BOUNDARY-VALUE PROBLEMS IN RECTANGULAR COORDINATES

13.1 SEPARABLE PARTIAL DIFFENTIAL EQUATIONS

Linear Partial Differential Equation

u denotes the dependent variable and x and y the independent variables, then the general form of a linear second-order partial differential equation is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial u} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \qquad (1)$$

$A, B, C, \ldots, G$ are constants or functions of x and y. When $G(x,y)=0$. If eq. (1) is said to be homogeneous; otherwise, it's nonhomogeneous.

Solution of a PDE

↳ E.g. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ * and $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = xy$ *   * 2nd order partial differential equations

A solution of a linear partial DE (1) is a function $u(x,y)$ of 2 independent variables that pocesses all partial derivatives occuring in the equation that satisfies the equation in some region of the xy-plane.

Separable of Variables

In the method of separation of variables, a particular solution is found, a linear second-order PDE in which the independent variables are x and y, then a particular solution in the form of a function x and a function of y:

$$\therefore u(x,y) = X(x)Y(y)$$

$$\frac{\partial u}{\partial x} = X'(x)Y(y) \qquad \frac{\partial u}{\partial y} = X(x)Y'(y), \qquad \frac{\partial^2 u}{\partial x^2} = X''(x)Y(y) \qquad \frac{\partial^2 u}{\partial y^2} = X(x)Y''(y)$$

Superposition Principle

Theorem 13.1.1 Superposition Theorem

If $u_1, u_2, \ldots, u_k$ are solutions of a homogenous linear partial differential equation, then the linear combination

$$u = c_1u_1 + c_2u_2 + \ldots + c_ku_k$$

where the c_i, $i = 1, 2, \ldots, k$ are constants, is a solution.

We assume that we have an infinite set $u_1, u_2, u_3, \ldots$ of solutions of a homogenous linear equation, we can construct another solution u by forming the infinite series

$$u = \sum_{i=1}^{\infty} c_k u_k$$

Classification of Equations

A linear second-order partial differential equation in 2 independent variables with constant coefficients can be classified as 1 of 3 types.

Def 13.1.1 Classification of Equations

The linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where A, B, C, D, E, F, G are real constants, is said to be

hyperbolic if $B^2 - 4AC > 0$

parabolic if $B^2 - 4AC = 0$

elliptic if $B^2 - 4AC < 0$

13.3 HEAT EQUATION

Introduction

Consider a thin rod of length L with an initial temperature $f(x)$ throughout and whose ends are held at temperature zero for all time $t > 0$. The temperature $u(x,t)$ in the rod is determined from the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < L, \ t > 0 \qquad (1)$$

$$u(0,t) = 0, \ u(L,t) = 0, \quad t > 0 \qquad (2)$$

$$u(x,0) = f(x), \quad 0 < x < L \qquad (3)$$

Solution of the BVP

$u(x,t) = X(x)T(t)$, and $-\lambda$ is the separation constant

$\therefore k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \longrightarrow kX''T = XT'$

$\frac{\partial u}{\partial t} = X \cdot T'$ $\qquad \frac{X''}{X} = \frac{T'}{kT} = -\lambda \quad (4)$ $\qquad \therefore X'' + \lambda X = 0 \quad (5)$

$\frac{\partial^2 u}{\partial x^2} = X''T$ $\qquad T' + \lambda T = 0 \quad (6)$

The boundary conditions become $u(0,t)=X(0)T(t)=0$ and $u(L,t)=X(L)T(t)=0$. We have $X(0)=0$ and $X(L)=0$ for $t>0$.

$$\therefore \boxed{X''+\lambda X=0,\ X(0)=0,\ X(L)=0} \qquad (7)$$

We consider 3 possible cases for the parameter λ: zero, +ve, -ve.

$\lambda=0$: $\quad X(x)=c_1+c_2x \qquad (8)$

$\lambda=-\alpha^2<0$: $\quad X(x)=c_1e^{\alpha x}+c_2e^{-\alpha x} \qquad (9)$

the boundary conditions are applied to eq (8) & (9) and we are left $\boxed{u=0}$

$\lambda=\alpha^2>0$: $\quad \boxed{X(x)=c_1\cos\alpha x+c_2\sin\alpha x} \qquad (10)$

When $X(0)=0$ is applied to (10): $c_1=0 \quad \therefore X(x)=c_2\sin\alpha x$.

When $X(L)=0$ is applied to (10): $X(L)=\underset{\neq 0}{c_2}\sin\alpha L=0 \qquad (11)$ when $\sin\alpha L=0$

$$\therefore \alpha L=n\pi$$

$$\alpha=\frac{n\pi}{L}, \text{ where } n=1,2,3,\dots$$

$$\therefore \boxed{X(x)=c_2\sin\frac{n\pi}{L}x,\ n=1,2,3,\dots} \qquad (12)$$

Back to eq (6): $T'+\lambda T=0$

The general solution of (6) is: $T(t)=c_0e^{-k\alpha^2t}$ (case 3)

$$\boxed{T(t)=c_0e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t}}$$

By the superposition principle,

$$u(x,t)=\sum_{n=1}^{\infty} c_0e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t}\cdot c_2\sin\left(\frac{n\pi}{L}x\right)$$

$$\boxed{u(x,t)=\sum_{n=1}^{\infty} A_n e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t}\cdot\sin\left(\frac{n\pi}{L}x\right)}$$

Substitute $t=0$: $\quad u(x,0)=f(x)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{L}x\right)$, where $\boxed{A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}$

We conclude that a solution of the boundary-value problem is given by the infinite series:

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}A_n e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t}\sin\left(\frac{n\pi}{L}x\right),\ \text{where } A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}$$

SPECIAL CASE: $u(x,0)=100$, $L=\pi$, and $k=1$

$$\boxed{A_n=\frac{200}{\pi}\left[\frac{1-(-1)^n}{n}\right],\quad u(x,t)=\frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]e^{-n^2t}\sin(nx)}$$

13.4 Wave Equation

Introduction

The vertical displacement $u(x,t)$ of a string of length L that is freely vibrating in the vertical plane is determined from

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0<x<L,\ t>0 \qquad (1)$$

$$u(0,t)=0,\ u(L,t)=0, \quad t>0 \qquad (2)$$

$$u(x,0)=f(x),\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x), \quad 0<x<L \qquad (3)$$

Solution of the BVP

With the assumption $u(x,t)=X(x)T(t)$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t) \qquad \therefore\ a^2 X''(x)T(t) = X(x)T''(t) \qquad X''+\lambda X=0 \qquad (4)$$

$$\frac{\partial^2 u}{\partial t^2} = X(x)T''(t) \qquad \frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda \qquad T''+\lambda T=0 \qquad (5)$$

The boundary conditions translate into:

$$X''+\lambda X=0,\ X(0)=0,\ X(L)=0 \qquad (6)$$

Of the usual 3 possibilities for the parameter λ:
- $\lambda=0$
- $\lambda=-\alpha^2<0$
- $\lambda=\alpha^2>0$

only the last choice leads to nontrivial solutions

$\therefore$ The general solution: $X(x)=c_1\cos\alpha x + c_2 \sin\alpha x$

$X(0)=0$: $c_1=0$

$X(L)=0$: $c_2 \sin\alpha L = 0$ $\therefore\ \alpha L = n\pi$

$$\alpha = \frac{n\pi}{L}$$

$$\therefore\ X(x) = c_2 \sin\left(\frac{n\pi}{L}x\right), \text{ where } n=1,2,3,\dots$$

The general solution of the 2^{nd}-order equation (5):

$$T(t) = c_3\cos\left(\frac{n\pi a}{L}t\right) + c_4 \sin\left(\frac{n\pi a}{L}t\right)$$

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi a}{L}t\right) + B_n\sin\left(\frac{n\pi a}{L}t\right)\right]\sin\left(\frac{n\pi}{L}\right)x \qquad (8)$$

Setting $t=0$: $u(x,0)=f(x)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$

$$\therefore\ A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

To determine B_n, we differentiate (8) w.r.t t and set $t=0$:

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = \sum_{n=1}^{\infty}\left(B_n \frac{n\pi a}{L}\right)\sin\left(\frac{n\pi}{L}x\right)$$

In order for this last series to be the half-range sine expansion of the initial velocity g on the interval, the total coefficient $B_n \frac{n\pi a}{L}$ must be given by the form:

$$B_n \frac{n\pi a}{\cancel{L}} = \frac{2}{\cancel{L}} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$\boxed{B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right)} \qquad (10)$$

NOTE: When the string is released from rest, then $g(x) = 0$ for every x in the interval $[0, L]$ and consequently $B_n = 0$.