

Definitions and synopsis. Let \mathcal{H}_n be a Hilbert space of vectors ψ that is spanned by a preferred basis that is given as an orthonormal set of n -factor tensor products (in physical terms, such a basis is variously called an n -particle, n -spin, or n -qudit basis). Allow the component vectors of the n -factor product to have various dimensions (in physical terms, allow the individual particles to carry various quantum numbers). Let \mathbf{l} be any linear Hilbert operator that when given in the n -factor basis acts solely upon one particle (for example, \mathbf{l} can raise/lower single-particle quantum numbers) and associate to \mathbf{l} the (bilinear) symbol function $i_{\mathcal{H}}: \mathcal{H} \rightarrow \mathbb{C}$ that is given by $i_{\mathcal{H}}(\bar{\psi}, \psi) = \langle \bar{\psi} | \mathbf{l} | \psi \rangle$. Similarly, let \mathbf{H} be any general linear Hilbert operator (for example, \mathbf{H} can be a multi-particle Hamiltonian) and associate to \mathbf{H} the symbol function $h_{\mathcal{H}}(\bar{\psi}, \psi) = \langle \bar{\psi} | \mathbf{H} | \psi \rangle$. Let \mathcal{K}_1 be the subset of vectors in \mathcal{H}_n that are n -factor products in the n -particle basis (algebraic geometers call \mathcal{K}_1 an n -factor Segre variety). Then by definition the rank- r Kronecker join \mathcal{K}_r is the immersed join of r copies of \mathcal{K}_1 (algebraic geometers call \mathcal{K}_r an $(r-1)$ 'th secant variety). Thus from the initial (non-entangled) state-space \mathcal{K}_1 there arises a rank-indexed stratification of Kronecker joins $\mathcal{K}_1 \subseteq \mathcal{K}_2 \subseteq \dots \subseteq \mathcal{K}_r \subseteq \dots \subseteq \mathcal{K}_{\infty} = \mathcal{H}_n$ (in physical terms, the rank r is a measure of quantum entanglement). Finally, let $\omega_{\mathcal{H}}$ be \mathcal{H}_n 's canonical (Kählerian) symplectic form, let $\iota_r: \mathcal{K}_r \hookrightarrow \mathcal{H}_n$ be \mathcal{K}_r 's inclusion map, and let ι_r^* be the pullback induced by ι_r . In summary, immersed in any n -particle Hilbert space \mathcal{H}_n is a geometrically and algebraically natural stratification of Kronecker joins \mathcal{K}_r , that in physical terms is a rank-indexed classical-to-quantum succession of state-spaces $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_r, \dots$, each of which supports the natural pullback $\iota_r^*: (\omega_{\mathcal{H}}, i_{\mathcal{H}}, h_{\mathcal{H}}) \rightarrow (\omega_{\mathcal{K}}, i_{\mathcal{K}}, h_{\mathcal{K}})$ of the symplectic forms and symbol functions that specify Hamiltonian dynamical flows.

Theorem 1 (QUANTUM PULLBACK THEOREM). For all n -particle Hilbert spaces \mathcal{H}_n , at all points of all ranks of \mathcal{H}_n 's natural stratification of Kronecker joins \mathcal{K}_r , this diagram commutes:

