEGRAL TRANSFORM METHOD

15.4 Fourier Transforms

Fourier Transform Pairs

Def 15.4.1 Fourier Transform Pairs

i) Fourier transform: $\boxed{\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx = F(\alpha)}$ (5)

Inverse Fourier transform: $\boxed{\mathcal{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}\,d\alpha = f(x)}$ (6)

ii) Fourier sine transform: $\boxed{\mathcal{F}_s\{f(x)\} = \int_0^{\infty} f(x)\sin(\alpha x)\,dx = F(\alpha)}$ (7)

Inverse Fourier sine transform: $\boxed{\mathcal{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^{\infty} F(\alpha)\sin(\alpha x)\,d\alpha = f(x)}$ (8)

iii) Fourier cosine transform: $\boxed{\mathcal{F}_c\{f(x)\} = \int_0^{\infty} f(x)\cos(\alpha x)\,dx = F(\alpha)}$ (9)

Inverse Fourier cosine transform: $\boxed{\mathcal{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^{\infty} F(\alpha)\cos(\alpha x)\,d\alpha = f(x)}$ (10)

Fourier Transform

Suppose that f is continuous and integrable on the interval $(-\infty;\infty)$ and f' is piecewise continuous on every finite interval. If $f(x)\to 0$ as $x\to\pm\infty$.

$$\therefore \mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)e^{i\alpha x}\,dx$$
$$= f(x)e^{i\alpha x}\Big]_{-\infty}^{\infty} - i\alpha\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx$$
$$= -i\alpha\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx$$

$\boxed{\mathcal{F}\{f'(x)\} = -i\alpha F(\alpha)}$ (11)

In general, under conditions analogous to those leading to (11), we have

$\boxed{\mathcal{F}\{f^{(n)}(x)\} = (-i\alpha)^n\mathcal{F}\{f(x)\}}$, where $n = 0,1,2,3\ldots$ (12)

Sine and cosine transforms are not suitable for transforming any derivative of odd order.

$\therefore \boxed{\mathcal{F}_s\{f'(x)\} = -\alpha\mathcal{F}_c\{f(x)\}}$ and $\boxed{\mathcal{F}_c\{f'(x)\} = \alpha\mathcal{F}_c\{f(x)\} - f(0)}$

Fourier Sine Transform

Suppose that f and f' are continuous, f is absolutely integrable on the interval $[0,\infty)$, and f'' is piecewise continuous on every finite interval. If $f\to 0$ and $f'\to 0$ as $x\to\infty$.

$$\mathcal{F}_s\{f''(x)\} = \int_0^{\infty} f''(x)\sin(\alpha x)\,dx$$

$\boxed{\mathcal{F}_s\{f''(x)\} = \alpha f(0) - \alpha^2\mathcal{F}_s\{f(x)\}}$ (13)

Fourier Cosine Transform

Under the same assumptions that lead to (9), we find the Fourier cosine transform of $f''(x)$ to be

$$\boxed{\mathcal{F}_c\{f''(x)\} = -\alpha^2 \mathcal{F}_c\{f(x)\} - f'(0)}$$

How do we know which transform to use on a given boundary-value problem?

$\mathcal{F}$: * variable ranges over $(-\infty, \infty)$

$\mathcal{F}_s$:
- variable ranges over $(0, \infty)$
- we don't have a formula for first derivative
- for the 2^{nd} derivative formula, we need to know the value of the function at 0.

$\mathcal{F}_c$:
- variable ranges over $(0, \infty)$
- we don't have a formula for first derivative
- for the 2^{nd} derivative formula, we need to know the value of the derivative of the function at 0.